On the Asymptotic Behavior of Internal Layer Solutions of Advection-Diffusion-Reaction Equations

Karl R. Knaub

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Signature Karl R. Kunert

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University of Washington

Graduate School

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Karl R. Knaub

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examining committee have been made.

Chair of Supervisory Committee:

Robert E. O'Malley, Jr.

Robert E. O'Malley, Jr.

Reading Committee:

J. Kevorkian

Jirair Kevorkian

Kenneth P. Bube

Kenneth P. Bube

Date: July 10, 2001

University of Washington

Abstract

On the Asymptotic Behavior of Internal Layer Solutions of Advection-Diffusion-Reaction Equations

by Karl R. Knaub

Chair of Supervisory Committee

Professor Robert E. O'Malley, Jr.
Applied Mathematics

We study the behavior of solutions of certain parabolic partial differential equations of the form $u_t = \epsilon^2 u_{xx} + \epsilon g(u) u_x + h(u)$ in the limit $\epsilon \to 0^+$. Solutions of advection-diffusion and reaction-diffusion equations are specifically considered. These solutions possess slowly moving internal layers, the positions of which are often of physical interest. Previous studies have focused on solutions which exhibit exponential asymptotics; we broaden the class studied to include the more common algebraic asymptotics. Metastability and supersensitivity are also considered in both cases.

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For the many important friends I've omitted, the check is in the mail.

DEDICATION

For Mom and Dad, who provided opportunities - and role models

and

For Jennifer, who seems to recognize truth when she sees it, and who has the most beautiful soul

and

For Hans, who kept me company while I learned calculus.

Much Love to all of you.

Quoi que vous fassiez, écrasez l'infâme, et aimez qui vous aime.
- Voltaire, letter to M. d'Alembert, November 28, 1762

Chapter 1

INTRODUCTION

This dissertation concerns certain classes of singularly perturbed parabolic-type partial differential equations on finite spatial domains, as well as the related steady-state problems obtained when the time t becomes unbounded; in particular, we study initial-boundary value problems (in the case of PDEs) and boundary value problems (in the case of the associated ODEs). Most generally, these are problems of the form

$$u_t = \epsilon^2 \nabla^2 u + f(t, \vec{x}, u, \epsilon \nabla u), \ \vec{x} \in \Omega, \ t > 0, \ 0 < \epsilon \ll 1 \ , \tag{1.1}$$

for an unknown function $u(\vec{x}, t)$ on a (possibly multidimensional) bounded domain Ω, together with some reasonable boundary and initial conditions. We have assumed that the diffusion-like term is density-independent, as well as explicitly independent of space and time, and that the diffusion is small relative to the other terms in the equation. We also assume that f depends weakly on the gradient ∇u. Here, we shall restrict attention to problems involving one spatial dimension, for which the function f does not depend explicitly on t or on $\vec{x}$, and which are *quasilinear*:

$$u_t = \epsilon^2 u_{xx} + \epsilon g(u) u_x + h(u), \ 0 < x < 1, \ t > 0, \ 0 < \epsilon \ll 1. \tag{1.2}$$

We take g and h to be of class C^1. We shall also concentrate on problems with Dirichlet boundary conditions:

$$u(0,t) = u_L \quad \text{and} \quad u(1,t) = u_R \quad \text{with} \quad u_L < u_R. \tag{1.3}$$

Analogous problems with Neumann or Robin boundary conditions can be similarly approached. Such equations have been studied in an amazing variety of contexts, where the small positive parameter ϵ corresponds (after nondimensionalization) to a diffusion coefficient or viscosity. Texts such as [19], [25], [47], and [60] demonstrate just a few of the many applications in which these equations (or systems of such equations) arise, such as population biology, geochemistry, epidemiology, and pattern formation. Here, we shall concentrate most heavily on equations of advection-diffusion (or viscous shock) form

$$u_t = \epsilon u_{xx} + g(u)u_x \tag{1.4}$$

(for an ϵ-rescaled time t) and on those of reaction-diffusion form

$$u_t = \epsilon^2 u_{xx} + h(u), \tag{1.5}$$

though we shall present new results for the combined advection-reaction-diffusion equation (1.2).

These problems can often be approached through the usual techniques of singular perturbation theory: "inner" and "outer" solutions are found in various sub-domains of Ω by determining which terms of the equation are most important on each sub-domain and then rescaling variables locally as necessary. Matching between bordering sub-domains generally removes any indeterminacies that arise during the procedure. However, certain subclasses of equation (1.2) cannot be completely solved by these techniques; typically, such a failure manifests itself as a constant or a function which is left undetermined after the matching process. This phenomenon was perhaps first observed in an ODE example (which is a linear steady-state version of (1.2)) in the well-known context of Ackerberg-O'Malley resonance [2]. Much has been written about similar nonlinear PDE problems, see for example [15], [20], [30], [36], [50], [53], [54], and [62]. However, one should not make the mistake of thinking the steady-state version is easy to analyze - see [6], [7], [8], [9], [14], [28], [41], [43], or [49].

The problems we study share some typical features. First, referring back to equation (1.1), they are often singular in the sense that the function $f(t, x, u, \epsilon u_x)$ lowers its order or vanishes within the domain Ω. For example, in [2], Ackerberg and O'Malley study the linear ODE

$$\begin{cases} \epsilon y'' + g(x;\epsilon)y' + h(x;\epsilon)y = 0, \ -a < x < b \\ y(-a) \text{ and } y(b) \text{ given; } a, b > 0; \ 0 < \epsilon \ll 1 . \end{cases} \tag{1.6}$$

They require that $g(x;\epsilon)$ has a single simple zero on $[-a, b]$, say at x_0. x_0 is referred to as a *turning point* of the equation, and analogous turning points play a vital role in the behavior of the problems we study. Note that Matkowsky [46] discusses an example where $g(x;\epsilon)$ has higher-order zeros, a feature that shall prove important to us later.

These problems can be singular in more subtle ways. In [36], Laforgue and O'Malley discuss the nonlinear Burgers' equation

$$\begin{cases} u_t = \epsilon u_{xx} + 2uu_x, \ -1 < x < 1, \ t > 0, \ 0 < \epsilon \ll 1 \\ u(\pm 1, t) = \pm 1, \ u(x, 0) \text{ given.} \end{cases} \tag{1.7}$$

The boundary conditions show that any continuous solution $u(x,t)$ will pass through zero at some interior point x_ϵ, which can move with time. This location $x_\epsilon(t)$, which functions as a moving "turning point" for the equation, turns out to be the location of a slowly moving internal shock or transition layer within which the solution jumps quite abruptly from nearly -1 to nearly $+1$. Note that, for most other boundary values, an endpoint layer occurs; thus the given boundary values are specially "balanced" in (1.7) to provide an internal transition layer for the limiting solution.

A second feature that our problems share is the presence of large-scale structures in the solutions, such as internal layers or spikes. (These phenomena are sometimes referred to as *contrast structures*; see, for example, [56] or [59].) Both (1.6) and (1.7) have internal layers, and the following famous stationary problem due to Carrier and Pearson [13] (see also [11] and [40]) involves asymptotic solutions with regularlyspaced spikes:

$$\begin{cases} \epsilon^2 u_{xx} + u^2 = 1, \ -1 < x < 1, \ 0 < \epsilon \ll 1 \\ u(\pm 1) = 0 \,. \end{cases}$$

The difficulty in using matched asymptotic expansions for such problems is that the method usually leaves the location of these persistent structures, which also tend to be the most physically interesting aspects of the solutions, undetermined. In this case, Carrier and Pearson point out the additional danger of obtaining *spurious asymptotics*, false "solutions" constructed by the application of matched asymptotic expansions in the usually dependable manner, suggesting large-scale structures that don't actually occur. Carrier [11] and MacGillivray et al. [44] go on to consider related, but more challenging, problems for nonautonomous equations. We restrict ourselves to problems whose solutions display a single internal layer. (See, however, [15], [23], [26], and [61] for more complicated possibilities.)

Many of the problems (1.2) that have recently received attention fall under the rubric of *exponential asymptotics*. This term is introduced here to emphasize that simple Poincaré power series expansions of the solution (with respect to the small parameter ϵ) may miss important characteristics of the solution. Thus, a simple application of matched asymptotic expansions fails for these problems. (When a power series expansion is sufficient, we say the problem exhibits *algebraic asymptotics*). The implication is that exponentially small terms must be included for the asymptotic formalism to succeed. Exponential asymptotics can also refer to a certain related sensitivity of the problems to perturbations; this implies that the numerical solution of these problems will consequently be undependable at best. For example, $O(1)$ changes in the solution can sometimes be caused by asymptotically exponentially small changes in the coefficients of the governing differential equation or in the prescribed boundary values! (In other cases, we shall show that a corresponding algebraic $O(\epsilon^n)$ perturbation creates an analogous, noticeable effect on the limiting solution.) This phenomenon is sometimes referred to as *supersensitivity*. Yet another use of the term exponential asymptotics is to refer to certain singular perturbation problems in which, when the solution is expanded in terms of the eigenfunctions of the corresponding linearized problem, the principal eigenvalue is both negative and asymptotically exponentially small as $\epsilon \to 0^+$. In time-dependent problems this can result in extremely slow, indeed exponentially slow, convergence to the steady-state solution, a phenomenon known as *dynamic metastability*. (We shall see that an analogous algebraic metastability also exists.) One must expect to encounter extreme difficulty when attempting to numerically solve problems exhibiting such supersensitivity or severe ill-conditioning. Thus, asymptotic analysis becomes of great practical relevance, both as an analytic alternative to less dependable numerical approximations and as a guide for developing more reliable special-purpose computational algorithms.

This thesis studies (1.2), (1.4), and (1.5) under hypotheses that guarantee having a single dynamic internal layer in the asymptotic solution, or at least having such a layer evolve from the prescribed initial data after an $O(1)$ time. One of the assumptions will be that the boundary values (1.3) are carefully "balanced", as they were in (1.7), so that we need not bother with endpoint layers in addition to the internal layer. We will also discuss the corresponding steady-state problem, considering the resulting ODE as the attractive limit for the PDE solution as $t \to \infty$. The location of the internal layer is left undetermined by the normal method of matched asymptotic expansions, and two methods have been developed in the literature to determine the motion of this layer: the *projection method* of Reyna and Ward [52] and the *travelling wave Ansatz* (our terminology) of Laforgue and O'Malley [33]-[38]. While the projection method seems more elegant and possesses some advantages over the travelling wave Ansatz, we have found classes of problems for which the former method may be ineffectual.

The overriding goals of this dissertation are the following:

- To provide, using the travelling wave Ansatz, limiting solutions for (1.2)-(1.3) in the case of *both* algebraic and exponential asymptotics. In particular, we seek the position of the internal layer as a function of time, as well as the eventual steady-state location of the stable shock.

- To demonstrate the new phenomena of *algebraic supersensitivity* and *algebraic metastability.*

- To demonstrate and explain the limitations of the projection method in the case of algebraic asymptotics.

- To present a new method for the asymptotic evaluation of a certain class of integrals arising in this work.

Chapter 2

THE PROJECTION METHOD, THE TRAVELLING WAVE ANSATZ, AND SOME LINEAR ALGEBRA

We now describe the projection method and the travelling wave Ansatz in some generality. Application of the travelling wave Ansatz to specific classes of differential equations will occur in subsequent chapters.

2.1 Analogy to Linear Algebra

In [62] Ward uses a truly ingenious linear algebra metaphor to describe the projection method. We repeat Ward's analogy now and will return to it to make our points clearer as we progress.

Let A_ϵ be an $n \times n$ symmetric matrix depending upon a small parameter $\epsilon > 0$. We consider the linear system

$$A_\epsilon \mathbf{x}_\epsilon = \mathbf{b}_\epsilon. \tag{2.1}$$

Here $\mathbf{b}_\epsilon \in \mathbb{R}^n$ also depends on ϵ; thus the unknown $\mathbf{x}_\epsilon \in \mathbb{R}^n$ does too. Assume that $\mathbf{b}_\epsilon$ additionally depends on a single unknown parameter q in a known way. (The method can be generalized to any $1 < m < n$, rather than one, unknown parameters q_j, $j = 1, \ldots, m$.) It is unusual in linear algebra to let the typically fully-known vector $\mathbf{b}_\epsilon$ depend upon another unknown; so we can think of q as a solvability parameter that allows (2.1) to be solved when A_ϵ is singular, or more importantly, nearly singular. Thus, if our system arises from a physical problem, we will select q to correspond to a physically valid solution.

Let Φ_j and λ_j be the corresponding orthonormal eigenvectors and the eigenvalues of A_ϵ. By completeness, we can uniquely write the solution vector $\mathbf{x}_\epsilon$ as

$$\mathbf{x}_\epsilon = \sum_{j=0}^{n-1} \frac{c_j}{\lambda_j}\Phi_j \quad \text{with} \quad c_j = \Phi_j^T \mathbf{b}_\epsilon. \tag{2.2}$$

The question is how and under what circumstances we can also determine the hidden q. If we assume that the eigenvalue $\lambda_0 = 0$ for all values of ϵ and that the rest of the eigenvalues are non-zero, then (2.1) has a solution if and only if $c_0 = \Phi_0^T \mathbf{b}_\epsilon = 0$. This is an algebraic equation through which we might determine the unknown parameter q.

Ward supposes, instead, that $\lambda_0 = O(e^{-k/\epsilon})$ as $\epsilon \to 0^+$ for some constant $k > 0$ independent of ϵ. Then, parallel to the already expressed solvability condition $\Phi_0^T \mathbf{b}_\epsilon = 0$ for $\lambda_0 = 0$, we now have the limiting solvability condition $\Phi_0^T \mathbf{b}_\epsilon \to 0$ as $\epsilon \to 0^+$. This then provides an asymptotic equation from which one again seeks to extract q. The term *projection method* comes from this idea of eliminating the projection of $\mathbf{b}_\epsilon$ onto the eigenvector associated with the asymptotically exponentially small eigenvalue of A_ϵ.

As an analogy to the PDE work to come, we'd like to raise an important point. The above argument does not require the eigenvalue λ_0 to decay to zero *exponentially* quickly as $\epsilon \to 0^+$. In fact, the same conclusion results if λ_0 decays merely *algebraically* quickly. This is the crucial feature of problems we shall be discussing later. Though the projection method still works in theory, we will then encounter algebraic difficulties which seem to preclude its practical application.

We now discuss the linear algebra analogy to the *travelling wave Ansatz* of Laforgue and O'Malley [36], extending Ward's metaphor. We reconsider (2.1), seeking the unknown parameter q. We continue to suppose that A_ϵ has a single eigenvalue λ_0 such that $\lambda_0 \to 0$ as $\epsilon \to 0^+$. The size of this eigenvalue could be either algebraically or exponentially small in the limit. In either case the method requires that we expand the coefficient matrix A_ϵ, the vector $\mathbf{b}_\epsilon$, and the solution vector $\mathbf{x}_\epsilon$ in compatible asymptotic series, the eponymous Ansatz:

$$A_\epsilon \sim A_0 + A_1 + A_2 + \cdots$$
$$\mathbf{b}_\epsilon \sim \mathbf{b_0} + \mathbf{b_1} + \mathbf{b_2} + \cdots$$
$$\mathbf{x}_\epsilon \sim \mathbf{x_0} + \mathbf{x_1} + \mathbf{x_2} + \cdots$$

(cf. Olver [48]). We assume that q is explicitly related to $\mathbf{b_0}$. Then, to the highest order, we must solve the system

$$A_0\mathbf{x_0} = \mathbf{b_0}.$$

Our hypothesis on the eigenvalue λ_0 ensures that A_0 is singular, and we then determine q so that this system has solutions. This is similar to the solvability condition invoked above.

Why should we expect A_0 to be singular? Let's presume A_ϵ has the spectral decomposition $A_\epsilon = S\Lambda S^{-1}$, where Λ is a diagonal matrix containing its eigenvalues and the columns of S are the associated eigenvectors. In general, both Λ and S will depend upon ϵ. We can then recast (2.1) as

$$\Lambda\mathbf{y}_\epsilon = S^{-1}\mathbf{b}_\epsilon,$$

where $\mathbf{y}_\epsilon \equiv S^{-1}\mathbf{x}_\epsilon$. By our hypothesis on λ_0, this system is singular in the limit $\epsilon \to 0^+$, so A_0 must be singular. We stress however that one does not actually use such a spectral decomposition of A_ϵ in the actual implementation of the method.

We comment on the relative merits of the two methods. First, the projection method depends on our ability to find the eigenvalues and eigenvectors of A_ϵ, or at least to asymptotically approximate some of these quantities. This will turn out to be the method's "Achilles' heel" for the class of problems in which we're interested. The travelling wave Ansatz will not require explicit knowledge of the eigenpairs. Second, the travelling wave Ansatz requires an asymptotic series expansion of the matrix and vectors under consideration. Typically, it requires some educated guess to provide such an Ansatz. This may be fairly trivial for a linear system, but can be quite difficult for the related PDE problems that we really wish to solve. The projection method avoids this difficulty, and thus it seems more elegant in application, when it succeeds.

We include somewhat trivial applications of each method to make the preceding discussion more explicit. To illustrate the projection method, consider the system

$$A_\epsilon \mathbf{x}_\epsilon = \begin{bmatrix} \epsilon - 2 & 2 \\ -4 & \epsilon + 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+q \\ \epsilon \end{bmatrix} = \mathbf{b}_\epsilon. \tag{2.3}$$

Here, A_ϵ has an eigenvalue ϵ with a corresponding eigenvector $\Phi_0 = [1 \quad 1]^T$. The projection method then requires that the inner product

$$\Phi_0^T \mathbf{b}_\epsilon = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1+q \\ \epsilon \end{bmatrix} = 1 + q + \epsilon \to 0$$

as $\epsilon \to 0^+$. Thus, it specifies that $q = -1$. Despite the fact that the small eigenvalue vanished only algebraically quickly in the limit, the projection method worked well.

We next apply the travelling wave Ansatz to (2.3). It is easy to see that the matrix A_ϵ decomposes as the sum

$$A_\epsilon = \begin{bmatrix} \epsilon - 2 & 2 \\ -4 & \epsilon + 4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & 4 \end{bmatrix} + \epsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_0 + \epsilon A_1.$$

Likewise expanding the right-hand side, we obtain the limiting equation

$$A_0 \mathbf{x}_0 = \begin{bmatrix} -2 & 2 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 1+q \\ 0 \end{bmatrix} = \mathbf{b}_0.$$

Clearly, this system will only have solutions if $q = -1$.

2.2 The Shock Profile

Consider the PDE (1.2) under an assumption we shall commonly take:

$$h(u_L) = h(u_R) = 0.$$

A naive application of standard asymptotic methods finds that, for long times, the limit of the solution $u(x,t)$ can be approximated by the constant function $u = u_L$ in some vicinity of the left boundary and by the constant function $u = u_R$ in some vicinity of the right boundary, since these constants satisfy the reduced version of (1.2) as well as the appropriate boundary condition. We shall seek an inner solution to connect the two constant outer solutions at some location, corresponding to either a boundary layer or an internal layer. (We assume in this work that we require only one such internal layer.) We denote the location of the "jump" by $x_\epsilon(t)$, noting that this location can move with time, and specifically define

$$u(x_\epsilon(t),t) \equiv \frac{u_L + u_R}{2}. \tag{2.4}$$

Using a stretched local variable

$$\bar{z} \equiv \frac{x - x_\epsilon(t)}{\epsilon}, \tag{2.5}$$

we obtain a nonlinear equation that contains, at least, all the terms on the right-hand side of (1.2), appropriately rescaled. We call the solution of this problem $u_s(\bar{z})$ where s denotes "stretched". See Figure 2.1. We can then write a uniform asymptotic expansion (cf. Van Dyke [55]) for $u(x,t)$ as

$$\begin{aligned} u(x,t) \;&\sim\; u_L + u_s(\bar{z}) + u_R - (\text{matching term between } u_L \text{ and } u_s(\bar{z})) \\ &\qquad - (\text{matching term between } u_R \text{ and } u_s(\bar{z})) \\ &\sim\; u_s(\bar{z}). \end{aligned} \tag{2.6}$$

So, the solution is approximated, to first-order, simply by the inner solution $u_s(\bar{z})$, which we have yet to determine; the shock location $x_\epsilon(t)$ also remains unknown. It turns out that, for the problems we consider, $u_s(\bar{z})$ is given by the *shock profile function* associated with (1.2) to be defined by (2.7).

Both the projection method and the travelling wave Ansatz rely on the existence of a shock profile function; some conditions on the functions $g(u)$ and $h(u)$ in (1.2) that guarantee the existence of such a shock profile function will be outlined later. In general, the monotonic shock profile function $u_p(z)$ will satisfy the stretched boundary value problem

$$\begin{cases} u_p'' + g(u_p)u_p' + h(u_p) = 0 \\ u_p(z) \to u_L \text{ as } z \to -\infty,\ u_p(0) = \frac{u_L+u_R}{2}, \text{ and } u_p(z) \to u_R \text{ as } z \to \infty. \end{cases} \tag{2.7}$$

The condition at $z = 0$ ensures that we are defining a unique heteroclinic orbit. Figure 2.2 illustrates a typical shock profile function. The ODE in (2.7) is equivalent to a first-order Abel's equation of the first kind; we have more to say about this in Chapter 6.

We are interested in solutions of (1.2)-(1.3) which exhibit a single internal layer in the limit. Recalling (2.6), both the projection method and the travelling wave Ansatz naturally seek such solutions in the additive form

$$u(x,t) = u_p(z) + \delta(\epsilon)v(z,\sigma). \tag{2.8}$$

Here z is the ϵ-stretched variable

$$z \equiv \frac{x - x_\epsilon(\sigma)}{\epsilon} \tag{2.9}$$

and σ is the slow time

$$\sigma \equiv \phi(\epsilon)t. \tag{2.10}$$

The unspecified gauge functions $\delta(\epsilon) = o(1)$ and $\phi(\epsilon) = o(1)$ will be chosen such that the motion of the internal layer is $O_s(1)$ on the σ-time scale, at least when the shock is near its steady-state location. (We shall say $f = O_s(g)$ holds if both $f = O(g)$ and $g = O(f)$ in the limit under consideration.)

Authors emphasizing geometrical singular perturbation theory have previously proposed the Ansatz (2.8)-(2.10) by considering the one-dimensional manifold of functions $\mathcal{M}$ given by $u_p(z)$ and parameterized by $x_\epsilon(\sigma)$. One can then give the solution

$u(x,t)$ in terms of the coordinates $(x_\epsilon(\sigma), \delta(\epsilon)v(z,\sigma))$, where we think of $\delta(\epsilon)v(z,\sigma)$ as being the transversal direction to the 1-manifold $\mathcal{M}$. One must then show that the solution remains near $\mathcal{M}$ and that the PDE uniquely determines (for each point in time) an element of $\mathcal{M}$. See, for example, [17], [18], and [32].

We, however, shall use more formal methods. If $u_p(z)$ is defined by (2.7) and we place mild hypotheses on the initial function $u(x,0)$, then $u_p(z)$ closely describes the ultimate limiting solution, for which we shall have

$$\delta(\epsilon)v(z,\sigma) \ll u_p(z).$$

Thus, the well-chosen Ansatz (2.8)-(2.10) is of considerable value, since it yields an excellent first-approximation $u_p(z)$ to the solution with its small error principally occurring at the spatial boundaries where the stretched variable $|z|$ becomes unbounded. We shall see that, although these errors are small, they are significant: though $u_p(z)$ approaches its end-values quickly as $|z| \to \infty$, the asymptotic tail behavior of $u_p(z)$ will turn out to be crucial in determining the observed shock motion.

We shall be primarily concerned with the asymptotic description of the solution $u(x,t)$ after the $O(1)$ time t_i in which the shock layer becomes well-developed, determining an initial value x_i for the location of the shock layer where $u(x_i,t_i) = \frac{1}{2}(u_L + u_R)$. To study the solution prior to this reset time t_i in order to obtain an initial location for the limiting shock, one must solve the PDE numerically or follow, for example, the work of Il'in [29], who uses asymptotic matching, based upon the method of characteristics, to construct an asymptotic solution for $0 \le t \le t_i$.

The non-trivial difficulty associated with utilizing (2.8)-(2.10) is determining the function $x_\epsilon(\sigma)$, which specifies the limiting shock location, but which cannot be discovered through the usual processes of asymptotic matching. (This is analogous to finding the undetermined parameter q in Section 2.1.) The projection method and the travelling wave Ansatz determine the shock location $x_\epsilon(\sigma)$ in significantly different fashions. However, both rely on the assumption that, for $t > t_i$, $\delta(\epsilon)v(z,\sigma) \ll u_p(z)$, since this allows us to then safely linearize the solution u about the profile function $u_p(z)$. Essentially, the Ansatz (2.8) guarantees that the problem for $v(z,\sigma)$ is *nearly linear*. To see this, insert (2.8) into (1.2) to obtain

$$\begin{aligned}-\frac{1}{\epsilon}\phi(\epsilon)\frac{dx_\epsilon}{d\sigma}u_p' + \delta(\epsilon)\phi(\epsilon)\left[-\frac{1}{\epsilon}\frac{dx_\epsilon}{d\sigma}v_z + v_\sigma\right] \\ = \left(u_p'' + \delta(\epsilon)v_{zz}\right) + g(u_p+\delta(\epsilon)v)\left(u_p' + \delta(\epsilon)v_z\right) + h(u_p+\delta(\epsilon)v).\end{aligned} \tag{2.11}$$

This exact equation is the starting point in both the projection method and the travelling wave Ansatz for obtaining approximations to the shock location $x_\epsilon(\sigma)$. Assume that the bracketed terms on the left-hand side of (2.11) are sufficiently asymptotically small that the left-hand side is well-ordered as an asymptotic series with respect to $\delta(\epsilon)$. Expanding g and h in Taylor series about $u_p(z)$, utilizing (2.7), and linearizing, we obtain

$$\begin{aligned}-\frac{1}{\epsilon}\frac{\phi(\epsilon)}{\delta(\epsilon)}\frac{dx_\epsilon}{d\sigma}u_p' + O\left(\frac{\phi(\epsilon)}{\epsilon}\right) &= v_{zz} + vu_p'g'(u_p) + v_z g(u_p) + vh'(u_p) + O(\delta(\epsilon)) \\ &= v_{zz} + [vg(u_p)]_z + vh'(u_p) + O(\delta(\epsilon))\end{aligned} \tag{2.12}$$

when we expose the first two terms of the series. Better approximations to (2.11) can certainly be obtained by using more terms in the Taylor series. Thus we obtain the important approximate equation

$$\mathcal{L}[v] \equiv v_{zz} + [vg(u_p)]_z + vh'(u_p) \sim -\frac{1}{\epsilon}\frac{\phi(\epsilon)}{\delta(\epsilon)}\frac{dx_\epsilon}{d\sigma}u_p' \tag{2.13}$$

for $v(z,\sigma)$, which we regard as a nonhomogeneous *linear* ODE in z in which σ simply appears as a parameter. We shall now require that

$$\frac{1}{\epsilon}\frac{\phi(\epsilon)}{\delta(\epsilon)}\frac{dx_\epsilon}{d\sigma} = O_s(1), \tag{2.14}$$

at least as the shock approaches its steady-state location. Indeed, we shall use this condition to select $\delta(\epsilon)$, once we have determined the appropriate time-scale $\phi(\epsilon)$ that we assume satisfies the implicit assumption that $\phi(\epsilon) = o(\epsilon)$. We also obtain corresponding asymptotic boundary conditions for (2.13):

$$\begin{cases} v\left(-\frac{x_\epsilon(\sigma)}{\epsilon},\sigma\right) = \frac{1}{\delta(\epsilon)}\left[u_L - u_p\left(-\frac{x_\epsilon(\sigma)}{\epsilon}\right)\right] \\ v\left(\frac{1-x_\epsilon(\sigma)}{\epsilon},\sigma\right) = \frac{1}{\delta(\epsilon)}\left[u_R - u_p\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}\right)\right]. \end{cases} \tag{2.15}$$

Finally, note that (2.7) implies that the differential operator $\mathcal{L}$ of (2.13) has u_p' as a null eigenfunction, i.e.

$$\mathcal{L}[u_p'] \equiv 0. \tag{2.16}$$

2.3 *The Projection Method, Dynamic Metastability, and Supersensitivity*

We now explain the projection method, as applied to differential equations, using our linear algebra metaphor. Though the PDEs to which the projection method can be applied are often nonlinear, the analogy remains appropriate due to the linearity of the approximation (2.13).

Both the projection method and the travelling wave Ansatz view the linear asymptotic equation (2.13) as an infinite-dimensional analog to the linear algebraic system (2.1), where the derivative of the unknown shock location $\frac{dx_\epsilon}{d\sigma}$ plays the role of the unknown $\mathbf{q}$. We shall again use a Fredholm alternative, or solvability condition, to obtain an equation involving the nonhomogeneity in (2.13) and to thereby determine $\frac{dx_\epsilon}{d\sigma}$. As a preliminary step, we recast equation (2.13) in self-adjoint form using the Liouville transformation

$$v(z,\sigma) \equiv \exp\left(-\frac{1}{2}\int_0^z g(u_p(s))ds\right) w(z,\sigma)$$

to obtain the limiting differential equation

$$\begin{aligned} \mathcal{S}[w] \equiv w_{zz} + \left[\frac{1}{2}g'(u_p(z))u_p'(z) - \frac{1}{4}\left(g(u_p(z))\right)^2 + h'(u_p(z))\right] w \\ \sim -\frac{1}{\epsilon}\frac{\phi(\epsilon)}{\delta(\epsilon)}\frac{dx_\epsilon}{d\sigma}u_p'(z)\exp\left(\frac{1}{2}\int_0^z g(u_p(s))ds\right) \end{aligned} \tag{2.17}$$

for w with corresponding transformed boundary conditions

$$\begin{cases} w\left(-\frac{x_\epsilon(\sigma)}{\epsilon},\sigma\right) = \frac{\exp\left(\frac{1}{2}\int_0^{-x_\epsilon(\sigma)/\epsilon} g(u_p(s))ds\right)}{\delta(\epsilon)}\left[u_L - u_p\left(-\frac{x_\epsilon(\sigma)}{\epsilon}\right)\right] \\ \\ w\left(\frac{1-x_\epsilon(\sigma)}{\epsilon},\sigma\right) = \frac{\exp\left(\frac{1}{2}\int_0^{(1-x_\epsilon(\sigma))/\epsilon} g(u_p(s))ds\right)}{\delta(\epsilon)}\left[u_R - u_p\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}\right)\right]. \end{cases} \tag{2.18}$$

Moreover, from (2.16) it follows that

$$\mathcal{S}\left[\exp\left(\frac{1}{2}\int_0^z g(u_p(s))ds\right)u_p'(z)\right] \equiv 0. \tag{2.19}$$

We now consider the stretched eigenvalue problem,

$$\mathcal{S}[\Phi(z)] = \lambda\Phi(z),\quad \Phi\left(-\frac{x_\epsilon(\sigma)}{\epsilon}\right) = \Phi\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}\right) = 0, \tag{2.20}$$

where we again view σ as a parameter.

Note that (2.20) is a regular Sturm-Liouville eigenvalue problem. The standard theory (cf. Hartman [27]) implies that (2.20) has a denumerable set of eigenpairs $(\lambda_j, \Phi_j(z))$ for $j = 0, 1, 2, \ldots$ such that the eigenfunctions form a complete system and are orthonormal under the inner product

$$(f, g) \equiv \int_{-\frac{x_\epsilon(\sigma)}{\epsilon}}^{\frac{1-x_\epsilon(\sigma)}{\epsilon}} f(s)g(s)ds. \tag{2.21}$$

Further, the real eigenvalues λ_j are bounded above and $\lambda_j \to -\infty$ as $j \to \infty$.

Just as in (2.2) we write the solution to the approximate equation (2.17) as the eigenfunction expansion

$$w(z, \sigma) = \sum_{j=0}^{\infty} \frac{c_j(\sigma)}{\lambda_j}\Phi_j(z). \tag{2.22}$$

In order to use the projection method we need (2.20) to have an eigenpair $(\Phi_\epsilon(z), \lambda_\epsilon)$ such that $\lambda_\epsilon \to 0$ as $\epsilon \to 0^+$. Assume that $u_p(z)$ approaches its end-values sufficiently quickly as $|z| \to \infty$; that is, g and u_p satisfy

$$\exp\left(\frac{1}{2}\int_0^z g(u_p(s))ds\right)u_p'(z) \to 0^+ \quad \text{as } |z| \to \infty. \tag{2.23}$$

Then, fortunately, (2.19) suggests that $\exp\left(\frac{1}{2}\int_0^z g(u_p(s))ds\right)u_p'(z)$ is a good estimate for $\Phi_\epsilon(z)$, up to a multiplicative normalization constant κ_0. Further, the monotonicity of $u_p(z)$ implies that $u_p'(z)$ is of one sign, so $\exp\left(\frac{1}{2}\int_0^z g(u_p(s))ds\right)u_p'(z)$ approximates the principal eigenfunction $\Phi_0(z)$. Typically one constructs the asymptotic approximation

$$\Phi_0(z) \sim \kappa_0\left(\exp\left(\frac{1}{2}\int_0^z g(u_p(s))ds\right)u_p'(z) + w_{lbl}(z,\sigma) + w_{rbl}(z,\sigma)\right), \tag{2.24}$$

where the added terms $w_{lbl}(z,\sigma)$ and $w_{rbl}(z,\sigma)$ are boundary layer corrections that force $\Phi_0(z)$ to vanish asymptotically at the prescribed endpoints.

Returning to (2.22) and using orthonormality, we obtain $(w, \Phi_0) = \frac{c_0(\sigma)}{\lambda_0}$. So,

$$\begin{aligned}
c_0(\sigma) &= (w, \lambda_0\Phi_0) \\
&= (w, S[\Phi_0]) \\
&= w(z,\sigma)\frac{d\Phi_0}{dz}\bigg|_{-\frac{x_\epsilon(\sigma)}{\epsilon}}^{\frac{1-x_\epsilon(\sigma)}{\epsilon}} + (\Phi_0, S[w]) \\
&= w(z,\sigma)\frac{d\Phi_0}{dz}\bigg|_{-\frac{x_\epsilon(\sigma)}{\epsilon}}^{\frac{1-x_\epsilon(\sigma)}{\epsilon}} \\
&\quad + \int_{-\frac{x_\epsilon(\sigma)}{\epsilon}}^{\frac{1-x_\epsilon(\sigma)}{\epsilon}} \Phi_0(z)\left(-\frac{1}{\epsilon}\frac{\phi(\epsilon)}{\delta(\epsilon)}\frac{dx_\epsilon}{d\sigma}u_p'(z)\exp\left(\frac{1}{2}\int_0^z g(u_p(s))ds\right)\right)dz.
\end{aligned} \tag{2.25}$$

Since $\Phi_\epsilon(z)$ is, by assumption, nearly a null eigenfunction, we expect that $\lambda_0 \to 0$ as $\epsilon \to 0^+$, as can be confirmed by using the Rayleigh quotient. Finally taking the projection step, the Fredholm alternative implies that $c_0(\sigma) \to 0$ as $\epsilon \to 0^+$, giving the limiting ODE

$$\frac{dx_\epsilon}{d\sigma} \sim \frac{\frac{\epsilon\delta(\epsilon)}{\phi(\epsilon)}\left[w\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}\right)\frac{d\Phi_0}{dz}\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}\right) - w\left(-\frac{x_\epsilon(\sigma)}{\epsilon}\right)\frac{d\Phi_0}{dz}\left(-\frac{x_\epsilon(\sigma)}{\epsilon}\right)\right]}{\int_{-\frac{x_\epsilon(\sigma)}{\epsilon}}^{\frac{1-x_\epsilon(\sigma)}{\epsilon}} \Phi_0(z)u_p'(z)\exp\left(\frac{1}{2}\int_0^z g(u_p(s))ds\right)dz} \tag{2.26}$$

for determining the shock motion $x_\epsilon(\sigma)$. We can now select $\phi(\epsilon)$ such that $\frac{dx_\epsilon}{d\sigma}$ is $O_s(1)$ (near steady-state) and then use (2.14) to determine $\delta(\epsilon)$. The remaining ODE (2.26) should be integrated using the initial condition $x_i = x_\epsilon(t_i)$ to find $x_\epsilon(\sigma)$ for all later times. The values of w in the above expression are provided by (2.18), so $x_\epsilon(\sigma)$ is almost entirely determined by the tail behavior of $u_p(z)$.

Since $\lambda_0 = o(1)$ is the principal eigenvalue, these problems exhibit *dynamic metastability*, meaning that, after an initial time interval, the solution evolves on an asymptotically long time scale before reaching a stable steady state (see, for example, [50] or [52]). The term has heretofore been reserved for evolution on an exponentially asymptotically long time scale, though we shall distinguish between the possibilities for exponential and algebraic metastability.

The fact that the linearized problem (2.17) has an eigenvalue close to zero implies that the given problem is severely ill-conditioned. Thus, we expect that asymptotically small changes in the coefficients of (1.2) or in the boundary values may cause an $O_s(1)$ change in the solution. This is termed *supersensitivity*, and we shall again distinguish between exponential and algebraic supersensitivity depending upon how large additive perturbations need to be to create an $O_s(1)$ change in the solution.

2.4 *The Travelling Wave Ansatz*

We now describe the travelling wave Ansatz, the approach we will be primarily concentrating upon. We return to the linearized problem (2.13)-(2.15). The method hinges upon having a third, interior condition for $v(z, \sigma)$. This condition arises from considering the representation (2.8) at $x = x_\epsilon$. With $u_p(z)$ defined by (2.7), we obtain the important identity

$$v(z = 0, \sigma) \equiv 0. \tag{2.27}$$

Combined with the boundary conditions in (2.15), we have three auxiliary conditions to impose on the second-order ODE (2.13). But for the unknown $\frac{dx_\epsilon}{d\sigma}$, we would generally have an over-determined system; instead we have precisely the correct amount of information to determine the limiting shock position.

We utilize (2.16) since it provides a solution to the homogeneous problem $\mathcal{L}[v_H] = 0$. Using reduction of order, we seek a second, linearly independent solution to $\mathcal{L}[v_H] = 0$ in the form $v_H(z) = \eta(z)u_p'(z)$. By substituting this product into $\mathcal{L}[v_H] = 0$, we obtain a linear equation that is first-order in η':

$$\eta'' u_p' + \eta'(2u_p'' + g(u_p)u_p') = 0.$$

Deviating from tradition, we multiply the equation by u_p' to obtain a first-order equation in $\eta'(z)(u_p'(z))^2$:

$$[\eta'(u_p')^2]' + g(u_p)[\eta'(u_p')^2] = 0.$$

It is now a simple matter to find a second homogeneous solution

$$v_H(z) = u_p'(z)\int_0^z \frac{ds}{\exp(\int_0^s g(u_p(r))dr)[u_p'(s)]^2}. \tag{2.28}$$

We then use variation of parameters to seek a particular solution $v_P(z,\sigma)$ to (2.13) in the form

$$v_P(z,\sigma) = u_p'(z)\left[F(z,\sigma) + G(z,\sigma)\int_0^z \frac{ds}{\exp(\int_0^s g(u_p(r))dr)[u_p'(s)]^2}\right].$$

Substituting this form into (2.13), we easily obtain asymptotic relations for F and G:

$$\begin{cases} F(z,\sigma) \sim \frac{1}{\epsilon}\frac{\phi(\epsilon)}{\delta(\epsilon)}\frac{dx_\epsilon}{d\sigma}\int_0^z\int_0^s \frac{[u_p'(s)]^2\exp(\int_0^s g(u_p(r))dr)}{[u_p'(q)]^2\exp(\int_0^q g(u_p(r))dr)}dq\,ds \\ G(z,\sigma) \sim -\frac{1}{\epsilon}\frac{\phi(\epsilon)}{\delta(\epsilon)}\frac{dx_\epsilon}{d\sigma}\int_0^z [u_p'(s)]^2\exp\left(\int_0^s g(u_p(r))dr\right)ds. \end{cases} \tag{2.29}$$

Using (2.29), we can write a general solution to (2.13):

$$v(z,\sigma) \sim u_p'(z)\left\{[\alpha(\sigma) + F(z,\sigma)] + \int_0^z \frac{ds}{\exp(\int_0^s g(u_p(r))dr)[u_p'(s)]^2}[\beta(\sigma) + G(z,\sigma)]\right\}. \tag{2.30}$$

The interior condition (2.27) yields $\alpha(\sigma) \equiv 0$. The boundary conditions (2.15) then provide two linear equations to solve for the unknowns $\frac{dx_\epsilon}{d\sigma}$ and, less importantly, $\beta(\sigma)$.

Let

$$\begin{cases} I_1(z) \equiv \int_0^z \frac{ds}{\exp\left(\int_0^s g(u_p(r))dr\right)[u_p'(s)]^2} \\ I_2(z) \equiv \int_0^z \int_0^s \frac{[u_p'(s)]^2 \exp\left(\int_0^s g(u_p(r))dr\right)}{[u_p'(q)]^2 \exp\left(\int_0^q g(u_p(r))dr\right)} dq ds \\ I_3(z) \equiv \int_0^z [u_p'(s)]^2 \exp\left(\int_0^s g(u_p(r))dr\right) ds \\ D(z) \equiv I_3(z) - \frac{I_2(z)}{I_1(z)}. \end{cases} \tag{2.31}$$

Then the solution of the linear system yields

$$\frac{dx_\epsilon}{d\sigma} \sim \frac{\frac{\epsilon\delta(\epsilon)}{\phi(\epsilon)}\left[\frac{v\left(-\frac{x_\epsilon}{\epsilon},\sigma\right)}{v_H\left(-\frac{x_\epsilon}{\epsilon}\right)} - \frac{v\left(\frac{1-x_\epsilon}{\epsilon},\sigma\right)}{v_H\left(\frac{1-x_\epsilon}{\epsilon}\right)}\right]}{D\left(\frac{1-x_\epsilon}{\epsilon}\right) - D\left(-\frac{x_\epsilon}{\epsilon}\right)} = \frac{\frac{\epsilon}{\phi(\epsilon)}\left[\frac{u_L - u_p\left(-\frac{x_\epsilon}{\epsilon}\right)}{v_H\left(-\frac{x_\epsilon}{\epsilon}\right)} - \frac{u_R - u_p\left(\frac{1-x_\epsilon}{\epsilon}\right)}{v_H\left(\frac{1-x_\epsilon}{\epsilon}\right)}\right]}{D\left(\frac{1-x_\epsilon}{\epsilon}\right) - D\left(-\frac{x_\epsilon}{\epsilon}\right)} \tag{2.32}$$

and $\beta(\sigma)$, which fully determine the correction term $v(z,\sigma)$:

$$\begin{aligned} v(z,\sigma) \sim \frac{v_H(z)}{D\left(\frac{1-x_\epsilon}{\epsilon}\right) - D\left(-\frac{x_\epsilon}{\epsilon}\right)} \Bigg\{ & \frac{v\left(\frac{1-x_\epsilon}{\epsilon},\sigma\right)}{v_H\left(\frac{1-x_\epsilon}{\epsilon}\right)}\left[D(z) - D\left(-\frac{x_\epsilon}{\epsilon}\right)\right] \\ & - \frac{v\left(-\frac{x_\epsilon}{\epsilon},\sigma\right)}{v_H\left(-\frac{x_\epsilon}{\epsilon}\right)}\left[D(z) - D\left(\frac{1-x_\epsilon}{\epsilon}\right)\right]\Bigg\}. \end{aligned} \tag{2.33}$$

We can use (2.32) to determine $\phi(\epsilon)$ such that $\frac{dx_\epsilon}{d\sigma}$ is $O_s(1)$ near the steady-state shock location. Then (2.14) allows us to determine the gauge $\delta(\epsilon)$. For the specific cases in which either $h(u) \equiv 0$ or $g(u) \equiv 0$ it is easy to see that (2.32) predicts monotonic motion of the shock position toward its steady-state location; we shall have more to say about this in Chapters 3 and 5. One should consult [37], as well.

Note that, when using the travelling wave Ansatz, we did not have to approximate the principal eigenfunctions of a linear operator. Admittedly, however, there is added work in approximately solving the boundary-value problem for v, which was fortunately tractable. In any case, we expect the limiting equations for the shock motion given by (2.26) and (2.32) to agree, though it is difficult to show this without having a general representation for the eigenfunction in (2.26).

2.5 An Example: Burgers' Equation

We briefly discuss the important example of *Burgers'* equation, which has been studied extensively by Laforgue in [33], by Laforgue and O'Malley in [34]-[37] and by Laforgue, O'Malley, and Ward in [38], among many others. Burgers' equation is given by (1.2) with $g(u) = u - \frac{1}{2}$, $h(u) \equiv 0$, $u_L = 0$, and $u_R = 1$. An exact solution of the initial-boundary value problem for $u(x,t)$ can be found by using the Cole-Hopf transformation together with separation of variables for the resulting heat equation ([31], [37]).

A shock profile function $u_p(z)$ satisfying (2.7) exists when the balanced boundary values satisfy

$$u_L < \frac{1}{2} < u_R = 1 - u_L; \tag{2.34}$$

specifically, we then obtain $u_p(z) = \frac{1}{2} + (u_R - \frac{1}{2}) \tanh\left(\frac{(u_R - \frac{1}{2})z}{2}\right)$. Moreover, the corresponding steady-state problem has the exact solution

$$u_\epsilon(x) = \frac{1}{2} + k_\epsilon \tanh\left(\frac{k_\epsilon}{2\epsilon}\left(x - \frac{1}{2}\right)\right)$$

provided the constant of integration k_ϵ satisfies the transcendental equation

$$k_\epsilon \tanh\frac{k_\epsilon}{4\epsilon} = \frac{1}{2} - u_L \equiv k_0.$$

k_ϵ can be determined asymptotically as

$$k_\epsilon = k_0\left(1 + 2e^{-k_0/2\epsilon} + 2e^{-k_0/\epsilon} + \cdots\right), \tag{2.35}$$

a power series in the asymptotically exponentially small parameter $e^{-k_0/2\epsilon}$. Note that the limiting steady-state solution features an $O(\epsilon)$-thin shock layer about the midpoint $x = \frac{1}{2}$ where it jumps from the limit u_L to the symmetric limit u_R as x and the ϵ-stretched variable $\frac{1}{\epsilon}(x - \frac{1}{2})$ increase.

As (2.35) might suggest, Burgers' equation, under these conditions, exhibits exponential asymptotics, and the steady-state problem exhibits exponential supersensitivity. For example, if we change the right boundary value from $1 - u_L$ to $u(1) =$

$1 - u_L - 2k_0 e^{-a/\epsilon}$ for a constant a satisfying $0 < a < \frac{k_0}{2}$, the asymptotic solution features an analogous shock layer profile of steep change, but its center is now relocated from $x = \frac{1}{2}$ to $x = 1 - \frac{a}{k_0}$. Thus, an asymptotically negligible change in one boundary value moves the shock layer right an $O(1)$ distance. For $a > \frac{k_0}{2}$, the resulting endpoint layer corresponds to that typically occurring for unbalanced boundary values, i.e. ones not satisfying (2.34).

For the initial-boundary value problem, an exponentially metastable shock layer forms after some initial time, easily predicted via characteristics, with the shock moving monotonically and asymptotically exponentially slowly toward its steady-state with a layer, described asymptotically by the hyperbolic tangent function, centered at the midpoint. This motion has been successfully captured by using both the projection method and the travelling wave Ansatz, in agreement with the exact answer provided by use of the linearizing Cole-Hopf transformation.

2.6 An Example for which the Projection Method is Impracticable

We now present an illustrative application of the projection method. We consider

$$\begin{cases} u_t = \epsilon u_{xx} - [\sin^2(\pi u)]_x, \; 0 < x < 1, \; 0 < \epsilon \ll 1 \\ \\ u(0) = 0, \; u(1) = 1, \; u(x, 0) = u_0(x). \end{cases} \tag{2.36}$$

Here, $u_0(x)$ is smooth and compatible with the end values. We easily find that (2.36) has the associated shock profile

$$u_p(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(\pi z), \tag{2.37}$$

with far-field behavior satisfying

$$\begin{cases} u_p(z) \sim -\frac{1}{\pi^2 z} + \frac{1}{3\pi^4 z^3} - \frac{1}{5\pi^6 z^5} + O(z^{-7}) & \text{as } z \to -\infty \\ u_p(z) \sim 1 - \frac{1}{\pi^2 z} + \frac{1}{3\pi^4 z^3} - \frac{1}{5\pi^6 z^5} + O(z^{-7}) & \text{as } z \to \infty. \end{cases} \tag{2.38}$$

We linearize (2.36) about $u_p(z)$ by setting $u(x,t) = u_p(z) + \epsilon^2 v(z,\sigma)$, where $\sigma \equiv \epsilon^2 t$. We immediately obtain the approximate linear equation

$$v_{zz} - [\pi \sin(2\pi u_p)v]_z \sim -\frac{dx_\epsilon}{d\sigma}u_p'(z)$$

for v, in the form of (2.13). Using (2.37) and a plethora of trigonometric identities, we can simplify this to precisely show that

$$v_{zz} + \left[\frac{2\pi^2 zv}{1+\pi^2 z^2}\right]_z \sim -\frac{dx_\epsilon}{d\sigma}u_p'(z),$$

subject to the boundary conditions

$$\begin{cases} v(-\frac{x_\epsilon}{\epsilon},\sigma) = -\frac{1}{\epsilon^2}u_p\left(-\frac{x_\epsilon}{\epsilon}\right) \\ \\ v(\frac{1-x_\epsilon}{\epsilon},\sigma) = \frac{1}{\epsilon^2}\left(1 - u_p\left(\frac{1-x_\epsilon}{\epsilon}\right)\right). \end{cases}$$

We can convert this two-point problem to self-adjoint form through the Liouville transformation $w(z,\sigma) = \left[u_p'(z)\right]^{-1/2} v(z,\sigma) = (1+\pi^2 z^2)^{1/2} v(z,\sigma)$ to get

$$S[w] \equiv w_{zz} + \frac{\pi^2(1-2\pi^2 z^2)}{(1+\pi^2 z^2)^2}w \sim -\frac{dx_\epsilon}{d\sigma}u_p'(z)(1+\pi^2 z^2)^{1/2}, \tag{2.39}$$

together with corresponding boundary values

$$\begin{cases} w\left(-\frac{x_\epsilon}{\epsilon},\sigma\right) = -\frac{\pi x_0}{\epsilon}\left(1+\frac{\epsilon^2}{\pi^2 x_0^2}\right)^{1/2} u_p\left(-\frac{x_0}{\epsilon}\right) \\ w\left(\frac{1-x_\epsilon}{\epsilon},\sigma\right) = \frac{\pi(1-x_0)}{\epsilon}\left(1+\frac{\epsilon^2}{\pi^2(1-x_0)^2}\right)^{1/2}\left[1 - u_p\left(\frac{1-x_0}{\epsilon}\right)\right]. \end{cases}$$

The projection method now requires looking at the self-adjoint eigenvalue problem associated with (2.39):

$$\Phi_{zz} + \frac{\pi^2(1-2\pi^2 z^2)}{(1+\pi^2 z^2)^2}\Phi = \lambda\Phi,\ \Phi\left(-\frac{x_\epsilon}{\epsilon}\right) = \Phi\left(\frac{1-x_\epsilon}{\epsilon}\right) = 0,\ (\Phi,\Phi) = 1 \tag{2.40}$$

with the inner product as defined in (2.21). As can be checked, the derivative of the shock profile, when transformed under the Liouville transformation, is an approximate principal eigenfunction for (2.40), up to a multiplicative constant κ_0. That is, this eigenfunction

$$\Phi_\epsilon(z) \equiv \left[u_p'(z)\right]^{-1/2} u_p'(z) = (1+\pi^2 z^2)^{-1/2}$$

satisfies $\mathcal{S}[\Phi_\epsilon] = 0$, is of one sign, and is algebraically small at both the left and the right boundary. So, if $\Phi_0(z)$ denotes the principal eigenfunction of (2.40), we anticipate that it satisfies $\Phi_0(z) \sim \kappa_0 \Phi_\epsilon(z)$, except for algebraic corrections near the endpoints to meet its boundary conditions.

As usual, we have fortuitously found an approximate form for the principal eigenfunction, and the next step is to add boundary layer corrections to $\kappa_0\Phi_\epsilon$ so that it reaches the trivial endpoint values. It is imperative that we be able to simplify the eigenvalue problem at the endpoints in order to find appropriate boundary layer corrections. As evidenced by (2.26), the behavior of the principal eigenfunction at the boundaries is paramount; thus, if we cannot approximate this eigenfunction there, we cannot use the projection method.

The natural simplification of (2.40) near the endpoints is

$$z^2\Phi_{zz} - 2\Phi = \lambda z^2 \Phi \approx 0$$

as $|z| \to \infty$. Under the assumption that $\lambda z^2 = o(1)$, the limiting Euler equation $z^2\Phi_{zz} = 2\Phi$ has two independent solutions. The first, z^{-1}, grows without bound away from the boundaries, so it cannot be included in a local boundary layer solution. The second solution, z^2, cannot be asymptotically matched to Φ_ϵ. Thus, we cannot seem to find boundary layer corrections to the outer limit $\kappa_0\Phi_\epsilon$. Presumably, the correction needed is global and requires considering the full equation (2.40), which remains a difficult eigenvalue problem.

2.7 More Linear Algebra

We now return to our linear algebra analogy in order to better explain the limitation of the projection method just encountered. Recall that we are attempting to solve

$$A_\epsilon \mathbf{x}_\epsilon = \mathbf{b}_\epsilon,$$

where the forcing $\mathbf{b}_\epsilon$ depends upon an unknown parameter q. Let us assume a specific power series form for the first few terms in the asymptotic expansions with which we are working:

$$\begin{aligned} A_\epsilon &\sim A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots \\ \mathbf{b}_\epsilon &\sim \mathbf{b}_0 + \epsilon \mathbf{b}_1 + \epsilon^2 \mathbf{b}_2 + \cdots \\ \mathbf{x}_\epsilon &\sim \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \epsilon^2 \mathbf{x}_2 + \cdots . \end{aligned}$$

Later terms in these expansions could very well be exponentially, rather than algebraically, small.

Let us now attempt to use the projection method on this problem. Previously, we assumed that we knew the eigenpairs associated with A_ϵ, or at least could estimate them well. In parallel to the PDE problem just discussed, we will now assume that we merely know $\tilde{\bar{\Phi}}_{00}$, an approximation to Φ_0 that satisfies $A_0\tilde{\bar{\Phi}}_{00} = 0$. If we assume an asymptotic expansion for the principal eigenfunction Φ_0 in the form

$$\Phi_0 \sim \tilde{\bar{\Phi}}_{00} + \epsilon\tilde{\bar{\Phi}}_{01} + \epsilon^2\tilde{\bar{\Phi}}_{02} + \cdots ,$$

then clearly our first goal is to improve our initial approximation by finding $\tilde{\bar{\Phi}}_{01}$. We have, of course, the eigenvalue relationship

$$(A_0 + \epsilon A_1 + \cdots)(\tilde{\bar{\Phi}}_{00} + \epsilon\tilde{\bar{\Phi}}_{01} + \cdots) \sim \lambda_0(\tilde{\bar{\Phi}}_{00} + \epsilon\tilde{\bar{\Phi}}_{01} + \cdots).$$

This implies, by linearity, the asymptotic relationship

$$\epsilon A_1\tilde{\bar{\Phi}}_{00} + \epsilon A_0\tilde{\bar{\Phi}}_{01} \sim \lambda_0\tilde{\bar{\Phi}}_{00}. \tag{2.41}$$

Now, we discuss two cases. First, consider the situation when $\lambda_0 = o(\epsilon)$. This includes examples in which the principal eigenvalue is exponentially small. Then we can simplify (2.41) to the relatively simple linear system

$$A_0\tilde{\Phi}_{01} \sim -A_1\tilde{\Phi}_{00}. \tag{2.42}$$

However, consider the second case in which $\lambda_0 \sim O_s(\epsilon)$. Then we must retain all three terms in (2.41), so we are left to solve a non-standard kind of eigenvalue problem for $\tilde{\Phi}_{01}$!

This dilemma is analogous to what happened in our previous example. The principal eigenvalue associated with the problem was sufficiently asymptotically large that we had to solve a full eigenvalue problem to estimate the eigenfunction. We propose that, in such a situation, the travelling wave Ansatz should be attempted since it is more tractable than the difficult eigenvalue problems resulting from the projection method. These sorts of situations might arise whenever the shock profile function fails to satisfy the boundary conditions by asymptotically algebraically small amounts.

We emphasize that there is no theoretical flaw in the projection method. Rather, the difficulty emerges in applying the projection method to a certain class of problems when one cannot adequately approximate the needed eigenvectors or eigenfunctions. Otherwise, the projection method would again be efficient and preferred.

We again include two simple finite-dimensional examples to make our discussion clearer.

Example (with an exponentially small eigenvalue):

Consider

$$A_\epsilon = \begin{bmatrix} e^{-1/\epsilon} & 0 \\ \epsilon e^{-1/\epsilon} - \epsilon & 1 \end{bmatrix}.$$

By inspection, the eigenvalues of A_ϵ are $e^{-1/\epsilon}$ and 1. We wish to approximate the eigenvector associated with $e^{-1/\epsilon}$, which is easily computed to be $\Phi_0 = [1 \;\; \epsilon]^T$. Let's now assume that we only have a first approximation $\tilde{\Phi}_0 = [1 \;\; 0]^T$. Since $\tilde{\Phi}_0 \not\sim \Phi_0$, however, we need to improve upon this approximation.

Assume an asymptotic expansion $\Phi_0 \sim \tilde{\Phi}_0 + \epsilon\tilde{\Phi}_1 + \cdots$. Using the approximation in (2.42), the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_{11} \\ \tilde{\Phi}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

implies that $\tilde{\Phi}_{12} = 1$ and that we can take $\tilde{\Phi}_{11}$ to be any $O(1)$ quantity, say zero for simplicity. Thus we have arrived at an approximation (actually exact) to the eigenvector which we could then use in the projection method.

Example (with an algebraically small eigenvalue, but where the eigenvector approximation fails):

Now take

$$A_\epsilon = \begin{bmatrix} \epsilon & 0 \\ -\epsilon + \epsilon^2 & 1 \end{bmatrix}.$$

The eigenvalues of A_ϵ are ϵ and 1, and we again wish to approximate the eigenvector associated with the small eigenvalue ϵ, which is $\Phi_0 = [1 \;\; \epsilon]^T$. We again take a first approximation $\tilde{\Phi}_0 = [1 \;\; 0]^T$, which we wish to improve upon. Note that the correction that we would like to find is of the same asymptotic order as the associated eigenvalue.

Assume the asymptotic expansion $\Phi_0 \sim \tilde{\Phi}_0 + \epsilon\tilde{\Phi}_1 + \cdots$. Attempting to use the approximation in (2.42), we have

$$A_0\tilde{\Phi}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_{11} \\ \tilde{\Phi}_{12} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This equation, however, has no solution! Thus, we have no recourse but to solve the three-term asymptotic eigenvalue problem in (2.41). Here, this is not difficult, but when using the projection method for differential equations, these eigenvalue problems can prove to be quite intractable analytically. Thus, we demonstrate the utility of the travelling wave Ansatz, which avoids these issues altogether.

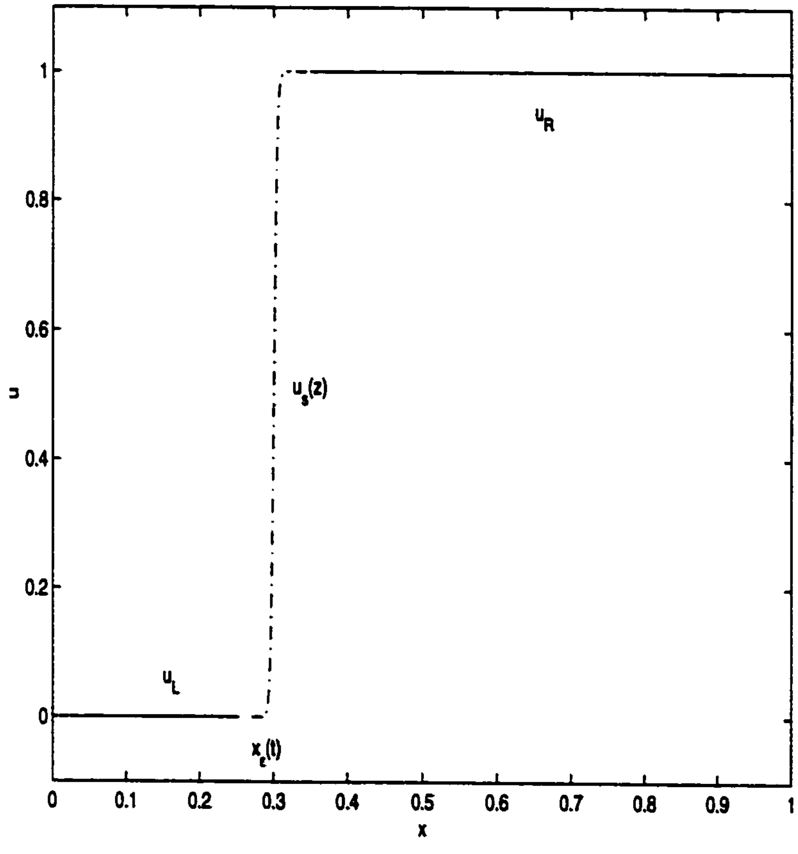

Figure 2.1: **Inner and Outer Solutions.** This figure illustrates the inner and outer solutions that one would normally asymptotically match to obtain a uniformly valid composite approximation to $u(x,t)$ in the case where $h(u_L) = h(u_R) = 0$. Note that, after subtracting off the matching values, one has $u(x,t) \sim u_s(z)$.

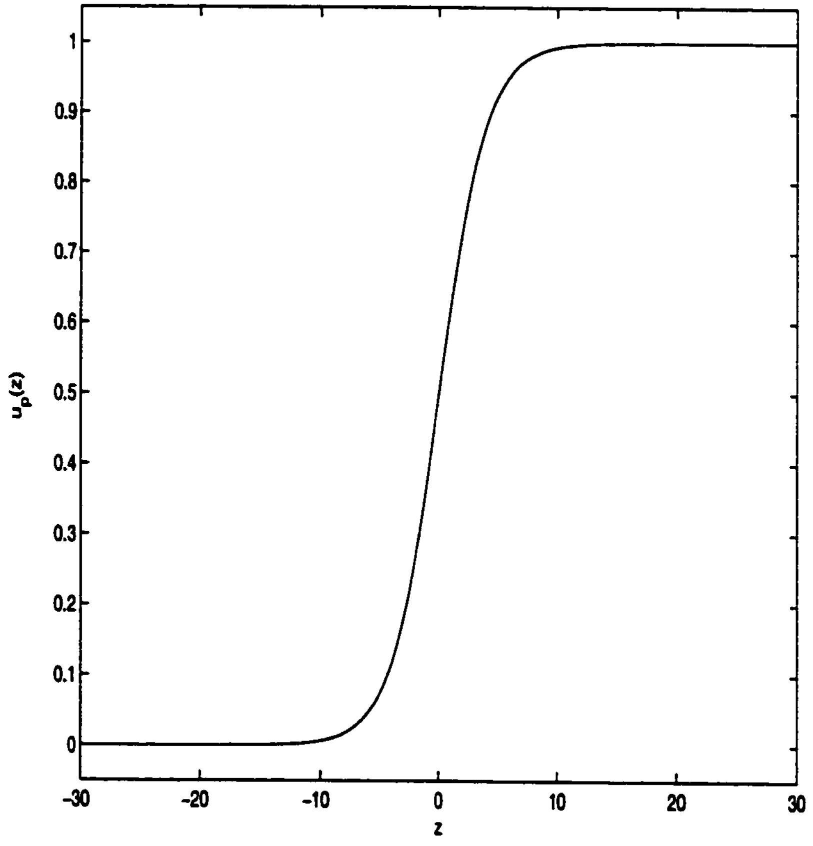

Figure 2.2: **A Typical Shock Profile Function** $u_p(z)$. This is a plot of the shock profile function for Burgers' equation, in which $h(u) \equiv 0$ and $g(u) = u - \frac{1}{2}$, with $u_L = 0$ and $u_R = 1$. The profile function is given explicitly by $u_p(z) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{z}{4}\right)$. Note how most of the transition occurs for $|z| < 10$.

Chapter 3

ADVECTION-DIFFUSION EQUATIONS

We shall now specifically consider the viscous shock, or advection-diffusion, problem

$$u_t = \epsilon u_{xx} + g(u)u_x \equiv (\epsilon u_x + G(u))_x\,,\; u(0,t) = u_L,\; u(1,t) = u_R,\; u(x,0) = u_I(x) \tag{3.1}$$

where $G(u) \equiv \int_{u_L}^{u} g(s)ds$ satisfies

$$G(u_R) = G(u_L) \tag{3.2}$$

and

$$G(u) < G(u_L) \quad \text{for } u_L < u < u_R. \tag{3.3}$$

The traditionally-used conditions (3.2) and (3.3) guarantee the existence of a monotonic shock profile function satisfying (2.7). Given such a $G(u)$, we introduce the critical order $m \geq 1$ to be the integer such that

$$\begin{cases} \frac{d^kG}{du^k}(u_L) = \frac{d^kG}{du^k}(u_R) = 0 \text{ for } 1 \leq k < m \\ \\ A_L \equiv \frac{d^mG}{du^m}(u_L) < 0, \quad (-1)^m A_R \equiv (-1)^m \frac{d^mG}{du^m}(u_R) < 0. \end{cases} \tag{3.4}$$

Thus, m tells us how "flat" the function $G(u)$ is at its endpoints u_L and u_R. (If $G(u)$ is more "flat" at one endpoint than the other, we then expect an endpoint layer, rather than a shock layer, to occur. Therefore, the carefully balanced end-values are necessary to obtain the shock.) A typical potential $G(u)$ is illustrated in Figure 3.1 for the case $m = 3$.

See Liu [42], Reyna and Ward [51], [52], and Ward [62], for example, for surveys of a wide variety of background literature. Liu, in particular, studies such equations on an infinite spatial interval, drawing the important connection to the hyperbolic conservation laws obtained when $\epsilon = 0$. Previous work has concentrated primarily on the case $m = 1$, with the notable exceptions of [15] and [16]. We shall see that analogous results can be obtained for $m > 1$, though there turn out to be interesting differences in the asymptotics required.

3.1 *An Example of Algebraic Asymptotics, Using the Travelling Wave Ansatz*

We reconsider an example from the last chapter with $m = 2$. Consider

$$G(u) = -\sin^2(\pi u),$$

which has second-order zeroes at $u = 0$ and $u = 1$. In particular we'll approximate the solution to

$$\begin{cases} u_t = \epsilon u_{xx} - [\sin^2(\pi u)]_x, \ 0 < x < 1, \ t > 0, \ 0 < \epsilon \ll 1 \\ u(0,t) = 0, \ u(1,t) = 1, \ u(x,0) = u_I(x). \end{cases} \tag{3.5}$$

First, consider the steady-state problem

$$\begin{cases} \epsilon u_{xx} - [\sin^2(\pi u)]_x = 0 \\ u(0) = 0, \ u(1) = 1. \end{cases} \tag{3.6}$$

Its steady-state shock location x_ϵ, satisfying $u(x_\epsilon) = \frac{1}{2}$, is easily determined. Indeed, since (3.6) is symmetric under the transformation $x \to 1 - x$ and $u \to 1 - u$, it is clear that $x_\epsilon = \frac{1}{2}$ for all ϵ.

Returning to (3.5), let us define the appropriate slow time

$$\sigma \equiv \epsilon^2 t$$

and the stretched variable $z \equiv \frac{x - x_\epsilon(\sigma)}{\epsilon}$. Here the important shock location $x_\epsilon(\sigma)$ for varying σ remains to be approximated. Recall that problem (3.5) has a corresponding unique monotonic stationary wave profile

$$u_p(z) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(\pi z). \tag{3.7}$$

In previously studied problems, $u_p(z)$ has typically approached its end-values exponentially quickly. Here, however, $u_p(z)$ reaches its far-field values algebraically quickly since

$$\begin{cases} u_p(z) \sim -\frac{1}{\pi^2 z} + \frac{1}{3\pi^4 z^3} - \frac{1}{5\pi^6 z^5} + O(z^{-7}) & \text{as } z \to -\infty \\ u_p(z) \sim 1 - \frac{1}{\pi^2 z} + \frac{1}{3\pi^4 z^3} - \frac{1}{5\pi^6 z^5} + O(z^{-7}) & \text{as } z \to \infty. \end{cases} \tag{3.8}$$

The profile function $u_p(z)$ is nearly a solution of the given advection-diffusion equation (3.1). The residual error in the PDE is $\epsilon u_p'(z)\frac{dx_\epsilon}{d\sigma}$, while the error in meeting both boundary conditions is $O_s(\epsilon)$. Thus, it seems reasonable to seek a solution to (3.5) using the combined multi-scale Ansatz

$$\begin{aligned} u(x,t) &\sim u_p(z) + \epsilon^2 v(z,\sigma) \sim u_p(z) + \epsilon^2\left[v_0(z,\sigma) + \epsilon v_1(z,\sigma) + \cdots\right] \\ x_\epsilon(\sigma) &\sim x_0(\sigma) + \epsilon x_1(\sigma) + \epsilon^2 x_2(\sigma) + \cdots . \end{aligned} \tag{3.9}$$

As is usual when using the travelling wave Ansatz, equation (2.27) holds, i.e. $v(0,\sigma) = 0$. The end conditions for v are also determined at $x = 0$ and $x = 1$. We note that, at some point, the form of the series (3.9) may have to be revised because, for example, switchback terms ultimately occur (see Lagerstrom [39]). Note that the use of an Ansatz like (3.9) would simplify the presentations of Laforgue and O'Malley ([34]-[37]) concerning exponential asymptotics.

Plugging the expansions (3.9) into (3.5) and using the fact that $u_p'(z) = \sin^2(u_p)$ by definition, we obtain, at $O(\epsilon^2)$ and $O(\epsilon^3)$, the respective *linear* ODEs

$$\begin{aligned} v_{0zz} - [2\pi\sin(2\pi u_p)\cos(2\pi u_p)v_0]_z = v_{0zz} + \left[\frac{2\pi^2 z v_0}{1+\pi^2 z^2}\right]_z &\sim -\dot{x}_0 u_p' \\ v_{1zz} - [2\pi\sin(2\pi u_p)\cos(2\pi u_p)v_1]_z = v_{1zz} + \left[\frac{2\pi^2 z v_1}{1+\pi^2 z^2}\right]_z &\sim -\dot{x}_1 u_p'. \end{aligned}$$

for v_0 and v_1, where we have used straightforward trigonometry to simplify the equations. It is also possible to determine the equations that need to be satisfied at higher orders, though they quickly become complicated. Since these two equations are fortuitously identical in structure, they can be solved simultaneously, though the asymptotic boundary values for v_0 will determine those for v_1. Noting that u_p' is a solution of the corresponding homogeneous equation, we can use reduction of order to find particular solutions. Specifically, we obtain the general solutions

$$v_{j-1}(z,\sigma) = \frac{1}{\pi^2 z^2+1}\left\{ c_{1(j-1)}(\sigma) + c_{2(j-1)}(\sigma) z\left(\frac{1}{3}\pi^2 z^2 + 1\right) \right. \\ \left. - \dot{x}_{j-1}\left[\frac{z}{\pi}\left(\frac{1}{3}\pi^2 z^2+1\right)\tan^{-1}(\pi z) - \frac{1}{6}z^2 - \frac{\log(\pi^2 z^2+1)}{3\pi^2}\right]\right\} \tag{3.10}$$

for "constants" $c_{i(j-1)}(\sigma)$ and for $j = 1$ and 2. The midpoint condition $v_{j-1}(0,\sigma) = 0$ immediately implies that $c_{1(j-1)} = 0$.

Now, we can use the two end conditions to determine the two unknowns $c_{2(j-1)}$ and, more importantly, $\dot{x}_{j-1}$. Using (3.8) and (3.9), we obtain

$$\begin{aligned} v_0\left(-\frac{x_\epsilon}{\epsilon},\sigma\right) &\sim -\frac{1}{\epsilon^2}u_p\left(-\frac{x_\epsilon}{\epsilon}\right) \sim -\frac{1}{\epsilon\pi^2 x_\epsilon} \\ v_0\left(\frac{1-x_\epsilon}{\epsilon},\sigma\right) &\sim \frac{1}{\epsilon^2}\left[1-u_p\left(\frac{1-x_\epsilon}{\epsilon}\right)\right] \sim \frac{1}{\epsilon\pi^2(1-x_\epsilon)}. \end{aligned}$$

These boundary values are unbounded as $\epsilon \to 0^+$, but this will not turn out to be problematic, since the scaled correction term $\epsilon^2 v$ in the solution $u(x,t)$ will be bounded. Applying these in (3.10) for $j = 1$, we get a system of two linear algebraic equations that we solve to find the unknowns

$$\dot{x}_0(\sigma) \sim \frac{3(1-2x_0)}{\pi^2 x_0^2(1-x_0)^2} \tag{3.11}$$

and

$$c_{20}(\sigma) \sim \frac{3(2x_0^2-2x_0+1)}{2\pi^2 x_0^2(1-x_0)^2}$$

and to thereby specify v_0, an $O(\epsilon)$ correction to the limiting solution $u_p(z)$ since v is unbounded like ϵ^{-1} near the endpoints. Clearly, finding the nonlinear differential equation (3.11) for the limiting shock location x_0 is the major achievement; we have asymptotically reduced the initial-boundary value problem for the highly nonlinear PDE of (3.5) to the much simpler problem of solving an initial-value problem for the ODE (3.11). First, of course, the whole PDE (3.5) would have to be numerically integrated for a short time until the shock corresponding to the profile $u_p(z)$ is well-developed, yielding an initial condition for x_ϵ at some time $t_i > 0$ for the ODE. Then, the ODE can either be separated as

$$\sigma - t_i \sim \frac{\pi^2}{3} \int_{x_\epsilon(t_i)}^{x_0} \frac{r^2(1-r)^2}{1-2r}$$

and inverted numerically, or simply numerically integrated from $t = t_i$. From inspection, (3.11) yields both the correct direction of the *algebraically* slow motion for the shock-location and the correct steady-state shock location $x_\epsilon(\infty) = \frac{1}{2}$, presuming the initial value $x_\epsilon(t_i)$ is within $(0, 1)$. We thereby see immediately that the shock location for problem (3.5) is *algebraically metastable.*

To determine a first correction to the limiting shock location (3.11), we consider (3.10) (with $j = 2$) subject to the boundary conditions

$$\begin{aligned} v_1\left(-\frac{x_\epsilon}{\epsilon}, \sigma\right) &\sim \frac{4\pi^2 x_1(1-x_0)^2 - 3(1-2x_0)}{2\epsilon\pi^4 x_0^2(1-x_0)^2} \\ v_1\left(\frac{1-x_\epsilon}{\epsilon}, \sigma\right) &\sim \frac{4\pi^2 x_1 x_0^2 - 3(1-2x_0)}{2\epsilon\pi^4 x_0^2(1-x_0)^2}. \end{aligned}$$

A simple calculation reveals that

$$\dot{x}_1(\sigma) \sim -\frac{6}{\pi^2}\frac{1-3x_0+3x_0^2}{[(1-x_0)x_0]^3}x_1 + \frac{9}{2\pi^4}\frac{1-2x_0}{[(1-x_0)x_0]^3} \tag{3.12}$$

and

$$c_{21}(\sigma) \sim -\frac{3}{4}\frac{4x_1\pi^2(1-3x_0+3x_0^2-2x_0^3) - 3(1-4x_0+4x_0^2)}{(1-x_0)^2 x_0^2 \pi^4}.$$

This information also provides v_1 asymptotically. On inspection of the linear ODE (3.12), we see that $x_1 \to 0$ as $x_0 \to \frac{1}{2}$ and $\sigma \to \infty$, in agreement with our earlier asymptotic work on the steady-state problem.

We now have an expansion for $\dot{x}_\epsilon(\sigma)$ valid to $O(\epsilon)$. Assuming that the next term in the expansion for $\dot{x}_\epsilon(\sigma)$ is of order ϵ^2, experience with two-timing suggests that we can only expect this two-term expansion to give an $O(1)$-accurate shock location for times as large as $\sigma = O(\epsilon^{-1})$ or $t = O(\epsilon^{-3})$. To extend our interval of accuracy, we might compute more terms in the expansion for $\dot{x}_\epsilon(t)$. While this is computationally straightforward, we do have to guess an Ansatz that provides the sizes of successive terms in the expansion. Experience with these problems shows that we will typically *not* encounter a simple power series expansion in ϵ; rather, we must expect so-called switchback terms, as the logarithmic terms in (3.10) suggest. Such switchback terms make successive terms somewhat more tedious to calculate, and one may sometimes need to revise the Ansatz being used. However, there is no evidence of logarithmic terms, which typically signal the necessity of switchback terms, at this point in the calculation. (For some related comments on the "prevalence of logarithms" in matched asymptotic expansions, see [55].)

3.2 The Steady-State Problem

We now consider whether the algebraic asymptotics arising from (3.5) are typical by studying (3.1) in general. First, consider the steady-state problem and its sensitivity for a general G. Supersensitivity of the steady-state problem implies supersensitivity of the time-dependent problem since the steady-state equation determines the equilibrium solution of the full PDE.

The steady-state problem associated with (3.1) is the two-point problem

$$(\epsilon u_x + G(u))_x = 0,\ u(0) = u_L,\ u(1) = u_R. \tag{3.13}$$

Integrating once implies

$$\epsilon u_x + G(u) = \epsilon u_x(0) + G(u_L) \equiv \gamma_m(\epsilon) + G(u_L) \tag{3.14}$$

where the shooting parameter $\gamma_m(\epsilon)$, satisfying $0 < \gamma_m(\epsilon) \ll 1$, remains to be determined. (Recall from (3.4) that the order m is determined by the Taylor series of $G(u)$ about both u_L and u_R.) The ODE (3.14) can be integrated to obtain

$$x = \epsilon \int_{u_L}^{u} \frac{dp}{\gamma_m(\epsilon) - G(p) + G(u_L)}.$$

This yields the two equations

$$1 = \epsilon \int_{u_L}^{u_R} \frac{dp}{\gamma_m(\epsilon) - G(p) + G(u_L)} \tag{3.15}$$

$$x_\epsilon = \epsilon \int_{u_L}^{\frac{u_L+u_R}{2}} \frac{dp}{\gamma_m(\epsilon) - G(p) + G(u_L)}. \tag{3.16}$$

We can anticipate that (3.15) determines $\gamma_m(\epsilon)$ and that (3.16) then determines the shock location x_ϵ. We can also rewrite (3.16) as

$$x(u) = \frac{\int_{u_L}^{u} \frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)}}{\int_{u_L}^{u_R} \frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)}}. \tag{3.17}$$

To more explicitly determine x_ϵ asymptotically, we must recall methods to evaluate integrals asymptotically (cf., e.g., Olver [48]). Since $\displaystyle\int_{u_L}^{u_R} \frac{dp}{\gamma_m(\epsilon) - G(p) + G(u_L)} = \frac{1}{\epsilon}$ is large, and since $G(u_L) > G(p)$ for $u_L < p < u_R$, the primary contributions to this integral must come from p values near both endpoints. For $\kappa > 0$ fixed, for example, with $u_L + \kappa < \frac{1}{2}(u_L + u_R)$,

$$\begin{aligned}
\int_{u_L}^{u_L+\kappa} \frac{dp}{\gamma_m(\epsilon) - G(p) + G(u_L)} &\sim \int_{u_L}^{u_L+\kappa} \frac{dp}{\gamma_m(\epsilon) - \frac{1}{m!}(p-u_L)^m A_L} \\
&= -\frac{m!}{A_L} \int_0^{\kappa} \frac{dq}{a^m + q^m} &(3.18) \\
&= \frac{\kappa}{m\gamma_m(\epsilon)} \Phi\left(\frac{A_L \kappa^m}{m!\gamma_m(\epsilon)}, 1, \frac{1}{m}\right). &(3.19)
\end{aligned}$$

Here, $a^m \equiv -\frac{m!\gamma_m(\epsilon)}{A_L} > 0$ and $\Phi(z,s,v)$ is the Lerch Phi function, appropriately defined for large negative values of z. (See Gradshteyn and Ryzhik [24], Section 9.55.)

While (3.19) is an exact representation of (3.18), this crude asymptotic approximation in terms of the special function $\Phi(z,s,v)$ may not be entirely satisfactory. A second alternative is to assume that, if (3.18) is independent of κ as $a \to 0^+$, then

$$-\frac{m!}{A_L}\int_0^{\kappa}\frac{dq}{a^m+q^m} \sim -\frac{m!}{A_L}\int_0^{\infty}\frac{dq}{a^m+q^m},$$

which can be integrated. One cannot, however, use this crude method to find higher order approximations to (3.18). We have also developed an alternate method for approximating integrals such as that in (3.18), which we shall discuss in Chapter 4.

Regardless of the method of approximation, we find that, for the case $m = 1$,

$$\int_{u_L}^{u_L+\kappa}\frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)} \sim \frac{1}{A_L}\log(\gamma_1(\epsilon))$$

for κ fixed and $\gamma_1(\epsilon) \to 0^+$, independent of κ. Obtaining the analogous estimate near $p = u_R$ and using (3.15) and (3.16), we find that

$$\gamma_1(\epsilon) \sim e^{-A/\epsilon}, \tag{3.20}$$

where

$$A \equiv \frac{-A_L A_R}{A_R - A_L}, \tag{3.21}$$

and, more critically, that the corresponding limiting interior shock location is

$$x_\epsilon \sim \frac{A_R}{A_R - A_L}. \tag{3.22}$$

This is in agreement with our earlier result for Burgers equation in Section 2.5.

For $m \geq 2$, we instead have that

$$\int_{u_L}^{u_L+\kappa}\frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)} \sim \frac{(m-1)!\pi}{-A_L}\csc\left(\frac{\pi}{m}\right)\left(\frac{m!\gamma_m(\epsilon)}{-A_L}\right)^{-\frac{m-1}{m}}. \tag{3.23}$$

Again, in combination with the analogous approximation near the right endpoint, we can easily find that

$$\gamma_m(\epsilon) \sim \alpha_m \epsilon^{\frac{m}{m-1}} \tag{3.24}$$

for the constant $\alpha_m \equiv (m!)^{-1}\left[(m-1)!\pi\csc\left(\frac{\pi}{m}\right)\left(|A_L|^{-1/m}+|A_R|^{-1/m}\right)\right]^{\frac{m}{m-1}}$, and this implies that the limiting shock location is given by the interior value

$$x_\epsilon \sim \frac{|A_R|^{1/m}}{|A_R|^{1/m}+|A_L|^{1/m}}, \tag{3.25}$$

generalizing the well-known result (3.22) for $m=1$. See Table 3.1 for a computational corroboration of this formula.

Notice that $\gamma_1(\epsilon)$ is exponentially asymptotically small and that $\gamma_m(\epsilon)$ is, instead, algebraically asymptotically small for any $m \geq 2$. Thus, we must expect that x_ϵ is sensitive to exponentially asymptotically small changes in the boundary conditions for $m=1$, as we saw for Burgers equation, but only to algebraically asymptotically small changes for $m \geq 2$.

We can show the latter result by considering

$$(\epsilon u_x + G(u))_x = 0,\ u(0) = u_L + \alpha\epsilon^{\frac{1}{m-1}},\ u(1) = u_R - \beta\epsilon^{\frac{1}{m-1}} \tag{3.26}$$

for $m \geq 2$, with positive, $O(1)$ constants α and β. In Chapter 4 we shall show that these algebraically asymptotically small changes move the shock location an $O(1)$ distance from its unperturbed location (3.25), though it is often difficult to obtain a closed-form expression for the new shock location.

3.3 The Time-Dependent Problem

We now return to the IBVP (3.1). Previous experience suggests that a shock layer solution will, after some initial $O(1)$ time, then take the asymptotic form

$$u_\epsilon(x,t) = u_p(z) + \delta_m(\epsilon)v(z,\sigma) \tag{3.27}$$

where z is the ϵ-stretched shock-layer variable

$$z = \frac{x - x_\epsilon(\sigma)}{\epsilon}, \tag{3.28}$$

$x_\epsilon(\sigma)$ is the shock-layer location, σ is the slow-time $\phi_m(\epsilon)t$, $\phi_m(\epsilon)$ and $\delta_m(\epsilon)$ are unknown $o(1)$ gauge functions, and the asymptotic profile $u_p(z)$ is the unique monotonically increasing function satisfying the autonomous stretched ordinary differential equation

$$\frac{d^2u_p}{dz^2} + \frac{dG(u_p)}{dz} = 0, \ -\infty < z < \infty \tag{3.29}$$

as well as the auxiliary conditions

$$u_p(-\infty) = u_L, \ u_p(0) = \frac{1}{2}(u_L + u_R), \ u_p(\infty) = u_R.$$

We assume that, once the shock is well-developed, $\delta_m(\epsilon)v(z,\sigma) = o(u_p(z))$. Integrating once, we obtain $\frac{du_p}{dz} = G(u_L) - G(u_p)$, so another integration provides the unique implicit solution

$$z = -\int_{u_p(0)}^{u_p} \frac{dw}{G(w) - G(u_L)} \tag{3.30}$$

for the profile. For example, in the special case that $G(u) = -\sin \pi u$ with $u_L = 0$ and $u_R = 1$, $m = 1$ and the explicit asymptotic profile is

$$u_p(z) = \frac{2}{\pi}\tan^{-1}(e^{\pi z}).$$

In the case that $G(u) = -\sin^2 \pi u$ with $u_L = 0$ and $u_R = 1$, $m = 2$ and the profile takes the form $u_p(z) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\pi z$.

The tail behavior of $u_p(z)$ will determine the shock motion. When $m = 1$, $u_p(z)$ approaches its limiting end-values exponentially quickly. To see this in general, rewrite (3.30) as

$$z = \int_{u_p(0)}^{u_p} \left[\frac{1}{G(u_r) - G(w)} - \frac{1}{A_R(u_R - w)}\right] dw - \frac{1}{A_R}\log\left(\frac{u_R - u_p}{u_R - u_p(0)}\right).$$

We have removed the simple pole from the integrand to allow us to integrate up to u_R. We immediately find that

$$u_p(z) \sim u_R - C_R e^{-A_R z} \quad \text{as } z \to \infty, \tag{3.31}$$

where

$$\log\left(\frac{C_R}{u_R - u_p(0)}\right) \sim A_R \int_{u_p(0)}^{u_R} \left[\frac{1}{G(u_R) - G(w)} - \frac{1}{A_R(u_R - w)}\right] dw$$

defines C_R. Likewise,

$$u_p(z) \sim u_L + C_L e^{-A_L z} \quad \text{as } z \to -\infty \tag{3.32}$$

where

$$\log\left(\frac{C_L}{u_p(0) - u_L}\right) \sim -A_L \int_{u_L}^{u_p(0)} \left[\frac{1}{G(u_L) - G(w)} - \frac{1}{A_L(u_L - w)}\right] dw.$$

For $m \geq 2$, $u_p(z)$ approaches its end-values algebraically. To see this in general, note that

$$\frac{d}{dz}(u_p - u_R) \sim -\frac{(u_p - u_R)^m}{m!} A_R$$

near $z = \infty$, so an integration shows that

$$u_p(z) \sim u_R - \left(\frac{m!}{|A_R|(m-1)z}\right)^{\frac{1}{m-1}} \quad \text{as } z \to \infty. \tag{3.33}$$

Likewise,

$$u_p(z) \sim u_L + \left(\frac{m!}{A_L(m-1)z}\right)^{\frac{1}{m-1}} \quad \text{as } z \to -\infty. \tag{3.34}$$

More complete asymptotic tail approximations could be obtained by direct substitution of a series representation into the differential equation (3.29), with $G(u)$ expanded in its Taylor series about the end-value. Again note the exponential asymptotics for $m = 1$ and the algebraic asymptotics for $m \geq 2$.

Observe that the function $u_p(z)$ satisfies the PDE (3.1) with a small residual $\frac{1}{\epsilon}\phi_m(\epsilon)u_p'(z)\frac{dx_\epsilon(\sigma)}{d\sigma}$, and, when $m = 1$, it satisfies the boundary condition at $x = 0$ with the limiting error $C_L e^{-A_L(x_\epsilon/\epsilon)}$ and that at $x = 1$ with the limiting error $C_R e^{-A_R((1-x_\epsilon)/\epsilon)}$. For $m > 1$, the boundary errors are $O(\epsilon^{\frac{1}{m-1}})$, and so less negligible than for $m = 1$, but still asymptotically small. This suggests that the Ansatz (3.27) remains reasonable, presuming $0 < x_\epsilon(\sigma) < 1$, i.e. we have an interior shock.

Let us now seek the asymptotic solution of (3.1) in the form

$$u_\epsilon(x,t) = u_p(z) + \delta_m(\epsilon)v(z,\sigma).$$

Thus, the correction term v must satisfy

$$\begin{aligned}-\frac{\phi_m(\epsilon)}{\epsilon}(u_p' + \delta_m(\epsilon)v_z)\frac{dx_\epsilon}{d\sigma} + \phi_m(\epsilon)\delta_m(\epsilon)v_\sigma \\ = \tfrac{1}{\epsilon}(u_p'' + \delta_m(\epsilon)v_{zz}) + \tfrac{1}{\epsilon}G'(u_p + \delta_m(\epsilon)v)(u_p' + \delta_m(\epsilon)v_z).\end{aligned}$$

With a little algebra, we find that the scaled correction v satisfies the nearly linear equation

$$v_{zz} + [G'(u_p)v]_z = -\frac{\phi_m(\epsilon)}{\delta_m(\epsilon)}\frac{dx_\epsilon}{d\sigma}u_p' + H[v] \tag{3.35}$$

where

$$\begin{aligned}H[v] \equiv& -[G'(u_p + \delta_m(\epsilon)v) - G'(u_p)]v_z \\ &- \delta_m^{-1}(\epsilon)u_p'[G'(u_p + \delta_m(\epsilon)v) - G'(u_p) - \delta_m(\epsilon)G''(u_p)v] - \phi_m(\epsilon)v_z\tfrac{dx_\epsilon}{d\sigma} + \epsilon\phi_m(\epsilon)v_\sigma\end{aligned}$$

is $o(1)$. We shall choose the unknown gauge functions such that $\frac{dx_\epsilon}{d\sigma} = O(1)$ and

$$\frac{\phi_m(\epsilon)}{\delta_m(\epsilon)}\frac{dx_\epsilon}{d\sigma} = O_s(1). \tag{3.36}$$

The boundary conditions for v are

$$\begin{cases} v\left(-\frac{x_\epsilon(\sigma)}{\epsilon}, \sigma\right) = \delta_m^{-1}\left(u_L - u_p\left(-\frac{x_\epsilon(\sigma)}{\epsilon}\right)\right) \\ \\ v\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}, \sigma\right) = \delta_m^{-1}\left(u_R - u_p\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}\right)\right) \end{cases} \tag{3.37}$$

and

$$v(0,\sigma) = 0. \tag{3.38}$$

For $m = 1$, (3.37) becomes

$$\begin{cases} v\left(-\frac{x_\epsilon(\sigma)}{\epsilon}, \sigma\right) \sim -\delta_1^{-1} C_L e^{A_L x_\epsilon/\epsilon} \\ v\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}, \sigma\right) \sim \delta_1^{-1} C_R e^{-A_R(1-x_\epsilon)/\epsilon}. \end{cases} \tag{3.39}$$

For $m \geq 2$, (3.37) instead becomes

$$\begin{cases} v\left(-\frac{x_\epsilon(\sigma)}{\epsilon}, \sigma\right) \sim -\delta_m^{-1} \left(-\frac{m!}{A_L(m-1)} \frac{\epsilon}{x_\epsilon}\right)^{\frac{1}{m-1}} \\ v\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}, \sigma\right) \sim \delta_m^{-1} \left(\frac{m!}{|A_R|(m-1)} \frac{\epsilon}{1-x_\epsilon}\right)^{\frac{1}{m-1}}. \end{cases} \tag{3.40}$$

A direct integration of (3.35) with respect to z shows that v satisfies the integro-differential equation

$$v_z + G'(u_p)v = \beta(\sigma) - \frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \frac{dx_\epsilon}{d\sigma} u_p + \int_0^z H[v(s,\sigma)]ds,$$

where $\beta(\sigma)$ is a constant of integration. Recalling that $u_p''(z) = -G'(u_p)u_p'$, we can integrate to obtain the integro-differential equation

$$\begin{aligned} v(z,\sigma) = \alpha(\sigma)u_p'(z) + \beta(\sigma)u_p'(z) \int_0^z \frac{ds}{u_p'(s)} - \frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \frac{dx_\epsilon}{d\sigma} u_p'(z) \int_0^z \frac{u_p(s)\,ds}{u_p'(s)} \\ +u_p'(z) \int_0^z \frac{1}{u_p'(s)} \int_0^s H[v(r,\sigma)]\,dr\,ds \end{aligned} \tag{3.41}$$

for constants of integration $\alpha(\sigma)$ and $\beta(\sigma)$. Using the interior condition (3.38), we find $\alpha(\sigma) \equiv 0$. We'd expect to solve the resulting equation by a successive approximations scheme, beginning with the approximation obtained by neglecting the final term. This yields the approximation

$$v(z,\sigma) \sim u_p'(z) \left[\beta(\sigma) \int_0^z \frac{ds}{u_p'(s)} - \frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \frac{dx_\epsilon}{d\sigma} \int_0^z \frac{u_p(s)\,ds}{u_p'(s)}\right]. \tag{3.42}$$

Since in many cases the profile function $u_p(z)$ will only be known implicitly, it would be useful to rewrite the integrals in (3.42) without reference to it. This can be accomplished by multiplying the integrands by $1 = \dfrac{u_p'(s)}{G(u_L) - G(u_p(s))}$ and converting equation (3.42) to

$$v(z,\sigma) \sim [G(u_L) - G(u_p(z))] \left[\beta(\sigma) \int_{u_p(0)}^{u_p(z)} \frac{dr}{[G(u_L) - G(r)]^2} \right.$$
$$\left. - \frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \frac{dx_\epsilon}{d\sigma} \int_{u_p(0)}^{u_p(z)} \frac{r dr}{[G(u_L) - G(r)]^2} \right].$$

There remain the two boundary conditions for the remaining unknowns $\beta(\sigma)$ and $\frac{dx_\epsilon}{d\sigma}$; thus, we have the approximate linear system of two equations in two variables:

$$\begin{cases} \beta(\sigma) \int_{u_p(0)}^{u_p\left(-\frac{x_\epsilon}{\epsilon}\right)} \frac{dr}{[G(u_L)-G(r)]^2} - \frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \frac{dx_\epsilon}{d\sigma} \int_{u_p(0)}^{u_p\left(-\frac{x_\epsilon}{\epsilon}\right)} \frac{r dr}{[G(u_L)-G(r)]^2} \sim \frac{v\left(-\frac{x_\epsilon}{\epsilon},\sigma\right)}{G(u_L)-G(u_p(-x_\epsilon/\epsilon))} \\ \\ \beta(\sigma) \int_{u_p(0)}^{u_p\left(\frac{1-x_\epsilon}{\epsilon}\right)} \frac{dr}{[G(u_L)-G(r)]^2} - \frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \frac{dx_\epsilon}{d\sigma} \int_{u_p(0)}^{u_p\left(\frac{1-x_\epsilon}{\epsilon}\right)} \frac{r dr}{[G(u_L)-G(r)]^2} \sim \frac{v\left(\frac{(1-x_\epsilon)}{\epsilon},\sigma\right)}{G(u_R)-G(u_p((1-x_\epsilon)/\epsilon))}. \end{cases}$$

Let $p_L(\sigma) \equiv u_p\left(-\frac{x_\epsilon}{\epsilon}\right) - u_L$ and $p_R(\sigma) \equiv u_R - u_p\left(\frac{1-x_\epsilon}{\epsilon}\right)$. Solving by using Cramer's rule then asymptotically provides an ODE for $\frac{dx_\epsilon}{d\sigma}$, namely

$$\frac{dx_\epsilon}{d\sigma} \sim \frac{\begin{vmatrix} \int_{u_p(0)}^{u_p(-x_\epsilon/\epsilon)} \frac{dr}{[G(u_L)-G(r)]^2} & \frac{v(-\frac{x_\epsilon}{\epsilon},\sigma)}{G(u_L)-G(u_p(-x_\epsilon/\epsilon))} \\ \int_{u_p(0)}^{u_p((1-x_\epsilon)/\epsilon)} \frac{dr}{[G(u_R)-G(r)]^2} & \frac{v(\frac{(1-x_\epsilon)}{\epsilon},\sigma)}{G(u_R)-G(u_p((1-x_\epsilon)/\epsilon))} \end{vmatrix}}{\begin{vmatrix} \int_{u_p(0)}^{u_p(-x_\epsilon/\epsilon)} \frac{dr}{[G(u_L)-G(r)]^2} & -\frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \int_{u_p(0)}^{u_p(-x_\epsilon/\epsilon)} \frac{rdr}{[G(u_L)-G(r)]^2} \\ \int_{u_p(0)}^{u_p((1-x_\epsilon)/\epsilon)} \frac{dr}{[G(u_L)-G(r)]^2} & -\frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \int_{u_p(0)}^{u_p((1-x_\epsilon)/\epsilon)} \frac{rdr}{[G(u_R)-G(r)]^2} \end{vmatrix}}$$

$$\sim \frac{\begin{vmatrix} -\left(\frac{m!}{A_L}\right)^2 \frac{1}{2m-1} (p_L(\sigma))^{1-2m} & \frac{v(-\frac{x_\epsilon}{\epsilon},\sigma)}{\frac{|A_L|}{m!}(p_L(\sigma))^m} \\ \left(\frac{m!}{A_R}\right)^2 \frac{1}{2m-1} (p_R(\sigma))^{1-2m} & \frac{v(\frac{1-x_\epsilon}{\epsilon},\sigma)}{\frac{|A_R|}{m!}(p_R(\sigma))^m} \end{vmatrix}}{\begin{vmatrix} -\left(\frac{m!}{A_L}\right)^2 \frac{1}{2m-1} (p_L(\sigma))^{1-2m} & \frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \left(\frac{m!}{A_L}\right)^2 \frac{u_L}{2m-1} (p_L(\sigma))^{1-2m} \\ \left(\frac{m!}{A_R}\right)^2 \frac{1}{2m-1} (p_R(\sigma))^{1-2m} & -\frac{\phi_m(\epsilon)}{\delta_m(\epsilon)} \left(\frac{m!}{A_R}\right)^2 \frac{u_R}{2m-1} (p_R(\sigma))^{1-2m} \end{vmatrix}}$$

$$= \frac{\delta_m(\epsilon)(2m-1)}{\phi_m(\epsilon)\, m!(u_R-u_L)} \left[A_L (p_L(\sigma))^{m-1} v\left(-\tfrac{x_\epsilon}{\epsilon},\sigma\right) - |A_R| (p_R(\sigma))^{m-1} v\left(\tfrac{1-x_\epsilon}{\epsilon},\sigma\right)\right] \tag{3.43}$$

using the same sort of analysis of integrals as we have used before. For example, for fixed appropriate $\kappa > 0$,

$$\begin{aligned}
\int_{u_p(0)}^{u_p(-x_\epsilon/\epsilon)} \frac{dr}{[G(u_L)-G(r)]^2} &\sim \int_{u_p(-x_\epsilon/\epsilon)+\kappa}^{u_p(-x_\epsilon/\epsilon)} \frac{dr}{[G(u_L)-G(r)]^2} \\
&\sim \int_{u_p(-x_\epsilon/\epsilon)+\kappa}^{u_p(-x_\epsilon/\epsilon)} \frac{dr}{\left[-\frac{A_L}{m!}(r-u_L)^m\right]^2} \\
&= -\left(\frac{m!}{A_L}\right)^2 \frac{1}{2m-1}\left[\frac{1}{(p_L(\sigma))^{2m-1}} - \frac{1}{(p_L(\sigma)+\kappa)^{2m-1}}\right] \\
&\sim -\left(\frac{m!}{A_L}\right)^2 \frac{1}{2m-1}(p_L(\sigma))^{1-2m}.
\end{aligned}$$

Then (3.39) and (3.40) provide

$$\frac{dx_\epsilon}{d\sigma} \sim \begin{cases} -\frac{1}{\phi_1(\epsilon)(u_R-u_L)}\left(A_L C_L e^{A_L x_\epsilon/\epsilon} + A_R C_R e^{-A_R(1-x_\epsilon)/\epsilon}\right) & \text{for } m=1 \\ \\ \frac{2m-1}{\phi_m(\epsilon)(u_R-u_L)}\left[\frac{\epsilon^m m!}{(m-1)^m}\right]^{\frac{1}{m-1}}\left[\left(\frac{1}{x_\epsilon^m|A_L|}\right)^{\frac{1}{m-1}} - \left(\frac{1}{(1-x_\epsilon)^m|A_R|}\right)^{\frac{1}{m-1}}\right] & \text{for } m \geq 2. \end{cases} \tag{3.44}$$

One could analogously determine $\beta(\sigma)$ and thereby obtain an $O(1)$ approximation for v. Note that the ODE obtained for x_ϵ is autonomous. Thus, one can integrate the PDE for a short time numerically to determine the initial value necessary to solve the ODE. Also, note that the motion of the shock layer toward its steady-state location is monotonic. The steady-state locations predicted by these ODEs corroborate our previous result in Section 3.2.

At this point, referring back to (3.39) and (3.40), it is a simple matter to choose appropriate gauge functions. For $m=1$, we cannot make the shock slow at all times, but we can choose it to be so when the shock is in the vicinity of its steady-state location $x_\epsilon(\infty) \sim \frac{A_R}{A_L+A_R}$. Thus a natural choice is

$$\phi_1(\epsilon) = e^{-A/\epsilon} \tag{3.45}$$

where A is defined by (3.21). Equation (3.36) then provides the scaling factor

$$\delta_1(\epsilon) = e^{-A/\epsilon}. \tag{3.46}$$

For $m \geq 2$, we use (3.40) to select

$$\phi_m(\epsilon) = \epsilon^{\frac{m}{m-1}} \tag{3.47}$$

so (3.36) implies

$$\delta_m(\epsilon) = \epsilon^{\frac{m}{m-1}}. \tag{3.48}$$

We emphasize the emergence of the pattern again. For $m = 1$, the shock moves on an exponentially asymptotically slow time-scale, at least near its rest location. For $m > 1$, the shock moves on an algebraically asymptotically slow time-scale. As m increases, the motion is faster. However, as $m \to \infty$, the motion approaches an $O(\epsilon^2)$ time-scale. As we might expect, the shooting parameter $\gamma_m(\epsilon)$ from Section 3.2 satisfies

$$\gamma_m(\epsilon) = \delta_m(\epsilon).$$

Finally, we'd like our shock motion (3.44) to be corroborated by our more general result (2.32), up to the undetermined time-scale in (2.32), taking into account that we've rescaled time by a factor of ϵ. Note that

$$\exp\left(\int_0^z g(u_p(r))dr\right) = \frac{G(u_L) - G(u_p(0))}{G(u_L) - G(u_p(z))} = \frac{u_p'(0)}{u_p'(z)}.$$

(Interestingly, this implies that (2.23) holds.) Referring back to (2.31), we now have

$$\begin{cases} I_1(z) = \frac{1}{G(u_L)-G(u_p(0)} \int_{u_p(0)}^{u_p(z)} \frac{dr}{[G(u_L)-G(r)]^2} \\ I_2(z) = u_p(z) \int_{u_p(0)}^{u_p(z)} \frac{dr}{[G(u_L)-G(r)]^2} - \int_{u_p(0)}^{u_p(z)} \frac{r\,dr}{[G(u_L)-G(r)]^2} \\ I_3(z) = [G(u_L) - G(u_p(0))][u_p(z) - u_p(0)]. \end{cases}$$

Plugging these into (2.32), we re-obtain (3.44).

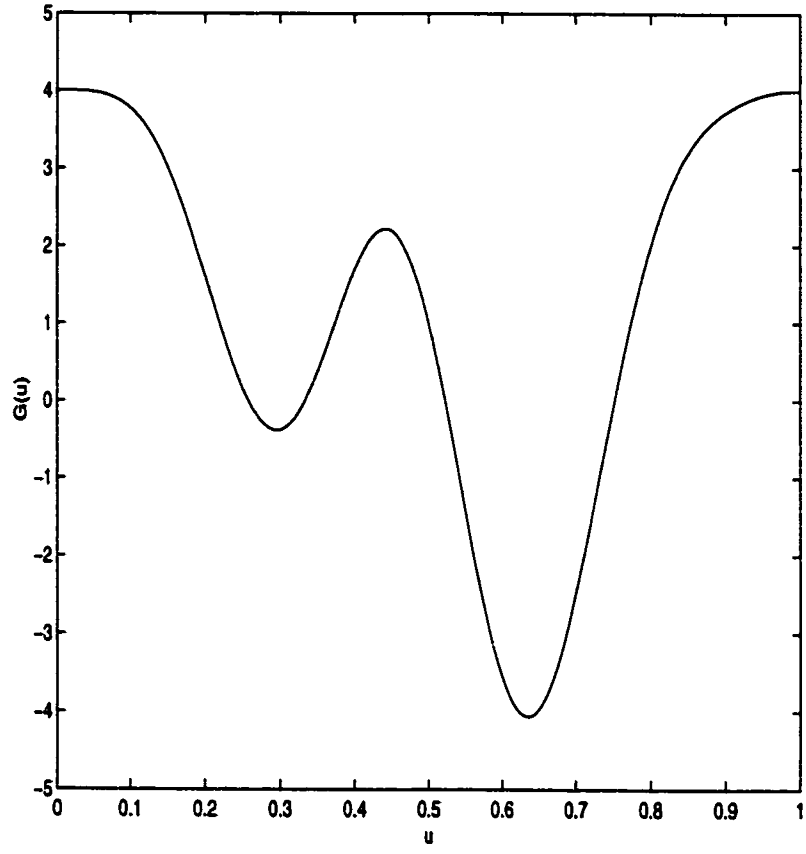

Figure 3.1: **A Typical Potential** $G(u)$**.** This a plot of the function $G(u) = 4 - 100(u-u^2)^3((u-\frac{1}{2})^7 - u^4 + 7u^3 - 2u^2 + 5\sin^2(7u) + 1)$. Since third-order zeros occur at the endpoints, $m = 3$.

Table 3.1: **The Shock Location of a Steady-State Advection-Diffusion Equation.** We compute the shock location x_ϵ of the steady-state viscous shock problem $\epsilon u_{xx} - [(u-u^3)^2]_x = 0$, $u(0) = 0$, $u(1) = 1$. Using bisection, we numerically find the integration constant $\gamma_2(\epsilon)$ via (3.15). Then we numerically integrate the quotient in (3.16) to determine x_ϵ. This corroborates (3.25) which predicts that $x_\epsilon \sim \frac{2}{3}$ as $\epsilon \to 0^+$. Note, however, the relatively slow approach to this limiting value as ϵ decreases.

ϵ	$\gamma_2(\epsilon)$	x_ϵ
0.1	$4.86875\,\epsilon^2$	0.560447
0.01	$5.799385\,\epsilon^2$	0.642740
0.001	$5.5969841\,\epsilon^2$	0.663108
0.0001	$5.55810085\,\epsilon^2$	0.666195

Chapter 4

ASYMPTOTIC EVALUATION OF ALGEBRAICALLY SINGULAR INTEGRALS FOR THE STEADY-STATE VISCOUS SHOCK PROBLEM

In applied mathematics we are often concerned with the asymptotic evaluation of integrals. In other words, we need to study the behavior of

$$I(\lambda) = \int_C F(z;\lambda)dz \tag{4.1}$$

as $\lambda \to \lambda_0$ for some known kernel $F(z;\lambda)$ and (possibly) complex contour C. Many techniques exist for classes of such problems; see [4], [5], [12], and [63] for a sample of the available literature. Most of them require the integrand to behave as a real or imaginary exponential in the limit $\lambda \to \lambda_0$. (Two exceptions are integration by parts and Mellin transform techniques.) We present here an additional technique for the asymptotic evaluation of integrals with algebraically singular kernels. Instead of a more formal presentation, we provide examples by discussing applications to material from Chapter 3.

In Section 3.2 we discussed the viscous shock problem at equilibrium, and this led to consideration of the important integral (3.18):

$$\int_{u_L}^{u_L+\kappa} \frac{dp}{\gamma_m(\epsilon) - G(p) + G(u_L)} \tag{4.2}$$

for fixed positive κ in the limit $\gamma_m(\epsilon) \to 0^+$. Here, $m \geq 2$. Let $a^m \equiv -\frac{m!\gamma_m(\epsilon)}{A_L}$. By approximating $G(p)$ near its endpoints, we easily see that

$$\int_{u_L}^{u_L+\kappa} \frac{dp}{\gamma_m(\epsilon) - G(p) + G(u_L)} \sim -\frac{m!}{A_L}\int_0^{\kappa} \frac{dq}{a^m + q^m}. \tag{4.3}$$

We need to approximate the latter integral as $a \to 0^+$, but the usual integral approximation techniques do not readily apply. Integration by parts seems only to create a more difficult integrand, and the non-exponential character of the integrand prevents the straightforward applicability of Laplace's method and the like. The simple approach of Section 3.2 yields a first approximation, but is not amenable to finding further improvements to this approximation, if desired. Thus, we have developed our own technique for approximating such integrals. The algebraic nature of the singularity of the integrand makes it convenient to carry out the analysis in the complex q-plane.

First, we convert to an integral on a semi-infinite contour. Let $q = \left(u + \frac{1}{\kappa}\right)^{-1}$, so

$$\int_0^{\kappa} \frac{dq}{a^m + q^m} = \int_0^{\infty} \frac{\left(u + \frac{1}{\kappa}\right)^{m-2} du}{1 + a^m \left(u + \frac{1}{\kappa}\right)^m} \equiv I. \tag{4.4}$$

Let Ω be the complex "keyhole" contour about the origin, extending out the positive real axis, and define

$$J \equiv \oint_{\Omega} \frac{(\log z)\left(z + \frac{1}{\kappa}\right)^{m-2} dz}{1 + a^m \left(z + \frac{1}{\kappa}\right)^m}. \tag{4.5}$$

We take the branch cut for the logarithm along the positive real axis. As usual, the contributions to J from the small interior circle of radius ρ and from the large exterior circle of radius R vanish as $\rho \to 0^+$ and $R \to \infty$. So,

$$\begin{aligned} J &= \int_0^{\infty} \frac{(\log u)\left(u + \frac{1}{\kappa}\right)^{m-2} du}{1 + a^m \left(u + \frac{1}{\kappa}\right)^m} - \int_0^{\infty} \frac{\left(\log\left(u\, e^{2\pi i}\right)\right)\left(u + \frac{1}{\kappa}\right)^{m-2} du}{1 + a^m \left(u + \frac{1}{\kappa}\right)^m} \\ &= -2\pi i I. \end{aligned} \tag{4.6}$$

Using the Cauchy Residue Theorem, however, we also have

$$J = 2\pi i \sum_{n=0}^{m-1} \operatorname{Res}\left(\frac{(\log z)(z + \frac{1}{\kappa})^{m-2}}{1 + a^m (z + \frac{1}{\kappa})^m}; z_n\right) \tag{4.7}$$

where z_n are the m roots of the polynomial $1 + a^m \left(z + \frac{1}{\kappa}\right)^m = 0$. Thus, $z_n = -\frac{1}{\kappa} + \frac{1}{a} e^{i\pi(1+2n)/m}$, where $n = 0, \ldots, m-1$. We are fortunate in this case that we can find the poles of the integrand exactly. In other cases, we must approximate their location asymptotically, which adds another layer of approximation to the problem but one that is not prohibitive.

Now, as $a \to 0^+$,

$$\begin{aligned}\operatorname{Res}\left(\frac{(\log z)(z+\frac{1}{\kappa})^{m-2}}{1+a^m(z+\frac{1}{\kappa})^m}; z_n\right) &= \left.\frac{\log z}{ma^m\left(z+\frac{1}{\kappa}\right)}\right|_{z=z_n} \\ &= \frac{\log\left(-\frac{1}{\kappa}+\frac{1}{a}e^{i\pi(1+2n)/m}\right)}{ma^{m-1}e^{i\pi(1+2n)/m}} \\ &\sim \frac{-\log a+\frac{i\pi(1+2n)}{m}}{ma^{m-1}e^{i\pi(1+2n)/m}}.\end{aligned} \tag{4.8}$$

So, using (4.7) and (4.8), as well as well-known formulae for the sums of these finite series [24],

$$\begin{aligned}J &\sim \frac{2\pi i}{ma^{m-1}}\sum_{n=0}^{m-1}\frac{\left(-\log a+\frac{i\pi(1+2n)}{m}\right)}{e^{i\pi(1+2n)/m}} \\ &= \frac{2\pi i e^{-i\pi/m}}{ma^{m-1}}\left[-\log a\sum_{n=0}^{m-1}\left(e^{-2\pi i/m}\right)^n+\frac{i\pi}{m}\sum_{n=0}^{m-1}(1+2n)\left(e^{-2\pi i/m}\right)^n\right] \\ &= -\frac{2\pi^2}{m^2a^{m-1}}e^{-\pi i/m}\left(-\frac{2m}{1-e^{-2\pi i/m}}\right) \\ &= -\frac{2\pi^2 i}{ma^{m-1}}\csc\left(\frac{\pi}{m}\right).\end{aligned}$$

Finally, we are able to approximate

$$I \sim \frac{\pi}{ma^{m-1}}\csc\left(\frac{\pi}{m}\right) = \frac{\pi}{m}\csc\left(\frac{\pi}{m}\right)\left(-\frac{A_L}{m!\gamma_m(\epsilon)}\right)^{\frac{m-1}{m}}.$$

Using the analogous estimate near the right endpoint, we obtain the estimates (3.24) and (3.25).

We can use the same techniques to find the steady-state shock location x_ϵ in (3.26). We again define x_ϵ by $u(x_\epsilon) = \frac{1}{2}(u_L+u_R)$. As in (3.17), we have the representation

$$x_\epsilon = \frac{\int_{u_L+\alpha\epsilon^{\frac{1}{m-1}}}^{\frac{u_L+u_R}{2}}\frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)}}{\int_{u_L+\alpha\epsilon^{\frac{1}{m-1}}}^{u_R-\beta\epsilon^{\frac{1}{m-1}}}\frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)}}, \tag{4.9}$$

where $\gamma_m(\epsilon)$ is to be determined by using the boundary data. Proceeding as above, we'll show that a major contribution to the integral in the numerator is changed by an $O(1)$ amount by the perturbations, which implies that the steady-state shock location will often be likewise changed. For an appropriate $O(1)$ κ, we have

$$\begin{aligned}
\int_{u_L+\alpha\epsilon^{\frac{1}{m-1}}}^{\frac{u_L+u_R}{2}} & \left[\frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)}\right] \\
&\sim \int_{u_L+\alpha\epsilon^{\frac{1}{m-1}}}^{u_L+\alpha\epsilon^{\frac{1}{m-1}}+\kappa} \left[\frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)}\right] \\
&\sim -\frac{m!}{A_L}\int_{\alpha\epsilon^{\frac{1}{m-1}}}^{\alpha\epsilon^{\frac{1}{m-1}}+\kappa} \frac{dq}{a^m+q^m} \quad \text{for } a^m \equiv -\tfrac{m!\gamma_m(\epsilon)}{A_L} > 0 \\
&= -\frac{m!}{A_L}\int_0^\infty \frac{\left(u+\frac{1}{\kappa}\right)^{m-2} du}{\left(1+\alpha\epsilon^{\frac{1}{m-1}}\left(u+\frac{1}{\kappa}\right)\right)^m + a^m\left(u+\frac{1}{\kappa}\right)^m} \\
&= \frac{m!}{A_L}\sum_{n=0}^{m-1} \operatorname{Res}\left(\frac{\left(z+\frac{1}{\kappa}\right)^{m-2}}{\left(1+\alpha\epsilon^{\frac{1}{m-1}}\left(z+\frac{1}{\kappa}\right)\right)^m + a^m\left(z+\frac{1}{\kappa}\right)^m}; z_n\right),
\end{aligned}$$

where the z_n, the m simple roots of the kernel's denominator, are given explicitly by

$$z_n = \frac{1}{ae^{i\pi(1+2n)/m} - \alpha\epsilon^{\frac{1}{m-1}}} - \frac{1}{\kappa}.$$

Now, approximate the nth residue:

$$\begin{aligned}
\operatorname{Res} & \left(\frac{(\log z)\left(z+\frac{1}{\kappa}\right)^{m-2}}{\left(1+\alpha\epsilon^{\frac{1}{m-1}}\left(z+\frac{1}{\kappa}\right)\right)^m + a^m\left(z+\frac{1}{\kappa}\right)^m}; z_n\right) \\
&\sim -\frac{\log a + \frac{i\pi(1+2n)}{m} + \log\left(1 - e^{i\pi(1+2n)/m}\frac{\alpha\epsilon^{\frac{1}{m-1}}}{a}\right)}{ma^{m-1}e^{-i\pi(1+2n)/m}}.
\end{aligned}$$

Summing the residues, we find that

$$\begin{aligned}
\int_{u_L+\alpha\epsilon^{\frac{1}{m-1}}}^{\frac{u_L+u_R}{2}} & \frac{dp}{\gamma_m(\epsilon)-G(p)+G(u_L)} \\
&\sim -\frac{m!}{A_L}\left[\frac{\pi}{ma^{m-1}}\csc\left(\frac{\pi}{m}\right)\right. \\
&\qquad \left. + \frac{e^{i\pi/m}}{ma^{m-1}}\sum_{n=0}^{m-1}\left(e^{2\pi i/m}\right)^n \log\left(1 - \alpha e^{i\pi(1+2n)/m}\frac{\alpha\epsilon^{\frac{1}{m-1}}}{a}\right)\right].
\end{aligned}$$

Obtaining the analogous estimate near the right endpoint, one easily sees that $a(\epsilon) = O_s\left(\epsilon^{\frac{1}{m-1}}\right)$, implying that $\gamma(\epsilon) = O_s\left(\epsilon^{\frac{m}{m-1}}\right)$, as before. More importantly, we see that $O_s\left(\epsilon^{\frac{1}{m-1}}\right)$ perturbations in the boundary values have an $O(1)$ effect in the steady-state shock location, sometimes even changing the internal layer to a boundary layer. (See Table 4.1.)

We illustrate this with a particular example, the problem studied in Sections 2.6 and 3.1:

$$\begin{cases} \epsilon u_{xx} - [\sin^2(\pi u)]_x = 0 \\ u(0) = \alpha\epsilon, \ u(1) = 1 - \beta\epsilon. \end{cases} \tag{4.10}$$

α and β are positive, $O(1)$-constants. Assume for this problem that $\gamma_2(\epsilon) \sim D^2\epsilon^2$, where D is a positive constant. Using (3.15), we have

$$\begin{aligned} \frac{1}{\epsilon} &= \int_{\alpha\epsilon}^{1-\beta\epsilon} \frac{dp}{\gamma_2(\epsilon) + \sin^2(\pi p)} \\ &\sim \int_{\alpha\epsilon}^{1-\beta\epsilon} \frac{dp}{D^2\epsilon^2 + \sin^2(\pi p)} \\ &= \frac{1}{\pi}\left(\int_{\pi\alpha\epsilon}^{\pi/2} + \int_{\pi\beta\epsilon}^{\pi/2}\right) \frac{dv}{D^2\epsilon^2 + \sin^2 v} \\ &\sim \frac{1}{D\epsilon} - \frac{\tan^{-1}\left(\frac{\pi\alpha}{D}\right)}{\pi D\epsilon} - \frac{\tan^{-1}\left(\frac{\pi\beta}{D}\right)}{\pi D\epsilon}, \end{aligned} \tag{4.11}$$

which is an algebraic equation that asymptotically determines D. Note that $D = 1$ for $\alpha = \beta = 0$, as we expect from the unperturbed case. Similarly, using (3.16), we have

$$\begin{aligned} x_\epsilon &\sim \epsilon \int_{\alpha\epsilon}^{1/2} \frac{dp}{D^2\epsilon^2 + \sin^2(\pi p)} \\ &\sim \frac{1}{2D} - \frac{\tan^{-1}\left(\frac{\pi\alpha}{D}\right)}{\pi D} \\ &\sim 1 - \frac{1}{2D} + \frac{\tan^{-1}\left(\frac{\pi\beta}{D}\right)}{\pi D}. \end{aligned}$$

Sensitivity to exponentially asymptotically small changes in the boundary values has been called *supersensitivity* or *exponential supersensitivity* and has been previously demonstrated (e.g. in [34], [36], [37]). For $m \geq 2$, we have demonstrated an *algebraic supersensitivity*, which may be more observable in physical systems, since algebraically small perturbations can be more accurately created. This algebraic supersensitivity is demonstrated in Table 4.1.

Table 4.1: **An Example of Algebraic Supersensitivity for an Advection-Diffusion Equation.** We tabulate the shock location x_ϵ of (4.10) for decreasing ϵ. We have taken $\alpha = 1$ and $\beta = 0$. Using a bisection method, we numerically find $\gamma_2(\epsilon)$ via (4.11). Then we numerically integrate the quotient in (4.9) to determine x_ϵ. Note that, as $\epsilon \to 0^+$, the interior layer at $x = \frac{1}{2}$ in the unperturbed problem destabilizes, leading to a left boundary boundary.

ϵ	$\gamma_2(\epsilon)$	x_ϵ
0.1	$0.305485\,\epsilon^2$	2.48771×10^{-2}
0.01	$0.3088026\,\epsilon^2$	3.60343×10^{-2}
0.001	$0.30883634\,\epsilon^2$	3.61387×10^{-2}
0.0001	$0.308836683\,\epsilon^2$	3.61397×10^{-2}

Chapter 5

REACTION-DIFFUSION EQUATIONS

We now turn to a different set of parabolic-type equations, those of reaction-diffusion type. We will consider scalar equations of the form

$$u_t = \epsilon^2 u_{xx} + h(u) \tag{5.1}$$

with

$$u(0,t) = u_L,\ u(1,t) = u_R,\ u(x,0) = u_I(x). \tag{5.2}$$

As noted in the introduction, entire texts have been devoted to the study of equation (5.1) and its many applications. Among this large body of research, we note that [10] and [21] dealt with the same equation under corresponding Neumann conditions and that [61] was highly successful under Robin (but nearly Dirichlet) conditions.

We shall restrict $h(u)$ to be of *bistable* type. Let u_C be a zero of h such that that $u_L < u_C < u_R$, where the smooth nonlinearity $h(u)$ and the boundary values satisfy

$$\begin{cases} h(u_L) = h(u_C) = h(u_R) = 0 \\ h'(u_C) > 0 \\ H(u) \equiv \int_{u_L}^{u} h(s)\,ds \begin{cases} < 0,\ u_L < u < u_R \\ = 0,\ u = u_R. \end{cases} \end{cases} \tag{5.3}$$

We take $h(u)$ to have only these three roots on $[u_L, u_R]$. Typically, it is assumed that $h'(u_L) < 0$ and that $h'(u_R) < 0$. In analogy to Chapter 3, we shall instead consider a broader set of problems. Given an h satisfying (5.3), define the order $n \geq 1$ to be the integer such that

$$\begin{cases} \frac{d^k h}{du^k}(u_L) = \frac{d^k h}{du^k}(u_R) = 0 \text{ for } 0 \le k < n \\ \\ B_L \equiv \frac{d^n h}{du^n}(u_L) < 0 \quad \text{and} \quad (-1)^n B_R \equiv (-1)^n \frac{d^n h}{du^n}(u_R) > 0. \end{cases} \tag{5.4}$$

Thus, n determines how "flat" $h(u)$ is at both end-values u_L and u_R. If the order of the zeros was not the same at both u_L and u_R we would expect endpoint layers, so we do not consider this case.

5.1 *An Example of Algebraic Asymptotics, Using the Travelling Wave Ansatz*

Consider the example

$$\begin{cases} u_t = \epsilon^2 u_{xx} + u(1-u^2)^2, \ 0 < x < 1, \ t > 0, \ 0 < \epsilon \ll 1 \\ u(x,0) = u_I(x), \ u(0,t) = -1, \ u(1,t) = 1. \end{cases} \tag{5.5}$$

Note the second-order zeros of $h(u)$ at both $u_L = -1$ and $u_R = 1$, implying that $n = 2$.

First, we discuss the steady-state problem

$$\begin{cases} \epsilon^2 u_{xx} + u(1-u^2)^2 = 0, \ 0 < x < 1, \ 0 < \epsilon \ll 1 \\ u(0) = -1, \ u(1) = 1. \end{cases} \tag{5.6}$$

The steady-state shock location is easily determined to be $u(x_\epsilon) = \frac{1}{2}$ for all ϵ, by symmetry, since the problem is invariant under the transformation $x \to 1 - x$ and $u \to -u$. We also have the implicit solution

$$\frac{x}{\epsilon} = \int_{-1}^{u} \frac{dv}{\sqrt{\frac{1}{3}(1-v^2)^3 + C(\epsilon)}}$$

where the constant of integration $C(\epsilon)$ must satisfy

$$\epsilon \int_{-1}^{1} \frac{dv}{\sqrt{\frac{1}{3}(1-v^2)^3 + C(\epsilon)}} = 1.$$

Define $z = \frac{x - x_\epsilon(\sigma)}{\epsilon}$ and the somewhat novel (and unexpected) slow-time

$$\sigma = \epsilon^7 t.$$

Returning to the time-dependent problem (5.5), we wish to compute the shock profile function $u_p(z)$ satisfying

$$\begin{cases} u_p'' + u_p(1 - u_p^2)^2 = 0 \\ u_p(z) \to -1 \text{ as } z \to -\infty, \; u_p(0) = 0, \text{ and } u_p(z) \to 1 \text{ as } z \to \infty. \end{cases}$$

Assuming that the solution $u_p(z)$ is strictly monotonically increasing, we can multiply the ODE by $u_p'(z)$ and integrate to obtain

$$u_p'(z) = \sqrt{-2\int_{-1}^{u_p(z)} s(1 - s^2)^2 ds} = \sqrt{\frac{1}{3}(1 - u_p^2)^3}.$$

Separating variables and integrating again, we obtain the asymptotic profile function

$$u_p(z) = \frac{z}{\sqrt{z^2 + 3}} \tag{5.7}$$

which, for large $|z|$, $u_p(z)$ has the behavior

$$\begin{cases} u_p(z) \sim -1 + \frac{3}{2z^2} - \frac{27}{8z^4} + \frac{135}{16z^6} + O(z^{-8}) & \text{as } z \to -\infty \\ u_p(z) \sim 1 - \frac{3}{2z^2} + \frac{27}{8z^4} - \frac{135}{16z^6} + O(z^{-8}) & \text{as } z \to \infty. \end{cases} \tag{5.8}$$

Thus, $u_p(z)$ decays algebraically to its rest-point values at $z = \pm\infty$. As usual, $u_p(z)$ is an approximate solution to (5.5) with a residual error of $\epsilon^6 u_p' \dot{x}_\epsilon$. So, for large t, we seek a solution to (5.5) using the Ansatz

$$u(x,t) \sim u_p(z) + \epsilon^6 v(z, \sigma, \epsilon) \tag{5.9}$$

under the condition $v(0, \sigma, \epsilon) = 0$ and boundary conditions determined by (5.8).

Plugging our Ansatz into (5.5), we obtain, to highest order, the limiting equation for $v(z, \sigma, \epsilon)$:

$$v_{zz} - (1 - u_p^2)(5u_p^2 - 1)v = v_{zz} - \frac{3(4z^2 - 3)}{(z^2 + 3)^2} v \sim -u_p' \frac{dx_\epsilon}{d\sigma} = -3(z^2 + 3)^{-3/2} \frac{dx_\epsilon}{d\sigma}.$$

We treat this as a second-order linear ODE for v. Since $u_p'(z)$ satisfies the homogeneous version of this equation, a straightforward application of reduction of order yields the solution

$$
\begin{aligned}
v(z,\sigma,\epsilon) &\sim C(\sigma)\frac{z(5z^6+63z^4+315z^2+945)}{(z^2+3)^{3/2}} \\
&\quad + \frac{3}{70(z^2+3)^{3/2}}\frac{dx_\epsilon}{d\sigma}\left[-\frac{1}{12}z^2(5z^4+58z^2+177)+18\log\left(\frac{z^2+3}{3}\right)\right. \\
&\quad \left. -\frac{\sqrt{3}z}{36}(5z^6+63z^4+315z^2+945)\arctan\left(\frac{z\sqrt{3}}{3}\right)\right] \\
&\sim \pm 5C(\sigma)z^4 - \frac{dx_\epsilon}{d\sigma}\frac{z^4\pi\sqrt{3}}{336} \qquad \text{as } z\to\pm\infty.
\end{aligned}
\tag{5.10}
$$

Here, $C(\sigma)$ is an unspecified constant of integration, and we have used $v(0,\sigma,\epsilon)\equiv 0$ to eliminate the other. From (5.8), $v(z,\sigma,\epsilon)\sim \pm\frac{3}{2z^2\epsilon^6}$ as $z\to\pm\infty$. Applying these boundary values in (5.10), we have two linear equations for the two unknowns $C(\sigma)$ and $\frac{dx_\epsilon}{d\sigma}$. These can be easily solved to provide

$$
\frac{dx_\epsilon}{d\sigma}\sim\frac{252}{\pi\sqrt{3}}\left[\frac{1}{x_\epsilon^6}-\frac{1}{(1-x_\epsilon)^6}\right] \tag{5.11}
$$

and

$$
C(\sigma)\sim\frac{3}{20}\left[\frac{1}{x_\epsilon^6}+\frac{1}{(1-x_\epsilon)^6}\right].
$$

Thus, we've found an approximation to the solution u of (5.5) accurate to $O(\epsilon^2)$, as well as a first approximation to the equation governing the motion of the slowly moving interior layer. Note that (5.11) correctly predicts the steady-state shock location, $x_\epsilon(\infty)=\frac{1}{2}$, as well as the monotonic decay to it.

5.2 The Steady-State Problem

We consider the steady-state problem for $u(x)$ associated with (5.1):

$$
\epsilon^2u''+h(u)=0, u(0)=u_L, u(1)=u_R. \tag{5.12}
$$

Assuming a strictly monotonic solution, we can multiply (5.12) by u' and integrate to find

$$u'(x) = \sqrt{[u'(0)]^2 - \frac{2}{\epsilon^2}H(u(x))}. \tag{5.13}$$

Note that this implies the symmetry condition

$$u'(0) = u'(1). \tag{5.14}$$

Separating variables and integrating, we obtain the implicit solution

$$x = \epsilon \int_{u_L}^{u} \frac{dv}{\sqrt{[\epsilon u'(0)]^2 - 2H(v)}} \tag{5.15}$$

with $u'(0)$ determined by

$$1 = \epsilon \int_{u_L}^{u_R} \frac{dv}{\sqrt{[\epsilon u'(0)]^2 - 2H(v)}}. \tag{5.16}$$

We are most interested in the shock location x_ϵ, given by

$$\begin{aligned} x_\epsilon &= \epsilon \int_{u_L}^{\frac{u_L+u_R}{2}} \frac{dv}{\sqrt{[\epsilon u'(0)]^2 - 2H(v)}} \\ &= \frac{\int_{u_L}^{\frac{u_L+u_R}{2}} \frac{dv}{\sqrt{[\epsilon u'(0)]^2-2H(v)}}}{\int_{u_L}^{u_R} \frac{dv}{\sqrt{[\epsilon u'(0)]^2-2H(v)}}}. \end{aligned} \tag{5.17}$$

To more explicitly determine x_ϵ as $\epsilon \to 0^+$, we must asymptotically estimate these integrals. Since $\displaystyle\int_{u_L}^{u_R} \frac{dv}{\sqrt{[\epsilon u'(0)]^2 - 2H(v)}} = \frac{1}{\epsilon}$ is large, and since $H(v) = 0$ only at $v = u_R$ and at $v = u_L$, the primary contributions to this integral must come from v values near both endpoints. For $\kappa > 0$ fixed, for example, with $u_L + \kappa < \frac{1}{2}(u_L + u_R)$,

$$\begin{aligned} \int_{u_L}^{u_L+\kappa} \frac{dv}{\sqrt{[\epsilon u'(0)]^2 - 2H(v)}} &\sim \int_{u_L}^{u_L+\kappa} \frac{dv}{\sqrt{[\epsilon u'(0)]^2 - \frac{2B_L}{(n+1)!}(v-u_L)^{n+1}}} \\ &= \sqrt{\frac{(n+1)!}{2|B_L|}} \int_0^{\kappa} \frac{dq}{\sqrt{a^{n+1} + q^{n+1}}} \\ &= a^{\frac{1-n}{2}} \sqrt{\frac{(n+1)!}{2|B_L|}} \int_0^{\kappa/a} \frac{ds}{\sqrt{1+s^{n+1}}}, \end{aligned} \tag{5.18}$$

where $a^{n+1} \equiv \frac{\epsilon^2(n+1)!(u'(0))^2}{2|B_L|}$. For $n = 1$, (5.18) reduces to

$$|B_L|^{-1/2} \int_0^{\kappa/a} \frac{ds}{\sqrt{1+s^2}} = |B_L|^{-1/2} \log\left(\frac{\kappa + \sqrt{\kappa^2 + a^2}}{a}\right) \sim -|B_L|^{-1/2} \log(a).$$

For $n \geq 2$, we instead have

$$\begin{aligned} a^{\frac{1-n}{2}} \sqrt{\frac{(n+1)!}{2|B_L|}} \int_0^{\kappa/a} \frac{ds}{\sqrt{1+s^{n+1}}} &\sim a^{\frac{1-n}{2}} \sqrt{\frac{(n+1)!}{2|B_L|}} \int_0^{\infty} \frac{ds}{\sqrt{1+s^{n+1}}} \\ &= \frac{a^{\frac{1-n}{2}}}{n+1} B\left(\frac{1}{n+1}, \frac{1}{2} - \frac{1}{n+1}\right) \\ &= \frac{a^{\frac{1-n}{2}}}{\sqrt{\pi}(n+1)} \Gamma\left(\frac{1}{n+1}\right) \Gamma\left(\frac{1}{2} - \frac{1}{n+1}\right) \end{aligned}$$

where $B(x, y)$ and $\Gamma(x)$ are the standard beta and gamma functions, respectively. Finally, we find that, for the case $n = 1$,

$$\int_{u_L}^{u_L+\kappa} \frac{dv}{\sqrt{[\epsilon u'(0)]^2 - 2H(v)}} \sim -|B_L|^{-1/2} \log(\epsilon u'(0))$$

for κ fixed and $\epsilon \to 0^+$. This estimate is independent of κ, provided $0 < \kappa < \frac{u_L+u_R}{2}$. Obtaining the analogous estimate near $v = u_R$ and using (5.17), we find the limiting shock location to be

$$x_\epsilon \sim \frac{|B_R|^{1/2}}{|B_R|^{1/2} + |B_L|^{1/2}}. \tag{5.19}$$

For $n \geq 2$,

$$\int_{u_L}^{u_L+\kappa} \frac{dv}{\sqrt{[\epsilon u'(0)]^2 - 2H(v)}} \sim (\epsilon u'(0))^{\frac{1-n}{n+1}} \left(\frac{(n+1)!}{2|B_L|}\right)^{\frac{1}{n+1}} \frac{\Gamma\left(\frac{1}{n+1}\right)\Gamma\left(\frac{1}{2} - \frac{1}{n+1}\right)}{\sqrt{\pi}(n+1)}. \tag{5.20}$$

Again, in combination with the analogous approximation over an interval near the right endpoint, we can easily find that the limiting shock location is given by

$$x_\epsilon \sim \frac{|B_R|^{\frac{1}{n+1}}}{|B_R|^{\frac{1}{n+1}} + |B_L|^{\frac{1}{n+1}}}. \tag{5.21}$$

See Table 5.1 for a confirmation of this asymptotic result.

For $n \geq 2$, we claim that the steady-state problem is algebraically supersensitive, just as it was for the advection-diffusion equation with $m \geq 2$. To see this, consider the problem (5.12) with perturbed boundary conditions:

$$\epsilon^2 u'' + h(u) = 0, u(0) = u_L + \alpha\epsilon^{\frac{2}{n-1}}, u(1) = u_R - \beta\epsilon^{\frac{2}{n-1}} \tag{5.22}$$

with α and β being positive, $O_s(1)$ constants.

Multiplying (5.22) through by $u'(x)$ and integrating twice we obtain the implicit solution

$$x = \epsilon \int_{u(0)}^{u(x)} \frac{dv}{\sqrt{C(\epsilon) - 2H(v)}}$$

where $C(\epsilon) \equiv 2H(u(0)) + (\epsilon u'(0))^2$. If we continue to define the shock location x_ϵ by $u(x_\epsilon) \equiv \frac{1}{2}(u_L + u_R)$, then we have

$$x_\epsilon = \epsilon \int_{u(0)}^{\frac{u_L+u_R}{2}} \frac{dv}{\sqrt{C(\epsilon) - 2H(v)}} \tag{5.23}$$

with $C(\epsilon)$ determined by

$$\frac{1}{\epsilon} = \int_{u(0)}^{u(1)} \frac{dv}{\sqrt{C(\epsilon) - 2H(v)}}. \tag{5.24}$$

As usual, the main contributions to the integrals in (5.23) and (5.24) come from v values near the endpoints $u(0)$ and $u(1)$ because the integrand is nearly singular there. If we consider the contribution near $v = u(0)$, we have, for an appropriately defined

κ,

$$\int_{u(0)}^{u(0)+\kappa} \frac{dv}{\sqrt{C(\epsilon) - 2H(v)}}$$

$$\sim \sqrt{\frac{(n+1)!}{2|B_L|}} \int_{\alpha\epsilon^{\frac{2}{n-1}}}^{\alpha\epsilon^{\frac{2}{n-1}}+\kappa} \frac{dq}{\sqrt{a^{n+1} + q^{n+1}}}$$

$$= \sqrt{\frac{(n+1)!}{2|B_L|a^{n+1}}}\, q\ F\left(\frac{1}{2}, \frac{1}{n+1}; \frac{n+2}{n+1}; -\left(\frac{q}{a}\right)^{n+1}\right)\Bigg|_{q=\alpha\epsilon^{\frac{2}{n-1}}}^{q=\alpha\epsilon^{\frac{2}{n-1}}+\kappa} \tag{5.25}$$

$$= \sqrt{\frac{(n+1)!}{2|B_L|a^{n+1}}}\, q\left(1 + \left(\frac{q}{a}\right)^{n+1}\right)^{-\frac{1}{n+1}}$$

$$\times\ F\left(\frac{1}{n+1}, \frac{n+3}{2(n+1)}; \frac{n+2}{n+1}; \frac{q^{n+1}}{q^{n+1} + a^{n+1}}\right)\Bigg|_{q=\alpha\epsilon^{\frac{2}{n-1}}}^{q=\alpha\epsilon^{\frac{2}{n-1}}+\kappa}, \tag{5.26}$$

where $a^{n+1} \equiv \frac{C(\epsilon)(n+1)!}{2|B_L|} > 0$, $\Gamma(z)$ is the usual gamma function, and $F(a, b; c; d)$ is the hypergeometric function (see [1], Chapter 15). We have used a well-known identity for the hypergeometric function ([1], Formula 15.3.5) to obtain (5.26) from (5.25). Assume that

$$a(\epsilon) = O_s(\epsilon^{\frac{2}{n-1}}). \tag{5.27}$$

Then (5.26) provides

$$\int_{u(0)}^{u(0)+\kappa} \frac{dv}{\sqrt{C(\epsilon)-2H(v)}}$$

$$\sim \sqrt{\frac{(n+1)!}{2|B_L|a^{n+1}}}\left\{a\,F\left(\frac{1}{n+1},\frac{n+3}{2(n+1)};\frac{n+2}{n+1};1\right)\right.$$

$$-\alpha\epsilon^{\frac{2}{n-1}}\left(1+\left(\frac{\alpha\epsilon^{\frac{2}{n-1}}}{a}\right)^{n+1}\right)^{-\frac{1}{n+1}}$$

$$\left.\times F\left(\frac{1}{n+1},\frac{n+3}{2(n+1)};\frac{n+2}{n+1};\frac{\left(\alpha\epsilon^{\frac{2}{n-1}}\right)^{n+1}}{\left(\alpha\epsilon^{\frac{2}{n-1}}\right)^{n+1}+a^{n+1}}\right)\right\} \tag{5.28}$$

$$\sim \sqrt{\frac{(n+1)!}{2|B_L|a^{n+1}}}\left\{\frac{a}{\sqrt{\pi}}\Gamma\left(\frac{n+2}{n+1}\right)\Gamma\left(\frac{n-1}{2n+2}\right)\right.$$

$$-\left(\frac{1}{\frac{1}{\left(\alpha\epsilon^{\frac{2}{n-1}}\right)^{n+1}}+\frac{1}{a^{n+1}}}\right)^{\frac{1}{n+1}}$$

$$\left.\times F\left(\frac{1}{n+1},\frac{n+3}{2(n+1)};\frac{n+2}{n+1};\frac{\left(\alpha\epsilon^{\frac{2}{n-1}}\right)^{n+1}}{\left(\alpha\epsilon^{\frac{2}{n-1}}\right)^{n+1}+a^{n+1}}\right)\right\}. \tag{5.29}$$

We have again used an identity to derive (5.29) from (5.28) ([1], Formula (15.1.20)), along with the facts that $\Gamma(0) = 1$ and that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Under the continued assumption (5.27), we see that both terms of (5.29) inside the braces are of the order $a = O_s(\epsilon^{\frac{2}{n-1}})$. Finally, we have that

$$\int_{u(0)}^{u(0)+\kappa} \frac{dv}{\sqrt{C(\epsilon)-2H(v)}} = O_s\left(a^{-(n+1)/2}\right) = O_s\left(\epsilon^{\frac{n-1}{n+1}}\right) \tag{5.30}$$

and that this limiting contribution depends on α.

Using the analogous assumption, we find a similar approximation in terms of β for the limiting contribution near $v = u(1)$. Using these results in (5.24), we find that $a(\epsilon) = O_s(\epsilon^{\frac{2}{n-1}})$, as assumed. This, also shows that $C(\epsilon)$ depends significantly on both

α and β. This suggests that the shock location in (5.23) is sensitive to perturbations of $O_s(\epsilon^{\frac{2}{n-1}})$, which is demonstrated numerically in Table 5.2.

5.3 The Time-Dependent Problem

We now return to the IBVP (5.1). Previous study of the case $n = 1$ suggests that a shock layer solution will, after some initial $O(1)$ time, then take the asymptotic form

$$u_\epsilon(x,t) = u_p(z) + \delta_n(\epsilon)v(z,\sigma) \tag{5.31}$$

where z is the ϵ-stretched shock-layer variable

$$z = \frac{x - x_\epsilon(\sigma)}{\epsilon}, \tag{5.32}$$

$x_\epsilon(\sigma)$ is the shock-layer location, $\delta_n(\epsilon) = o(1)$ is an unknown gauge function, σ is an unknown slow-time $\phi_n(\epsilon)t$, and the asymptotic profile $u_p(z)$ is the unique monotonically increasing function satisfying the autonomous stretched ODE

$$\frac{d^2u_p}{dz^2} + h(u_p) = 0, \; -\infty < z < \infty \tag{5.33}$$

as well as the auxiliary conditions

$$u_p(-\infty) = u_L, \; u_p(0) = \frac{1}{2}(u_L + u_R), \; u_p(\infty) = u_R.$$

We assume that, once the shock is well-developed, $\delta_n(\epsilon)v(z,\sigma) = o(u_p(z))$.

By assumption, $u_p(z)$ is monotonically increasing, so $u_p'(z) > 0$. Multiplying (5.33) through by $u_p'(z)$ and integrating once, we obtain

$$u_p'(z) = \sqrt{-2H(u_p(z))} \tag{5.34}$$

A second integration provides the implicit solution

$$z = \int_{u_p(0)}^{u_p} \frac{ds}{\sqrt{-2H(s)}} \tag{5.35}$$

for the profile. For example, in the special case that $h(u) = u(1-u^2)$ with $u_L = -1$ and $u_R = 1$, $n = 1$ and the explicit asymptotic profile is

$$u_p(z) = \tanh\left(\frac{z}{\sqrt{2}}\right).$$

In the case that $h(u) = u(1-u^2)^2$ with $u_L = -1$ and $u_R = 1$, $n = 2$ and the profile takes the form

$$u_p(z) = \frac{z}{\sqrt{z^2+3}}.$$

As usual, the tail behavior of $u_p(z)$ determines the shock motion. When $n = 1$, $u_p(z)$ approaches its limiting end-values exponentially quickly. To see this in general, first note that

$$\begin{aligned}\sqrt{-2H(u_p(z))} &\sim \sqrt{-2\left[H(u_{R,L}) + h(u_{R,L})(u_p(z)-u_{R,L}) + \frac{1}{2}B_{R,L}(u_p(z)-u_{R,L})^2\right]} \\ &= |B_{R,L}|^{1/2}|u_p(z)-u_{R,L}|\end{aligned}$$

as $z \to \pm\infty$, so the integral in (5.35) has a simple pole in these limits. Then, rewrite (5.35) as

$$z = \int_{u_p(0)}^{u_p}\left[\frac{1}{\sqrt{-2H(s)}} - \frac{1}{|B_R|^{1/2}(u_R-s)}\right]ds - \frac{1}{|B_R|^{1/2}}\log\left(\frac{u_R-u_p}{u_R-u_p(0)}\right)$$

in order to remove the simple pole from the integrand to allow us to integrate up to u_R. We immediately find that

$$u_p(z) \sim u_R - C_R e^{-|B_R|^{1/2}z} \quad \text{as } z \to \infty, \tag{5.36}$$

where

$$\log\left(\frac{C_R}{u_R-u_p(0)}\right) \sim |B_R|^{1/2}\int_{u_p(0)}^{u_R}\left[\frac{1}{\sqrt{-2H(s)}} - \frac{1}{|B_R|^{1/2}(u_R-s)}\right]ds$$

defines C_R. Likewise,

$$u_p(z) \sim u_L + C_L e^{|B_L|^{1/2}z} \quad \text{as } z \to -\infty \tag{5.37}$$

where

$$\log\left(\frac{C_L}{u_p(0)-u_L}\right) \sim |B_L|^{1/2}\int_{u_L}^{u_p(0)}\left[\frac{1}{\sqrt{-2H(s)}}-\frac{1}{|B_L|^{1/2}(s-u_L)}\right]ds.$$

For $n \geq 2$, $u_p(z)$ approaches its end-values algebraically. To see this in general, note that

$$\frac{d^2}{dz^2}(u_p-u_R) = -h(u_p) \sim -\frac{(u_p-u_R)^n}{n!}B_R$$

near $z=\infty$. Multiplication by $\frac{d}{dz}(u_p-u_R)$ and an integration show that

$$\frac{d}{dz}(u_p-u_R) \sim \left|\frac{2(u_p-u_R)^{n+1}}{(n+1)!}B_R\right|^{1/2}. \tag{5.38}$$

Separating variables and integrating once again, we finally get

$$u_p(z) \sim u_R - \left|\frac{2(n+1)!}{B_R(n-1)^2}\right|^{\frac{1}{n-1}}\frac{1}{z^{\frac{2}{n-1}}} \quad \text{as } z\to\infty. \tag{5.39}$$

Likewise,

$$\frac{d}{dz}(u_p-u_L) \sim \left|\frac{2(u_p-u_L)^{n+1}}{(n+1)!}B_L\right|^{1/2} \tag{5.40}$$

and

$$u_p(z) \sim u_L + \left|\frac{2(n+1)!}{B_L(n-1)^2}\right|^{\frac{1}{n-1}}\frac{1}{|z|^{\frac{2}{n-1}}} \tag{5.41}$$

as $z\to-\infty$.

More complete asymptotic tail approximations could be obtained by direct substitution of a series representation into the differential equation (5.33), with $h(u)$ expanded in its Taylor series about the end-values. Note the re-emergence of the pattern: exponential asymptotics for $n=1$ and algebraic asymptotics for $n\geq 2$.

Observe that the function $u_p(z)$ satisfies the PDE (5.1) with a small residual $\frac{1}{\epsilon}\phi_n(\epsilon)u_p'(z)\frac{dx_\epsilon(\sigma)}{d\sigma}$, and, when $n=1$, it satisfies the boundary conditions at $x=0$ and 1, respectively, with the limiting errors $C_L e^{-|B_L|^{1/2}(x_\epsilon/\epsilon)}$ and $C_R e^{-|B_R|^{1/2}((1-x_\epsilon)/\epsilon)}$.

For $n > 1$, the boundary errors are $O(\epsilon^{\frac{2}{n-1}})$, and so they are less negligible than for $n = 1$, but still asymptotically small. This suggests that the Ansatz (5.31) remains reasonable, presuming $0 < x_\epsilon(\sigma) < 1$, i.e. we have an interior shock.

Let us now seek the asymptotic solution of (5.1) in the form

$$u_\epsilon(x,t) = u_p(z) + \delta_n(\epsilon)v(z,\sigma),$$

so the correction term v must satisfy

$$-\frac{\phi_n(\epsilon)}{\epsilon}(u_p' + \delta_n(\epsilon)v_z)\frac{dx_\epsilon}{d\sigma} + \phi_n(\epsilon)\delta_n(\epsilon)v_\sigma = \left(u_p'' + \delta_n(\epsilon)v_{zz}\right) + h(u_p + \delta_n(\epsilon)v).$$

Thus, under the assumptions that $\phi_n(\epsilon) = o(\epsilon)$ and that we can choose $\delta_n(\epsilon)$ and $\phi_n(\epsilon)$ to satisfy $\frac{\phi_n(\epsilon)}{\epsilon\delta_n(\epsilon)}\frac{dx_\epsilon}{d\sigma} = O_s(1)$, the scaled correction v satisfies the nearly linear problem

$$v_{zz} + h'(u_p)v = -\frac{\phi_n(\epsilon)}{\epsilon\delta_n(\epsilon)}\frac{dx_\epsilon}{d\sigma}u_p' + \phi_n(\epsilon)H[v] \tag{5.42}$$

where

$$H[v] \equiv -\phi_n^{-1}(\epsilon)\delta_n^{-1}(\epsilon)[h(u_p + \delta_n(\epsilon)v) - h(u_p) - \delta_n(\epsilon)h'(u_p)v] - \frac{1}{\epsilon}v_z\frac{dx_\epsilon}{d\sigma} + v_\sigma$$

is bounded. The boundary conditions for v are

$$\begin{cases} v\left(-\frac{x_\epsilon(\sigma)}{\epsilon}, \sigma\right) = \delta_n^{-1}\left(u_L - u_p\left(-\frac{x_\epsilon(\sigma)}{\epsilon}\right)\right) \\ \\ v\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}, \sigma\right) = \delta_n^{-1}\left(u_R - u_p\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}\right)\right) \end{cases} \tag{5.43}$$

and

$$v(0,\sigma,\epsilon) = 0. \tag{5.44}$$

For $n = 1$, (5.43) becomes

$$\begin{cases} v\left(-\frac{x_\epsilon(\sigma)}{\epsilon}, \sigma\right) \sim -\delta_1^{-1}C_L e^{-|B_L|^{1/2}x_\epsilon/\epsilon} \\ \\ v\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}, \sigma\right) \sim \delta_1^{-1}C_R e^{-|B_R|^{1/2}(1-x_\epsilon)/\epsilon}. \end{cases} \tag{5.45}$$

For $n \geq 2$, (5.43) becomes

$$\begin{cases} v\left(-\frac{x_\epsilon(\sigma)}{\epsilon}, \sigma\right) \sim -\delta_n^{-1} \left|\frac{2(n+1)!\,\epsilon^2}{B_L(n-1)^2 x_\epsilon^2}\right|^{\frac{1}{n-1}} \\ \\ v\left(\frac{1-x_\epsilon(\sigma)}{\epsilon}, \sigma\right) \sim \delta_n^{-1} \left|\frac{2(n+1)!\,\epsilon^2}{B_R(n-1)^2(1-x_\epsilon)^2}\right|^{\frac{1}{n-1}}. \end{cases} \tag{5.46}$$

We proceed by seeking a first-approximation to (5.42) that satisfies the limiting equation

$$v_{zz} + h'(u_p)v \sim -\frac{\phi_n(\epsilon)}{\epsilon\delta_n(\epsilon)}\frac{dx_\epsilon}{d\sigma}u_p'. \tag{5.47}$$

Predictably, $u_p'(z) = \sqrt{-2H(u_p(z))}$ satisfies the homogeneous equation $v_{zz}+h'(u_p)v = 0$. We know that a second solution to the homogeneous equation will take the form $\eta(z)u_p'(z)$, where $\eta(z)$ satisfies

$$u_p'\eta'' + 2u_p''\eta' = 0.$$

Multiplying by the integrating factor $u_p'(z)$, solving the resulting exact equation for η', and integrating again, we find a second homogeneous solution

$$\eta(z)u_p'(z) \equiv u_p'(z)\int_0^z \frac{ds}{[u_p'(s)]^2} = \sqrt{-2H(u_p(z))}\int_{u_p(0)}^{u_p(z)} \frac{dv}{(-2H(v))^{3/2}}.$$

We can now solve (5.47), using variation of parameters, to find a limiting solution in the form

$$v(z,\sigma,\epsilon) \sim u_p'(z)\left\{[\alpha(\sigma,\epsilon) + F(z,\sigma,\epsilon)] + \int_0^z \frac{ds}{[u_p'(s)]^2}[\beta(\sigma,\epsilon) + G(z,\sigma,\epsilon)]\right\}$$

where

$$F(z,\sigma,\epsilon) \sim \frac{\phi_n(\epsilon)}{\epsilon\delta_n(\epsilon)}\frac{dx_\epsilon}{d\sigma}\int_0^z [u_p'(s)]^2 \int_0^s \frac{dq}{[u_p'(q)]^2}ds$$

and

$$G(z,\sigma,\epsilon) \sim -\frac{\phi_n(\epsilon)}{\epsilon\delta_n(\epsilon)}\frac{dx_\epsilon}{d\sigma}\int_0^z [u_p'(s)]^2 ds.$$

Since (5.44) immediately implies $\alpha(\sigma,\epsilon)\equiv 0$, we can further simplify our expression for v to

$$v(z,\sigma,\epsilon)\sim u_p'(z)\frac{\phi_n(\epsilon)}{\epsilon\delta_n(\epsilon)}\frac{dx_\epsilon}{d\sigma}\left\{I_2(z)+I_1(z)\left[\gamma(\sigma,\epsilon)-I_3(z)\right]\right\} \tag{5.48}$$

where

$$\begin{cases} I_1(z)\equiv\int_{u_p(0)}^{u_p(z)}\frac{dq}{(-2H(q))^{3/2}} \\ I_2(z)\equiv\int_{u_p(0)}^{u_p(z)}\sqrt{-2H(s)}\int_{u_p(0)}^{s}\frac{dq}{(-2H(q))^{3/2}}ds \\ I_3(z)\equiv\int_{u_p(0)}^{u_p(z)}\sqrt{-2H(v)}dv. \end{cases} \tag{5.49}$$

We are left with two unknowns, $\gamma(\sigma,\epsilon)$ and $\frac{dx_\epsilon}{d\sigma}$, that we can asymptotically find by applying the boundary conditions at $z=-\frac{x_\epsilon}{\epsilon}$ and at $z=\frac{1-x_\epsilon}{\epsilon}$:

$$\begin{cases} v\left(-\frac{x_\epsilon}{\epsilon}\right)\sim u_p'\left(-\frac{x_\epsilon}{\epsilon}\right)\frac{\phi_n(\epsilon)}{\epsilon\delta_n(\epsilon)}\frac{dx_\epsilon}{d\sigma}\left\{I_2\left(-\frac{x_\epsilon}{\epsilon}\right)+I_1\left(-\frac{x_\epsilon}{\epsilon}\right)\left[\gamma-I_3\left(-\frac{x_\epsilon}{\epsilon}\right)\right]\right\} \\ v\left(\frac{1-x_\epsilon}{\epsilon}\right)\sim u_p'\left(\frac{1-x_\epsilon}{\epsilon}\right)\frac{\phi_n(\epsilon)}{\epsilon\delta_n(\epsilon)}\frac{dx_\epsilon}{d\sigma}\left\{I_2\left(\frac{1-x_\epsilon}{\epsilon}\right)+I_1\left(\frac{1-x_\epsilon}{\epsilon}\right)\left[\gamma-I_3\left(\frac{1-x_\epsilon}{\epsilon}\right)\right]\right\} \end{cases} \tag{5.50}$$

Dividing through by $\frac{dx_\epsilon}{d\sigma}$ we see that these equations form a linear system for the two unknowns. Thus, we solve (5.50) to find the limiting differential equation

$$\frac{dx_\epsilon}{d\sigma}\sim\frac{\epsilon\delta_n(\epsilon)}{\phi_n(\epsilon)}\frac{\frac{v\left(-\frac{x_\epsilon}{\epsilon}\right)}{E\left(-\frac{x_\epsilon}{\epsilon}\right)}-\frac{v\left(\frac{1-x_\epsilon}{\epsilon}\right)}{E\left(\frac{1-x_\epsilon}{\epsilon}\right)}}{D\left(\frac{1-x_\epsilon}{\epsilon}\right)-D\left(-\frac{x_\epsilon}{\epsilon}\right)} \tag{5.51}$$

and the constant (with respect to z)

$$\gamma(\sigma,\epsilon)\sim\frac{u_p'\left(\frac{1-x_\epsilon}{\epsilon}\right)v\left(-\frac{x_\epsilon}{\epsilon}\right)D\left(\frac{1-x_\epsilon}{\epsilon}\right)-u_p'\left(-\frac{x_\epsilon}{\epsilon}\right)v\left(\frac{1-x_\epsilon}{\epsilon}\right)D\left(-\frac{x_\epsilon}{\epsilon}\right)}{v\left(\frac{1-x_\epsilon}{\epsilon}\right)E\left(-\frac{x_\epsilon}{\epsilon}\right)-v\left(-\frac{x_\epsilon}{\epsilon}\right)E\left(\frac{1-x_\epsilon}{\epsilon}\right)}, \tag{5.52}$$

where

$$D(z)\equiv I_3(z)-\frac{I_2(z)}{I_1(z)}\quad\text{and}\quad E(z)\equiv u_p'(z)I_1(z). \tag{5.53}$$

Clearly, the next step is to approximate $I_1(z)$, $I_2(z)$ and $I_3(z)$ for large $|z|$. First, consider $I_1(z)$ as $z \to \infty$. For appropriate fixed κ, as $u_p(z) \to u_R$,

$$\begin{aligned}
\int_{u_p(0)}^{u_p(z)} \frac{dq}{(-2H(q))^{3/2}} &\sim \int_{u_p(z)-\kappa}^{u_p(z)} \frac{dq}{(-2H(q))^{3/2}} \\
&\sim \int_{u_p(z)-\kappa}^{u_p(z)} \frac{dq}{\left[-\frac{2B_R}{(n+1)!}(q-u_R)^{n+1}\right]^{3/2}} \\
&\sim \frac{2}{3n+1}\left|\frac{(n+1)!}{2B_R}\right|^{3/2}\left(\frac{1}{u_R-u_p(z)}\right)^{\frac{3n+1}{2}}.
\end{aligned}$$

So, we can use (5.36) and (5.39) to find that, as $z \to \infty$,

$$I_1(z) \sim \begin{cases} \frac{1}{2C_R^2}\left|\frac{1}{B_R}\right|^{3/2} e^{2|B_R|^{1/2}z}, & \text{for } n=1 \\ \frac{2}{3n+1}\left|\frac{(n+1)!}{2B_R}\right|^{3/2}\left|\frac{B_R(n-1)^2}{2(n+1)!}z^2\right|^{\frac{3n+1}{2n-2}} & \text{for } n \geq 2. \end{cases} \tag{5.54}$$

Likewise, for $z \to -\infty$, we find

$$I_1(z) \sim \begin{cases} -\frac{1}{2C_L^2}\left|\frac{1}{B_L}\right|^{3/2} e^{-2|B_L|^{1/2}z}, & \text{for } n=1 \\ -\frac{2}{3n+1}\left|\frac{(n+1)!}{2B_L}\right|^{3/2}\left|\frac{B_L(n-1)^2}{2(n+1)!}z^2\right|^{\frac{3n+1}{2n-2}} & \text{for } n \geq 2. \end{cases} \tag{5.55}$$

Turning to $I_3(z)$, we find the bounded limit

$$\lim_{z\to\pm\infty} I_3(z) \sim \int_{u_p(0)}^{u_{R,L}} \sqrt{-2H(v)dv}. \tag{5.56}$$

The double integral $I_2(z)$ is more difficult to estimate for large z. However, if we use integration by parts to write

$$\begin{aligned}
I_2(z) &\equiv \int_{u_p(0)}^{u_p(z)} \sqrt{-2H(s)} \int_{u_p(0)}^{s} \frac{dq}{(-2H(q))^{3/2}} ds \\
&= \left(\int_{u_p(0)}^{u_p(z)} \frac{dq}{(-2H(q))^{3/2}}\right)\left(\int_{u_p(0)}^{u_p(z)} \sqrt{-2H(q)}dq\right) \\
&\quad - \int_{u_p(0)}^{u_p(z)} \frac{\int_{u_p(0)}^{s}\sqrt{-2H(q)}dq}{(-2H(s))^{3/2}} ds \\
&= I_1(z)I_3(z) - \int_{u_p(0)}^{u_p(z)} \frac{\int_{u_p(0)}^{s}\sqrt{-2H(q)}dq}{(-2H(s))^{3/2}} ds \\
&= I_1(z)I_3(z) - \int_{u_p(0)}^{u_p(z)} \frac{\left(\int_{u_p(0)}^{u_{R,L}} - \int_{s}^{u_{R,L}}\right)\sqrt{-2H(q)}dq}{(-2H(s))^{3/2}} ds.
\end{aligned}$$

Then, as $z \to \pm\infty$, we find the bound

$$I_2(z) \sim I_1(z)I_3(z) - [I_1(z)I_3(z) + o(I_1(z))] = o(I_1(z)).$$

This implies that

$$D(z) \sim I_3(z) \quad \text{as } z \to \pm\infty. \tag{5.57}$$

Using the accumulated information, we return to (5.51) to find

$$\frac{dx_\epsilon}{d\sigma} \sim \frac{\epsilon}{\phi_n(\epsilon)} \frac{3n+1}{(n+1)! \int_{u_L}^{u_R} \sqrt{-2H(v)dv}} \left[|B_L| \left(u_p\left(-\frac{x_\epsilon}{\epsilon}\right) - u_L \right)^{n+1} \right.$$
$$\left. - |B_R| \left(u_R - u_p\left(\frac{1-x_\epsilon}{\epsilon} \right) \right)^{n+1} \right].$$

For $n = 1$, this reduces to

$$\frac{dx_\epsilon}{d\sigma} \sim \frac{2\epsilon}{\phi_1(\epsilon) \int_{u_L}^{u_R} \sqrt{-2H(v)dv}} \left[|B_L|\, C_L^2 e^{-2|B_L|^{1/2} x_\epsilon/\epsilon} - |B_R|\, C_R^2 e^{-2|B_R|^{1/2}(1-x_\epsilon)/\epsilon} \right]. \tag{5.58}$$

For $n \geq 2$, however, we instead have

$$\frac{dx_\epsilon}{d\sigma} \sim \frac{\epsilon^{(3n+1)/(n-1)}}{\phi_n(\epsilon)} \frac{\omega_n}{\int_{u_L}^{u_R} \sqrt{-2H(v)}dv} \left(\frac{1}{(|B_L| x_\epsilon^{n+1})^{\frac{2}{n-1}}} - \frac{1}{(|B_R|(1-x_\epsilon)^{n+1})^{\frac{2}{n-1}}} \right) \tag{5.59}$$

where

$$\omega_n \equiv (3n+1) \left\{ \left[\frac{2}{(n-1)^2} \right]^{n+1} [(n+1)!]^2 \right\}^{\frac{1}{n-1}}.$$

One could analogously determine $\gamma(\sigma)$ and thereby obtain an $O(1)$ approximation for v. Note that the ODE obtained is autonomous. Thus, one can integrate the PDE for a short time numerically to determine the initial value needed to solve the ODE. Also, note that the motion of the shock layer toward its steady-state location is monotonic. The steady-state locations predicted by these ODEs corroborate our previous result in Section 5.2.

At this point, we choose $\phi_n(\epsilon)$ to ensure an $O(1)$ shock-speed on the σ-time scale. We then select $\delta_n(\epsilon) = O_s\left(\frac{\phi_n(\epsilon)}{\epsilon}\right)$. For $n = 1$, we choose

$$\phi_1(\epsilon) = \epsilon\, e^{-B/\epsilon} \tag{5.60}$$

and

$$\delta_1(\epsilon) = e^{-B/\epsilon} \tag{5.61}$$

where $B \equiv \frac{2|B_L B_R|^{1/2}}{|B_L|^{1/2}+|B_R|^{1/2}}$. This ensures that the shock-speed is $O(1)$ for all times after the shock forms. For $n \geq 2$, we choose

$$\phi_n(\epsilon) = \epsilon^{\frac{3n+1}{n-1}} \tag{5.62}$$

and

$$\delta_n(\epsilon) = \epsilon^{\frac{2(n+1)}{n-1}}. \tag{5.63}$$

The pattern appears again in the asymptotics. For $n = 1$, the shock moves on an exponentially asymptotically slow time-scale. For $n > 1$, however, the shock moves on an algebraically asymptotically slow time-scale. As n increases, we see that the motion is faster. However, as $n \to \infty$, the motion approaches an $O_s(\epsilon^3)$ time-scale.

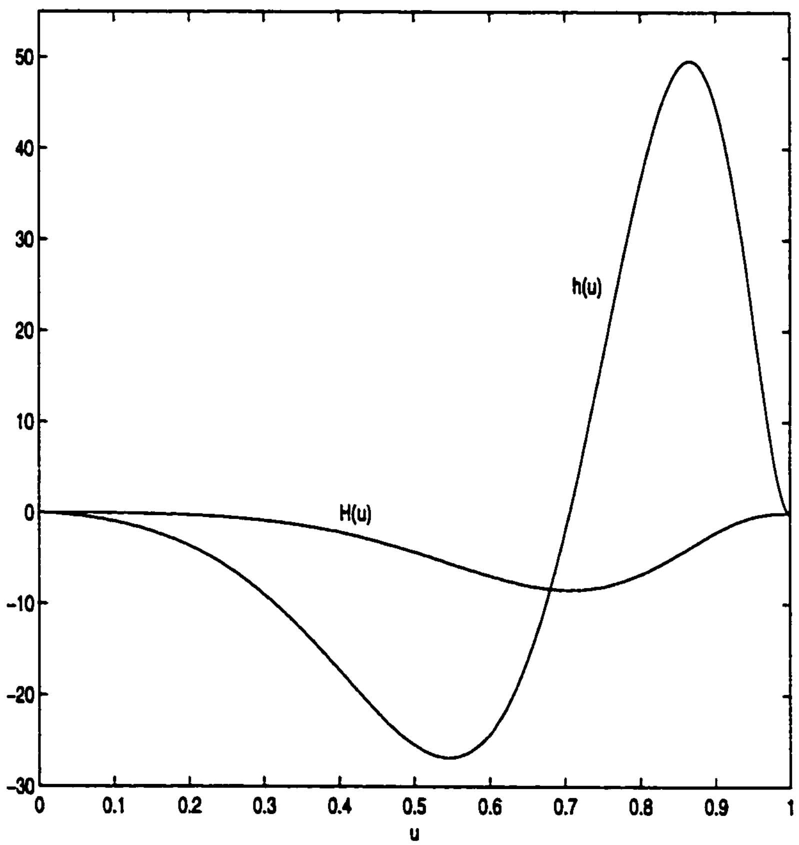

Figure 5.1: **A Typical Bistable Reaction Term $h(u)$ and Its Potential $H(u)$.** This is a plot of the function $H(u) = -1000(u - u^2)^3(u^2 + 0.25)^2(u^2 + 0.8)^2$ and its derivative $h(u)$. Here, $h(u)$ has second-order zeros at the endpoints, so $m = 2$.

Table 5.1: **The Shock Location of a Steady-State Reaction-Diffusion Equation.** We compute the shock location x_ϵ of the steady-state reaction-diffusion problem $\epsilon^2 u'' + (1-u^2)^3\,[1+16u-9u^2] = 0$, $u(0) = -1$, $u(1) = 1$. Using a bisection method, we numerically find $u'(0)$ via (5.16). Then we numerically integrate the quotient in (5.17) to determine x_ϵ. This corroborates (5.21) which predicts that $x_\epsilon \sim \frac{384^{1/4}}{384^{1/4}+1152^{1/4}} \approx 0.431765$ as $\epsilon \to 0^+$.

ϵ	$\epsilon u'(0)$	x_ϵ
0.1	2.12805	0.445779
0.01	1.924134	0.433865
0.001	1.8879852	0.432043
0.0001	1.8827932	0.431800

Table 5.2: **An Example of Algebraic Supersensitivity for a Reaction-Diffusion Equation.** We compute the shock location x_ϵ of the steady-state reaction-diffusion problem $\epsilon^2 u'' + (1-u^2)^3\,[1+16u-9u^2] = 0$, $u(0) = -1$, $u(1) = 1-\epsilon$, with a perturbed right boundary value. Using a bisection method, we numerically find $C(\epsilon)$ via (5.24). Then we numerically integrate (5.23) to determine x_ϵ. The perturbed boundary value has caused the shock layer to move from a location of $x_\epsilon \approx 0.43$ to $x_\epsilon \approx 0.82$.

ϵ	$C(\epsilon) = (\epsilon u'(0))^2$	x_ϵ
0.1	$0.370625\,\epsilon^4$	0.806736
0.01	$0.2832155\,\epsilon^4$	0.819572
0.001	$0.26994895\,\epsilon^4$	0.822787
0.0001	$0.26810135\,\epsilon^4$	0.823292

Chapter 6

ADVECTION-DIFFUSION-REACTION EQUATIONS

We now briefly discuss the full advection-diffusion-reaction equation

$$\begin{cases} u_t = \epsilon^2 u_{xx} + \epsilon g(u) u_x + h(u) = \epsilon^2 u_{xx} + \epsilon\left[G(u)\right]_x + h(u) \\ u(0,t) = u_L,\ u(1,t) = u_R,\ u(x,0) = u_I(x). \end{cases} \tag{6.1}$$

We combine the assumptions we made in Chapter 3 and Chapter 5:

$$\begin{cases} G(u_R) = G(u_L),\ G(u) < G(u_L) \quad \text{for } u_L < u < u_R \\ \frac{d^k G}{du^k}(u_L) = \frac{d^k G}{du^k}(u_R) = 0 \ \text{ for } \ 1 \le k < m \\ A_L \equiv \frac{d^m G}{du^m}(u_L) < 0,\ (-1)^m A_R \equiv (-1)^m \frac{d^m G}{du^m}(u_R) < 0 \\ h(u_L) = h(u_C) = h(u_R) = 0,\ h'(u_C) > 0 \\ H(u) \equiv \int_{u_L}^{u} h(s)\, ds \begin{cases} < 0,\ u_L < u < u_R \\ = 0,\ u = u_R \end{cases} \\ \frac{d^k h}{du^k}(u_L) = \frac{d^k h}{du^k}(u_R) = 0 \ \text{ for } \ 0 \le k < n \\ B_L \equiv \frac{d^n h}{du^n}(u_L) < 0,\ (-1)^n B_R \equiv (-1)^n \frac{d^n h}{du^n}(u_R) > 0. \end{cases} \tag{6.2}$$

Thus, we have a balanced advection nonlinearity $G(u)$ and a bistable reaction nonlinearity $h(u)$. (Note that, for $m > 1$, $g(u)$ satisfies the same hypotheses as $h(u)$, i.e. bistability.) We presume that these hypotheses will produce a slow-moving internal layer in the solutions obtained, as we saw in Chapters 3 and 5. Perhaps weaker hypotheses are sufficient, but these give an opportunity to study the situation.

6.1 Asymptotic Behavior of the Shock Profile Function

In Chapter 2, we used the travelling wave Ansatz to derive an explicit limiting equation for the shock motion. This shock motion was highly dependent on the asymptotic tail behavior of the shock profile function associated with the PDE. The question remains as to the nature of the shock profile when the PDE contains both advection and reaction terms.

The shock profile function $u_p(z)$ satisfies, by definition,

$$\begin{cases} u_p'' + g(u_p)u_p' + h(u_p) = 0 \\ u_p(z) \to u_L \text{ as } z \to -\infty,\ u_p(0) = \frac{u_L+u_R}{2}, \text{ and } u_p(z) \to u_R \text{ as } z \to \infty. \end{cases} \tag{6.3}$$

We assume here that a monotonic profile function satisfying the above conditions exists, though we cannot, in general, solve (6.3) or provide sufficient conditions for existence. Since $u_p(z)$ is strictly monotonic, we can consider the inverse function $z(u_p)$ which satisfies the simpler, but non-autonomous, first-order nonlinear ODE

$$q' = g(u_p)q^2 + h(u_p)q^3, \tag{6.4}$$

for $q(u_p) \equiv z'(u_p) \equiv \left(\frac{du_p}{dz}\right)^{-1}$. Equation (6.4) is an Abel equation of the first kind [64]; unfortunately, such equations generally lack exact solutions (see [45]). In the special case where $g(u_p) \equiv h(u_p)$, (6.4) is separable, and the resulting equation can be analyzed in the natural phase plane.

Though a solution of (6.3) for all z escapes us, we can often determine the tail behavior of $u_p(z)$ by approximating (6.4) near, say, the left end-point u_L:

$$q' \sim \frac{A_L(u_p - u_L)^{m-1}}{(m-1)!}q^2 + \frac{B_L(u_p - u_L)^n}{n!}q^3. \tag{6.5}$$

Using dominant balance arguments, we find that, for $n + 1 > 2m$,

$$q(u_p) \sim \frac{m!}{|A_L|}\frac{1}{(u_p - u_L)^m}$$

with analogous behavior near the right endpoint. Thus, the advection term then determines the tail behavior. For $m > 1$ we again have (3.34). If $m = 1$ and $n > 1$, we obtain (3.32) where the constant C_L would have to be determined. In this case, the higher-order zeros of the reaction term do not destroy the exponential decay, caused by the advection term, of the profile function's tails as $|z| \to \infty$.

For $n+1 < 2m$,

$$q(u_p) \sim \sqrt{\frac{(n+1)!}{2|B_L|}} \frac{1}{(u_p - u_L)^{(n+1)/2}},$$

again with analogous behavior near $u_p = u_R$. Here, the reaction term determines the tail behavior. If $n > 1$ we re-derive (5.41). However, when $n = 1$ and $m > 1$, we again have (5.37) with C_L to be determined. This is another possibility for exponentially decaying tails.

There remains the indeterminate case $n+1 = 2m$ in which all three terms of (6.5) balance. In particular we shall investigate the case $n = m = 1$, which we expect might also allow exponentially decaying tails. This is in fact true. To see this, we return to equation (6.3) and approximate near $u_p = u_L$, finding a nearly linear limiting equation

$$u_p'' \sim -A_L u_p' - B_L(u_p - u_L).$$

Changing variables to $\bar{u} \equiv u_p - u_L$, the resulting equation is easily solved as

$$\bar{u}(z) \sim c_+ e^{r_+ z} + c_- e^{r_- z},$$

where

$$r_\pm \equiv \frac{-A_L \pm \sqrt{A_L^2 - 4B_L}}{2} \gtrless 0.$$

Since $\bar{u} \to 0^+$ as $z \to -\infty$, we can discard the decaying exponential to find

$$u_p(z) \sim u_L + c_+ e^{r_+ z}$$

there. Of course, the constant $c_+ > 0$ would still have to be determined. Similarly,

$$u_p(z) \sim u_R - c_- e^{\gamma z}$$

as $z \to \infty$ for some $c_- > 0$ and for

$$\gamma \equiv \frac{-A_R - \sqrt{A_R^2 - 4B_R}}{2} < 0.$$

Thus, the rates of decay as $z \to \pm\infty$ are determined by both the advection and the diffusion terms. Putting this information together, we see the profile function $u_p(z)$ has exponentially decaying tails if $n = 1$ or $m = 1$ (or if both hold).

6.2 The Equation for the Shock Motion

The behavior of the tails of the shock profile function suggest the presence of either algebraic or exponential asymptotics in the solution to (6.1). Previously, we've seen algebraic metastability associated with algebraically decaying tails, and likewise for exponential metastability. We would like to confirm this by determining an ODE describing the shock motion. In Chapter 2 we determined such a limiting equation:

$$\frac{dx_\epsilon}{d\sigma} \sim \frac{\epsilon\left[\frac{v\left(-\frac{x_\epsilon}{\epsilon},\sigma\right)}{E\left(-\frac{x_\epsilon}{\epsilon}\right)} - \frac{v\left(\frac{1-x_\epsilon}{\epsilon},\sigma\right)}{E\left(\frac{1-x_\epsilon}{\epsilon}\right)}\right]}{D\left(\frac{1-x_\epsilon}{\epsilon}\right) - D\left(-\frac{x_\epsilon}{\epsilon}\right)} \tag{6.6}$$

with $v(z,\sigma)$, $D(z)$, and $E(z)$ as defined in Section 2.4. We can asymptotically determine $v(z,\sigma)$ for large $|z|$ using the tail behavior for $u_p(z)$. Unfortunately, the quantities $D(z)$ and $E(z)$ are defined in terms of integrals with require global knowledge of the shock profile function. We are left with two choices, both computational to some degree. Either one can solve (6.3) numerically for the profile function and use the result to integrate (6.6), or one can capitulate and immediately attempt to solve (6.1) numerically. However, given the supersensitivity and slow evolution of the solutions to these PDEs, one will likely get better results by solving the ODEs (6.3) and (6.6).

Chapter 7

COMMENTS ON NUMERICS AND COMPUTATIONAL RESULTS

We now briefly discuss comparisons between our asymptotic results and numerical solutions of these PDEs and of their associated steady-state ODEs. Several authors have found good agreement between the numerical work and the asymptotic solutions for the case $m = 1$ and $n = 1$ for the associated advection-diffusion and reaction-diffusion equations, as well as the steady-state problems. (See, for example, [35], [38], [51], and [52].) Thus, we won't dwell on these cases.

For $m > 1$ and $n > 1$, we expect agreement to be more problematic. As noted in Chapter 3, two-timing suggests a limit to the asymptotic accuracy of our approximate solutions, limited by the asymptotic order of the first terms dropped after the series is truncated. For example, in Section 3.1, we found an explicit expansion for the derivative of the shock layer location $\frac{dx_\epsilon}{d\sigma}$,

$$\frac{dx_\epsilon}{d\sigma} \sim \frac{dx_0}{d\sigma} + \epsilon\frac{dx_1}{d\sigma},$$

where $\sigma \equiv \epsilon^2 t$. Then, we expect this ODE to yield an $O(1)$-accurate solution for the shock location $x_\epsilon(\sigma)$ until $\sigma = O(\epsilon^{-1})$ or $t = O(\epsilon^{-3})$. (Sometimes, of course, we are luckier.) In other words, because of the (relatively-speaking) large error in discarding higher order terms, we'll only have an expansion valid for a (relatively-speaking) long time for extremely small ϵ. In contrast, for problems exhibiting exponential metastability, we generally have a solution valid for an exponentially asymptotically long time using just the *first* term in the asymptotic expansion for $\frac{dx_\epsilon}{d\sigma}$. Thus good agreement was maintained even for $\epsilon = 0.2$ in [36].

Therein lies the problem. We need to compare numerical results to asymptotic results using very small ϵ. However, all of these problems are singularly perturbed, and are thus extremely stiff numerically because of the steep internal layers. Adding in the issues of metastability and supersensitivity, these problems still pose significant computational issues as $\epsilon \to 0^+$. Properly, an adaptive grid-technique should be used so that more grid points are placed near the shock. Additionally, a splitting technique could be used to divide the PDEs into their hyperbolic and parabolic parts, and a conservative scheme like CLAWPACK could be used to solve the hyperbolic part. Stiff solvers for the associated steady-state ODEs also exist, for example COLNEW [3]. Forgoing such complexities in this research, we have used standard semi-implicit finite difference schemes, and have seen just the resulting problems we described above! The following plots do illustrate the slow convergence of the solutions to the predicted steady-state solutions, however.

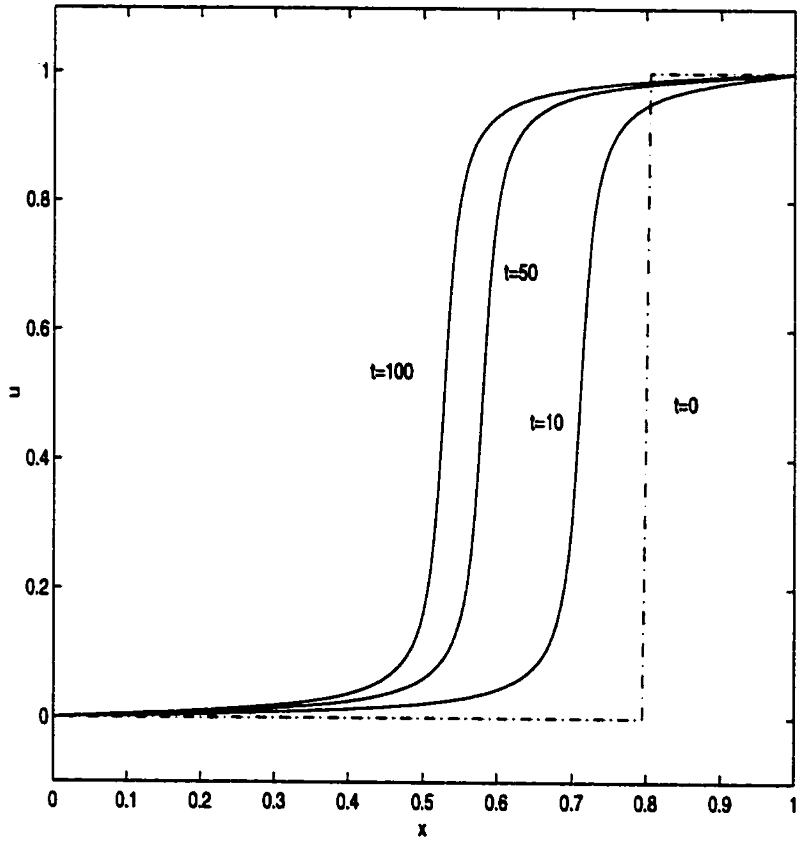

Figure 7.1: **A Slowly Moving Internal Layer for an Advection-Diffusion Equation.** This figure illustrates the evolution of the solution of the PDE $u_t = \epsilon u_{xx} - [\sin^2(\pi u)]_x$ with boundary conditions $u(0,t) = 0$, $u(1,t) = 1$, considered in detail in Section 3.1. The initial condition is a Heaviside function with a jump at $x = 0.8$. Here, $\epsilon = 0.05$. The computational space step and time step were both 0.005. The internal layer drifts slowly leftwards from 0.8 toward its steady-state location of $x_\epsilon(\infty) = 0.5$.

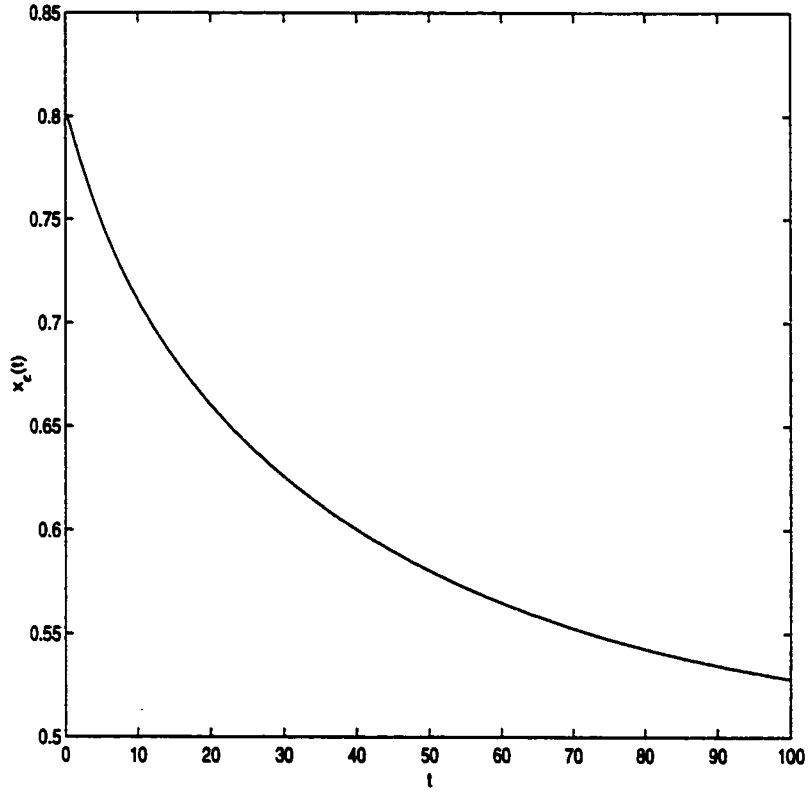

Figure 7.2: **The Shock Location of an Advection-Diffusion Equation.** We solve the problem of Figure 7.1, using the same method and parameters. This figure gives the shock location x_ϵ as a function of time.

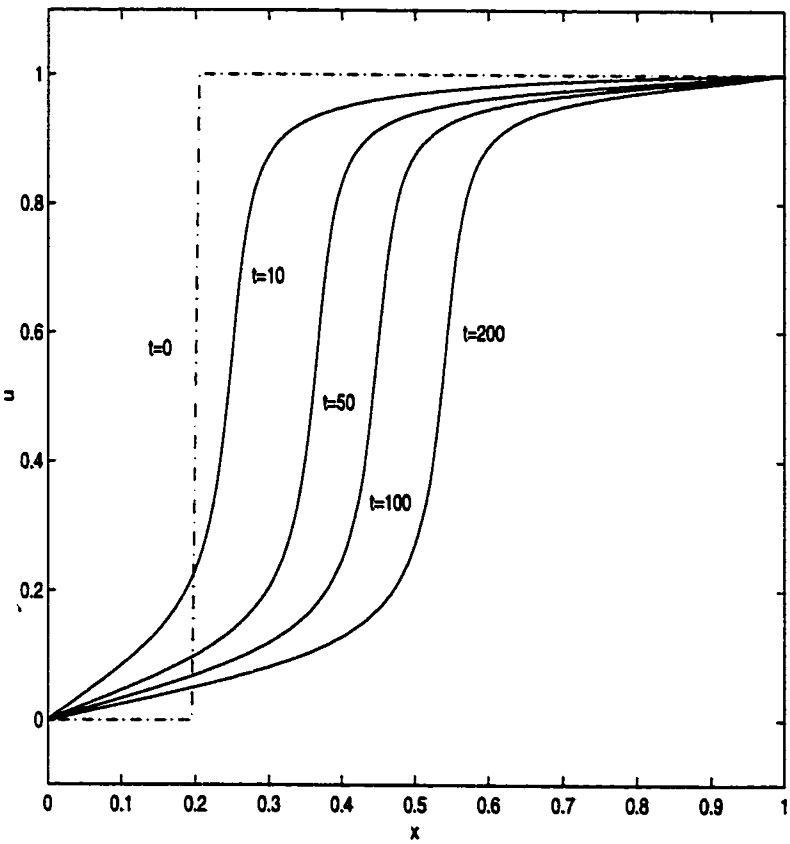

Figure 7.3: **The Evolution of an Internal Layer for an Advection-Diffusion Equation with $m = 3$.** This figure shows the time evolution of the solution of the PDE $u_t = \epsilon u_{xx} - [(u - u^3)^3]_x$ with boundary conditions $u(0,t) = 0$, $u(1,t) = 1$. The initial condition is a Heaviside function with a jump at $x = 0.2$. We have taken $\epsilon = 0.005$. The computational space and time steps were both 0.005. The internal layer drifts increasingly slowly toward its steady-state location of $x_\epsilon(\infty) = \frac{2}{3}$.

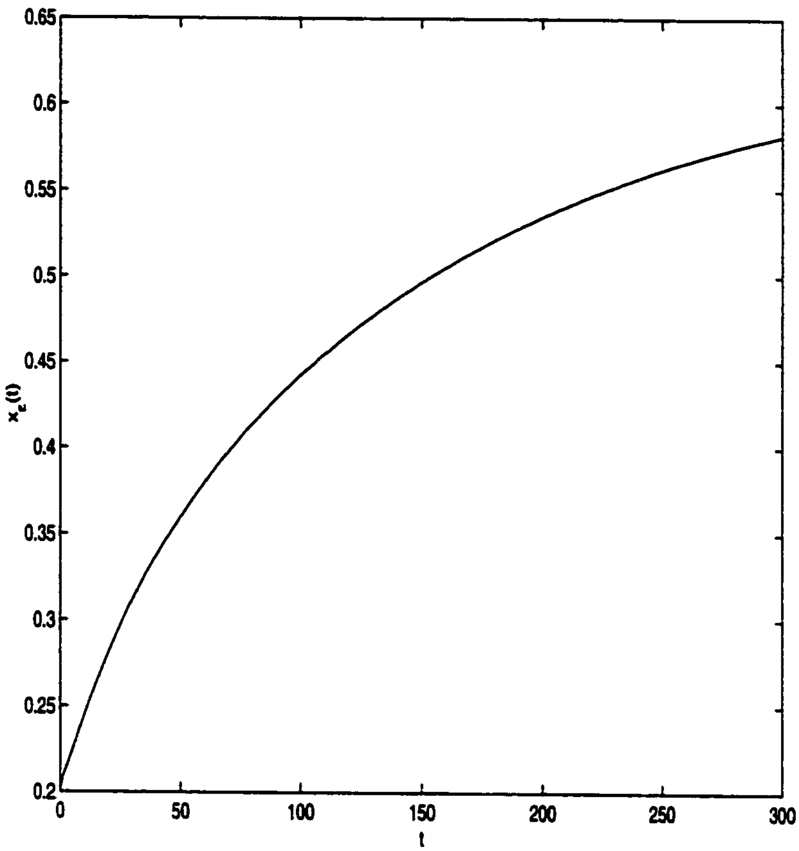

Figure 7.4: **The Shock Location x_ϵ for the Advection-Diffusion Equation in Figure 7.3 as a Function of Time.** We solve the problem of Figure 7.3, using the same method and parameters.

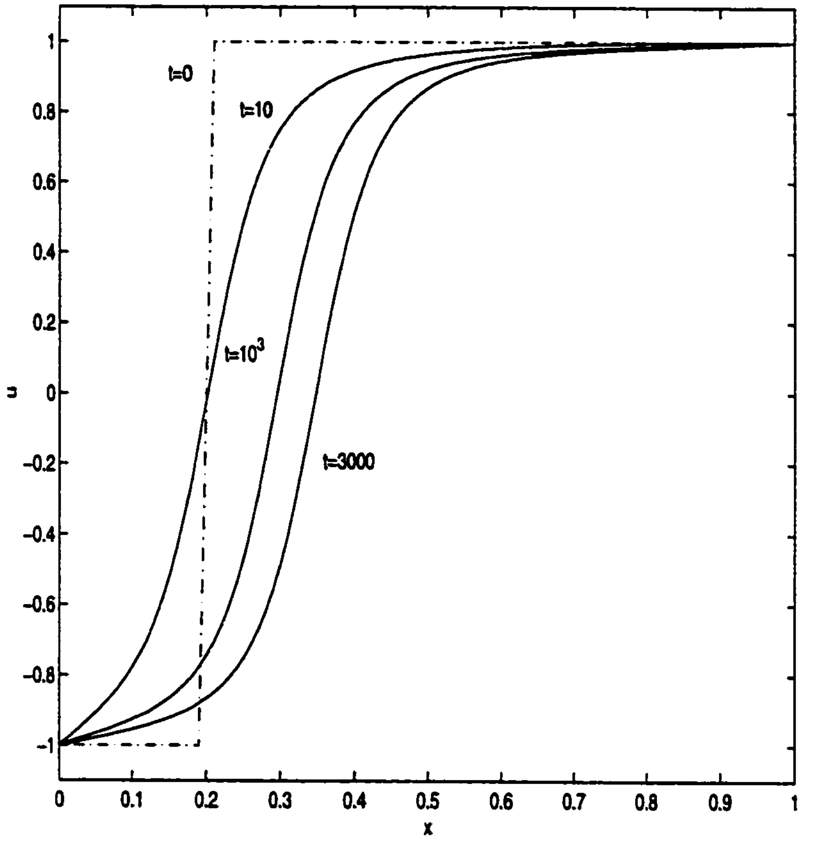

Figure 7.5: **A Slowly Moving Internal Layer for a Reaction-Diffusion Equation.** This figure illustrates the evolution of the solution of the PDE $u_t = \epsilon^2 u_{xx} - u(1-u^2)^2$ with boundary conditions $u(0,t) = -1$, $u(1,t) = 1$, considered in detail in Section 5.1. The initial condition is a Heaviside function with a jump at $x = 0.2$. Here, $\epsilon = 0.05$. The computational space and time steps were both 0.01. The internal layer drifts very slowly leftwards toward its steady-state location of $x_\epsilon(\infty) = 0.5$.

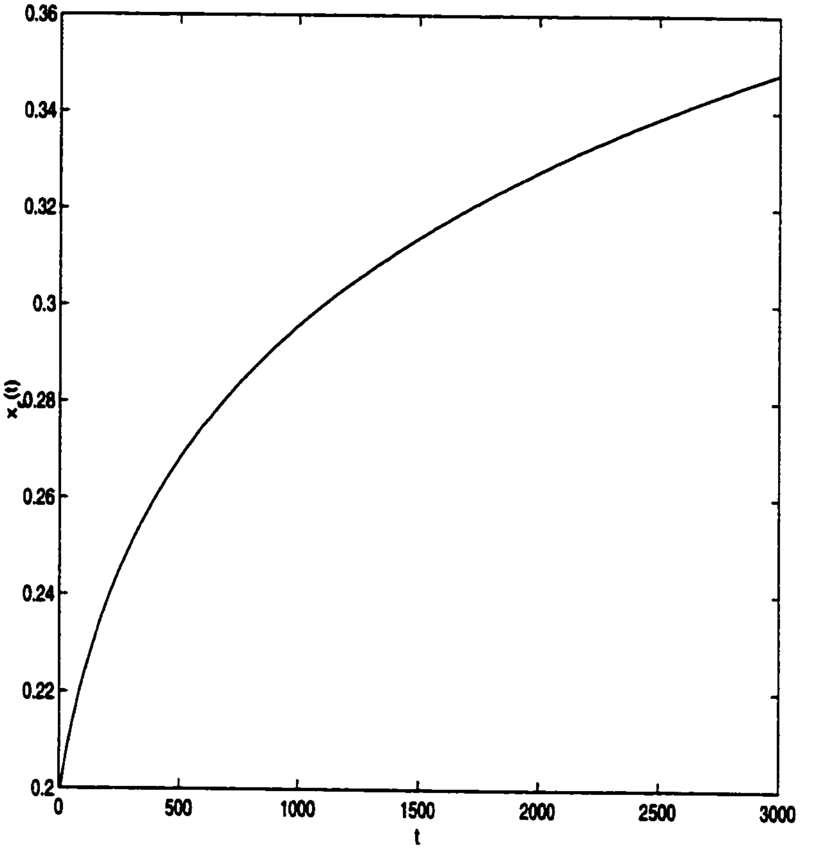

Figure 7.6: **The Shock Location for a Reaction-Diffusion Equation as a Function of Time.** We solve the problem of Figure 7.5, using the same method and parameters.

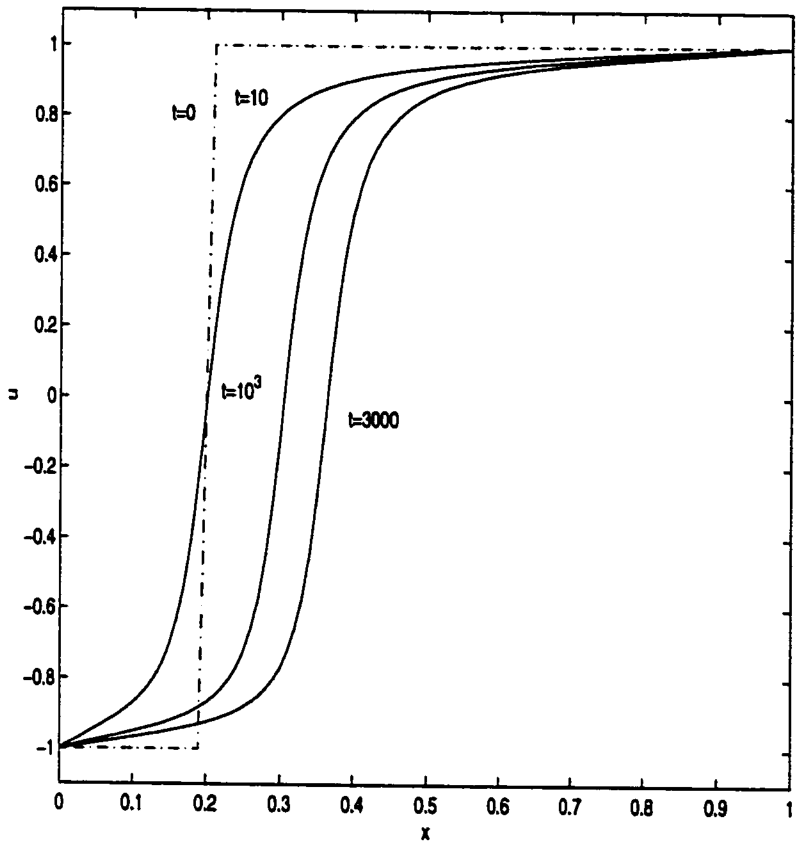

Figure 7.7: **The Evolution of an Internal Layer for a Reaction-Diffusion Equation with $n = 3$.** This figures shows the time evolution of the solution of the PDE $u_t = \epsilon^2 u_{xx} + (1-u^2)^3(1+16u-9u^2)$ with boundary conditions $u(0,t) = -1$, $u(1,t) = 1$. The initial condition is a Heaviside function with a jump at $x = 0.2$. We have taken $\epsilon = 0.1$. The computational space and time steps were both 0.01. The internal layer drifts increasingly slowly toward its steady-state location of $x_\epsilon(\infty) \approx 0.43$.

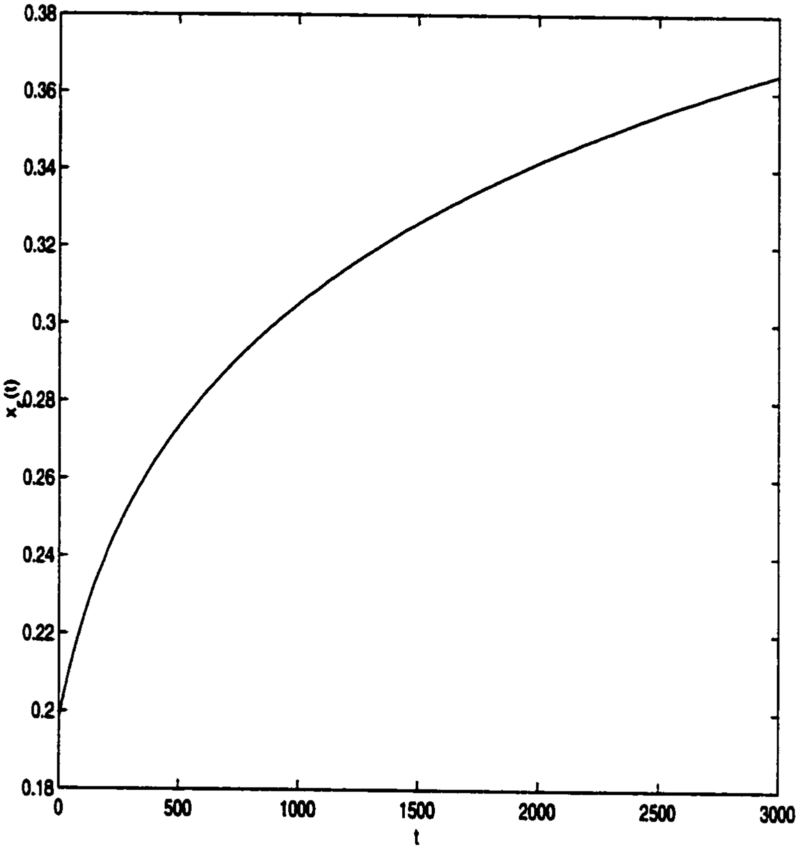

Figure 7.8: **The Motion of the Shock Location for the Reaction-Diffusion Equation in Figure 7.7.**

Chapter 8

CONCLUSIONS AND FUTURE WORK

In this dissertation we have developed asymptotic approximations to the solutions of certain subclasses of equation (1.2). Specifically, we have made the most progress with the advection-diffusion equation (1.4) and the reaction-diffusion equation (1.5), and achieved more limited results for the more general advection-diffusion-reaction equation (1.2). In particular, we derived the limiting ODE which describes the motion of the internal layer present in the solution to these PDEs. With the important orders n and m as defined in Chapters 3 and 5, we found that these layers were, in general, *algebraically* metastable for $m > 1$ and $n > 1$. This is a relatively new phenomenon; previous results (for $n = 1$ or for $m = 1$) found *exponential* metastability. We have also demonstrated algebraic supersensitivity, as contrasted with exponential supersensitivity. In light of the numerical issues involved in directly solving these stiff PDEs, we believe the asymptotic ODEs governing the layers to be important results, especially since the location of the layers tends to be the aspect of the solution most important in applications. Moreover, we believe that the demonstration of algebraic metastability and supersensitivity is important, since previously these phenomena were associated only with exponential asymptotics, which appears as a more restricted special case of our results. Algebraically metastable problems may be expected to have more analogs in the physical world, or at least more that are observable, since the time-scales involved are not so exceedingly long that they are impractical.

We have many directions in which to proceed in the future. An obvious start is to generalize to multidimensional spatial domains. For example, if we consider a rectangular spatial domain, the internal layer will manifest itself as a curve in the domain governed by a PDE, instead of an ODE [22]. Even fewer efficient numerical methods are available in two or more spatial dimensions, so asymptotic results become of even greater importance.

We can also generalize to multiple shock layers. In this case we expect to obtain a *system* of ODEs describing the motion of each layer ([15], [61]). In the case of exponential asymptotics, the ODEs demonstrate nearest-neighbor coupling [61], but in algebraic asymptotics perhaps this is not the case; so, these problems become more interesting.

Different boundary conditions can also be considered. Problems with Neumann or Robin boundary conditions can probably be addressed through slight variations on our methods. Additionally, it would be interesting to draw the connection to the WKB-method of Ward [38], which seems unavailable for dealing with Dirichlet boundary conditions.

Finally, we mention some interesting work by Vasil'eva and her collaborators ([57], [58]) in which we see layers of *alternating-type*. An example from [57] is

$$\begin{cases} \epsilon^2 u_t = \epsilon^2 u_{xx} + (1-u^2)(u - 0.4\,x \sin t), \ 0 < x < 1, \ t > 0, \ 0 < \epsilon \ll 1 \\ u(0,t) = 0, \ u(1,t) = 0, \ u(x,0) = \sin \pi x. \end{cases} \tag{8.1}$$

In these problems, a boundary layer may be stable near one boundary for a period of time, then an internal layer forms and drifts to the other side, then this happens in the reverse direction. This can happen a finite or an infinite number times, depending on the nonlinearities involved. While directly connected to our work involving dynamic interior layers, dynamical systems approaches are necessary to determine the stability of the boundary layers, i.e. when the large-scale changes in the solution will occur. These problems, as well as those mentioned above, provide a rich source of research for the future which is of technological importance.

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Karl R. Knaub

Education:

- Ph.D. in Applied Mathematics (anticipated 2001), University of Washington, Seattle, Washington.
- M.S. in Applied Mathematics (1998), University of Washington, Seattle, Washington.
- B.A. with majors in Mathematics and Biology (1995), Swarthmore College, Swarthmore, Pennsylvania.

Experience and Honors:

- Assistant Professor (August 2001 - the foreseeable future), Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania.
- Predoctoral Lecturer (March 2000 - June 2000), Department of Applied Mathematics, University of Washington, Seattle, Washington.
- Co-organizer, primary mathematics educational workshop (Summer 1999), University of Washington, Seattle, Washington.
- Departmental Award for Teaching and Service (November 1999), Department of Applied Mathematics, University of Washington, Seattle, Washington.
- NSF VIGRE Research Fellowship (September 1999 - June 2001), Department of Applied Mathematics, University of Washington, Seattle, Washington.

- Predoctoral Lecturer (June 1998 - August 1998), Center for Quantitative Science, University of Washington, Seattle, Washington.

- Teaching Assistant (September 1995 - December 2000), Departments of Mathematics and Applied Mathematics, University of Washington, Seattle, Washington.