Generalized Matrix-fractional Functions and Their Applications

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Abstract

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The support function of a closed convex set is a central object in convex geometry as it completely identifies the underlying set. For a particular class of sets – the graph of matrix valued mapping $Y \mapsto -\frac{1}{2}YY^T$ over an affine manifold $\{Y \in \mathbb{R}^{n\times m} \mid AY = B\}$, their support functions are named generalized matrix-fractional (GMF) functions, and were first introduced by Burke and Hoheisel as a tool for unifying a wide range of applications including variational properties of linear constrained quadratic optimization problems, generalized Ky Fan norms, variational Gram functions (VGF), the Aitken's theorem and Gauss-Markov theorem in statistical estimation, and many topics in machine learning such as K-means clustering, support vector machines and multi-task learning.

In the first part of the thesis we study the convex geometry of GMF functions and dramatically simplify their original representations. Second part of the thesis is devoted to the study of partial infimal projections of the sum of GMF functions and various classes of convex functions, where most applications arise.

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Chapter 1

INTRODUCTION

Consider the quadratic optimization problem

$$\max_{u\in\mathbb{R}^n} -\frac{1}{2}u^TVu + x^Tu \tag{1.1}$$

The optimal value function for (1.1) is the pseudo matrix-fractional function [17, Ex. 3.5.0.0.2] defined as

$$\gamma(x,V) := \begin{cases} \frac{1}{2}x^TV^\dagger x & \text{if} \quad V \succeq 0,\ x \in \operatorname{rge} V, \\ +\infty & \text{else.} \end{cases}$$

It is the closure [10, Prop. 5.5] of the well known matrix-fractional function [1, 9, 29, 31, 32] defined as

$$\phi(x,V) := \begin{cases} \frac{1}{2}x^TV^{-1}x & \text{if} \quad V \succ 0, \\ +\infty & \text{else.} \end{cases}$$

The function $\gamma(x,V)$ is proper, closed and sublinear. In fact, from (1.1) it is not difficult to see that the optimal value function $\gamma(x,V)$ coincides with the support function of the set $\left\{(u, -\frac{1}{2}uu^T) \mid u \in \mathbb{R}^n\right\}$, i.e., the graph of a quadratic function.

The Generalized Matrix-fractional (GMF) function [10, 13, 14] is a further generalization of $\gamma(x,V)$. In particular, it is defined as the support function of the graph of the following matrix-valued quadratic mapping over an affine-manifold

$$\mathcal{D}(A,B) := \left\{\left(Y, -\frac{1}{2}YY^T\right) \in \mathbb{E} \;\middle|\; Y \in \mathbb{R}^{n\times m} : AY = B\right\}.$$

An explicit formula for the GMF function is given in [10, Theorem 4.1]:

$$\sigma_{\mathcal{D}(A,B)}(X,V) = \begin{cases} \frac{1}{2}\operatorname{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B}\right) & \text{if} \ \operatorname{rge}\binom{X}{B} \subset \operatorname{rge} M(V),\ V \in \mathcal{K}_A, \\ +\infty & \text{else,} \end{cases} \tag{1.2}$$

where $\mathcal{K}_A := \left\{V \in \mathbb{S}^n \;\middle|\; u^T V u \geq 0 \ (u \in \ker A)\right\}$ and $M(V)^\dagger$ is the Moore-Penrose pseudo inverse of the matrix

$$M(V) := \begin{pmatrix} V & A^T \\ A & 0 \end{pmatrix}.$$

When $A = B = 0$, the GMF function

$$\gamma(X, V) := \begin{cases} \frac{1}{2}\mathrm{tr}\,(X^T V^\dagger X) & \text{if} \quad V \succeq 0,\ \mathrm{rge}\, X \subset \mathrm{rge}\, V, \\ +\infty & \text{else}, \end{cases}$$

generalizes $\gamma(x, V)$ to matrix variables.

The geometry of the set $\mathcal{D}(A, B)$, especially its closed convex hull, is central to the understanding of GMF function. Burke and Hoheisel [10, Lemma 4.2] proposed a complex representation based on Carathéodory's theorem. In Chapter 2, by introducing a new class of cones (see Proposition 2.2.1), we provide a much simpler and intuitive representation of the set (see Theorem 2.2.2). Based on the new representation, we also compute various related geometric objects that were previously unavailable. This paves the way to Chapter 3, where we focus on partial infimal projection of the form

$$\inf_{V \in \mathbb{S}^n} \sigma_{\mathcal{D}(A,B)}(X, V) + h(V), \tag{1.3}$$

where $h : \mathbb{S}^n \to \overline{\mathbb{R}}$ is convex. This form is of particular interest, because it connects the GMF function to a diverse range of applications. For example, in machine learning, prior information about the model parameters X is usually encoded as structural constraints on X. The problem class (1.3) is extremely expressive in forming these structural specifications, through different choices of h. This includes various types of matrix norms, such as Frobenius norm, weighted nuclear norm (see Corollary 3.4.7), *Ky Fan* norm (see Section 3.5.3) and their linear combinations [10, Corollary 5.9]. The major contribution of Chapter 3 are constraint qualification for (1.3) as well as the formulas for its convex conjugate and subdifferential (see Section 3.3). Many results and their proofs that previously required significant effort are now obtained simple special cases of the results in Chapter 3. This includes, for instance, a complete characterization of variational Gram functions [30] (see Section 3.5.2).

In the following, we first start with notations and background materials related to the research. Section 1.4 illustrates a few selected applications of the GMF function in statistical estimation and machine learning.

1.1 Notations

Let $\mathcal{E}$ be a finite dimensional Euclidean space with inner product denoted by $\langle\cdot,\cdot\rangle$ and accompanying induced norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. For example, on the space of matrices, the Frobenius norm induced by the trace inner procuct. The closed ϵ-ball about a point $x \in \mathcal{E}$ denoted by $B_\epsilon(x)$.

Let $S \subset \mathcal{E}$ be nonempty. The (topological) *closure* and *interior* of S are denoted by $\mathrm{cl}\, S$ and $\mathrm{int}\, S$, respectively. The *(linear) span* of S will be denoted by $\mathrm{span}\, S$.

The *convex hull* of S is the set of all convex combinations of elements of S and is denoted by $\mathrm{conv}\, S$. Its closure (the *closed convex hull*) is $\overline{\mathrm{conv}}\, S := \mathrm{cl}\,(\mathrm{conv}\, S)$. The *conical hull* of S is the set

$$\mathrm{pos}\, S := \mathbb{R}_+ \cdot S := \{\lambda x \mid x \in S,\ \lambda \geq 0\}.$$

The *convex conical hull* of S is

$$\mathrm{cone}\ S := \left\{ \sum_{i=1}^{r} \lambda_i x_i \mid r \in \mathbb{N},\ x_i \in S,\ \lambda_i \geq 0 \right\}.$$

It is easily seen that $\mathrm{cone}\ S = \mathrm{pos}\,(\mathrm{conv}\, S) = \mathrm{conv}\,(\mathrm{pos}\, S)$. The closure of the latter is $\overline{\mathrm{cone}}\, S := \mathrm{cl}\,(\mathrm{cone}\ S)$. The *affine hull* of S, denoted by $\mathrm{aff}\, S$, is the smallest affine space that contains S.

The *relative interior* of a convex set $C \subset \mathcal{E}$ is its interior relative to its affine hull, i.e.

$$\mathrm{ri}\, C = \{x \in C \mid \exists \varepsilon > 0 :\ B_\varepsilon(x) \cap \mathrm{aff}\, C \subset C\}.$$

It is well known, see e.g. [6, Section 6.2], that the points $x \in \mathrm{ri}\, C$ are characterized through

$$\mathrm{pos}\,(C - x) = \mathrm{span}\,(C - x), \tag{1.4}$$

where the latter is the (unique) subspace parallel to $\operatorname{aff} C$. In particular, we have $\operatorname{pos} C = \operatorname{aff} C = \operatorname{span} C$ if and only if $0 \in \operatorname{ri} C$.

The *polar set* of S is defined by

$$S^\circ := \{v \in \mathcal{E} \mid \langle v, x\rangle \leq 1 \ (x \in S)\}.$$

Moreover, we define the *bipolar set* of S by $S^{\circ\circ} := (S^\circ)^\circ$. It is well known that $S^{\circ\circ} = \overline{\operatorname{cone}}\,(S \cup \{0\})$. If $K \subset \mathcal{E}$ is a cone (i.e. $\operatorname{pos} K \subset K$) it can be seen by a homogeneity argument that

$$K^\circ = \{v \in \mathcal{E} \mid \langle v, x\rangle \leq 0 \ (x \in K)\} =: K^-,$$

and if $\mathcal{S} \subset \mathcal{E}$ is a subspace, $\mathcal{S}^\circ$ is the orthogonal subspace $\mathcal{S}^\perp$.

The *horizon cone* of $S \subset \mathcal{E}$ is the set

$$S^\infty := \{v \in \mathcal{E} \mid \exists \{\lambda_k\} \downarrow 0, \ \{x_k \in S\} : \ \lambda_k x_k \to v\}$$

which is always a closed cone. For a cone $K \subset \mathcal{E}$, we have $K^\infty = \operatorname{cl} K$. Moreover, for a convex set $C \subset \mathcal{E}$, C^∞ coincides with the *recession cone* of the closure of C, i.e.

$$C^\infty = \{v \mid x + tv \in \operatorname{cl} C \ (t \geq 0, \ x \in C)\} = \{y \mid C + y \subset C\}. \tag{1.5}$$

For $f : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ its *domain* and *epigraph* are given by

$$\operatorname{dom} f := \{x \in \mathcal{E} \mid f(x) < +\infty\} \quad \text{and} \quad \operatorname{epi} f := \{(x, \alpha) \in \mathcal{E} \times \mathbb{R} \mid f(x) \leq \alpha\}.$$

We call f *convex* if its epigraph $\operatorname{epi} f$ is a convex set, and we call it closed (or lower semicontinuous) if $\operatorname{epi} f$ is closed. If f is proper, we call it positively homogeneous if $\operatorname{epi} f$ is a cone, and sublinear if $\operatorname{epi} f$ is a convex cone.

In what follows we use the following abbreviations:

$$\begin{aligned} \Gamma(\mathcal{E}) &:= \{f : \mathcal{E} \to \mathbb{R} \cup \{+\infty\} \mid f \text{ proper, convex}\}, \\ \Gamma_0(\mathcal{E}) &:= \{f \in \Gamma(\mathcal{E}) \mid f \text{ closed}\}. \end{aligned}$$

When the underlying space $\mathcal{E}$ is apparent from the context, we omit it from the notation and simply write Γ and Γ_0, respectively.

The *lower semicontinuous hull* $\mathrm{cl}\, f$ and the *horizon function* f^∞ of f are defined through the relations

$$\mathrm{cl}\,(\mathrm{epi}\, f) = \mathrm{epi}\,\mathrm{cl}\, f \quad \text{and} \quad \mathrm{epi}\, f^\infty = (\mathrm{epi}\, f)^\infty,$$

respectively. For $f \in \Gamma_0$ the horizon function f^∞ coincides with the *recession function*, see e.g. [36, p. 66], and all of the respective results apply. Note that also the moniker *asymptotic function* is used for the horizon function, see e.g. [5, 26].

The *horizon cone of a function* f is defined as

$$\mathrm{hzn}\, f := \{x \mid f^\infty(x) \le 0\}.$$

For a convex function $f : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ its *subdifferential* at a point $\bar{x} \in \mathrm{dom}\, f$ is given by

$$\partial f(\bar{x}) := \{v \in \mathcal{E} \mid f(x) \ge f(\bar{x}) + \langle v,\, x - \bar{x}\rangle\}.$$

Recall that, for a convex function f, we always have

$$\mathrm{ri}\,(\mathrm{dom}\, f) \subset \mathrm{dom}\, \partial f \subset \mathrm{dom}\, f,$$

see e.g. [36, p. 227], where $\mathrm{dom}\, \partial f := \{x \in \mathcal{E} \mid \partial f(x) \neq \emptyset\}$ is the *domain of the subdifferential.*

Given a function $f : \mathcal{E} \to \overline{\mathbb{R}}$ its *(Fenchel) conjugate* $f^* : \mathcal{E} \to \overline{\mathbb{R}}$ is defined by

$$f^*(y) := \sup_{x \in \mathcal{E}} \{\langle x,\, y\rangle - f(x)\}.$$

The conjugate f^* is always a closed convex function as it is the pointwise supremum of a collection of linear functions. Given $f \in \Gamma$, $(f^*)^* = \mathrm{cl}\, f$. In particular, $f \in \Gamma_0$ if and only if $f = f^{**} := (f^*)^*$. The definition of the conjugate function yields the *Fenchel-Young inequality*

$$f(x) + f^*(y) \ge \langle x,\, y\rangle \quad (x, y \in \mathcal{E}). \tag{1.6}$$

Recall the following characterization of subgradients of closed, proper, convex functions.

Proposition 1.1.1 ([36, Theorem 23.5]) *Let $f : \mathcal{E} \to \overline{\mathbb{R}}$ be closed, proper and convex and let $x, y \in \mathcal{E}$. Then the following are equivalent:*

i) $y \in \partial f(x)$;

ii) $f(x) + f^*(y) = \langle x,\, y\rangle$

iii) $x \in \partial f^*(y)$.

Note that for the equivalence of i) and ii) no closedness of f is needed.

Proposition 1.1.2 *Let $f : \mathcal{E} \to \overline{\mathbb{R}}$ be convex. If $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$ is such that $f(\bar{x}) \in \mathbb{R}$, then f is proper and we have $\partial f(\bar{x}) \neq \emptyset$. In this case, $\langle y,\, \bar{x}\rangle = f(\bar{x}) + f^*(y)$ for all $y \in \partial f(\bar{x})$ and $\partial f(\bar{x}) = \mathrm{argmax}_y\{\langle \bar{x},\, y\rangle - f^*(y)\}$.*

Proof: By [36, Lemma 7.3],

$$\mathrm{ri}\,(\mathrm{epi}\, f) = \{(x, \mu) \in \mathcal{E} \times \mathbb{R} \mid x \in \mathrm{ri}\,(\mathrm{dom}\, f) \;\text{ and }\; f(x) < \mu\}\,.$$

Hence $(\bar{x}, f(\bar{x})) \notin \mathrm{ri}\,(\mathrm{epi}\, f)$. Since $(\bar{x}, f(\bar{x})) \in \mathcal{E} \times \mathbb{R}$, [36, Theorem 11.2] tells us that there exists $(\bar{y}, \gamma) \in \mathcal{E} \times \mathbb{R}$ such that $\langle (\bar{y}, \gamma),\, (\bar{x}, f(\bar{x}))\rangle < \langle (\bar{y}, \gamma),\, (x, \mu)\rangle$ for all $(x, \mu) \in \mathrm{ri}\,(\mathrm{epi}\, f)$, or equivalently,

$$\langle \bar{y},\, x - \bar{x}\rangle < \gamma(f(\bar{x}) - \mu) \quad \forall\; (x, \mu) \in \mathrm{ri}\,(\mathrm{epi}\, f).$$

By setting $x = \bar{x}$, we find that $\gamma < 0$. Consequently, $\bar{y}/|\gamma| \in \partial f(\bar{x})$.

Next let $y \in \partial f(\bar{x})$. In particular, this implies that f has an affine minorant so that f never takes the value $-\infty$. Hence, $\langle y,\, \bar{x}\rangle - f(\bar{x}) \geq \langle y,\, x\rangle - f(x)$ for all $x \in \mathrm{dom}\, f$. Consequently, $\langle y,\, \bar{x}\rangle - f(\bar{x}) \geq f^*(y)$ which shows that $\langle y,\, \bar{x}\rangle = f(\bar{x}) + f^*(y)$.

Finally, note that the equivalence $\langle y,\, \bar{x}\rangle = f(\bar{x}) + f^*(y)$ for all $y \in \partial f(\bar{x})$ implies that $\partial\partial f(\bar{x}) \subset \mathrm{argmax}_y\{\langle \bar{x},\, y\rangle - f^*(y)\}$. Conversely, if $\bar{y} \in \mathrm{argmax}_y\{\langle \bar{x},\, y\rangle - f^*(y)\}$, then $\langle \bar{x},\, y\rangle - f^*(y) \leq \langle \bar{x},\, \bar{y}\rangle - f^*(\bar{y})$ for all y so that $f(\bar{x}) \leq f^{**}(\bar{x}) \leq \langle \bar{x},\, \bar{y}\rangle - f^*(\bar{y})$. $\square$

Given a nonempty set $S \subset \mathcal{E}$, its *(convex) indicator function* $\delta_S : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\delta_S(x) := \begin{cases} 0 & \text{if} \quad x \in S, \\ +\infty & \text{if} \quad x \notin S. \end{cases}$$

The indicator of S is convex if and only if S is a convex set, in which case the *normal cone* of S at $\bar{x} \in S$ is given by

$$N_S(\bar{x}) := \partial \delta_S(\bar{x}) = \{v \in \mathcal{E} \mid \langle v,\, x - \bar{x}\rangle \leq 0 \; (x \in S)\}.$$

The *support function* $\sigma_S : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ and the *gauge function* $\gamma_S : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ of a nonempty set $S \subset \mathcal{E}$ are given by

$$\sigma_S(x) := \sup_{v \in S} \langle v,\, x\rangle \quad \text{and} \quad \gamma_S(x) := \inf\{t \geq 0 \mid x \in tS\},$$

respectively. Here we use the standard convention that $\inf \emptyset = +\infty$. It is easy to see that

$$\sigma_S = \sigma_{\overline{\text{conv}}\, S}. \tag{1.7}$$

Moreover, given two (nonempty) sets $S, T \subset \mathcal{E}$ and $x \in \mathcal{E}$, we have

$$\sup_{v \in S} \langle v,\, x\rangle + \sup_{w \in T} \langle w,\, x\rangle = \sup_{(v,w) \in S \times T} \langle v + w,\, x\rangle = \sup_{z \in S+T} \langle z,\, x\rangle,$$

thus

$$\sigma_S + \sigma_T = \sigma_{S+T}. \tag{1.8}$$

Suppose $C \subset \mathcal{E}$ is closed and convex. Then its *barrier cone* is defined by $\text{bar}\, C := \text{dom}\, \sigma_C$. The closure of the barrier cone of C and the horizon cone are paired in polarity, i.e.

$$(\text{bar}\, C)^\circ = C^\infty \quad \text{and} \quad \text{cl}\,(\text{bar}\, C) = (C^\infty)^\circ. \tag{1.9}$$

1.2 Background

In this section we provide some background materials used throughout the study.

1.2.1 Support Functions

One of the most far-reaching results in convex geometry is that a closed convex set coincides with the intersection of all its supporting halfspaces. This idea can be made precise via support functions:

$$S = \bigcap_x \{y \mid \langle x, y\rangle \leq \sigma_S(x)\}.$$

The isomorphism between closed convex sets and their support functions has profound implications. In particular, the usual isomorphism between $\mathbb{R}^n$ and linear functionals on $\mathbb{R}^n$ is recovered by taking S to be a singleton set. Another important example is $\operatorname{epi} f := \{(x, v) \mid f(x) \leq v\}$, the epigraph of a convex function f. Properties of $\sigma_{\operatorname{epi} f}(\cdot)$ directly translates to properties of the set $\operatorname{epi} f$ and in turn to the function f. In fact, $\sigma_{\operatorname{epi} f}((y, -1))$ is the famous Legendre-Fenchel transform of f, one of the fundamental frameworks for convex duality theory.

Many widely used functions are known to be support functions via different choices of the underlying set. In the following section, we give some of those examples.

Examples of Support Functions

For any norm $\|\cdot\|$ in $\mathbb{R}^n$, its dual norm is defined as $\|\cdot\|^d = \sup_{\|y\|\leq 1}\langle \cdot, y\rangle$. In other words, any norm is the support function of its dual norm ball. In a support function $\sigma_S(x)$, if we take x as a unit vector, then the support function can be seen as the Euclidean distance from the origin to the supporting hyperplane of S whose normal direction is specified by x. This observation tells us that the dual norm of $\|\cdot\|_2$ is itself.

Let $e_1, e_2, \ldots, e_n$ be the standard unit coordinate basis in $\mathbb{R}^n$. Then $\operatorname{conv}\{e_1, e_2, \ldots, e_n\} := \Delta_{n-1}$, is the unit simplex and $\sigma_{\Delta_{n-1}}(x) = \max\{x_1, x_2, \ldots, x_n\}$. This is closely related to the duality between $\|\cdot\|_1$ and $\|\cdot\|_\infty$, as Δ_{n-1} is a face of the 1-norm ball.

Calculus with Support Functions

Many algebraic structures on closed convex sets in $\mathbb{R}^n$ translate to operations on support functions. This allows us to establish a set of calculus rules on support functions. Below we list some of these rules. More details can be found at [26, Chap. 3.3].

- Order preserving. $S_1 \subset S_2 \Leftrightarrow \sigma_{S_1}(\cdot) \leq \sigma_{S_2}(\cdot)$.

- Conic combinations. For closed convex sets S_1 and S_2, $t_1, t_2 \geq 0$,

$$\sigma_{\mathrm{cl}\,(t_1 S_1 + t_2 S_2)}(\cdot) = t_1 \sigma_{S_1}(\cdot) + t_2 \sigma_{S_2}(\cdot)$$

- Unions and Intersections. Let $\{S_i\}_{i \in I}$ be a set of nonempty closed convex sets. Then

$$\sigma_{\overline{\mathrm{conv}} \bigcup S_i}(\cdot) = \sup_{i \in I} \sigma_{S_i}(\cdot)$$

  and

$$\sigma_{\bigcap S_i}(\cdot) = \overline{\mathrm{conv}} \inf_{i \in I} \sigma_{S_i}(\cdot), \quad \text{when} \quad \bigcap S_i \neq \emptyset.$$

- Linear mapping. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ an linear operator, S nonempty, then $\sigma_{\mathrm{cl}\,A(S)}(x) = \sigma_S\left(A^T x\right)$.

1.2.2 Fenchel Conjugation and Infimal Convolution

Recall in the previous section we define $f^*(y) := \sup_x \langle x, y \rangle - f(x)$ as the Fenchel conjugate of a convex function f. Then, for any $S \subset \mathbb{R}^n$,

$$\sigma_S^*(x) = \delta_{\overline{\mathrm{conv}}\,S}(x).$$

This support-indicator conjugation again reflects the duality between closed convex sets and support functions. As a special case, the conjugate of a norm is the indicator of its dual norm ball. We also have $(\frac{1}{2}\|x\|^2)^* = \frac{1}{2}(\|x\|^d)^2$. In particular, the function $\frac{1}{2}\|\cdot\|_2^2$ is the only function that is invariant under conjugation.

The infimal convolution of two functions f and g is defined as

$$(f\Box g)(x) = \inf\{f(x-y)+g(y) \mid y \in \mathbb{R}^n\}. \tag{1.10}$$

We have $\operatorname{epi}_{f\Box g} = \operatorname{epi}_f + \operatorname{epi}_g$, so the operation is also termed epi-sum. It then follows that $(f\Box g)^* = \sigma_{\operatorname{epi}_{f\Box g}}((\cdot,-1)) = \sigma_{\operatorname{epi}_f}((\cdot,-1))+\sigma_{\operatorname{epi}_g}((\cdot,-1)) = f^*+g^*$. See Theorem 1.2.1 a) for a generalization of this result. In particular, if we take $g = \frac{1}{2}\|x\|^2$, $e_\lambda f := f\Box\frac{1}{2}\|x\|_2^2$ is called the Moreau envelope of f. Taking $f = \delta_C(x)$, then we get $(\frac{1}{2}\operatorname{dist}(\cdot \mid C)^2)^* = \sigma_C(\cdot)+\frac{1}{2}\|x\|_2^2$. Similarly, $\operatorname{dist}(\cdot \mid C)^* = (\delta_C(\cdot)\,\Box\,\|\cdot\|_2)^* = \sigma_C(\cdot)+\delta_{\mathbb{B}(0,1)}(\cdot)$. These results can be generalized to arbitrary norms.

1.2.3 Extended Sum Rule and Partial Conjugates

In what follows we use the *direct sum* of functions $f_i \in \mathcal{E}$ which is defined by

$$\oplus_{i=1}^m f_i : \mathcal{E}^m \to \mathbb{R}\cup\{+\infty\}, \quad \oplus_{i=1}^m f_i(x_1,\ldots,x_m) = \sum_{i=1}^m f_i(x_i).$$

Theorem 1.2.1 (Extended sum rule) *Let $f_i \in \Gamma_0(\mathcal{E})$ $(i=1,\ldots,m)$ and set $f := \sum_{i=1}^m f_i$. Then the following hold:*

a) (Attouch-Brézis) It holds that $f^* = \operatorname{cl}(f_1^*\,\Box\, f_2^*\,\Box\cdots\Box\, f_m^*)$. Under the qualification condition

$$\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom} f_i) \neq \emptyset \tag{1.11}$$

we have $f^* = f_1^*\,\Box\, f_2^*\,\Box\cdots\Box\, f_m^*$ which is closed, proper and convex and

$$\emptyset \neq \mathcal{T}(z) := \operatorname{argmin}\left\{\sum_{i=1}^m f_i^*(z^i) \;\middle|\; z = \sum_{i=1}^m z^i\right\} \quad (z \in \operatorname{dom} f^*).$$

b) If $\bar{z} \in \sum_{i=1}^m \partial f_i(\bar{x})$, then $\mathcal{T}(\bar{z}) \neq \emptyset$ and

$$\mathcal{T}(\bar{z}) = \left\{(z^1,\ldots,z^m) \;\middle|\; \bar{z} = \sum_{i=1}^m z^i,\ z^i \in \partial f_i(\bar{x}),\ i=1,\ldots,m\right\}.$$

c) Under (1.11) *we have* $\partial f = \sum_{i=1}^m \partial f_i$, $\operatorname{dom} \partial f = \bigcap_{i=1}^m \operatorname{dom} \partial f_i$, *and*

$$\partial f(\bar{x}) = \left\{ \sum_{i=1}^{m} z^i \;\middle|\; z^i \in \partial f_i(\bar{x}), i = 1, \ldots, m \right\} \quad (\bar{x} \in \operatorname{dom} \partial f)$$
$$= \left\{ \bar{z} \;\middle|\; (z^1, \ldots, z^m) \in \mathcal{T}(\bar{z}) \;\; \text{and} \;\; z^i \in \partial f_i(\bar{x}) \; i = 1, \ldots m \right\}.$$

d) Under (1.11), $f^* = f_1^* \,\square\, f_2^* \,\square \cdots \square\, f_m^*$, $\operatorname{dom} \partial f^* = \{z \mid \emptyset \neq \mathcal{T}(z)\} \neq \emptyset$, *and*

$$\partial f^*(\bar{z}) = \left\{ \bigcap_{i=1}^{m} \partial f_i^*(z^i) \;\middle|\; \bar{z} = \sum_{i=1}^{m} z^i \right\} \quad (\bar{z} \in \operatorname{dom} \partial f^*).$$

Proof: a) See [36, Theorem 16.4]).

b) Let $L : \mathcal{E}^m \to \mathcal{E}$ be defined by $L(z^1, \ldots, z^m) = \sum_{i=1}^m z^i$. Then its adjoint $L^* : \mathcal{E} \to \mathcal{E}^m$ is given by $L^*(x) = (x, \ldots, x)$ $(x \in \mathcal{E})$. Let $\bar{z} \in \sum_{i=1}^m \partial f_i(\bar{x})$, and take any $z^i \in \partial f_i(\bar{x})$ $(i = 1, \ldots, m)$ such that $\bar{z} = \sum_{i=1}^m z^i$. By Proposition [36, Theorem 23.5], $\bar{x} \in \partial f_i^*(z^i)$ $(i = 1, \ldots, m)$. Hence, by [36, Theorem 23.8, 23.9] and [6, Proposition 16.8] we obtain

$$0 \in \operatorname{rge} L^* + \partial f_1^*(z^1) \times \cdots \times \partial f_m^*(z^m) \subset \partial(\delta_{\{0\}}\left(L(\cdot) - \bar{z}\right) + \oplus_{i=1}^m f_i^*)(z^1, \ldots, z^m).$$

Hence, $(z^1, \ldots, z^m) \in \mathcal{T}(\bar{z})$. This establishes that

$$\emptyset \neq \left\{ (z^1, \ldots, z^m) \;\middle|\; \bar{z} = \sum_{i=1}^{m} z^i, \; z^i \in \partial f_i(\bar{x}), \; i = 1, \ldots, m \right\} \subset \mathcal{T}(\bar{z}).$$

To see the reverse inclusion, let $(z^1, \ldots, z^m) \in \mathcal{T}(\bar{z})$. By assumption and again [36, Theorem 23.8], we have $\bar{z} \in \sum_{i=1}^m \partial f_i(\bar{x}) \subset \partial f(\bar{x})$. By Proposition [36, Theorem 23.5] and the fact that $f^*(\bar{z}) = \sum_{i=1}^m f_i^*(z^i)$, we have

$$\sum_{i=1}^{m} \left\langle z^i, \bar{x} \right\rangle = \langle \bar{z}, \bar{x} \rangle = f^*(\bar{z}) + f(\bar{x}) = \sum_{i=1}^{m} (f_i^*(z^i) + f_i(\bar{x})),$$

so that

$$0 = \sum_{i=1}^{m} (f_i^*(z^i) + f_i(\bar{x}) - \left\langle z^i, \bar{x} \right\rangle).$$

By the Fenchel-Young inequality, $f_i^*(z^i) + f_i(\bar{x}) - \langle z^i, \bar{x} \rangle \geq 0$ $(i = 1, \ldots, m)$, hence equality must hold for each $i = 1, \ldots, m$, or equivalently $z^i \in \partial f_i(\bar{x})$ $(i = 1, \ldots, m)$. This establishes the reverse inclusion.

c) The first two consequences follow from [36, Theorem 23.8]. For the third, the first equivalence simply follows from the fact that $\partial f = \sum_{i=1}^m \partial f_i$. To see the second equivalence, let $\bar z \in \partial f(\bar x)$. Then, by part b), $\mathcal{T}(\bar z) \neq \emptyset$, and, for every $(z^1, \dots, z^m) \in \mathcal{T}(\bar z)$, we have $z^i \in \partial f_i(\bar x),\ i = 1, \dots, m$. Hence,

$$\partial f(\bar x) \subset \left\{ \bar z \;\middle|\; (z^1, \dots, z^m) \in \mathcal{T}(\bar z),\ z^i \in \partial f_i(\bar x),\ i = 1, \dots, m \right\}.$$

The reverse inclusion follows from the first equivalence.

d) By part a), $f^* = f_1^* \,\square\, f_2^* \,\square \cdots \square\, f_m^*$ is closed, proper, convex, and $\mathcal{T}(z) \neq \emptyset$ for all $z \in \operatorname{dom} f^*$.

Let us first suppose that $\bar z \in \operatorname{dom} \partial f^* \subset \operatorname{dom} f^*$, then $\mathcal{T}(\bar z) \neq \emptyset$. Let $\bar x \in \partial f^*(\bar z)$. By [36, Theorem 23.5], $\bar z \in \partial f(\bar x)$. By part c), this is equivalent to the existence of $z^i \in \partial f_i(\bar x)$ such that $\bar z = \sum_{i=1}^m z^i$, which, by [36, Theorem 23.5], is equivalent to $\bar x \in \{\bigcap_{i=1}^m \partial f_i^*(z^i) \mid \bar z = \sum_{i=1}^m z^i\}$. Hence $\partial f^*(\bar z) \subset \{\bigcap_{i=1}^m \partial f_i^*(z^i) \mid \bar z = \sum_{i=1}^m z^i\}$.

On the other hand, let $\bar x \in \{\bigcap_{i=1}^m \partial f_i^*(z^i) \mid \bar z = \sum_{i=1}^m z^i\}$. Then, by [36, Theorem 23.5] we have $\bar z \in \partial f(\bar x)$. But then, again by [36, Theorem 23.5], $\bar x \in \partial f^*(\bar y)$. Finally, suppose that $(z^1, \dots, z^m) \in \mathcal{T}(\bar z) \neq \emptyset$. Then, as in part a), $0 \in \operatorname{rge} L^* + \partial f_1^*(z^1) \times \cdots \times \partial f_m^*(z^m)$, or equivalently, there is an $\bar x$ such that $\bar x \in \bigcap_{i=1}^m \partial f_1^*(z^i)$ with $\bar z = \sum_{i=1}^m z^i$, i.e., $\bar x \in \partial f^*(\bar z)$. This completes the proof. □

An interesting consequence of Proposition 1.2.1 a) is the following result.

Corollary 1.2.2 (Partial conjugates) *Let $f \in \Gamma(\mathcal{E}_1 \times \mathcal{E}_2)$ and $\bar x \in \mathcal{E}_1$ such that $\bar g := f(\bar x, \cdot)$ is proper. Then $\bar g^*$ is the closure of the function*

$$w \mapsto \inf_{z:(z,w)\in \operatorname{dom} f^*} \{f^*(z, w) - \langle \bar x,\, z\rangle\}.$$

If $\bar x \in \operatorname{ri} L(\operatorname{dom} f)$, where $L : (x, v) \mapsto x$, then the closure can be dropped.

Proof: We use Proposition 1.2.1 a) throughout: Observe that

$$\begin{aligned}\bar{g}^*(w) &= \sup_v\{\langle v,\, w\rangle - f(\bar{x}, w)\}\\ &= \sup_{(x,v)}\{\langle (x,v),\, (0,w)\rangle - (f + \delta_{\{\bar{x}\}\times\mathcal{E}_2})(x,v)\}\\ &= (f + \delta_{\{\bar{x}\}\times\mathcal{E}_2})^*(0,w)\\ &= \mathrm{cl}\,(f^* \,\square\, \sigma_{\{\bar{x}\}\times\mathcal{E}_2})(0,w).\end{aligned}$$

Now notice that $\sigma_{\{\bar{x}\}\times\mathcal{E}_2} = \langle \bar{x},\, \cdot\rangle \oplus \delta_{\{0\}}$. Hence

$$\begin{aligned}(f^* \,\square\, \sigma_{\{\bar{x}\}\times\mathcal{E}_2})(0,w) &= \inf_{(z,u)}\{f^*(z,u) + \langle \bar{x},\, 0 - z\rangle + \delta_{\{0\}}(w-u)\}\\ &= \inf_{z:(z,w)\in \mathrm{dom}\, f^*}\{f^*(z,w) - \langle \bar{x},\, z\rangle\}.\end{aligned}$$

This proves the first statement. Note that the closure can be dropped if $\mathrm{ri}\,(\mathrm{dom}\, f)$ and $\mathrm{ri}\,(\mathrm{dom}\,\delta_{\{\bar{x}\}\times\mathcal{E}_2}) = \{\bar{x}\}\times\mathcal{E}_2$ intersect, which is equivalent to the condition stated.

This concludes the proof.

□

1.2.4 Perturbation Duality Framework

Duality is a key concept in optimization. For a given optimization problem, there are multiple ways to construct its duals, each of which provides a different viewpoint of the original problem (the primal), and sometimes these viewpoints can provide significant computational benefits. In fact, many modern numerical algorithms take advantage of primal-dual pairs, and uses the duality gap as a signal for convergence. In this section, we review the perturbation framework for duality [37, Theorem 11.39] or [5, Chapter 5].

In a convex minimization problem $\inf_x \varphi(x)$, we consider *lifting* the function $\varphi(x) : \mathbb{R}^n \to \overline{\mathbb{R}}$ to $f(x,u) : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, and without loss of generality set $\varphi(x) = f(x,0)$. Denote $p(u) := \inf_x f(x,u)$ as the value function for the primal problem, then we have

$$p^*(y) = \sup_u\{\langle y,u\rangle - \inf_x f(x,u)\} = \sup_{x,u}\{\langle (0,y),(x,u)\rangle - f(x,u)\} = f^*(0,y).$$

The primal problem

$$p(0) = \inf_x f(x, 0) = \inf_x \varphi(x)$$

is then paired with a dual problem

$$p^{**}(0) = \sup_y -f^*(0, y) := \sup_y \psi(y),$$

where $\psi(y) := -f^*(0, y)$ with $\psi : \mathbb{R}^m \to \overline{\mathbb{R}}$ is called the dual objective function. Denote the negative of the perturbed dual problem as $q(v) := \inf_y f^*(v, y)$ Then under the condition that f is proper, lsc and convex, either $0 \in \operatorname{int} \operatorname{dom} p$ or $0 \in \operatorname{int} \operatorname{dom} q$ ensures a zero duality gap $p(0) = p^{**}(0)$ [37, Theorem 11.39]. Next we list some examples.

Example 1.2.3 (Lagrange duality) *Consider the following nonlinear convex optimization problem in standard form*

$$\begin{aligned} &\mathit{min}\ g_0(x) \\ &\mathit{s.t.}\ g_i(x) \leq 0,\ \ i \in \{1, \ldots, m\} \\ &\qquad h_i(x) = 0,\ \ i \in \{1, \ldots, n\} \end{aligned}$$

with g_i and h_i all proper, convex and lsc. Utilizing indicator functions we have $\inf_x \varphi(x) := g_0(x) + \sum_{i=1}^m \delta_{g_i(x) \leq 0}(x) + \sum_{i=1}^n \delta_{h_i(x)=0}(x)$ *as the primal problem. Lagrange duality is equipped with the following choice of perturbation function*

$$f(x, u, v) := g_0(x) + \sum_{i=1}^m \delta_{g_i(x)+u_i \leq 0}(x) + \sum_{i=1}^n \delta_{h_i(x)+v_i=0}(x)$$

and $p(u, v) := \inf_x f(x, u, v)$. *Computing the dual of f gives*

$$\begin{aligned} f^*(0, \mu, \lambda) &= \sup_{x,u,v} \sum_{i=1}^m \mu_i u_i + \sum_{i=1}^n \lambda_i v_i - g_0(x) - \sum_{i=1}^m \delta_{g_i(x)+u_i \leq 0}(x) - \sum_{i=1}^n \delta_{h_i(x)+v_i=0}(x) \\ &= \sup_{\substack{g_i(x)+u_i \leq 0 \\ h_i(x)+v_i=0}} \sum_{i=1}^m \mu_i u_i + \sum_{i=1}^n \lambda_i v_i - g_0(x) \\ &= \sup_{g_i(x)+u_i \leq 0} \sum_{i=1}^m \mu_i u_i - \sum_{i=1}^n \lambda_i h_i(x) - g_0(x). \end{aligned}$$

This reveals the dual problem

$$\begin{aligned}\sup_{\mu,\lambda}\ -f^*(0,\mu,\lambda) &= \sup_{\mu,\lambda}\ \inf_{g_i(x)+u_i\le 0}\ g_0(x)-\sum_{i=1}^m \mu_i u_i+\sum_{i=1}^n \lambda_i h_i(x)\\ &= \sup_{\mu\ge 0,\lambda}\ \inf_x\ g_0(x)+\sum_{i=1}^m \mu_i g_i(x)+\sum_{i=1}^n \lambda_i h_i(x).\end{aligned}$$

Moreover, Slater's condition on the primal problem implies $0\in\operatorname{int}\operatorname{dom} p$*, which is a sufficient condition for strong duality.*

Example 1.2.4 (Fenchel duality) *Optimization problems of the form* $\inf_x \varphi(x) := g(x)+h(Ax)$*, where* g *and* h *are proper, convex and lsc, can be perturbed as*

$$f(x,u) := g(x)+h(Ax+u)$$

with $p(u) := \inf_x f(x,u)$*. In this case the full conjugate of* f *is available as*

$$\begin{aligned}f^*(v,y) &= \sup_{x,u}\ \langle x,\, v\rangle+\langle u,\, y\rangle-g(x)-h(Ax+u)\\ &= \sup_{x,w}\ \langle x,\, v\rangle+\langle w-Ax,\, y\rangle-g(x)-h(w)\\ &= \sup_x\ \left\langle x,\, v-A^Ty\right\rangle-g(x)+\sup_w\ \langle w,\, y\rangle-h(w)\\ &= g^*(v-A^Ty)+h^*(y).\end{aligned}$$

This gives the Fenchel dual problem

$$\sup_y \psi(y) := \sup_y\ -f^*(0,y) = \sup_y\ -g^*(-A^Ty)-h^*(y),$$

and the negative of the value function for the dual problem $q(v) := \inf_y f^*(v,y) = \inf_y g^*(v-A^Ty)+h^*(y)$*. Also we have strong duality when* $0\in\operatorname{int}\operatorname{dom} p \iff 0\in\operatorname{int}(\operatorname{dom} h - A\operatorname{dom} g)$*, or when* $0\in\operatorname{int}\operatorname{dom} q \iff 0\in\operatorname{int}(-A^T\operatorname{dom} h-\operatorname{dom} g)$*.*

1.3 The Bordered Gramian Matrix

For $V\in\mathbb{S}^n_+$, $A\in\mathbb{R}^{p\times n}$, define the matrix

$$M(V) := \begin{pmatrix} V & A^T\\ A & 0\end{pmatrix}. \tag{1.12}$$

$M(V)$ is called the *bordered Gramian matrix* [15, 20, 34], and it plays an important role in the study of $\mathcal{D}(A,B)$. The following theorem gives the Moore-Penrose pseudo-inverse of $M(V)$ [34, Chap. 3, Thm. 21]:

Theorem 1.3.1 *Let $M(V)$ be given by (1.12). Then*

$$M(V)^\dagger = \begin{pmatrix} D & E^T \\ E & -F \end{pmatrix},$$

where $D = N^\dagger - N^\dagger A^T C^\dagger A N^\dagger$, $E = (N^\dagger A^T C^\dagger)^T$, $F = C^\dagger - CC^\dagger$, and $N = V + A^T A$, $C = AN^\dagger A^T$.

As a special case when $A = I - VV^\dagger$ is the orthogonal projection onto $\ker V$. Then we have the following observation:

Lemma 1.3.2

$$M(V^\dagger)^\dagger = M(V) = \begin{pmatrix} V & A \\ A & 0 \end{pmatrix}.$$

The following proposition shows the condition under which $M(V)$ is invertible:

Proposition 1.3.3 *[15, Thm. 7] $M(V)$ is invertible if and only if $\operatorname{rank} A = p$ and $V \succ_{\ker A} 0$, and in that case*

$$M(V)^{-1} = \begin{pmatrix} P(P^TVP)^{-1}P^T & \left(I - P(P^TVP)^{-1}P^TV\right)A^\dagger \\ (A^\dagger)^T\left(I - VP(P^TVP)^{-1}P^T\right) & (A^\dagger)^T\left(VP(P^TVP)^{-1}P^TV - V\right)A^\dagger \end{pmatrix},$$

where $P \in \mathbb{R}^{n\times(n-p)}$ is any matrix whose columns form an orthonormal basis of $\ker A$.

1.4 Applications

The GMF function is of particular interest, because it establishes a connection between topics in many different areas, including variational properties of linear constrained quadratic optimization problems mentioned in the beginning of this chapter, generalized Ky Fan norms, matrix gauge functionals and many interesting results in statistical estimation and machine learning. In this section we present some applications where the GMF function arise.

Example 1.4.1 (Support Vector Machines) *Consider a binary classification problem where n labeled training data $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^m \times \{-1, 1\}$ are given, in which y_i is the class label for x_i. The two classes of points are called linearly separable if there exists $w \in \mathbb{R}^m, b \in \mathbb{R}$, such that $y_i(\langle w,\, x_i\rangle + b) \geq 1$ for all i. The points lying on these two hyperplanes $\langle w,\, x\rangle + b = \pm 1$ are called support vectors, and the space between these two hyperplanes is called the margin. The support vector machine (SVM) [16, 24, 40] is a classification model that finds w, b which maximizes $2/\|w\|$, the distance between these two hyperplanes.*

In the case when the points are not linearly separable, one can relax the constraint for point i to $y_i(\langle w,\, x_i\rangle + b) \geq \beta_i$, where $\beta_i \leq 1$, thereby allowing point i to lie within (when $-1 < \beta_i < 1$), or even beyond (when $\beta_i \leq -1$) the margin. For a given $\beta \in \mathbb{R}^n$, we can formulate the SVM optimization problem as

$$\min_{w,b}\ \frac{1}{2}\|w\|_2^2, \qquad s.t. \qquad y \circ (Xw + b\mathbf{1}_n) \geq \beta, \tag{1.13}$$

where $X \in \mathbb{R}^{n\times m}$ is the data matrix. Note that the hard margin SVM is a special case of (1.13) *when $\beta = \mathbf{1}_n$, the vector of all ones of length n. The Lagrangian for* (1.13) *is*

$$\begin{aligned} L(w, b, \alpha) &= \frac{1}{2}\|w\|_2^2 + \langle \alpha,\, \beta - y \circ (Xw + b\mathbf{1}_n)\rangle \\ &= \frac{1}{2}\|w\|_2^2 + \langle \alpha,\, \beta\rangle - \langle \alpha,\, YXw\rangle - \langle \alpha,\, Y\mathbf{1}_n\rangle\, b, \end{aligned}$$

where $\alpha \in \mathbb{R}^n_+$ and $Y \in \mathbb{S}^n$ with $Y = \operatorname{diag}(y)$. Note that $L_w = w - X^TY\alpha$, and $L_b = \mathbf{1}_n^TY\alpha$. Therefore the dual problem is

$$\max_{\alpha\in C}\, \langle \alpha,\, \beta\rangle - q(\alpha), \tag{1.14}$$

with $q(\alpha) := \frac{1}{2}\alpha^T(YXX^TY)\alpha$ and $C := \left\{\alpha \in \mathbb{R}^n \;\middle|\; y^T\alpha = 0, \alpha \geq 0\right\} = \mathbb{R}^n_+ \cap \ker(y^T)$ being a cone. Denote

$$h(\alpha) := q(\alpha) + \delta_C(\alpha),$$

The optimal value of (1.14) *is simply $h^*(\beta)$. Using the fact that $(YXX^TY)^\dagger = Y(XX^T)^\dagger Y$ and $\operatorname{rge}(YXX^TY) = \operatorname{rge}(YX)$, we have*

$$q^*(\alpha) = \frac{1}{2}\alpha^TY(XX^T)^\dagger Y\alpha + \delta_{\operatorname{rge} YX}(\alpha).$$

Now apply Theorem 1.2.1 a), we have

$$h^*(\beta) = (q^* \,\Box\, \delta^*_C)(\beta) = \inf_{\omega} \frac{1}{2}\omega^T Y(XX^T)^{\dagger} Y\omega + \delta_{\mathrm{rge}\,(YX)}(\omega) + \delta_{C^\circ}(\beta - \omega),$$

where $C^\circ = \mathbb{R}^n_- + \mathrm{span}\,(y)$. *Note that* $\beta - \omega \in C^\circ$ *if and only if* $\omega \in \beta + \mathbb{R}^n_+ + \mathrm{span}\,(y)$. *Furthermore, define* $\eta := Y\omega$, *then the condition* $\omega \in (\beta + \mathbb{R}^n_+ + \mathrm{span}\,(y)) \cap \mathrm{rge}\,(YX)$ *translates to* $\eta \in (Y\beta + O_y + \mathrm{span}\,(\mathbf{1}_n)) \cap \mathrm{rge}\,X$, *where* $O_y := Y\mathbb{R}^n_+$ *is the closed orthant that contains* y. *Denote the set* $S_\beta := Y\beta + O_y + \mathrm{span}\,(\mathbf{1}_n)$, *then the above equation is equivalent to*

$$h^*(\beta) = \inf_{\eta \in S_\beta \cap \mathrm{rge}\,X} \frac{1}{2}\eta^T (XX^T)^{\dagger}\eta = \inf_{\eta \in S_\beta} \gamma(\eta, XX^T). \tag{1.15}$$

As a special case, the dual optimal value of the hard margin SVM is

$$h^*(\mathbf{1}_n) = \inf_{\eta \in y + O_y + \mathrm{span}\,(\mathbf{1}_n)} \gamma(\eta, XX^T).$$

Example 1.4.2 (Multi-task Learning) *In multi-task learning [3, 4, 19],* T *sets of labelled training data* (x_{t1}, y_{t1}), …, $(x_{tn}, y_{tn}) \in \mathbb{R}^m \times \mathbb{R}$, $t = 1, \ldots, T$ *are given, representing* T *learning tasks. The assumption is that all these tasks share a common set of features* $h_i(x) = \langle u_i, x\rangle$, $i = 1, \ldots, m$, $u_i \in \mathbb{R}^m$, *so that the estimator for each task* t *is given by* $f_t = \langle a_t, h\rangle$, *where* $a_t \in \mathbb{R}^m$. *If we further assume* u_i *are orthonormal vectors and denote* $U \in \mathbb{R}^{m\times m}$ *as the matrix whose columns are* u_i, *then the multi-task learning problem can be formulated as*

$$\min_{a_t, U} \sum_{t=1}^{T} \sum_{i=1}^{m} L\left(y_{ti}, \langle a_t, U^T x_{ti}\rangle\right),$$

where $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a loss function that is convex in the second argument. Note that in this formulation the number of common features is assumed to be* m, *the size of the dimension. In reality, features shared by the tasks are usually sparse, so a penalization term is added to the objective,*

$$\min_{A, U} \sum_{t=1}^{T} \sum_{i=1}^{m} L\left(y_{ti}, \langle a_t, U^T x_{ti}\rangle\right) + \mu \|A\|^2_{2,1}, \tag{1.16}$$

where $\mu \geq 0$, $A = (a_1, \ldots, a_T) \in \mathbb{R}^{m\times T}$. *Here* $\|\cdot\|_{2,1}$ *is defined as the sum of the Euclidean norms of the columns of the matrix. This norm acts as a Tikhonov regularization for each* *task and at the same time it encourages sparsity of the shared features. Denote* $W = UA$*, then the nonconvex problem (1.16) is equivalent to the following convex problem [3, Thm 3.1]:*

$$\min_{W,D} \sum_{t=1}^{T} \sum_{i=1}^{m} L\left(y_{ti}, \langle w_t, x_{ti}\rangle\right) + 2\mu\gamma(W, D) \quad s.t. \quad \operatorname{tr} D \leq 1. \tag{1.17}$$

Convexity of (1.17) is easily established as $\gamma(W, D)$ *is a support functional. Moreover, as Corollary 3.4.7 later shows that the optimization in* D *has an analytical solution, and (1.17) reduces to*

$$\min_{W} \sum_{t=1}^{T} \sum_{i=1}^{m} L\left(y_{ti}, \langle w_t, x_{ti}\rangle\right) + \mu\|W\|_*^2.$$

Example 1.4.3 (K-means Clustering) *Consider* n *points in* $\mathbb{R}^m$*, the objective is to partition the points into* k *clusters such that the total distances from the points to their corresponding cluster centers are minimized. k-means clustering [33] solves*

$$K(X) := \min_{C,E} \frac{1}{2}\|X - EC\|_2^2,$$

where $X \in \mathbb{R}^{n\times m}$*,* $C \in \mathbb{R}^{k\times m}$ *represents the* k *centers, and* E *is a* $n \times k$ *matrix where each row is one of* $e_1^T, \ldots, e_k^T$ *that represents cluster assignments. For a fixed cluster assignment matrix* E*, the center matrix* C *is easily determined, where each center is simply the mean of those points in the same cluster. This can be seen by solving* C *in the above optimization problem:*

$$C = (E^T E)^{-1} E^T X.$$

Define $P = E(E^T E)^{-1} E^T$*, then* P *is an orthogonal projection since* P *is symmetric and it is easy to verify* $P^2 = P$*. [41] considered rewriting the k-means objective as*

$$K(X) = \min_{P} \frac{1}{2}\|(I - P)X\|_2^2 = \frac{1}{2}\min_{P} \operatorname{tr}\left((I - P)XX^T\right) = \frac{1}{2}\|X\|_2^2 - \sigma_{\mathcal{P}}\left(\frac{1}{2}XX^T\right),$$

where $\mathcal{P} = \{E(E^T E)^{-1} E^T\}$*. Through this viewpoint, [41] obtains a lower bound on the objective of k-means by relaxing* $\mathcal{P}$ *to be the set of rank-k orthogonal projection matrices* $\mathcal{P}' = \{UU^T | U \in \mathbb{R}^{n\times k}, U^T U = I_k\} \supset \mathcal{P}$*, in which case the support function enjoys an* *analytical form* $\sigma_{P'}(XX^T) = \sum_{i=1}^{k} \sigma_i^2(X)$. *Note that he optimal value function can also be rewritten as*

$$\begin{aligned} K(X) &= \min_{\mathcal{P}} \frac{1}{2}\|X - PX\|_2^2 \\ &= \min_{\mathcal{P}} \frac{1}{2}\langle (I-P)X, X^T(I-P)\rangle \\ &= \min_{\mathcal{P}} \frac{1}{2}\mathrm{tr}\,(X^T(I-P)X) \\ &= \min_{\mathcal{P}} \frac{1}{2}\mathrm{tr}\,(X^T(I-P)^{\dagger}X). \end{aligned}$$

Here we use the fact that $I - P$ *is an orthogonal projection. Note that* $\mathrm{tr}\,(I-P) = n-k$, $\forall P \in \mathcal{P}$, *so if we denote* $\mathcal{Q} = \{Q \in \mathbb{S}^n \mid \mathrm{tr}\, Q \le n-k\}$, *then Corollary 3.4.7 gives us the following lower bound on the objective of k-means:*

$$K(X) \ge \min_{\mathcal{Q}} \gamma(X, Q) = \frac{1}{2(n-k)}\|X\|_*^2.$$

It is possible to derive other bounds by utilizing the geometric centering matrix *[17, App. B.4.1]*

$$V_n = I_n - \frac{1}{n}\mathbf{e_n}\mathbf{e_n}^T.$$

Note that the geometric center of all the n *points is given by* $c = \frac{1}{n}\mathbf{e_n}^T X$. *If we were to translate their geometric center to the origin, then the operation* $X - \mathbf{e_n}c = X - \frac{1}{n}\mathbf{e_n}\mathbf{e_n}^T X = V_n X$ *will do. That's where the name* geometric centering matrix *comes from. The matrix* V_n *itself is an orthogonal projection, and enjoys the following property:*

Proposition 1.4.4 $V_n P = P V_n = P - I + V_n$.

Proof: *To see the first equality,*

$$\begin{aligned} V_n P &= (I - \frac{1}{n}\mathbf{e_n}\mathbf{e_n}^T)P \\ &= P - \frac{1}{n}\mathbf{e_n}\mathbf{e_n}^T P \\ &= P - \frac{1}{n}\mathbf{e_n}\mathbf{e_n}^T \quad (P\mathbf{e_n} = \mathbf{e_n}) \\ &= P(I - \frac{1}{n}\mathbf{e_n}\mathbf{e_n}^T) \quad (P\mathbf{e_n} = \mathbf{e_n}) \\ &= PV_n. \end{aligned}$$

The second equality follows because

$$P - I + V_n = P - I + I - \frac{1}{n}\mathbf{e_n}\mathbf{e_n}^T = PV_n.$$

□ *Intuitively translating the points will not affect the k-means objective, and this can be made precise by the following proposition:*

Proposition 1.4.5 $V_n(I-P)V_n = I - P$, *which allows us to deduce* $K(X) = K(V_nX)$.

Proof: *The proof can be done by applying 1.4.4:* $V_n(I-P)V_n = (V_n - V_nP)V_n = V_n - V_nPV_n = V_n - PV_n = V_n - (P - I + V_n) = I - P$. □ *Finally we show the following proposition regarding* V_n*:*

Proposition 1.4.6 *For* $\lambda \neq 1$, $(I + \frac{\lambda}{1-\lambda}P)V_n(I - \lambda P) = (I - \lambda P)V_n(I + \frac{\lambda}{1-\lambda}P) = V_n$.

Proof:

$$\begin{aligned}(I + \frac{\lambda}{1-\lambda}P)V_n(I-\lambda P) &= (V_n + \frac{\lambda}{1-\lambda}PV_n)(I - \lambda P)\\ &= V_n + \frac{\lambda}{1-\lambda}PV_n - \lambda V_nP - \frac{\lambda^2}{1-\lambda}PV_nP\\ &= V_n + \left(\frac{\lambda}{1-\lambda} - \lambda - \frac{\lambda^2}{1-\lambda}\right)PV_n\\ &= V_n.\end{aligned}$$

□ *This leads us to the following lemma, which allows us to rewrite the k-means objective:*

Lemma 1.4.7 *For* $\lambda \neq 1$, *define* $\Sigma = V_n(I + \frac{\lambda}{1-\lambda}P)V_n$, *then* $\Sigma^\dagger = V_n(I - \lambda P)V_n$.

Proof: *Using 1.4.6, we have* $\Sigma\Sigma^\dagger = V_n(I + \frac{\lambda}{1-\lambda}P)V_nV_n(I-\lambda P)V_n = V_n(I + \frac{\lambda}{1-\lambda}P)V_n(I - \lambda P)V_n = V_n$. $\Sigma^\dagger\Sigma = V_n(I-\lambda P)V_nV_n(I + \frac{\lambda}{1-\lambda}P)V_n = V_n(I-\lambda P)V_n(I + \frac{\lambda}{1-\lambda}P)V_n = V_n$. □ *In particular, take* $\lambda = \frac{1}{2}$, *one can deduce*

$$(V_n(I+P)V_n)^\dagger = V_n(I - \frac{1}{2}P)V_n.$$

Therefore we can rewrite the objective

$$\begin{aligned} K(X) &= \min_P \frac{1}{2}\mathrm{tr}\,(X^T(I-P)X) \\ &= \min_P \frac{1}{2}\mathrm{tr}\,(X^T V_n(I-P)V_n X) \\ &= -\frac{1}{2}\|V_n X\|_2^2 + \min_P \mathrm{tr}\,\left(X^T V_n(I-\frac{1}{2}P)V_n X\right) \\ &= -\frac{1}{2}\|V_n X\|_2^2 + \min_P \mathrm{tr}\,\left(X^T (V_n(I+K)V_n)^\dagger X\right) \end{aligned}$$

Note that $\mathrm{tr}\,(V_n(I+P)V_n) = n+k-2$*, so if we denote* $\mathcal{Q} = \{Q \in \mathbb{S}^n \mid \mathrm{tr}\,Q \leq n+k-2\}$*, then Corollary 3.4.7 provides the following lower bound on the objective of k-means:*

$$K(X) \geq -\frac{1}{2}\|V_n X\|_2^2 + 2\min_{\mathcal{Q}} \gamma(X,Q) = -\frac{1}{2}\|V_n X\|_2^2 + \frac{1}{n+k-2}\|X\|_*^2.$$

Example 1.4.8 (Minimum Variance Affine Unbiased Estimator) *For a linear regression model* $y = A^T\beta + \epsilon$ *where* $\epsilon \sim \mathcal{N}(0, \sigma^2 V)$*, and a given matrix* B*, an affine unbiased estimator of* $B^T\beta$ *is an estimator of the form* $\hat\theta = X^T y + c$ *satisfying* $\mathrm{E}\hat\theta = B^T\beta$*. Among all affine unbiased estimators, the* minimum variance affine unbiased estimator *is* $\hat\theta^*$ *such that* $\mathrm{Var}(\hat\theta^*) \preceq \mathrm{Var}(\hat\theta)$*,* $\forall\hat\theta$*. Note that an affine estimator* $\hat\theta = X^T y + c$ *is unbiased if and only if* $AX = B$ *and* $c = 0$*. Since* $\mathrm{Var}(\hat\theta) = \sigma^2 X^T V X$*, one way to calculate the minimum variance affine unbiased estimator* $\hat\theta^*$ *is to solve the following optimization problem [34, Chap. 13]*

$$v(A,B,V) := \min_{X:AX=B} \frac{1}{2}\mathrm{tr}\, X^T V X. \tag{1.18}$$

If a solution X^* *to (1.18) exists and is unique, then* $\hat\theta^* = (X^*)^T y$*. Observe that* $v(A,B,V) = -\sigma_{\mathcal{D}(A,B)}(0,V)$*. The optimal solution* X^* *satisfies*

$$M(V)\begin{pmatrix} X^* \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ B \end{pmatrix}.$$

As a special case when X *is full rank and* $V \succ 0$*,* $M(V)$ *is invertible and we recover Aitken's Theorem. If we further restrict* $V = I$ *we obtain the Gauss-Markov Theorem.*

In the study of GMF functions, a key step is to establish the closed convex hull of the set $\mathcal{D}(A,B)$ and this is obtained in [10, Lemma 4.2] based on Carathéodory's Theorem.

In Chapter 2, through the study of the convex geometry of GMF functions a much more elegant representation of $\mathcal{D}(A, B)$ is discovered. The new representation also facilitates the computation of infimal projections of the GMF functions which are considered in Chapter 3.

Chapter 2

CONVEX GEOMETRY OF GENERALIZED MATRIX-FRACTIONAL FUNCTIONS

2.1 Introduction

Generalized matrix-fractional (GMF) functions were introduced in [10] as a means to unify a range of seemingly divergent tools in matrix optimization related to inverse problems, regularization and machine learning. Somewhat surprisingly GMF functions coincide with the negative of the optimal value function for affinely constrained quadratic programs, and are representable as support functions on the matrix space $\mathbb{E} := \mathbb{R}^{n\times m} \times \mathbb{S}^n$, where $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ are the vector spaces of real $n \times m$ and symmetric $n \times n$ matrices, respectively. The most significant challenge in [10] is the derivation of an expression for the closed convex set associated with the support function representation. Unfortunately, the representation given in [10] is exceedingly complicated. The main contribution of this paper is to provide a simple, elegant, and intuitive representation for this set. We then use this representation to provide a simplified expression for the subdifferential of a GMF function and to compute various related geometric objects that were previously unavailable. These representations dramatically simplify the use of these tools in a wide range of applications [14]. Before proceeding, we review the definition of a GMF function.

Given $(A, B) \in \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ with $\operatorname{rge} B \subset \operatorname{rge} A$, the graph of the matrix valued mapping $Y \mapsto -\frac{1}{2}YY^T$ over an affine manifold $\{Y \in \mathbb{R}^{n\times m} \mid AY = B\}$ is given by

$$\mathcal{D}(A, B) := \left\{ \left(Y, -\frac{1}{2}YY^T\right) \in \mathbb{E} \;\middle|\; Y \in \mathbb{R}^{n\times m} :\; AY = B \right\}. \tag{2.1}$$

The associated GMF function is the support function of the set $\mathcal{D}(A, B)$:

$$\sigma_{\mathcal{D}(A,B)}(X, V) = \sup_{(Y,W)\in\mathcal{D}(A,B)} \langle (X, V),\, (Y, W)\rangle\,,$$

where we use the Frobenius inner product on $\mathbb{E}$,

$$\langle (Y,W),\,(X,V)\rangle = \mathrm{tr}\,(Y^TX) + \mathrm{tr}\,WV = \mathrm{tr}\,(XY^T + WV).$$

In [10, Theorem 4.1], it is shown that

$$\sigma_{\mathcal{D}(A,B)}(X,V) = \begin{cases} \frac{1}{2}\mathrm{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B}\right) & \text{if } \ \mathrm{rge}\binom{X}{B} \subset \mathrm{rge}\, M(V),\ V \in \mathcal{K}_A, \\ +\infty & \text{else,} \end{cases} \tag{2.2}$$

where $\mathcal{K}_A := \left\{V \in \mathbb{S}^n \;\middle|\; u^TVu \geq 0 \ (u \in \ker A)\right\}$ and $M(V)^\dagger$ is the Moore-Penrose pseudo inverse of the matrix

$$M(V) = \begin{pmatrix} V & A^T \\ A & 0 \end{pmatrix}.$$

In particular, this implies that

$$\begin{aligned} \mathrm{dom}\,\sigma_{D(A,B)} &= \mathrm{dom}\,\partial\sigma_{D(A,B)} \\ &= \left\{(X,V) \in \mathbb{R}^{n\times m} \times \mathbb{S}^n \;\middle|\; \mathrm{rge}\begin{pmatrix} X \\ B \end{pmatrix} \subset \mathrm{rge}\, M(V),\ V \in \mathcal{K}_A\right\}. \end{aligned} \tag{2.3}$$

Note that $\mathrm{dom}\,\sigma_{D(A,B)}$ is clearly not a closed set. To see this consider the case $A = B = 0$ and $V = \eta I$ so that any $X \neq 0$ has $\mathrm{rge}\, X \in \mathrm{rge}\, V$. But as $\eta \downarrow 0$ it is not the case that $\mathrm{rge}\, X \subset \mathrm{rge}\, 0$. Consequently, the statement in [10, Theorem 4.1] that this domain is closed is clearly false. This error does not affect the validity of the other results in [10] since none of them require that the set $\mathrm{dom}\,\sigma_{D(A,B)}$ be closed.

The representation (2.2) is the basis for the name *generalized matrix-fractional function* since the matrix-fractional functions [9, 17, 25, 29] are obtained when the matrices A and B are both taken to be zero.

The paper is organized as follows: Section 2.2 begins with a study of the cones $\mathcal{K}_A$ defined in (3.6) and their polars. This is immediately followed by deriving the new representation of the set $\Omega(A,B) := \overline{\mathrm{conv}}\,\mathcal{D}(A,B)$ in Theorem 2.2.2. With this representation in hand, we derive new simplified descriptions for the normal cone $N_{\Omega(A,B)}$ and the subdifferential $\partial\sigma_{\Omega(A,B)}$ in Section 2.3. In Section 2.4 we explore the convex geometry of the set $\Omega(A,B)$, and conclude in Section 2.5 with the important special case where $B = 0$ and $\sigma_{\Omega(A,0)}$ is a gauge function.

2.2 *New Representation of* $\overline{\text{conv}}\,\mathcal{D}(A, B)$

In view of (1.7), in order to obtain a complete understanding of the variational properties of σ_S, it is critical to have a useful description of the closed convex hull $\overline{\text{conv}}\,S$. This is often a non-trivial task. In [10, Proposition 4.3], a representation for $\overline{\text{conv}}\,\mathcal{D}(A, B)$ is obtained after great effort, and the representation is arduous. Although it is successfully used in [10, Section 5] in several important situations, the representation is an obstacle to a deeper understanding of the function $\sigma_{\mathcal{D}(A,B)}$ as well as its ease of use in applications. The focus of this section is to provide a new and intuitively appealing representation that dramatically facilitates the use of $\sigma_{\mathcal{D}(A,B)}$. The key to this new representation is the class of cones

$$\mathcal{K}_{\mathcal{S}} := \left\{ V \in \mathbb{S}^n \;\middle|\; u^T V u \geq 0,\ (u \in \mathcal{S}) \right\}, \tag{2.4}$$

where $\mathcal{S}$ is a subspace of $\mathbb{R}^n$, that is, $\mathcal{K}_{\mathcal{S}}$ is the set of all symmetric matrices that are positive semidefinite with respect to the given subspace $\mathcal{S}$. Observe that if $P \in \mathbb{S}^n$ is the orthogonal projection onto $\mathcal{S}$, then

$$\mathcal{K}_{\mathcal{S}} = \{ V \in \mathbb{S}^n \mid PVP \succeq 0 \}. \tag{2.5}$$

Clearly, $\mathcal{K}_{\mathcal{S}}$ is a convex cone, and, for $\mathcal{S} = \mathbb{R}^n$, it reduces to $\mathbb{S}^n_+$. Given a matrix $A \in \mathbb{R}^{p\times n}$, the cones $\mathcal{K}_{\ker A}$ play a special role in our analysis. For this reason, we simply write $\mathcal{K}_A$ to denote $\mathcal{K}_{\ker A}$, i.e. $\mathcal{K}_A := \mathcal{K}_{\ker A}$.

Proposition 2.2.1 ($\mathcal{K}_{\mathcal{S}}$ and its polar) *Let $\mathcal{S}$ be a nonempty subspace of $\mathbb{R}^n$ and let P be the orthogonal projection onto $\mathcal{S}$. Then the following hold:*

a) $\mathcal{K}_{\mathcal{S}}^\circ = \text{cone}\,\left\{ -vv^T \mid v \in \mathcal{S} \right\} = \{ W \in \mathbb{S}^n \mid W = PWP \preceq 0 \}$.

b) $\text{int}\,\mathcal{K}_{\mathcal{S}} = \left\{ V \in \mathbb{S}^n \;\middle|\; u^T V u > 0\ (u \in \mathcal{S} \setminus \{0\}) \right\}$.

c) $\text{aff}\,(\mathcal{K}_{\mathcal{S}}^\circ) = \text{span}\,\left\{ vv^T \mid v \in \mathcal{S} \right\} = \{ W \in \mathbb{S}^n \mid \text{rge}\,W \subset \mathcal{S} \}$.

d) $\text{ri}\,(\mathcal{K}_{\mathcal{S}}^\circ) = \left\{ W \in \mathcal{K}_{\mathcal{S}}^\circ \;\middle|\; u^T W u < 0\ \ (u \in \mathcal{S} \setminus \{0\}) \right\}$ *when* $\mathcal{S} \neq \{0\}$ *and*

$\text{ri}\,(\mathcal{K}_{\{0\}}^\circ) = \{0\}$ *(since* $\mathcal{K}_{\{0\}} = \mathbb{S}^n$*).*

Proof:

a) Put $B := \{-ss^T \mid s \in \mathcal{S}\} \subset \mathbb{S}^n_-$ and observe that

$$\operatorname{cone} B = \left\{ -\sum_{i=1}^{r} \lambda_i s_i s_i^T \mid r \in \mathbb{N}, s_i \in \mathcal{S}, \lambda_i \geq 0 \ (i = 1, \dots, r) \right\}.$$

We have $\operatorname{cone} B = \{W \in \mathbb{S}^n_- \mid W = PWP\}$: To see this, first note that $\operatorname{cone} B \subset \{W \in \mathbb{S}^n_- \mid W = PWP\}$. The reverse inclusion invokes the spectral decomposition of $W = \sum_{i=1}^n \lambda_i q_i q_i^T$ for $\lambda_1, \dots, \lambda_n \leq 0$. In particular, this representation of cone B shows that it is closed. We now prove the first equality in a): To this end, observe that

$$\begin{aligned} \mathcal{K}_{\mathcal{S}} &= \left\{ V \in \mathbb{S}^n \;\middle|\; s^T V s \geq 0 \ (s \in \mathcal{S}) \right\} \\ &= \left\{ V \in \mathbb{S}^n \;\middle|\; \left\langle V, -ss^T \right\rangle \leq 0 \ (s \in \mathcal{S}) \right\} \\ &= (\operatorname{cone} B)^\circ, \end{aligned}$$

where the third equality uses simply the linearity of the inner product in the second argument. Polarization then gives

$$\mathcal{K}_{\mathcal{S}}^\circ = (\operatorname{cone} B)^{\circ\circ} = \overline{\operatorname{cone}} B = \operatorname{cone} B.$$

b) The proof is straightforward and follows the pattern of proof for $\operatorname{int} \mathbb{S}^n_+ = \mathbb{S}^n_{++}$.

c) With B as defined above, observe that

$$\operatorname{aff} \mathcal{K}_{\mathcal{S}}^\circ = \operatorname{span} \mathcal{K}_{\mathcal{S}}^\circ = \operatorname{span} B,$$

since $0 \in \mathcal{K}_{\mathcal{S}}^\circ$, which shows the first equality. It is hence obvious that $\operatorname{aff} \mathcal{K}_{\mathcal{S}} \subset \{W \in \mathbb{S}^n \mid \operatorname{rge} W \subset \mathcal{S}\}$. On the other hand, every $W \in \mathbb{S}^n$ such that $\operatorname{rge} W \subset \mathcal{S}$ has a decomposition $W = \sum_{i=1}^{\operatorname{rank} W} \lambda_i q_i q_i^T$ where $\lambda_i \neq 0$ and $q_i \in \operatorname{rge} W \subset \mathcal{S}$ for all $i = 1, \dots, \operatorname{rank} W$, i.e. $W \in \operatorname{span} B = \operatorname{aff} \mathcal{K}_{\mathcal{S}}^\circ$.

d) Set $R := \left\{ W \in \mathcal{K}_{\mathcal{S}}^{\circ} \;\middle|\; u^T W u < 0 \;\; (u \in \mathcal{S} \setminus \{0\}) \right\}$ and let $W \in \mathrm{ri}\,(\mathcal{K}_{\mathcal{S}}^{\circ}) \setminus R \subset \mathcal{K}_{\mathcal{S}}^{\circ}$. Then there exists $u \in \mathcal{S}$ with $\|u\| = 1$ such that $u^T W u = 0$. Then for every $\varepsilon > 0$ we have $u^T(W + \varepsilon u u^T)u = \varepsilon > 0$. Therefore $W + \varepsilon u u^T \in (B_\varepsilon(W) \cap \mathrm{aff}\,(\mathcal{K}_{\mathcal{S}}^{\circ})) \setminus \mathcal{K}_{\mathcal{S}}^{\circ}$ for all $\varepsilon > 0$, and hence $W \notin \mathrm{ri}\,(\mathcal{K}_{\mathcal{S}}^{\circ})$, which contradicts our assumption. Hence, $\mathrm{ri}\,(\mathcal{K}_{\mathcal{S}}^{\circ}) \subset R$.

To see the reverse implication assume there were $W \in R \setminus \mathrm{ri}\,(\mathcal{K}_{\mathcal{S}}^{\circ})$, i.e. for all $k \in \mathbb{N}$ there exists $W_k \in B_{\frac{1}{k}}(W) \cap \mathrm{aff}\,(\mathcal{K}_{\mathcal{S}}^{\circ}) \setminus \mathcal{K}_{\mathcal{S}}^{\circ}$. In particular, there exists $\{u_k \in \mathcal{S} \mid \|u_k\| = 1\}$ such that $u_k^T W_k u_k \geq 0$ for all $k \in \mathbb{N}$. W.l.o.g. we can assume that $u_k \to u \in \mathcal{S} \setminus \{0\}$. Letting $k \to \infty$, we find that $u^T W u \geq 0$ since $W_k \to W$. This contradicts the fact that $W \in R$.

□

We are now in a position to prove the main result of this chapter which gives a new, simplified description of the closed convex hull of $\Omega(A, B)$.

Theorem 2.2.2 *Let $\mathcal{D}(A, B)$ be as given by* (3.2)*, then $\overline{\mathrm{conv}}\, \mathcal{D}(A, B) = \Omega(A, B)$, where*

$$\Omega(A, B) := \left\{ (Y, W) \in \mathbb{E} \;\middle|\; AY = B \text{ and } \frac{1}{2} Y Y^T + W \in \mathcal{K}_A^{\circ} \right\}. \tag{2.6}$$

Proof: We first show that $\Omega(A, B)$ is itself a closed convex set. Obviously, $\Omega(A, B)$ is closed since $\mathcal{K}_A^{\circ}$ is closed and the mappings $Y \mapsto AY$ and $(Y, W) \mapsto \frac{1}{2} Y Y^T + W$ are continuous.

So we need only show that $\Omega(A, B)$ is convex: To this end, let $(Y_i, W_i) \in \Omega(A, B),\ i = 1, 2$ and $0 \leq \lambda \leq 1$. Then there exist $M_i \in \mathcal{K}_A^{\circ},\ i = 1, 2$ such that $W_i = -\frac{1}{2} Y_i Y_i^T + M_i$. Observe that $A((1 - \lambda) Y_1 + \lambda Y_2) = B$. Moreover, we compute that

$$\begin{aligned}
&\frac{1}{2}((1-\lambda)Y_1 + \lambda Y_2)((1-\lambda)Y_1 + \lambda Y_2)^T + ((1-\lambda)W_1 + \lambda W_2) \\
=&\frac{1}{2}((1-\lambda)Y_1 + \lambda Y_2)((1-\lambda)Y_1 + \lambda Y_2)^T + \left((1-\lambda)(-\frac{1}{2}Y_1Y_1^T + M_1) + \lambda(-\frac{1}{2}Y_2Y_2^T + M_2)\right) \\
=&\frac{1}{2}\lambda(1-\lambda)(-Y_1Y_1^T + Y_1Y_2^T + Y_2Y_1^T - Y_2Y_2^T) + (1-\lambda)M_1 + \lambda M_2 \\
=&\lambda(1-\lambda)\left(-\frac{1}{2}(Y_1 - Y_2)(Y_1 - Y_2)^T\right) + (1-\lambda)M_1 + \lambda M_2.
\end{aligned}$$

Since $\operatorname{rge}(Y_1-Y_2) \subset \ker A$, this shows $\lambda(1-\lambda)\left(-\frac{1}{2}(Y_1-Y_2)(Y_1-Y_2)^T\right)+(1-\lambda)M_1+\lambda M_2 \in \mathcal{K}_A^\circ$. Consequently, $\Omega(A,B)$ is a closed convex set.

Next note that if $(Y,-\frac{1}{2}YY^T) \in \mathcal{D}(A,B)$, then $(Y,-\frac{1}{2}YY^T) \in \Omega(A,B)$ since $0 \in \mathcal{K}_A^\circ$. Hence, $\overline{\operatorname{conv}}\,\mathcal{D}(A,B) \subset \Omega(A,B)$.

It therefore remains to establish the reverse inclusion: For these purposes, let $(Y,W) \in \Omega(A,B)$. By Carathéodory's theorem, there exist $\mu_i \geq 0, v_i \in \ker A$ $(i=1,\ldots,N)$ such that

$$W = -\frac{1}{2}YY^T - \sum_{i=1}^{N} \mu_i v_i v_i^T,$$

where $N = \frac{n(n+1)}{2}+1$. Let $0<\epsilon<1$. Set $\lambda_1 := 1-\epsilon$ and $\lambda_2 = \ldots = \lambda_{N+1} = \lambda := \epsilon/N$. Denote $Y_1 := Y/\sqrt{1-\epsilon}$. Take $Z_i \in \mathbb{R}^{n\times m},\ i=1,\ldots,N$ such that $AZ_i = B$. Finally, set

$$V_i = \left[\sqrt{\frac{2\mu_i}{\lambda}}v_i, 0, \ldots, 0\right] \in \mathbb{R}^{n\times m} \quad \text{and} \quad Y_{i+1} = Z_i + V_i,\ (i=1,\ldots,N).$$

Observe that

$$\sum_{i=1}^{N+1}\lambda_i Y_i = \sqrt{1-\epsilon}Y + \frac{\epsilon}{N}\sum_{i=2}^{N+1} Y_i = \sqrt{1-\epsilon}Y + \frac{\epsilon}{N}\sum_{i=1}^{N} Z_i + \sqrt{\frac{\epsilon}{N}}\sum_{i=1}^{N}\bar{V}_i,$$

where $\bar{V}_i = [\sqrt{2\mu_i}v_i, 0, \ldots, 0],\ i=1,\ldots,N$, and

$$\begin{aligned} -\frac{1}{2}\sum_{i=1}^{N+1}\lambda_i Y_i Y_i^T &= -\frac{1}{2}YY^T - \frac{1}{2}\sum_{i=1}^{N}\frac{\epsilon}{N}\left(Z_iZ_i^T + Z_iV_i^T + V_iZ_i^T\right) - \sum_{i=1}^{N}\mu_i v_i v_i^T \\ &= W - \sum_{i=1}^{N}\frac{1}{2}\left(\frac{\epsilon}{N}Z_iZ_i^T + \sqrt{\frac{\epsilon}{N}}Z_i\bar{V}_i^T + \sqrt{\frac{\epsilon}{N}}\bar{V}_iZ_i^T\right), \end{aligned}$$

Therefore

$$\begin{aligned} &\left(\sqrt{1-\epsilon}Y + \frac{\epsilon}{N}\sum_{i=1}^{N} Z_i + \sqrt{\frac{\epsilon}{N}}\sum_{i=1}^{N}\bar{V}_i,\quad W - \sum_{i=1}^{N}\frac{1}{2}\left(\frac{\epsilon}{N}Z_iZ_i^T + \sqrt{\frac{\epsilon}{N}}Z_i\bar{V}_i^T + \sqrt{\frac{\epsilon}{N}}\bar{V}_iZ_i^T\right)\right) \\ &\qquad = \left(\sum_{i=1}^{N+1}\lambda_i Y_i,\quad -\frac{1}{2}\sum_{i=1}^{N+1}\lambda_i Y_i Y_i^T\right). \end{aligned} \tag{2.7}$$

Set $\kappa := \dim \mathbb{E}$. By Carathéodory's theorem,

$$\operatorname{conv}\mathcal{D}(A,B) = \left\{\left(\sum_{i=1}^{\kappa+1}\lambda_i Y_i, -\frac{1}{2}\sum_{i=1}^{\kappa+1}\lambda_i Y_i Y_i^T\right) \,\middle|\, \begin{array}{c} \lambda \in \mathbb{R}_+^{\kappa+1}, \sum_{i=1}^{\kappa+1}\lambda_i = 1,\ Y_i \in \mathbb{R}^{n\times m} \\ AY_i = B\ (i=1,\ldots,\kappa+1) \end{array}\right\}.$$

By letting $\epsilon \downarrow 0$ in (2.7), we find $(Y,W) \in \overline{\operatorname{conv}}\,\mathcal{D}(A,B)$ thereby concluding the proof. □

2.3 Normal cone of $\Omega(A, B)$ and the subdifferential of $\sigma_{\mathcal{D}(A,B)}$

The new representation for $\overline{\operatorname{conv}}\,\mathcal{D}(A, B)$ allows us to dramatically simplify the representation for the subdifferential of $\sigma_{\mathcal{D}(A,B)}$ given in [10, Theorem 4.8]. For this we use the well-established relation

$$\partial\sigma_C(x) = \{z \in \overline{\operatorname{conv}}\, C \mid x \in N_{\overline{\operatorname{conv}}\, C}(z)\}, \tag{2.8}$$

where $C \subset \mathbb{E}$ is nonempty and convex.

Proposition 2.3.1 (The normal cone to $\Omega(A, B)$) *Let $\Omega(A, B)$ be as given by (3.2) and let $(Y, W) \in \Omega(A, B)$. Then*

$$N_{\Omega(A,B)}(Y, W) = \left\{ (X, V) \in \mathbb{E} \;\middle|\; \begin{array}{c} V \in \mathcal{K}_A,\ \left\langle V,\, \frac{1}{2}YY^T + W \right\rangle = 0 \\ \text{and}\ \ \operatorname{rge}(X - VY) \subset (\ker A)^{\perp} \end{array} \right\}$$

Proof: Observe that $\Omega(A, B) = C_1 \cap C_2 \subset \mathbb{E}$ where

$$C_1 := \left\{Y \in \mathbb{R}^{n\times m} \mid AY = B\right\} \times \mathbb{S}^n \quad \text{and} \quad C_2 := \{(Y, W) \mid F(Y, W) \in \mathcal{K}_A^{\circ}\},$$

with $F(Y, W) := \frac{1}{2}YY^T + W$. Clearly, C_1 is affine, hence convex, and C_2 is also convex, which can be seen by an analogous reasoning as for the convexity of $\Omega(A, B)$ (cf. the proof of Theorem 2.2.2). Therefore, [36, Corollary 23.8.1] tells us that

$$N_{\Omega(A,B)}(Y, W) = N_{C_1}(Y, W) + N_{C_2}(Y, W), \tag{2.9}$$

where

$$N_{C_1}(Y, W) = \left\{R \in \mathbb{R}^{n\times m} \;\middle|\; \operatorname{rge} R \subset (\ker A)^{\perp}\right\} \times \{0\}.$$

We now compute $N_{C_2}((Y, W))$. First recall that for any nonempty closed convex cone $C \subset \mathcal{E}$, we have $N_C(x) = \{z \in C^{\circ} \mid \langle z,\, x\rangle = 0\}$ for all $x \in C$. Next, note that

$$\nabla F(Y, W)^* U = (UY,\ U) \quad (U \in \mathbb{S}^n),$$

so that $\nabla F(Y,W)^* U = 0$ if and only if $U = 0$. Hence, by [37, Exercise 10.26 Part (d)],

$$N_{C_2}(Y,W) = \left\{ (VY, V) \;\middle|\; V \in \mathcal{K}_A,\ \left\langle V,\ \frac{1}{2}YY^T + W \right\rangle = 0 \right\}.$$

Therefore, by (2.9), $N_{\Omega(A,B)}(Y,W)$ is given by

$$\left\{ (X,V) \;\middle|\; \operatorname{rge}(X - VY) \subset (\ker A)^\perp,\ V \in \mathcal{K}_A,\ \left\langle V,\ \frac{1}{2}YY^T + W \right\rangle = 0 \right\},$$

which proves the result. □

By combining (2.8) and Proposition 2.3.1 we obtain a simplified representation of the subdifferential of the support function $\sigma_{\mathcal{D}}(A,B)$.

Corollary 2.3.2 (The subdifferential of $\sigma_{\mathcal{D}(A,B)}$) *Let $\mathcal{D}(A,B)$ be as given in* (3.2). *Then, for all $(X,V) \in \operatorname{dom} \sigma_{\mathcal{D}(A,B)}$ (see* (3.8)*) we have*

$$\partial \sigma_{\mathcal{D}(A,B)}(X,V) = \left\{ (Y,W) \in \Omega(A,B) \;\middle|\; \begin{array}{l} \exists Z \in \mathbb{R}^{p\times m}:\ X = VY + A^T Z, \\ \left\langle V,\ \frac{1}{2}YY^T + W \right\rangle = 0 \end{array} \right\}.$$

Proof: This follows directly from the normal cone description in Proposition 2.3.1 and the relation (2.8). □

2.4 The geometry of $\Omega(A,B)$

We first compute the relative interior and the affine hull of $\Omega(A,B)$. For these purposes, we recall an established result on the relative interior of a convex set in a product space.

Proposition 2.4.1 ([36, Theorem 6.8]) *Let $C \subset \mathbb{E}_1 \times \mathbb{E}_2$. For each $y \in \mathbb{E}_1$ we define $C_y := \{z \in \mathbb{E}_2 \mid (y,z) \in C\}$ and $D := \{y \mid C_y \neq \emptyset\}$. Then*

$$\operatorname{ri} C = \{(y,z) \mid y \in \operatorname{ri} D,\ z \in \operatorname{ri} C_y\}.$$

We use this result to get a representation for the relative interior of $\Omega(A,B)$ directly, and then mimic its technique of proof to tackle the affine hull.

Lemma 2.4.2 *Let $A, B \subset \mathbb{E}$ be convex with* $\mathrm{ri}\, A \cap \mathrm{ri}\, B \neq \emptyset$. *Then* $\mathrm{aff}\,(A \cap B) = \mathrm{aff}\, A \cap \mathrm{aff}\, B$.

Proof: The inclusion $\mathrm{aff}\,(A \cap B) \subset \mathrm{aff}\, A \cap \mathrm{aff}\, B$ is clear since the latter set is affine and contains $A \cap B$.

For proving the reverse inclusion, we can assume w.l.o.g. that $0 \in \mathrm{ri}\, A \cap \mathrm{ri}\, B = \mathrm{ri}\,(A \cap B)$, where for the latter equality we refer to [36, Theorem 6.5]. In particular we have

$$\mathrm{aff}\, A = \mathbb{R}_+ A, \;\; \mathrm{aff}\, B = \mathbb{R}_+ B \;\; \text{and} \;\; \mathrm{aff}\,(A \cap B) = \mathbb{R}_+(A \cap B), \tag{2.10}$$

see (1.4) and the discussion afterwards. Now, let $x \in \mathrm{aff}\, A \cap \mathrm{aff}\, B$. If $x = 0$ there is nothing to prove. If $x \neq 0$, by (2.10), we have $x = \lambda a = \mu b$ for some $\lambda, \mu > 0$ and $a \in A, b \in B$. W.l.o.g we have $\lambda > \mu$, and hence, by convexity of B, we have

$$a = \left(1 - \frac{\mu}{\lambda}\right) 0 + \frac{\mu}{\lambda} b \in B.$$

Therefore $x = \lambda a \in \mathbb{R}_+(A \cap B) = \mathrm{aff}\,(A \cap B)$, see (2.10). □

We now prove a result analogous to Proposition 2.4.1.

Proposition 2.4.3 *In addition to the assumptions of Proposition 2.4.1 assume that D is affine. Then $(y, z) \in \mathrm{aff}\, C$ if and only if $y \in D$ and $z \in \mathrm{aff}\, C_y$.*

Proof: We imitate the proof of [36, Theorem 6.8]: Let $L : (y, z) \mapsto z$. Since D is assumed to be affine (hence $D = \mathrm{aff}\, D = \mathrm{ri}\, D$), we have

$$D = L(C) = L(\mathrm{ri}\, C) = L(\mathrm{aff}\, C), \tag{2.11}$$

where we invoke the fact that linear mappings commute with the relative interior and the affine hull, see [36, Theorem 6.7 and p. 8].

Now fix $y \in D = \mathrm{ri}\, D$ and define the affine set $M_y := \{(y, z) \mid z \in \mathbb{E}_2\} = \{y\} \times \mathbb{E}_2$. Then, by (2.11), there exists $z \in \mathbb{E}_2$ such that $y = L(y, z)$ and $(y, z) \in \mathrm{ri}\, C$. Hence, $\mathrm{ri}\, M_y \cap \mathrm{ri}\, C \neq \emptyset$ and we can invoke Lemma 2.4.2 to obtain

$$\mathrm{aff}\, M_y \cap \mathrm{aff}\, C = \mathrm{aff}\,(M_y \cap C) = \mathrm{aff}\,(\{y\} \times C_y) = \{y\} \times \mathrm{aff}\, C_y.$$

Hence, if $y \in D, z \in \operatorname{aff} C_y$, we have $(y,z) \in \{y\} \times \operatorname{aff} C_y = M_y \cap \operatorname{aff} C \subset \operatorname{aff} C$.

In turn, for $(y,z) \in C$, we have $(y,z) \in M_y \cap \operatorname{aff} C = \{y\} \times C_y$, hence $z \in C_y \neq \emptyset$, so $y \in D$. □

We are now in a position to prove the desired result on the relative interior and the affine hull of $\Omega(A,B)$.

Proposition 2.4.4 *For $\Omega(A,B)$ given by (3.2) the following hold:*

a) $\operatorname{ri}\Omega(A,B) = \left\{(Y,W) \in \mathbb{E} \;\middle|\; AY = B \text{ and } \frac{1}{2}YY^T + W \in \operatorname{ri}(\mathcal{K}_A^\circ)\right\}.$

b) $\operatorname{aff}\Omega(A,B) = \left\{(Y,W) \in \mathbb{E} \;\middle|\; AY = B \text{ and } \frac{1}{2}YY^T + W \in \operatorname{span}\mathcal{K}_A^\circ\right\},$

where $\operatorname{span}\mathcal{K}_A^\circ = \operatorname{span}\left\{vv^T \mid v \in \ker A\right\}$.

Proof: We apply the format of Proposition 2.4.1 and 2.4.3, respectively, for $C := \Omega(A,B)$. Then

$$D = \{Y \mid AY = B\} \quad \text{and} \quad C_y = \begin{cases} \mathcal{K}_A^\circ - \frac{1}{2}YY^T, & \text{if} \quad AY = B, \\ \emptyset, & \text{else.} \end{cases} \qquad (Y \in \mathbb{R}^{n\times m}),$$

a) Apply Proposition 2.4.1 and observe that $\operatorname{ri}(\mathcal{K}_A^\circ - \frac{1}{2}YY^T) = \operatorname{ri}(\mathcal{K}_A^\circ) - \frac{1}{2}YY^T$.

b) Apply Proposition 2.4.3 and observe that D is affine, and that $\operatorname{aff}(\mathcal{K}_A^\circ - \frac{1}{2}YY^T) = \operatorname{aff}(\mathcal{K}_A^\circ) - \frac{1}{2}YY^T$.

□

As a direct consequence of Propositions 2.2.1 and 2.4.4, we obtain the following result for the special case $(A,B) = (0,0)$.

Corollary 2.4.5 *It holds that*

$$\overline{\operatorname{conv}}\left\{(Y, -\frac{1}{2}YY^T) \mid Y \in \mathbb{R}^{n\times m}\right\} = \left\{(Y,W) \in \mathbb{E} \;\middle|\; W + \frac{1}{2}YY^T \preceq 0\right\},$$

and

$$\operatorname{int}\left(\overline{\operatorname{conv}}\left\{(Y, -\frac{1}{2}YY^T) \mid Y \in \mathbb{R}^{n\times m}\right\}\right) = \left\{(Y,W) \in \mathbb{E} \;\middle|\; W + \frac{1}{2}YY^T \prec 0\right\}.$$

We conclude this section by giving representations for the horizon cone and polar of $\Omega(A, B)$.

Proposition 2.4.6 (The polar of $\Omega(A,B)$) *Let $\Omega(A, B)$ be as given in (3.2). Then*

$$\Omega(A,B)^\circ = \left\{ (X,V) \;\middle|\; \begin{array}{c} \operatorname{rge}\binom{X}{B} \subset \operatorname{rge} M(V),\ V \in \mathcal{K}_A, \\ \frac{1}{2}\operatorname{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B}\right) \le 1 \end{array} \right\}.$$

Moreover,

$$\Omega(A,B)^\infty = \{0_{n\times m}\} \times \mathcal{K}_A^\circ \tag{2.12}$$

and

$$(\Omega(A,B)^\circ)^\infty = \left\{ (X,V) \;\middle|\; \begin{array}{c} \operatorname{rge}\binom{X}{B} \subset \operatorname{rge} M(V),\ V \in \mathcal{K}_A, \\ \frac{1}{2}\operatorname{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B}\right) \le 0 \end{array} \right\}. \tag{2.13}$$

Proof: Given any nonempty closed convex set $C \subset \mathbb{E}$, it is easily seen that $C^\circ = \{z \mid \sigma_C(z) \le 1\}$. Consequently, our expression for $\Omega(A, B)^\circ$ follows from (2.2).

To see (2.12), let $(Y, W) \in \Omega(A, B)$ and recall that $(S, T) \in \Omega(A, B)^\infty$ if and only if $(Y + tS, W + tT) \in \Omega(A, B)$ for all $t \ge 0$. In particular, for $(S, T) \in \Omega(A, B)^\infty$, we have $A(Y + tS) = B$ and

$$\frac{1}{2}\left[YY^T + t(SY^T + YS^T) + \frac{t^2}{2}SS^T\right] + (W + tT) \in \mathcal{K}_A^\circ \quad (t > 0). \tag{2.14}$$

Consequently, $AS = 0$ and, if we divide (2.14) by t^2 and let $t \uparrow \infty$, we see that $SS^T \in \mathcal{K}_A^\circ$. But $SS^T \in \mathcal{K}_A$ since $\operatorname{rge} S \subset \ker A$, so we must have $S = 0$. If we now divide (2.14) by t and let $t \uparrow \infty$, we find that $T \in \mathcal{K}_A^\circ$. Hence the set on the left-hand side of (2.12) is contained in the one on the right. To see the reverse inclusion, simply recall that $\mathcal{K}_A^\circ$ is a closed convex cone so that $\mathcal{K}_A^\circ + \mathcal{K}_A^\circ \subset \mathcal{K}_A^\circ$.

Finally, we show (2.13). Since $(0, 0) \in \Omega(A, B)^\circ$, we have $(S, T) \in (\Omega(A, B)^\circ)^\infty$ if and only if $(tS, tT) \in \Omega(A, B)^\circ$ for all $t > 0$, or equivalently, for all $t > 0$,

$$tT \in \mathcal{K}_A \ \text{ and } \ \exists\, (Y_t, Z_t) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times m} \ \text{ s.t. } \begin{pmatrix} tS \\ B \end{pmatrix} = M(tT)\begin{pmatrix} Y_t \\ Z_t \end{pmatrix}$$
$$\text{with } \ \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} Y_t \\ Z_t \end{pmatrix}^T M(tT)\begin{pmatrix} Y_t \\ Z_t \end{pmatrix}\right) \le 1,$$

or equivalently, by taking $\widehat{Z}_t := t^{-1}Z_t$,

$$T \in \mathcal{K}_A \ \text{ and } \ \exists\ (Y_t, \widehat{Z}_t) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times m} \ \text{ s.t. } \begin{pmatrix} S \\ B \end{pmatrix} = M(T)\begin{pmatrix} Y_t \\ \widehat{Z}_t \end{pmatrix}$$
$$\text{with } \ \frac{t}{2}\mathrm{tr}\left(\begin{pmatrix} Y_t \\ \widehat{Z}_t \end{pmatrix}^T M(T)\begin{pmatrix} Y_t \\ \widehat{Z}_t \end{pmatrix}\right) \leq 1.$$

If we take $\binom{Y_t}{\widehat{Z}_t} := M(T)^\dagger\binom{S}{B}$, we find that $(S,T) \in (\Omega(A,B)^\circ)^\infty$ if and only if

$$T \in \mathcal{K}_A \ \text{ and } \ \frac{t}{2}\mathrm{tr}\left(\begin{pmatrix} S \\ B \end{pmatrix}^T M(T)^\dagger\begin{pmatrix} S \\ B \end{pmatrix}\right) \leq 1 \quad (t > 0),$$

which proves the result. □

2.5 $\sigma_{\Omega(A,0)}$ as a gauge

Note that the origin is an element of $\Omega(A,B)$ if and only if $B = 0$. In this case the support function of $\Omega(A,0)$ equals the gauge of $\Omega(A,0)^\circ$. Gauges are important in a number of applications and they posses their own duality theory [21, 22, 23]. An explicit representation for both $\gamma_{\Omega(A,0)^\circ}$ and $\gamma_{\Omega(A,0)}$ will be given in the following theorem.

Theorem 2.5.1 ($\sigma_{\mathcal{D}(A,0)}$ is a gauge) *Let $\Omega(A,B)$ be as given in (3.2). Then*

$$\sigma_{\Omega(A,0)}(X,V) = \gamma_{\Omega(A,0)^\circ}(X,V) = \gamma^\circ_{\Omega(A,0)}(X,V), \tag{2.15}$$

and

$$\gamma_{\Omega(A,0)}(Y,W) = \sigma_{\Omega(A,0)^\circ}(Y,W)$$
$$= \begin{cases} \frac{1}{2}\sigma_{\min}^{-1}(-Y^\dagger W(Y^\dagger)^T) & \text{if } \operatorname{rge} Y \subset \ker A \cap \operatorname{rge} W, W \in \mathcal{K}_A^\circ, \\ +\infty & \text{else,} \end{cases} \tag{2.16}$$

where $\sigma_{\min}(-Y^\dagger W(Y^\dagger)^T)$ is the smallest nonzero singular-value of $-Y^\dagger W(Y^\dagger)^T$ when such an eigenvalue exists and $+\infty$ otherwise, e.g. when $Y = 0$. Here we interpret $\frac{1}{\infty}$ as 0 $(0 = \frac{1}{\infty})$, and so, in particular, $\gamma_{\Omega(A,0)}(0,W) = \delta_{\mathcal{K}_A^\circ}(W)$.

Proof: The expression (2.15) follows from [36, Theorem 14.5]. To show (2.16), first observe that

$$t\,\Omega(A,0)=\left\{(Y,W)\;\middle|\; AY=0 \text{ and } \frac{1}{2}YY^T+tW\in\mathcal{K}_A^\circ\right\}, \tag{2.17}$$

whose straightforward proof is left to the reader.

Given $\bar{t}\geq 0$, by (2.17), $(Y,W)\in t\Omega(A,0)$ for all $t>\bar{t}$ if and only if $AY=0$ and $\frac{1}{2}YY^T+tW\in\mathcal{K}_A^\circ$ for all $t>\bar{t}$. By Proposition 2.2.1 a), this is equivalent to $AY=0$ and

$$\frac{1}{2}YY^T+tW=P\left(\frac{1}{2}YY^T+tW\right)P\preceq 0 \quad (t>\bar{t}), \tag{2.18}$$

where, again, P is the orthogonal projection onto $\ker A$. Dividing this inequality by t and taking the limit as $t\uparrow\infty$ tells us that $W=PWP\preceq 0$. Since YY^T is positive semidefinite, inequality (2.18) also tells us that $\ker W\subset\ker Y^T$, i.e. $\operatorname{rge}Y\subset\operatorname{rge}W$. Consequently,

$$\operatorname{dom}\gamma_{\Omega(A,0)}\subset\{(Y,W)\mid \operatorname{rge}Y\subset\ker A\cap\operatorname{rge}W, W\in\mathcal{K}_A^\circ\}.$$

Now suppose $(Y,W)\in\operatorname{dom}\gamma_{\Omega(A,0)}$. Let $Y=U\Sigma V^T$ be the reduced singular-value decomposition of Y where Σ is an invertible diagonal matrix and U,V have orthonormal columns. Since $\operatorname{rge}Y\subset\operatorname{rge}W=(\ker W)^\perp$, we know that U^TWU is negative definite, and so $\Sigma^{-1}U^TWU\Sigma^{-1}$ is also negative definite. Multiplying (2.18) on the left by $\Sigma^{-1}U^T$ and on the right by $U\Sigma^{-1}$ gives

$$\mu I\preceq -2\Sigma^{-1}U^TWU\Sigma^{-1}\quad (0<\mu\leq\bar{\mu}),$$

where $\bar{\mu}=\bar{t}^{-1}$. The largest $\bar{\mu}$ satisfying this inequality is

$$\sigma_{\min}(-2Y^\dagger W(Y^\dagger)^T)=\sigma_{\min}(-2\Sigma^{-1}U^TWU\Sigma^{-1})>0,$$

or equivalently, the smallest possible $\bar{t}$ in (2.18) is $1/\sigma_{\min}(-2Y^\dagger W(Y^\dagger)^T)$, which proves the result. □

2.6 Conclusions

The representation $\Omega(A, B)$ for the closed convex hull of the set $\mathcal{D}(A, B)$ in Theorem 2.2.2 is a dramatic simplification of the one given in [10]. As a consequence, we also obtain simplified expressions for both the normal cone to $\Omega(A, B)$ and the subdifferential for generalized matrix-fractional functions in Section 2.3. In addition, representations for several important geometric objects related to the set $\Omega(A, B)$ are computed in Section 2.4. These results provide the key to the applications discussed in [10], and open the door to the numerous further applications discussed in the next chapter.

Chapter 3

INFIMAL PROJECTIONS

3.1 Introduction

In this chapter we greatly expand the number of applications of generalized matrix-fractional functions to include all *Ky Fan norms*, matrix *gauge functionals*, and *variational Gram functions* [30]. Our analysis includes descriptions of the variational properties of these functions such as formulas for their convex conjugates and their subdifferentials.

In what follows, $\mathbb{E} := \mathbb{R}^{n\times m} \times \mathbb{S}^n$ where $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ are the linear spaces of real $n \times m$ matrices and (real) symmetric $n\times n$ matrices, respectively. Given $(A, B) \in \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ with $\operatorname{rge} B \subset \operatorname{rge} A$, recall that the GMF function φ is defined as the support function of the graph of the matrix valued mapping $Y \mapsto -\frac{1}{2}YY^T$ over the manifold $\{Y \in \mathbb{R}^{n\times m} \mid AY = B\}$, i.e., $\varphi : \mathbb{E} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is

$$\varphi(X, V) := \sup \{\langle (Y, W),\, (X, V)\rangle \mid (Y, W) \in \mathcal{D}(A, B)\}\,, \tag{3.1}$$

where

$$\mathcal{D}(A, B) := \left\{ \left(Y, -\frac{1}{2}YY^T\right) \in \mathbb{E} \;\Big|\; Y \in \mathbb{R}^{n\times m} :\; AY = B \right\}. \tag{3.2}$$

A closed form expression for φ is derived in [10] and displayed in (2.2). In [10] it is also shown that φ is smooth on the (nonempty) interior of its domain.

Our study focuses on functions $p : \mathbb{R}^{n\times m} \to \overline{\mathbb{R}}$ representable as the partial infimal projection of the form

$$p(X) := \inf_{V\in\mathbb{S}^n} \varphi(X, V) + h(V), \tag{3.3}$$

where $h : \mathbb{S}^n \to \overline{\mathbb{R}}$ is convex. Different functions h illuminate different variational properties of the matrix X. For example, when $h := \langle U, \cdot\rangle$ for $U \in \mathbb{S}^n_{++}$, and when both A and

B are zero, then p is a weighted nuclear norm where the weights depend on any "square-root" of U (see Corollary 3.4.7). Among the consequences of the representation (3.3) are conditions under which p is closed and proper as well as formulas for the ready computation of both p^* and ∂p (Section 3.3). As an application of our general results, we give more detailed explorations in the cases where h is a support function (Section 3.4) or an indicator function (Section 3.5). We illustrate these results with specific instances. For example, we obtain all weighted squared gauges on $\mathbb{R}^{n\times m}$, cf. Corollary 3.5.9, as well as a complete characterization of variational Gram functions [30] and their conjugates. In addition, we show that all variational Gram funtions are representable as squares of gauges, cf. Proposition 3.5.10. Other choices yield weighted sums of Frobenius and nuclear norms, see [10, Corollary 5.9]. The scope of applications is large and the range of variational properties is fascinating and fundamental.

Beyond the variational results of this chapter, there is a compelling but unexplored computational aspect of this representation. Hsieh and Olsen [29] show that (3.3) with $h = \frac{1}{2}\mathrm{tr}\,(\cdot)$ yields a smoothing approach to optimization problems involving the nuclear norm. More generally, observe that many matrix optimization problems often take the form

$$\min_{X\in\mathbb{R}^{n\times m}} f(X) + p(X), \tag{P}$$

where $f, p : \mathbb{R}^{n\times m} \to \overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$. The function f is thought of as the primary objective and is often smooth or convex while p is typically a structure inducing convex function. Using the representation (3.3), the problem (P) can be written as

$$\min_{(X,V)\in\mathbb{E}} f(X) + \varphi(X, V) + h(V).$$

This reformulation allows one to exploit the smoothness of φ on the interior of its domain. For example, if both f and h are smooth, one can employ a damped Newton, or path following approach to solving (P). We emphasize, that this is not the goal or intent of this chapter, however, our results provide the basis for future investigations along a variety of such numerical and theoretical avenues.

This chapter is organized as follows: In Section 3.2 we provide some basic properties of the GMF function. Section 3.3 contains the general theory for partial infimal projections of the form (3.3). In Section 3.4 we specify h in (3.3) to be a support function of some closed, convex set $\mathcal{V} \subset \mathbb{S}^n$. In Section 3.5 we choose h to be the indicator of such set. In particular, this yields powerful results on variational Gram functions and Ky Fan norms, see Section 3.5.2-3.5.3. We close out with some final remarks in Section 3.6.

Notation: For a linear transformation L, we write $\mathrm{rge}\, L$ and $\ker L$ for its *range* and *kernel*, respectively. For $A \in \mathbb{R}^{p\times n}$, we abuse notation somewhat and write $\mathrm{rge}\, A$ and $\ker A$ for its *range* and *kernel*, respectively, when A is considered as a linear transformation between $\mathbb{R}^n$ and $\mathbb{R}^p$. Again, for $A \in \mathbb{R}^{p\times n}$, we set

$$\begin{aligned}\mathrm{Ker}_r A &:= \left\{X \in \mathbb{R}^{n\times r} \mid AX = 0\right\} = \left\{X \in \mathbb{R}^{n\times r} \mid \mathrm{rge}\, X \subset \ker A\right\},\\ \mathrm{Rge}_r A &:= \left\{Y \in \mathbb{R}^{p\times r} \;\middle|\; \exists\, X \in \mathbb{R}^{n\times r} \,:\, Y = AX\right\} = \left\{Y \in \mathbb{R}^{p\times r} \mid \mathrm{rge}\, Y \subset \mathrm{rge}\, A\right\}\end{aligned}$$

and write $\mathrm{Ker}A$ or $\mathrm{Rge}A$ when the choice of r is clear from the context. Observe that $\mathrm{Ker}_1 A = \ker A$, $\mathrm{Rge}_1 A = \mathrm{rge}\, A$, and $(\mathrm{Ker}_r A)^\perp = \mathrm{Rge}_r A^T$, where we equip any matrix space with the (Frobenius) inner product $\langle X, Y\rangle := \mathrm{tr}\,(X^T Y)$. The *Moore-Penrose pseudoinverse* of A, see e.g. [27], is denoted by $A^\dagger$. The set of all symmetric matrices of dimension n is given by $\mathbb{S}^n$. The positive and negative semidefinite cone are denoted by $\mathbb{S}^n_+$ and $\mathbb{S}^n_-$, respectively.

For two sets S, T in the same real linear space their *Minkowski sum* is $S+T := \{s + t \mid s \in S,\ t \in T\}$. For $K \subset \mathbb{R}$ we also put $K \cdot S := \{\lambda s \mid \lambda \in K,\ s \in S\}$.

3.2 Preliminaries

As noted in the introduction, the GMF function is the support function of $\mathcal{D}(A, B)$ given in (3.2). Hence, we write

$$\sigma_{\mathcal{D}(A,B)}(X, V) = \varphi(X, V) \tag{3.4}$$

and also refer to $\sigma_{\mathcal{D}(A,B)}$ as the GMF function. From [10, 13] and Chapter 2, we obtain the formula given in (2.2):

$$\varphi(X,V)=\begin{cases}\frac{1}{2}\mathrm{tr}\left(\binom{X}{B}^T M(V)^\dagger\binom{X}{B}\right) & \text{if} \quad \mathrm{rge}\binom{X}{B}\subset \mathrm{rge}\, M(V),\ V\in\mathcal{K}_A,\\ +\infty & \text{else,}\end{cases} \tag{3.5}$$

where $(A,B)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}$ with $\mathrm{rge}\, B\subset \mathrm{rge}\, A$ and $\mathcal{K}_A$ is the cone of all symmetric matrices that are positive semidefinite with respect to the subspace $\ker A$, i.e.

$$\mathcal{K}_A := \left\{V\in\mathbb{S}^n \;\middle|\; u^TVu\geq 0\ (u\in\ker A)\right\}, \tag{3.6}$$

and $M(V)^\dagger$ is the Moore-Penrose pseudoinverse of the *bordered matrix*

$$M(V)=\begin{pmatrix} V & A^T\\ A & 0\end{pmatrix}. \tag{3.7}$$

The *matrix-fractional function* [9, 17] is obtained by setting the matrices A and B to zero.

A detailed analysis of the GMF function appears in the papers [10, 13] whose contents were discussed in Chapter 2. In particular, it is shown that

$$\begin{aligned}\mathrm{dom}\,\sigma_{\mathcal{D}(A,B)} &= \mathrm{dom}\,\partial\sigma_{\mathcal{D}(A,B)}\\ &= \left\{(X,V)\in\mathbb{R}^{n\times m}\times\mathbb{S}^n \;\middle|\; \mathrm{rge}\begin{pmatrix}X\\ B\end{pmatrix}\subset\mathrm{rge}\, M(V),\ V\in\mathcal{K}_A\right\}.\end{aligned} \tag{3.8}$$

For the study of the convex-analytical properties of the support function $\sigma_{\mathcal{D}(A,B)}$ the computation of the closed convex hull of the (nonconvex) set $\mathcal{D}(A,B)$ has been critical. A representation of $\overline{\mathrm{conv}}\,\mathcal{D}(A,B)$ relying mainly on Carathéodory's theorem was obtained in [10, Proposition 4.3]. A refined and more versatile expression was presented in Chapter 2 [13]. The key object for this expression is the (closed, convex) cone $\mathcal{K}_A$ defined in (3.6), which reduces to $\mathbb{S}^n_+$ for $A=0$.

The basic geometric and topological properties of $\mathcal{K}_A$ are given in Proposition 2.2.1, and which follow from [13, Proposition 1] (by setting $\mathcal{S}=\ker A$). We recall this proposition below.

Proposition 3.2.1 *For $A \in \mathbb{R}^{p\times n}$ let $P \in \mathbb{R}^{n\times n}$ be the orthogonal projection onto* $\ker A$ *and let $\mathcal{K}_A$ be given by* (3.6)*. Then the following hold:*

a) $\mathcal{K}_A = \{V \in \mathbb{S}^n \mid PVP \succeq 0\}$.

b) $\mathcal{K}_A^\circ = \mathrm{cone}\,\left\{-vv^T \mid v \in \ker A\right\} = \{W \in \mathbb{S}^n \mid W = PWP \preceq 0\}$.

c) $\mathrm{int}\,\mathcal{K}_A = \left\{V \in \mathbb{S}^n \;\middle|\; u^TVu > 0 \;(u \in A \setminus \{0\})\right\}$.

The central result from Chapter 2 (Theorem 2.2.2) is the following characterization of $\overline{\mathrm{conv}}\,\mathcal{D}(A,B)$:

$$\overline{\mathrm{conv}}\,\mathcal{D}(A,B) = \Omega(A,B) := \left\{(Y,W) \in \mathbb{E} \;\middle|\; AY = B \text{ and } \frac{1}{2}YY^T + W \in \mathcal{K}_A^\circ\right\}.$$

In particular, Theorem 2.2.2 in combination with (1.7) implies that $\sigma_{\mathcal{D}(A,B)} = \sigma_{\Omega(A,B)}$, an identity which we will employ throughout.

3.3 Infimal projections of the generalized matrix-fractional function

We now focus on infimal projections involving the GMF function. For these purposes consider the function $\psi : \mathbb{E} \to \overline{\mathbb{R}}$, given by

$$\psi(X,V) := \sigma_{\Omega(A,B)}(X,V) + h(V), \tag{3.9}$$

where $h \in \Gamma(\mathbb{S}^n)$ and $\Omega(A,B)$ is given by Theorem 2.2.2. Our primary objective is the infimal projection of the sum ψ from (3.9) in the variable V, i.e. we analyze the marginal function $p : \mathbb{R}^{n\times m} \to \overline{\mathbb{R}}$ defined by

$$p(X) := \inf_{V\in\mathbb{S}^n} \psi(X,V). \tag{3.10}$$

We lead with an elementary observation.

Lemma 3.3.1 (Domain of p) *Let p defined by* (3.10)*. Then the following hold:*

a) p is convex.

b) $\operatorname{dom} p = \{X \in \mathbb{R}^{n\times m} \mid \exists V \in \mathcal{K}_A \cap \operatorname{dom} h : \ \operatorname{rge}\left(\begin{smallmatrix} X \\ B \end{smallmatrix}\right) \subset \operatorname{rge} M(V)\}$. *In particular, p is proper if and only if* $\operatorname{dom} h \cap \mathcal{K}_A$ *is nonempty.*

Moreover, if $\operatorname{dom} p \neq \emptyset$ *then the following hold:*

c) If $B = 0$ (e.g. if $A = 0$) then $\operatorname{dom} p$ *is a subspace, hence relatively open.*

d) If $\operatorname{rank} A = p$ *(full row rank) then* $\operatorname{dom} p = \mathbb{R}^{n\times m}$*, hence open.*

Proof: a) The convexity follows from, e.g., [37, Proposition 2.22].

b) The formula for $\operatorname{dom} p$ follows from the definition of p and the representation of $\operatorname{dom} \sigma_{\Omega(A,B)}$ in (3.8) which also gives the properness exactly when $\operatorname{dom} h \cap \mathcal{K}_A \neq \emptyset$.

c) If $B = 0$, note that, $X \in \operatorname{dom} p$ if and only if $\operatorname{span}\{X\} \subset \operatorname{dom} p$. Since $\operatorname{dom} p$ is also convex, it is a subspace, see, e.g., [37, Proposition 3.8].

d) The bordered matrix $M(V)$ from (3.7) is invertible if (and only if) $\operatorname{rank} A = p$. In this case the condition $\operatorname{rge}\left(\begin{smallmatrix} X \\ B \end{smallmatrix}\right) \subset \operatorname{rge} M(V)$ is trivially satisfied for any $X \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times m}$. Therefore the statement follows from b). □

The following example shows that the domain of p may not be relatively open (hence not a subspace) if $B \neq 0$, which proves that this assumption in Lemma 3.3.1 c) is not redundant.

Example 3.3.2 ($\operatorname{dom} p$**)** *Let* $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ *and* $b = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$*. Then*

$$\ker A = \operatorname{span}\left\{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\right\} \quad \text{and} \quad \mathcal{K}_A = \left\{\left(\begin{smallmatrix} v & w \\ w & u \end{smallmatrix}\right) \mid v + u \geq 2w\right\}.$$

Moreover, put $\bar{V} := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *and define*

$$\mathcal{V} := [0,1] \cdot \bar{V} = \left\{\left(\begin{smallmatrix} 2w & w \\ w & 0 \end{smallmatrix}\right) \mid w \in [0,1]\right\} \subset \mathbb{S}^2.$$

Then $\mathcal{V}$ is clearly convex and compact. Now let $h \in \Gamma_0(\mathbb{S}^2)$ be any function with $\operatorname{dom} h = \mathcal{V}$ *(e.g. $h := \delta_{\mathcal{V}}$). Note that*

$$\operatorname{dom} h \cap \mathcal{K}_A = \mathcal{V}.$$

We hence infer that

$$\begin{aligned} x \in \operatorname{dom} p &\iff \exists w \in [0,1] : \ \binom{x}{b} \in \operatorname{rge} \begin{pmatrix} w\bar{V} & A^T \\ A & 0 \end{pmatrix} \\ &\iff \exists w \in [0,1], r, s \in \mathbb{R}^2 : \ \begin{array}{rcl} x &=& w\bar{V}r + A^T s, \\ b &=& Ar \end{array} \\ &\iff \exists w \in [0,1], \lambda, \mu \in \mathbb{R} : \ x = w \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \left[\binom{0}{1} + \lambda \binom{1}{-1} \right] + \mu \binom{1}{1} \\ &\iff \exists w \in [0,1], \gamma \in \mathbb{R} : \ x = w \binom{1}{0} + \gamma \binom{1}{1}. \end{aligned}$$

Therefore, we find that

$$\operatorname{dom} p = [0,1] \cdot \binom{1}{0} + \operatorname{span}\left\{\binom{1}{1}\right\},$$

and hence

$$\operatorname{ri}(\operatorname{dom} p) = (0,1) \cdot \binom{1}{0} + \operatorname{span}\left\{\binom{1}{1}\right\},$$

so that $\operatorname{dom} p$ *is clearly not relatively open.*

As mentioned above, the former example shows that $\operatorname{dom} p$ may fail to be a subspace if $B \neq 0$. Lemma 3.3.1 d) and Example 3.3.17 a), on the other hand, illustrate that $\operatorname{dom} p$ might still be a subspace even if $B \neq 0$, hence the condition $B = 0$ is only sufficient for $\operatorname{dom} p$ to be a subspace (if nonempty).

3.3.1 ψ, ψ^*, and their subdifferentials

Our study of the infimal projection p given in (3.10) requires a thorough understanding of the properties of the functions ψ, ψ^*, and their subdifferentials with ψ defined in (3.9). For this we make extensive use of the condition

$$\operatorname{ri}(\operatorname{dom} h) \cap \operatorname{int} \mathcal{K}_A \neq \emptyset,$$

which we refer to as the *conjugate constraint qualification* (CCQ).

Lemma 3.3.3 (Conjugate of ψ) *Let ψ be given as in* (3.9) *and define*

$$\eta : (Y, W) \in \mathbb{E} \mapsto \inf_{(Y,T) \in \Omega(A,B)} h^*(W - T).$$

Then

$$\operatorname{dom}\eta = \left\{ (Y,W) \;\middle|\; AY = B,\ \left(-\frac{1}{2}YY^T + \mathcal{K}_A^\circ\right) \cap (W - \operatorname{dom}h^*) \neq \emptyset \right\} \tag{3.11}$$

and the following hold:

a) ψ *is closed and convex.*

b) If $\operatorname{dom}h \cap \mathcal{K}_A \neq \emptyset$ *then* $\psi, \psi^* \in \Gamma_0(\mathbb{E})$ *with* $\psi^* = \operatorname{cl}\eta$*.*

c) Under CCQ , we have $\psi^* = \eta$*. Moreover, in this case, the infimum in the definition of* η *is attained on the whole domain, i.e.*

$$\begin{aligned} \mathcal{T}(\bar{Y},\bar{W}) &:= \underset{(Y,W)}{\operatorname{argmin}} \left\{ h^*(\bar{W}-W) \;\middle|\; (Y,W) \in \Omega(A,B),\ Y = \bar{Y} \right\} \\ &= \left\{ (\bar{Y},W) \;\middle|\; (\bar{Y},W) \in \Omega(A,B),\ \psi^*(\bar{Y},\bar{W}) = h^*(\bar{W}-W) \right\}. \end{aligned} \tag{3.12}$$

is nonempty for all $(\bar{Y},\bar{W}) \in \operatorname{dom}\psi^*$ *.*

d) Under CCQ, $\operatorname{dom}\partial\psi^* = \{(Y,W) \mid \emptyset \neq \mathcal{T}(Y,W)\}$ *and, for every* $(Y,W) \in \operatorname{dom}\partial\psi^*$*, we have*

$$\partial\psi^*(Y,W) = \left\{ (X,V) \;\middle|\; \begin{array}{l} \exists\, T \in \mathbb{S}^n :\ V \in \partial h^*(W-T) \cap \mathcal{K}_A, \\ \left\langle V,\ \frac{1}{2}YY^T + T \right\rangle = 0,\ \operatorname{rge}(X - VY) \subset (\operatorname{Ker}A)^\perp \end{array} \right\}.$$

Proof: Note that $\eta(Y,W) < +\infty$ if and only if there is a $T \in \mathbb{S}^n$ such that $(Y,T) \in \Omega(A,B)$ and $W - T \in \operatorname{dom}h^*$, or equivalently, $AY = B,\ T \in -\frac{1}{2}YY^T + \mathcal{K}_A^\circ$ and $T \in W - \operatorname{dom}h^*$, which proves (3.11).

Define $\hat{h} : \mathbb{E} \to \overline{\mathbb{R}}$ by $\hat{h}(X,V) := h(V)$. Then $\operatorname{dom}\hat{h} = \mathbb{R}^{n\times m} \times \operatorname{dom}h$ and $\psi = \sigma_{\Omega(A,B)} + \hat{h}$.

a) The sum of two closed and convex functions is always closed and convex.

b) The sum of two proper functions is proper if and only if the domains of both functions intersect. Here, note that

$$\operatorname{dom}\hat{h} \cap \operatorname{dom}\sigma_{\mathcal{D}(A,B)} \neq \emptyset \quad \Longleftrightarrow \quad \operatorname{dom}h \cap \mathcal{K}_A \neq \emptyset.$$

Therefore, ψ is proper if (and only if) the latter condition holds. Combined with a) this shows ψ is closed, proper, and convex, and hence, so is its conjugate ψ^*.

Moreover, from Theorem 1.2.1 a) in Appendix B, we infer

$$\psi^*(Y,W) = \mathrm{cl}\,\left(\delta_{\Omega(A,B)} \,\Box\, \hat{h}^*\right)(Y,W).$$

Since $\hat{h}^*(Y,W) = \delta_{\{0\}}(Y) + h^*(W)$, we have

$$(\delta_{\Omega(A,B)} \,\Box\, \hat{h}^*)(Y,W) = \inf_{(Y,T)\in\Omega(A,B)} h^*(W-T),$$

which proves $\psi^* = \mathrm{cl}\,\eta$.

c) We have $\mathrm{ri}\,(\mathrm{dom}\,\hat{h}) = \mathbb{R}^{n\times m}\times \mathrm{ri}\,(\mathrm{dom}\,h)$. Also, by [10, Theorem 4.1], we have $\mathrm{int}\,(\mathrm{dom}\,\sigma_{\Omega(A,B)}) = \{(X,V) \mid V \in \mathrm{int}\,\mathcal{K}_A\}$. Hence

$$\mathrm{ri}\,(\mathrm{dom}\,\hat{h}) \cap \mathrm{ri}\,(\mathrm{dom}\,\sigma_{\mathcal{D}(A,B)}) \neq \emptyset \quad\Longleftrightarrow\quad \mathrm{ri}\,(\mathrm{dom}\,h) \cap \mathrm{int}\,\mathcal{K}_A \neq \emptyset. \tag{3.13}$$

Hence Theorem 1.2.1 a) (applied to $\sigma_{\Omega(A,B)}$ and $\hat{h}$), CCQ, and (3.13) imply $\psi^* = \eta$ with

$$\emptyset \neq \mathcal{T}(\bar{Y},\bar{W}) := \operatorname*{argmin}_{(Y,W)} \left\{ h^*(\bar{W}-W) \;\middle|\; (Y,W)\in\Omega(A,B),\ Y=\bar{Y} \right\}.$$

d) Observe that $\partial\sigma^*_{\mathcal{D}(A,B)} = N_{\Omega(A,B)}$ and $\partial\hat{h}^* = \mathbb{R}^{n\times m}\times\partial h^*$. Then part c) and Theorem 1.2.1 d) (applied to $\sigma_{\Omega(A,B)}$ and $\hat{h}$) yield

$$\begin{aligned}\partial\psi^*(Y,W) &= \left\{ (X,V) \;\middle|\; \begin{array}{l} (X,V)\in\partial\sigma^*_{\mathcal{D}(A,B)}(Y_1,W_1)\cap\partial\hat{h}^*(Y_2,W_2), \\ (Y,W) = (Y_1,W_1)+(Y_2,W_2) \end{array} \right\} \\ &= \left\{ (X,V) \;\middle|\; \exists T\in\mathbb{R}^{n\times m}:\ (X,V)\in N_{\Omega(A,B)}(Y,T),\ V\in\partial h^*(W-T) \right\}.\end{aligned}$$

The claim follows from the representation for $N_{\Omega(A,B)}(Y,T)$ in 2.3.1 and [13, Proposition 3].
□

We now turn our attention to the subdifferential of ψ which will be used for computing the subdifferential of its infimal projection p.

Corollary 3.3.4 (Subdifferential of ψ) *Let ψ be given by (3.9) and $\mathcal{T}(\cdot,\cdot)$ by (3.12). Then the following hold:*

a) *If* $(\bar{Y},\bar{W})\in\partial\sigma_{\Omega(A,B)}(\bar{X},\bar{V})+\{0\}\times\partial h(\bar{V})$*, then* $\mathcal{T}(\bar{Y},\bar{W})\neq\emptyset$ *and*

$$\mathcal{T}(\bar{Y},\bar{W})=\left\{\bar{T}\in\mathbb{S}^n\;\middle|\;\bar{W}-\bar{T}\in\partial h(\bar{V}),\;(\bar{Y},\bar{T})\in\partial\sigma_{\Omega(A,B)}(\bar{X},\bar{V})\right\}. \tag{3.14}$$

b) *Under CCQ we have*

$$\operatorname{dom}\partial\psi=\left\{(X,V)\;\middle|\;V\in\operatorname{dom}\partial h\cap\mathcal{K}_A,\;\operatorname{rge}\begin{pmatrix}X\\B\end{pmatrix}\subset\operatorname{rge}M(V)\right\}.$$

Moreover, for all $(\bar{X},\bar{V})\in\operatorname{dom}\partial\psi$ *and all* $(\bar{Y},\bar{W})\in\partial\psi(\bar{X},\bar{V})$*, we have* $\mathcal{T}(\bar{Y},\bar{W})\neq\emptyset$ *and*

$$\begin{aligned}\partial\psi(\bar{X},\bar{V})&=\partial\sigma_{\Omega(A,B)}(\bar{X},\bar{V})+\{0\}\times\partial h(\bar{V})\\&=\left\{(\bar{Y},\bar{W})\in\mathbb{E}\;\middle|\;\mathcal{T}(\bar{Y},\bar{W})\neq\emptyset\right\}.\end{aligned} \tag{3.15}$$

Proof: Set $f_1(X,V):=\sigma_{\Omega(A,B)}(X,V)$ and $f_2(X,V):=h(V)$. Then part a) follows from Theorem 1.2.1 b), and part b) follows from Theorem 1.2.1 c). □

3.3.2 Infimal projection I

We are now in position to prove our first main result about the infimal projection p defined in (3.10).

Theorem 3.3.5 (Conjugate of p and properties under CCQ) *Let p be given by* (3.10). *Moreover, let* $q:\mathbb{R}^{n\times m}\to\overline{\mathbb{R}}$ *be given by*

$$q:Y\mapsto\inf_{(Y,-W)\in\Omega(A,B)}h^*(W).$$

Then the following hold:

a) $\operatorname{dom}q=\left\{Y\in\mathbb{R}^{n\times m}\;\middle|\;AY=B,\;\left(\tfrac{1}{2}YY^T-\mathcal{K}_A^\circ\right)\cap\operatorname{dom}h^*\neq\emptyset\right\}.$

b) $p^*=\operatorname{cl}q$*, hence* $\operatorname{dom}q\subset\operatorname{dom}p^*$.

c) *If CCQ holds for p, then we have:*

I) $p^* = q$, *i.e.*

$$p^*(Y) = \inf_{(Y,-W)\in\Omega(A,B)} h^*(W). \tag{3.16}$$

Moreover, for all $Y \in \operatorname{dom} p^*$*, the infimum is a minimum, i.e. there exists* $W \in \operatorname{dom} h^*$ *with* $(Y,-W) \in \Omega(A,B)$ *such that* $p^*(Y) = h^*(W)$*. In particular,* p^* *is closed, proper, and convex with* $\operatorname{dom} p^* = \operatorname{dom} q$*.*

II) $p \in \Gamma_0(\mathbb{R}^{n\times m})$ *is finite-valued (hence locally Lipschitz).*

Proof: a) Obvious.

b) The expression for p^* (without CCQ) follows from [37, Theorem 11.23 c)] and Lemma 3.3.3 b). The domain containment is clear as $p^* = \operatorname{cl} q \le q$.

c.I) From [37, Theorem 11.23 c)] we have $p^* = \psi^*(\cdot, 0)$, hence Lemma 3.3.3 c) gives the claimed statements.

c.II) p is convex by Lemma 3.3.1 a), and it does not take the value $-\infty$ as p^* is proper by I). To prove the desired statement it therefore suffices to see that $\operatorname{dom} p = \mathbb{R}^{n\times m}$. To this end, observe, see Lemma 3.3.1, that

$$\operatorname{dom} p = L(\operatorname{dom} \sigma_{\Omega(A,B)} \cap \mathbb{R}^{n\times m} \times \operatorname{dom} h),$$

where $L : (X, V) \mapsto X$. By CCQ we have $\operatorname{ri}(\operatorname{dom} h) \cap \operatorname{int} \mathcal{K}_A \neq \emptyset$, hence

$$\begin{aligned}
\operatorname{ri}(\operatorname{dom} \sigma_{\Omega(A,B)} \cap \mathbb{R}^{n\times m} \times \operatorname{dom} h) &= \operatorname{int}(\operatorname{dom} \sigma_{\Omega(A,B)}) \cap \mathbb{R}^{n\times m} \times \operatorname{ri}(\operatorname{dom} h) \\
&= \mathbb{R}^{n\times m} \times \operatorname{int} \mathcal{K}_A \cap \mathbb{R}^{n\times m} \times \operatorname{ri}(\operatorname{dom} h) \\
&= \mathbb{R}^{n\times m} \times \operatorname{int} \mathcal{K}_A \cap \operatorname{ri}(\operatorname{dom} h),
\end{aligned}$$

where we use [10, Theorem 4.1] to represent $\operatorname{int}(\operatorname{dom} \sigma_{\Omega(A,B)})$. This now gives

$$\operatorname{ri}(\operatorname{dom} p) = L\left[\operatorname{ri}(\operatorname{dom} \sigma_{\Omega(A,B)} \cap \mathbb{R}^{n\times m} \times \operatorname{dom} h)\right] = \mathbb{R}^{n\times m}.$$

□

We now take a broader perspective on infimal projection by embedding it into a *pertubation duality framework* in the sense of [37, Theorem 11.39] or [5, Chapter 5].

Given $\bar{X} \in \mathbb{R}^{n\times m}$, we define $\psi_{\bar{X}}$ by

$$\psi_{\bar{X}}(X,V) := \psi(X+\bar{X},V) \quad ((X,V)\in\mathbb{E}).$$

Moreover define $p_{\bar{X}}$ by

$$p_{\bar{X}}(X) := \inf_{V\in\mathbb{S}^n} \psi_{\bar{X}}(X,V) \quad (X\in\mathbb{R}^{n\times m}). \tag{3.17}$$

Then

$$\psi^*_{\bar{X}}(Y,W) = \psi^*(Y,W) - \left\langle \bar{X},\, Y\right\rangle \quad ((Y,W)\in\mathbb{E}),$$

see [37, Equation 11(3)]. Defining

$$q_{\bar{X}}(W) := -\sup_Y\{\left\langle \bar{X},\, Y\right\rangle - \psi^*(Y,W)\} \quad (W\in\mathbb{S}^n), \tag{3.18}$$

then $q_{\bar{X}}$ is a proper (see Lemma 3.3.7 for its domain) and convex function and we have a natural duality pairing of $p_{\bar{X}}$ and $q_{\bar{X}}$ with weak duality reading

$$p_{\bar{X}}(0) \geq -q_{\bar{X}}(0) \quad (\bar{X}\in\mathbb{R}^{n\times m}).$$

Applying the general pertubation duality to our scenario yields the following result.

Proposition 3.3.6 (Shifted duality for p) *Let p be defined by (3.10), let $\bar{X}\in\operatorname{dom} p$ and $q_{\bar{X}}$ be defined by (3.18). Then the following hold:*

a) *If $0\in\operatorname{ri}(\operatorname{dom} q_{\bar{X}})$ then $p(\bar{X}) = -q_{\bar{X}}(0)\in\mathbb{R}$, $\operatorname{argmax}\psi(\bar{X},\cdot)\neq\emptyset$, and $\partial q_{\bar{X}}(0)\neq\emptyset$.*

b) *If $\bar{X}\in\operatorname{ri}(\operatorname{dom} p)$ then $p(\bar{X}) = -q_{\bar{X}}(0)\in\mathbb{R}$, $\operatorname{argmax}_Y\{\left\langle \bar{X},\, Y\right\rangle - \psi^*(Y,W)\}\neq\emptyset$, and $\partial p(\bar{X})\neq\emptyset$.*

c) *Under either condition $0\in\operatorname{ri}(\operatorname{dom} q_{\bar{X}})$ or $\bar{X}\in\operatorname{ri}(\operatorname{dom} p)$, p is lsc at $\bar{X}$ and $-q_{\bar{X}}$ is lsc at 0.*

d) We have

$$\left.\begin{array}{rl} & p(\bar{X}) \\ = & \psi(\bar{X},\bar{V}), \\ = & \left\langle \bar{X},\, \bar{Y} \right\rangle - \psi^*(\bar{Y},0), \\ = & -q_{\bar{X}}(0) \end{array}\right\} \Longleftrightarrow (\bar{Y},0)\in\partial\psi(\bar{X},\bar{V}) \Longleftrightarrow (\bar{Y},0)\in\partial\psi^*(\bar{X},\bar{V}).$$

Proof: Let $\bar{X} \in \operatorname{dom} p$ and observe that

$$p(X+\bar{X}) = p_{\bar{X}}(X) \quad (X \in \mathbb{R}^{n\times m}),$$

hence, in particular, $p(\bar{X}) = p(0) \in \mathbb{R}$. Applying [5, Theorem 5.1.2–5.1.5, Corollary 5.1.2] to the duality pair $p_{\bar{X}}$ and $q_{\bar{X}}$ and translating from $p_{\bar{X}}$ at 0 to p at $\bar{X}$ gives all the desired statements. □

The domain of $q_{\bar{X}}$ is given below. Here, the set

$$\mathcal{C}(A,B) := \{W \in \mathbb{S}^n \mid \exists Y:\ (Y,W) \in \Omega(A,B)\}\,, \tag{3.19}$$

which will play a crucial role in what follows, occurs naturally.

Lemma 3.3.7 (Domain of $q_{\bar{X}}$) *Let $\bar{X} \in \mathbb{R}^{n\times m}$ and $q_{\bar{X}}$ defined by* (3.18)*. Then*

$$\operatorname{dom} q_{\bar{X}} = \mathcal{C}(A,B) + \operatorname{dom} h^*.$$

Proof: a) Using Lemma 3.3.3, observe that

$$\begin{aligned} q_{\bar{X}}(W) &= \inf_Y \left\{\psi^*(Y,W) - \left\langle \bar{X},\, Y \right\rangle\right\} \\ &= \inf_Y \left\{\eta(Y,W) - \left\langle \bar{X},\, Y \right\rangle\right\} \\ &= \inf_{(Y,T)\in\Omega(A,B)} \left\{h^*(W-T) - \left\langle \bar{X},\, Y \right\rangle\right\}. \end{aligned}$$

Therefore, we have

$$\operatorname{dom} q_{\bar{X}} = \{W \mid \exists (Y,T) \in \Omega(A,B) : W - T \in \operatorname{dom} h^*\} = \mathcal{C}(A,B) + \operatorname{dom} h^*.$$

□

Before we proceed with our analysis, we will discuss various constraint qualifications for the optimization problem defining p in the next section.

3.3.3 Constraint qualifications

We start our analysis with a result about the set $\mathcal{C}(A,B)$ from (3.19), which was used in Lemma 3.3.7 to represent the domain of $q_{\bar{X}}$.

Lemma 3.3.8 (Properties of $\mathcal{C}(A,B)$) *Let $\mathcal{C}(A,B)$ be as in (3.19). Then we have:*

a) $\mathcal{C}(A,B)$ is closed and convex with $\mathcal{C}(A,B)^\infty = \mathcal{K}_A^\circ$.

b) $\mathcal{C}(A,B) = \mathrm{dom}\,\sigma_{\Omega(A,B)}(\bar{X},\cdot)^*$ for all $\bar{X}$ such that $\sigma_{\Omega(A,B)}(\bar{X},\cdot)$ is proper.

c) We have

$$\begin{aligned}\mathrm{ri}\,\mathcal{C}(A,B) &= \left\{W \;\middle|\; \exists Y:\ AY = B,\ \frac{1}{2}YY^T + W \in \mathrm{ri}\,(\mathcal{K}_A^\circ)\right\} \\ &= \mathrm{ri}\,(\mathrm{dom}\,\sigma_{\Omega(A,B)}(\bar{X},\cdot)^*)\end{aligned}$$

for all $\bar{X}$ such that $\sigma_{\Omega(A,B)}(\bar{X},\cdot)$ is proper.

Proof: a) With the linear map $T:(Y,W)\mapsto W$ we have $\mathcal{C}(A,B) = T(\Omega(A,B))$. Therefore $\mathcal{C}(A,B)$ is convex. By 2.4.6 we have $\Omega(A,B)^\infty = \{0\}\times\mathcal{K}_A^\circ$. Therefore, $\ker T \cap \Omega(A,B)^\infty = \{0\}$. Hence [37, Theorem 3.10] gives the rest of a).

b) Apply Corollary 1.2.2 to $\bar{g} := \sigma_{\Omega(A,B)}(\bar{X},\cdot)$ to infer that

$$\bar{g}^*(W) = \inf_{Y:(Y,W)\in\Omega(A,B)} \left\langle -\bar{X},\, Y\right\rangle \quad (W\in\mathbb{S}^n).$$

This proves the claim.

c) Observe that $\mathrm{ri}\,\mathcal{C}(A,B) = \mathrm{ri}\,T(\Omega(A,B) = T(\mathrm{ri}\,\Omega(A,B))$ and use 2.3.1 to get the first representation. The second one follows from b). $\square$

We now define the constraint qualifications central to our study. Note that CCQ was already defined earlier.

Definition 3.3.9 (Constraint qualifications) *Let p be given by (3.10). We say that p satisfies* *i)* PCQ *if* $0 \in \mathrm{ri}\,(\mathrm{dom}\, h^* + \mathcal{C}(A,B))$*;*

ii) strong PCQ (SPCQ) *if* $0 \in \mathrm{int}\,(\mathrm{dom}\, h^* + \mathcal{C}(A,B))$*;*

iii) boundedness PCQ (BPCQ) *if* $\mathrm{dom}\, h \cap \mathcal{K}_A \neq \emptyset$ *and* $(\mathrm{dom}\, h)^\infty \cap \mathcal{K}_A = \{0\}$*;*

iv) CCQ *if* $\mathrm{ri}\,(\mathrm{dom}\, h) \cap \mathrm{int}\, \mathcal{K}_A \neq \emptyset$*.*

Note that PCQ stands for *primal constraint qualification* and CCQ for *conjugate constraint qualification.*

The next results clarify the relations between the various constraint qualifications. We lead with characterizations of PCQ and BPCQ.

Lemma 3.3.10 (Characterizations of (B)PCQ) *Let* p *be given by* (3.10) *and let*

$$f_{\bar{X}} := \psi(\bar{X}, \cdot) \quad (\bar{X} \in \mathbb{R}^{n\times m}). \tag{3.20}$$

Let $\bar{X} \in \mathrm{dom}\, p$*. Then the following hold:*

a) The following are equivalent:

i) $0 \in \mathrm{ri}\,(\mathrm{dom}\, f^*_{\bar{X}})$*;*

ii) PCQ holds for p*;*

iii) $\exists Y \in \mathbb{R}^{n\times m} : \; AY = B, \quad \frac{1}{2}YY^T \in \mathrm{ri}\,(\mathcal{K}^\circ_A + \mathrm{dom}\, h^*)$*.*

In addition, similar characterizations of SPCQ hold by substituting the relative interior for the interior.

b) BPCQ holds for p *if and only if* $\mathrm{dom}\, h \cap \mathcal{K}_A$ *is nonempty and bounded.*

Proof: a) Defining $g_{\bar{X}} := \sigma_{\Omega(A,B)}(\bar{X}, \cdot)$, we find that $f^*_{\bar{X}} = \mathrm{cl}\,(g^*_{\bar{X}} \,\square\, h^*)$ and therefore $\mathrm{ri}\,(\mathrm{dom}\, f^*_{\bar{X}}) = \mathrm{ri}\,(\mathrm{dom}\, g^*_{\bar{X}} + \mathrm{dom}\, h^*) = \mathrm{ri}\,(\mathcal{C}(A,B) + \mathrm{dom}\, h^*)$, see Lemma 3.3.8 c). This proves the first two equivalences. The third follows readily from the representation of $\mathrm{ri}\,(\Omega(A,B))$ from 2.3.1.

b) Follows readily from [37, Theorem 3.5, Proposition 3.9]. □

We point out that, under PCQ, Lemma 3.3.10 shows that the objective functions $\psi(\bar{X},\cdot)$ ($\bar{X} \in \mathrm{dom}\, p$) occuring in the definition of p in (3.10) are *weakly coercive* when proper, see [5, Theorem 3.2.1]. The latter reference tells us that the infimum in (3.10) is attained under PCQ if finite, a fact that will be stated again (and derived alternatively) in Theorem 3.3.14. Under SPCQ, the objective functions $\psi(\bar{X},\cdot)$ ($\bar{X} \in \mathrm{dom}\, p$) are *level-bounded* (or *coercive*), in which case the argmin $\psi(\bar{X},\cdot)$ is nonempty and compact (and clearly convex).

The next result shows the relations between the different notions of PCQ.

Lemma 3.3.11 *Let p be given by* (3.10)*. Then the following hold:*

a) BPCQ $\implies$ SPCQ $\implies$ PCQ.

b) If $\mathrm{int}\,(\mathrm{dom}\, h^*) \cap \mathrm{int}\,(-\mathcal{C}(A,B)) \neq \emptyset$ then PCQ and SPCQ are equivalent.

Proof: a) The first implication can be seen as follows: If BPCQ holds then $\mathrm{dom}\, f_{\bar{X}} \subset \mathrm{dom}\, h \cap \mathcal{K}_A$ is bounded (and nonempty exactly if $\bar{X} \in \mathrm{dom}\, p$). Therefore $f_{\bar{X}}$ is level-bounded for all $\bar{X} \in \mathrm{dom}\, p$, i.e. $0 \in \mathrm{int}\,(\mathrm{dom}\, f^*_{\bar{X}})$ ($\bar{X} \in \mathrm{dom}\, p$), see e.g. [37, Theorem 11.8]. In view of Lemma 3.3.10 a) this implies that SPCQ holds.

The second implication is trivial.

b) Obvious from the definitions. □

We now provide characterizations for CCQ.

Lemma 3.3.12 (Characterizations of CCQ) *Let p be given by* (3.10)*. Then*

i) $\mathrm{dom}\, h \cap \mathrm{int}\, \mathcal{K}_A \neq \emptyset \iff$ *ii) CCQ holds for p* $\iff$ *iii)* $(-\mathcal{K}_A^\circ) \cap \mathrm{hzn}\, h^* = \{0\}$.

Proof: The first equivalence is a direct consequence of the *line segment principle* (cf. [36, Theorem 6.1]): The fact that ii) implies i) is obvious. For the converse direction let $y \in$

$\operatorname{dom} h \cap \operatorname{int} \mathcal{K}_A$ and pick $x \in \operatorname{ri}(\operatorname{dom} h)$. Then $z_\lambda := \lambda x + (1-\lambda)y \in \operatorname{ri}(\operatorname{dom} h)$ for all $\lambda \in (0,1]$. Letting $\lambda \downarrow 0$ we find that $z_\lambda \in \operatorname{ri}(\operatorname{dom} h) \cap \operatorname{int} \mathcal{K}_A$ for all $\lambda \in (0,1]$ sufficiently small, which proves that $\operatorname{ri}(\operatorname{dom} h) \cap \operatorname{int} \mathcal{K}_A \neq \emptyset$.

The second equivalence can be seen as follows: We apply [36, Corollary 16.2.2] (to $f_1 := h$ and $f_2 := \delta_{\mathcal{K}_A}$). This result tells us that $\operatorname{ri}(\operatorname{dom} h) \cap \operatorname{int} \mathcal{K}_A \neq \emptyset$ if and only if there does not exist a matrix $W \in \mathbb{S}^n$ such that

$$(h^*)^\infty(W) + \sigma_{\mathcal{K}_A}(-W) \leq 0 \quad \text{and} \quad (h^*)^\infty(-W) + \sigma_{\mathcal{K}_A}(W) > 0. \tag{3.21}$$

Since $\sigma_{\mathcal{K}_A}(-W) = \delta_{\mathcal{K}_A^\circ}(-W)$, the first of these conditions is equivalent to the condition $W \in (-\mathcal{K}_A^\circ) \cap \operatorname{hzn} h^*$. In particular, we can infer that $(-\mathcal{K}_A^\circ) \cap \operatorname{hzn} h^* = \{0\}$ gives the inconsistency of (3.21) and thus establishes iii)$\Rightarrow$ii).

The second condition in (3.21) implies $W \neq 0$. Thus, in view of Proposition 3.2.1 b), $0 \neq -W \in \mathcal{K}_A^\circ \subset \mathbb{S}^n_+$, and hence $W \notin \mathcal{K}_A^\circ$. Thus, every nonzero element of the set $(-\mathcal{K}_A^\circ) \cap \operatorname{hzn} h^*$ satisfies (3.21). Thus, the nonexistence of a W satisfying (3.21) implies that $(-\mathcal{K}_A^\circ) \cap \operatorname{hzn} h^* = \{0\}$, which altogether proves the result. $\square$

We note that for any proper, convex function f we always have $\operatorname{hzn} f \subset (\operatorname{dom} f)^\infty$ which, in view of Lemma 3.3.12, implies that the condition

$$(-\mathcal{K}_A^\circ) \cap (\operatorname{dom} h^*)^\infty = \{0\} \tag{3.22}$$

is stronger than CCQ. However, we do not use it in our subsequent study.

Moreover, since $\mathcal{K}_A = \mathbb{S}^n$ if (and only if) A has full column rank we have

$$\operatorname{rank} A = n \quad \Longrightarrow \quad \text{CCQ}.$$

3.3.4 Infimal projection II

We return to our analysis of the infimal projection defining p in (3.10). The following result reveals that the two critical conditions $0 \in \operatorname{ri}(\operatorname{dom} q_{\bar{X}})$ and $\bar{X} \in \operatorname{ri}(\operatorname{dom} p)$, respectively, that occured in (3.3.6), embed nicely into our constraint qualifications studied in Section 3.3.3.

Corollary 3.3.13 *Let p be defined by* (3.10)*, let $\bar{X} \in \operatorname{dom} p$ and $q_{\bar{X}}$ be defined by* (3.18)*. Then the following hold:*

a) PCQ holds for p if and only if $0 \in \operatorname{ri}(\operatorname{dom} q_{\bar{X}})$;

b) If CCQ holds then $\bar{X} \in \operatorname{ri}(\operatorname{dom} p)$.

Proof: a) Follows immediately from Lemma 3.3.7 and the definition of PCQ.

b) Under CCQ we have $\operatorname{dom} p = \mathbb{R}^{n \times m}$, see Theorem 3.3.5, hence b) follows. □

As a consequence of Corollary 3.3.13 and Proposition 3.3.6 we can add to the properties of p proven in Theorem 3.3.5.

Theorem 3.3.14 (Properties of p under PCQ) *Let p be defined by* (3.10) *such that PCQ is satisfied and let $q_{\bar{X}}$ be given by* (3.18)*. Then the following hold:*

a) $p \in \Gamma_0(\mathbb{R}^{n \times m})$;

b) $\operatorname{argmin}_V \psi(\bar{X}, V) \neq \emptyset \quad (\bar{X} \in \operatorname{dom} p)$ (primal attainment);

c) $p(\bar{X}) = q_{\bar{X}}(0) \quad (\bar{X} \in \operatorname{dom} p)$ (strong duality).

Proof: a) Under PCQ, by Corollary 3.3.13, we have $0 \in \operatorname{ri}(\operatorname{dom} q_{\bar{X}})$ for all $\bar{X} \in \operatorname{dom} p$. Hence, by Proposition 3.3.6 c), p is lsc at $\bar{X} \in \operatorname{dom} p$. Since p is proper and convex, see Lemma 3.3.1, this shows that $p \in \Gamma_0$.

b), c) Follows readily from Corollary 3.3.13 and Proposition 3.3.6 a). □

We note that Theorem 3.3.14 could have been proven entirely without using the shifted duality framework from Proposition 3.3.6, but by using the following approach: With the linear projection $L : (X, V) \to X$ which has been used implicitly throughout our study, it can be seen that $p = L\psi$ is a *linear image* in the sense of [36, p. 38]. Then [36, Theorem 9.2] gives all statements from Proposition 3.3.14. This can be seen after realizing that the constraint qualification from the latter reference, which for $p = L\psi$ reads

$$\psi(0, V) > 0 \quad \text{or} \quad \psi^\infty(0, -V) \leq 0 \quad (V \in \mathbb{S}^n),$$

Consequence\Hypothesis	-	PCQ	SPCQ	BPCQ	CCQ	PCQ + CCQ
$p \in \Gamma$	✓	✓	✓	✓	✓	✓
$p \in \Gamma_0$		✓	✓	✓	✓	✓
$p(\bar{X}) = -q_{\bar{X}}(0)$		✓	✓	✓	✓	✓
$\operatorname{argmin} \psi(\bar{X}, \cdot) \neq \emptyset$		✓	✓	✓		✓
$\operatorname{argmin} \psi(\bar{X}, \cdot)$ compact			✓	✓[1]		✓
$\operatorname{dom} p = \mathbb{R}^{n\times m}$					✓	✓
$p = p^{**}$		✓	✓	✓	✓	✓
$\underset{(\bar{Y},T)\in\Omega(A,B)}{\operatorname{argmin}} h^*(-T) \neq \emptyset$					✓	✓

Table 3.1: Constraint qualifications for p and their implications

as $\ker L = \{0\} \times \mathbb{S}^n$, is exactly PCQ, which, however, also takes some effort. For the sake of uniformity, we have chosen to derive Theorem 3.3.14 from the shifted duality scheme, which will also be serviceable for our subsequent subdifferential analysis.

The next result follows readily from the foregoing analysis.

Corollary 3.3.15 *Let p be given by* (3.10). *If PCQ and CCQ are satisfied for p then the following hold:*

a) $p \in \Gamma_0(\mathbb{R}^{n\times m})$ is finite-valued and for all $\bar{X} \in \mathbb{R}^{n\times m}$ there exists $\bar{V}$ such that $p(\bar{X}) = \psi(\bar{X}, \bar{V})$.

b) $p^* = q$ and for all $\bar{Y} \in \operatorname{dom} p^*$ there exists $\bar{W}$ such that $(\bar{Y}, \bar{W}) \in \Omega(A, B)$ and $p^*(\bar{Y}) = h^*(-\bar{W})$.

Proof: Follows from Theorem 3.3.5. □

Table 3.1 below summarizes most of our findings so far. Here $\bar{X} \in \operatorname{dom} p$ and $\bar{Y} \in \operatorname{dom} p^*$.

[1] $\operatorname{dom} \psi(\bar{X}, \cdot)$ is bounded.

In view of Proposition 3.3.6 b) and Corollary 3.3.13 one might be inclined to think that using CCQ instead of the pointwise condition $0 \in \operatorname{ri}(\operatorname{dom} p)$ is excessively strong. However, computing the relative interior of $\operatorname{dom} p$ without CCQ is problematic, cf. the derivations in the proof of Theorem 3.3.5 c.II) under CCQ. Moreover, CCQ is exactly what is needed to establish desirable properties of p^*, see Theorem 3.3.5 c.I). Hence, we do not consider constraint qualifications weaker than CCQ.

We now turn our attention to subdifferentiation of p.

Proposition 3.3.16 (Subdifferential of p) *Let p be given by* (3.10). *Then the following hold:*

a) Under CCQ we have

$$\partial p(\bar{X}) = \operatorname*{argmax}_{Y}\{\langle \bar{X}, Y\rangle - \inf_{(Y,T)\in\Omega(A,B)} h^*(-T)\}, \tag{3.23}$$

which is nonempty and compact.

b) Under PCQ equation (3.23) *holds, and, for $\bar{X} \in \operatorname{dom} p$, we have*

$$\begin{aligned}\partial p(\bar{X}) &= \{\bar{Y} \mid \exists \bar{V} :\ (\bar{Y}, 0) \in \partial\psi(\bar{X}, \bar{V})\} \\ &= \{\bar{Y} \mid \exists \bar{V} :\ (\bar{X}, \bar{V}) \in \partial\psi^*(\bar{Y}, 0)\} \\ &= \{\bar{Y} \mid \exists \bar{V} : p(\bar{X}) = \psi(\bar{X}, \bar{V}) = \langle \bar{X}, \bar{Y}\rangle - p^*(\bar{Y})\}.\end{aligned}$$

c) Under PCQ and CCQ, we have

$$\partial p(\bar{X}) = \{Y \mid \exists \bar{V}, \bar{T} :\ -\bar{T} \in \partial h(\bar{V}),\ (Y, \bar{T}) \in \partial\sigma_{\Omega(A,B)}(\bar{X}, \bar{V})\},$$

which is compact and nonempty.

Proof: a) Under CCQ, p is convex and finite-valued (hence closed and proper), therefore (3.23) follows from [36, Theorem 23.5] and the fact that the closure for p^* can be dropped in the argmax problem.

Moreover, we have $\operatorname{dom} p = \mathbb{R}^{n\times m}$, which gives the remaining statements in a).

b) Under PCQ we also have that $p \in \Gamma_0$, hence the same reasoning as in a) gives (3.23). We now prove the remainder: For the first identity notice that (see e.g. [**?**, Chapter D, Corollary 4.5.3])

$$\partial p(\bar{X}) = \left\{ Y \;\middle|\; (Y,0) \in \partial \psi(\bar{X},\bar{V}) \right\} \quad (\bar{V} \in \operatorname*{argmin}_V \psi(\bar{X},V)),$$

the latter argmin set being nonempty due to what was argued above. The '$\subset$'-inclusion is hence clear. For the reverse inclusion invoke also [37, Example 10.12] to see that if $(Y,0) \in \psi(\bar{X},\bar{V})$ then $\bar{V} \in \operatorname{argmin}_V \psi(\bar{X},V)$.

The second identity in c) is clear from [36, Theorem 23.5] as $\psi \in \Gamma_0(\mathbb{E})$.

The third follows from Proposition 3.3.6 in combination with Corollary 3.3.13 and recalling that $\psi^*(\bar{Y},0) = p^*(\bar{Y})$.

c) Apply Corollary 3.3.4 to the first representation in b). $\square$

For $\bar{X} \in \operatorname{rbd}(\operatorname{dom} p)$ the subdifferential $\partial p(\bar{X})$ can be empty. Moreover, it is unbounded if $\bar{X} \notin \operatorname{int}(\operatorname{dom} p)$. The latter may even occur under BPCQ as the following example shows.

Example 3.3.17 *Let* $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and* $b = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ *so that*

$$\mathcal{K}_A = \left\{ \left(\begin{smallmatrix} v & w \\ w & u \end{smallmatrix}\right) \mid u \geq 0 \right\}.$$

Defining $h := \delta_{\mathcal{V}}$ *for*

$$\mathcal{V} := \left\{ \left(\begin{smallmatrix} v & 0 \\ 0 & u \end{smallmatrix}\right) \mid u \leq 0, v \in [0,1] \right\}$$

we hence find that

$$\operatorname{dom} h \cap \mathcal{K}_A = \left\{ \left(\begin{smallmatrix} v & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid v \in [0,1] \right\} \quad \text{and} \quad \operatorname{dom} h \cap \operatorname{int} \mathcal{K}_A = \emptyset,$$

so that CCQ is violated but BPCQ (hence (S)PCQ) holds. We find that

$$\begin{aligned}
x \in \operatorname{dom} p \iff & \exists V \in \mathcal{V} \cap \mathcal{K}_A : \left(\begin{smallmatrix} x \\ b \end{smallmatrix}\right) \in \operatorname{rge} \left(\begin{smallmatrix} V & A^T \\ A & 0 \end{smallmatrix}\right) \\
\iff & \exists v \in [0,1], r,s \in \mathbb{R}^2 : \begin{array}{rcl} x & = & \left(\begin{smallmatrix} v & 0 \\ 0 & 0 \end{smallmatrix}\right) r + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) s, \\ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) & = & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) r \end{array} \\
\iff & \exists v \in [0,1], \rho, \sigma \in \mathbb{R} : x = \left(\begin{smallmatrix} v & 0 \\ 0 & 0 \end{smallmatrix}\right) \left[\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + \rho \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \right] + \sigma \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \\
\iff & x \in \operatorname{span} \left\{ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \right\}.
\end{aligned}$$

Therefore we have $\operatorname{dom} p = \operatorname{span}\{\binom{1}{0}\}$. *In particular,* $\operatorname{dom} p$ *is a proper subspace of* $\mathbb{R}^2$, *hence relatively open with empty interior. Therefore* $\partial p(x)$ *is nonempty and unbounded for any* $x \in \operatorname{dom} p$.

3.4 *h* is a support function

We now study the case where h is a support function. Concretely, given a closed, convex set $\mathcal{V} \subset \mathbb{S}^n$, we consider the function $p : \mathbb{R}^{n\times m} \to \overline{\mathbb{R}}$ given by

$$p(X) := \inf_{V\in\mathbb{S}^n} \sigma_{\Omega(A,B)}(X,V) + \sigma_{\mathcal{V}}(V). \tag{3.24}$$

Recall that, by *Hörmander's Theorem*, see e.g. [36, Corollary 13.2.1], this covers exactly the cases where h is positively homogeneous (and closed, proper, convex).

We commence by analyzing the constraint qualifications from Section 3.3.3 in the case that h is a support function. Here, and for the remainder of this section, observe that the choice $h = \sigma_{\mathcal{V}}$ implies that $\operatorname{dom} h = \operatorname{bar} \mathcal{V}$ and $\operatorname{dom} h^* = \mathcal{V}$.

Lemma 3.4.1 (Constraint qualifications for (3.24)**)** *Let p be given by* (3.24). *Then the following hold:*

a) (CCQ) The conditions

$$\operatorname{bar} \mathcal{V} \cap \operatorname{int} \mathcal{K}_A \neq \emptyset, \tag{3.25}$$

$$\mathcal{V}^\infty \cap (-\mathcal{K}_A^\circ) = \{0\}, \tag{3.26}$$

$$\operatorname{cl}(\operatorname{bar} \mathcal{V}) - \mathcal{K}_A = \mathbb{S}^n \tag{3.27}$$

are each equivalent to CCQ for p.

b) (PCQ) PCQ holds for p if and only if

$$\operatorname{pos}(\mathcal{C}(A,B) + \mathcal{V}) = \operatorname{span}(\mathcal{C}(A,B) + \mathcal{V}). \tag{3.28}$$

c) (BPCQ) The conditions

$$\operatorname{bar}\mathcal{V}\cap\mathcal{K}_A\neq\emptyset \quad \textit{and} \quad \operatorname{cl}(\operatorname{bar}\mathcal{V})\cap\mathcal{K}_A=\{0\}, \tag{3.29}$$

$$\operatorname{bar}\mathcal{V}\cap\mathcal{K}_A\neq\emptyset \ \textit{is bounded}, \tag{3.30}$$

$$\operatorname{bar}\mathcal{V}\cap\mathcal{K}_A\neq\emptyset \quad \textit{and} \quad \mathcal{V}^\infty+\mathcal{K}_A^\circ=\mathbb{S}^n \tag{3.31}$$

are each equivalent to BPCQ for p, hence imply (3.28)*.*

Proof: Observe that with $h=\sigma_{\mathcal{V}}$ we have $\operatorname{dom}h=\operatorname{bar}\mathcal{V}$ and $\operatorname{hzn}h^*=\mathcal{V}^\infty$.

a) (3.25) is condition i) in Lemma 3.3.12 for $h=\sigma_{\mathcal{V}}$, while (3.26) is condition iii).

Employing [8, Section 3.3, Exercise 16]) we have

$$(3.26) \iff \operatorname{cl}(\operatorname{bar}\mathcal{V}-\mathcal{K}_A)=\mathbb{S}^n.$$

This completes the proof of a).

b) This is just an application of (1.4).

c) Using (1.9), we see that (3.29) is exactly BPCQ (for $h=\sigma_{\mathcal{V}}$), while the equivalence to (3.30) follows from Lemma 3.3.10 b). The equivalence of (3.31) to the former follows from the fact that

$$(3.29) \iff \operatorname{cl}(\mathcal{V}^\infty+\mathcal{K}_A^\circ)=\mathbb{S}^n,$$

see [8, Section 3.3, Exercise 16]), where the closure can be dropped by interpreting [36, Theorem 6.3] accordingly. □

By the additivity of support functions, see (1.8), we find that

$$p(X)=\inf_{V\in\mathbb{S}^n}\sigma_\Sigma(X,V) \quad (X\in\mathbb{R}^{n\times m}), \tag{3.32}$$

where

$$\Sigma:=\Sigma(A,B,\mathcal{V}):=\Omega(A,B)+\{0\}\times\mathcal{V}\subset\mathbb{E}. \tag{3.33}$$

This facilitates some of the analysis.

Proposition 3.4.2 *Let p be given by* (3.24)*. Then the following hold:*

a) $p \in \Gamma_0$ *(i.e.* $p = p^{**}$*) under any of the conditions in* (3.25)-(3.27) *or* (3.28). *In particular this holds under any condition* (3.29)-(3.31). *Under any of the conditions* (3.25)-(3.27) p *also finite-valued.*

b) $p^* = \delta_{\mathrm{cl}\,\Sigma}(\cdot, 0)$ *where the closure is superfluous (i.e.* Σ *is closed), in particular, under any condition* (3.25)-(3.27).

Proof:a) Follows respectively from Lemma 3.4.1, Theorem 3.3.5 c) and Theorem 3.3.14.

b) By [37, Exercise 3.12], Σ is closed if $(-\mathcal{K}_A^\circ) \cap \mathcal{V}^\infty = \{0\}$, i.e. under any condition in (3.25)-(3.27), see Lemma 3.4.1 a). The rest follows from [37, Proposition 11.23 (c)]. □

We are now interested in computing refined representations for the conjugate of p given by (3.24).

Corollary 3.4.3 *Consider the function* p *from* (3.24) *with* $\mathcal{V} \subset \mathbb{S}^n$ *nonempty, closed and convex. Under any condition* (3.25)-(3.27) *we have*

$$p^* = \delta_{\Xi(A,B)}$$

where

$$\begin{aligned} \Xi(A,B) &:= \{Y \mid \exists W \in \mathcal{V} : (Y, -W) \in \Omega(A,B)\} \\ &= \left\{ Y \;\middle|\; AY = B,\ \left(\frac{1}{2}YY^T - \mathcal{K}_A^\circ\right) \cap \mathcal{V} \neq \emptyset \right\}. \end{aligned}$$

In particular, we have $p = \sigma_{\Xi(A,B)}$ *which is finite-valued.*

Proof: By Theorem 3.3.5 c) and Lemma 3.4.1 we find that

$$p^*(Y) = \inf_{(Y,-W)\in\Omega(A,B)} \delta_{\mathcal{V}}(W) = \begin{cases} 0 & \text{if} \quad \exists W \in \mathcal{V} : (Y, -W) \in \Omega(A,B), \\ +\infty & \text{else,} \end{cases}$$

which shows that $p^* = \delta_\Xi(A, B)$. The fact about p follows from Proposition 3.4.2 a).

□

3.4.1 The case $B = 0$

We now consider the case when $B = 0$. Recall from 2.5.1 that this implies that $\sigma_{\Omega(A,0)}$ is a gauge function. Similarly, if $0 \in \mathcal{V}$, then $\sigma_{\mathcal{V}}$ is also a gauge, in fact, $\sigma_{\mathcal{V}} = \gamma_{\mathcal{V}^\circ}$, cf. [37, Example 11.19].

This combination of assumptions has interesting consequences when the geometries of the sets $\mathcal{V}$ and $-\mathcal{K}_A^\circ$ are compatible in the following sense.

Definition 3.4.4 (Cone compatible gauges) *Given a closed, convex cone $K \subset \mathcal{E}$, we define an ordering on $\mathcal{E}$ by $x \preceq_K y$ if and only if $y - x \in K$. A gauge γ on $\mathcal{E}$ is said to be compatible with this ordering if and only if*

$$\gamma(x) \leq \gamma(y) \quad \text{whenever} \quad 0 \preceq_K x \preceq_K y.$$

The following lemma provides a characterization of cone compatible gauges.

Lemma 3.4.5 (Cones and compatible gauges) *Let $0 \in C \subset \mathcal{E}$ be a closed, convex set, and let $K \subset \mathcal{E}$ be a closed, convex cone. Then γ_C is compatible with the ordering $\preceq_K$ if and only if*

$$K \cap (y - K) \subset C \quad (y \in K \cap C). \tag{3.34}$$

Proof: Note that, for $y \in K$, we have

$$K \cap (y - K) = \{x \mid 0 \preceq_K x \preceq_K y\}.$$

Suppose that γ_C is compatible with K, and let $y \in C \cap K$. If $x \in K \cap (y - K)$, then $\gamma_C(x) \leq \gamma_C(y) \leq 1$, and, consequently, $K \cap (y - K) \subset C$.

Next suppose (3.34) holds, and let $x, y \in \mathcal{E}$ be such that $0 \preceq_K x \preceq_K y$. Then, $y \in K$ and $x \in K \cap (y - K)$. We need to show that $\gamma_C(x) \leq \gamma_C(y)$. If $\gamma_C(y) = +\infty$, this is trivially the case, so we may as well assume that $\gamma_C(y) =: \bar{t} < +\infty$. If $\bar{t} > 0$, then $\bar{t}^{-1}y \in C \cap K$ and $\bar{t}^{-1}x \in K \cap (\bar{t}^{-1}y - K) \subset C$. Hence, $\gamma_C(\bar{t}^{-1}y) = 1 \geq \gamma_C(\bar{t}^{-1}x)$, and so, $\gamma_C(x) \leq \gamma_C(y)$ as desired. In turn, if $\bar{t} = 0$, then $ty \in K \cap C$ $(t > 0)$, so that $tx \in K \cap (ty - K) \subset C$ $(t > 0)$, i.e., $x \in C^\infty$ and so $\gamma_C(x) = 0$. $\square$

Corollary 3.4.6 (Infimal projection with a gauge function) *Let p be given by (3.24) where $\mathcal{V}$ is a nonempty, closed, convex subset of $\mathbb{S}^n$. Suppose that $B = 0$. Then the following hold:*

a) Under any of the conditions (3.25)-(3.27) we have

$$p^* = \delta_{\left\{Y \,\middle|\, AY=0,\ \exists W\in\mathcal{V}:\, AW=0,\ \frac{1}{2}YY^T\preceq W\right\}}. \tag{3.35}$$

b) If $0 \in \mathcal{V}$ and $\gamma_{\mathcal{V}}$ is compatible with the ordering induced by $-\mathcal{K}_A^\circ$ then

$$\begin{aligned} p^*(Y) &= \delta_{\left\{Y \,\middle|\, AY=0,\ \gamma_{\mathcal{V}}\left(\frac{1}{2}YY^T\right)\le 1\right\}}(Y) \\ &= \delta_{(-\mathcal{K}_A^\circ)\cap\mathcal{V}}\left(\frac{1}{2}YY^T\right). \end{aligned} \tag{3.36}$$

Proof: a) Follows readily from Corollary 3.4.3 by setting $B = 0$ and using the representation of $\mathcal{K}_A$ in Proposition 3.2.1.

b) First observe that $-\mathcal{K}_A^\circ = \left\{W \in \mathbb{S}^n_+ \mid \operatorname{rge} W \subset \ker A\right\}$, see Proposition 3.2.1 b), recall that $\operatorname{rge} Y = \operatorname{rge} YY^T$ ($Y \in \mathbb{R}^{n\times m}$) and $V \in \mathcal{V}$ if and only if $\gamma_{\mathcal{V}}(V) \le 1$. Exploiting these facts, we see that

$$\begin{aligned} & AY = 0,\ \exists W \in \mathcal{V} : AW = 0, \frac{1}{2}YY^T \preceq W \\ \iff\ & AY = 0,\ \exists W \in \mathcal{V} :\ \gamma_{\mathcal{V}}(W) \ge \gamma_{\mathcal{V}}\left(\frac{1}{2}YY^T\right) \\ \iff\ & AY = 0,\ \gamma_{\mathcal{V}}\left(\frac{1}{2}YY^T\right) \le 1 \\ \iff\ & AY = 0,\ \frac{1}{2}YY^T \in \mathcal{V} \\ \iff\ & \operatorname{rge} YY^T \subset \ker A,\ \frac{1}{2}YY^T \in \mathcal{V} \\ \iff\ & \frac{1}{2}YY^T \in (-\mathcal{K}_A^\circ) \cap \mathcal{V}. \end{aligned}$$

Therefore b) follows from a). $\square$

Linear functionals are special instances of support functions. We hence obtain the following remarkable result as a consequence of our more general analysis above. Here $\|\cdot\|_*$ denotes the nuclear norm[2].

Corollary 3.4.7 (h linear) *Let $p : \mathbb{R}^{n\times m} \to \overline{\mathbb{R}}$ be defined by*

$$p(X) = \inf_{V\in\mathbb{S}^n} \sigma_{\Omega(A,0)}(X,V) + \langle \bar{U}, V\rangle$$

for some $\bar{U} \in \mathbb{S}^n_+ \cap \mathrm{Ker}_n A$ and $C(\bar{U}) := \left\{ Y \;\middle|\; \frac{1}{2}YY^T \preceq \bar{U} \right\}$. Then we have:

a) $p^* = \delta_{C(\bar{U})\cap \mathrm{Ker}_n A}$ is closed, proper, convex.

b) $p = \sigma_{C(\bar{U})\cap \mathrm{Ker}_n A} = \gamma_{C(\bar{U})^\circ + \mathrm{Rge}_n A^T}$ is sublinear, finite-valued, nonnegative and symmetric (i.e. a seminorm).

c) If $\bar{U} \succ 0$ with $2\bar{U} = LL^T$ ($L \in \mathbb{R}^{n\times n}$) and $A = 0$ then

$$p = \sigma_{C(\bar{U})} = \|L^T(\cdot)\|_*,$$

i.e. p is a norm with $C(\bar{U})^\circ$ as its unit ball and $\gamma_{C(\bar{U})}$ as its dual norm.

d) If $\bar{U}$ is positive definite, $C(\bar{U})$ and $C(\bar{U})^\circ$ are compact, convex, symmetric[3] with 0 in their interior, thus $\mathrm{pos}\, C(\bar{U}) = \mathrm{pos}\, C(\bar{U})^\circ = \mathbb{S}^n$.

Proof: a) Observe that $h := \langle \bar{U}, \cdot\rangle = \sigma_{\{\bar{U}\}}$. Hence the machinery from above applies with $\mathcal{V} = \{\bar{U}\}$. As $\mathcal{V}$ is bounded, CCQ is trivially satisfied (cf. (3.25)-(3.27)) and the representation of p^* follows from Corollary 3.4.6 a).

[2]For a matrix T the nuclear norm $\|T\|_*$ is the sum of its singular values.

[3]We say the set $S \subset \mathcal{E}$ symmetric if $S = -S$.

b) We have

$$\begin{aligned} p &= p^{**} \\ &= \sigma_{C(\bar{U})\cap \mathrm{Ker}_n A} \\ &= \gamma_{(C(\bar{U})\cap \mathrm{Ker}_n A)^\circ} \\ &= \gamma_{\mathrm{cl}\,(C(\bar{U})^\circ + \mathrm{Rge}_n A^T)} \\ &= \gamma_{C(\bar{U})^\circ + \mathrm{Rge}_n A^T}. \end{aligned}$$

As CCQ holds, the first identity is due to Proposition 3.4.2. The second uses a), the third follows from [36, Theorem 14.5]. The sublinearity of p is clear. The finite-valuedness follows from Proposition 3.4.2. Since $0 \in C(\bar{U})$ the nonnegativity follows as well, and the symmetry is due to the symmetry of $C(\bar{U})$.

c) Consider the case $\bar{U} = \frac{1}{2}I$: By part a), we have $p^* = \delta_{\{Y \mid YY^T \preceq I\}}$. Observe that

$$\left\{ Y \;\middle|\; YY^T \preceq I \right\} = \{ Y \mid \|Y\|_2 \le 1 \} =: \mathbb{B}_\Lambda$$

is the closed unit ball of the spectral norm. Therefore, $p = \sigma_{\mathbb{B}_\Lambda} = \|\cdot\|_{\mathbb{B}_\Lambda^\circ} = \|\cdot\|_*$.

To prove the general case suppose that $2\bar{U} = LL^T$. Then it is clear that $C(\bar{U}) = \left\{ Y \;\middle|\; L^{-1}Y \in C(\frac{1}{2}I) \right\}$, and therefore

$$\begin{aligned} p(X) &= \sigma_{C(\bar{U})}(X) \\ &= \sup_{Y : L^{-1}Y \in C(\frac{1}{2}I)} \langle Y, X \rangle \\ &= \sup_{L^{-1}Y \in C(\frac{1}{2}I)} \left\langle L^{-1}Y, L^T X \right\rangle \\ &= \sigma_{C(\frac{1}{2}I)}(L^T X) \\ &= \|L^T X\|_*. \end{aligned}$$

Here the first identity is due to part b) (with $A = 0$) and the last one follows from the special case considered above.

d) Follows from c) using [36, Theorem 15.2]. □

We point out that Corollary 3.4.7 generalizes the nuclear norm smoothing result by Hsieh and Olsen [29, Lemma 1] and complements [10, Theorem 5.7].

3.5 h is an indicator function

We now suppose that the function h in (3.9) is given by $h := \delta_{\mathcal{V}}$ for some nonempty, closed, and convex set $\mathcal{V} \in \mathbb{S}^n$, i.e., in this section, the infimal projection $p : \mathbb{R}^{n\times m} \to \overline{\mathbb{R}}$ is given by

$$p(X) = \inf_{V\in\mathbb{S}^n} \sigma_{\mathcal{D}(A,B)}(X, V) + \delta_{\mathcal{V}}(V). \tag{3.37}$$

We first want to discuss the constraint qualifications from Section 3.3.3 in this particular case. Here, and for the remainder of this section, observe that the choice $h = \delta_{\mathcal{V}}$ implies that $\operatorname{dom} h = \mathcal{V}$ and $\operatorname{dom} h^* = \operatorname{bar} \mathcal{V}$.

Lemma 3.5.1 (Constraint qualifications for (3.37)**)** *Let p be given by* (3.37)*. Then the following hold:*

a) (CCQ) The conditions

$$\mathcal{V} \cap \operatorname{int} \mathcal{K}_A \neq \emptyset, \tag{3.38}$$

$$\overline{\operatorname{cone}}\, \mathcal{V} - \mathcal{K}_A = \mathbb{S}^n \tag{3.39}$$

are each equivalent to CCQ for p.

b) (PCQ) The PCQ holds for p if and only if

$$\operatorname{pos} \mathcal{C}(A, B) + \operatorname{bar} \mathcal{V} = \operatorname{span} (\mathcal{C}(A, B) + \operatorname{bar} \mathcal{V}). \tag{3.40}$$

c) (BPCQ) The qualification conditions

$$\mathcal{V} \cap \mathcal{K}_A \neq \emptyset \quad \text{and} \quad \mathcal{V}^\infty \cap \mathcal{K}_A = \{0\}, \tag{3.41}$$

$$\mathcal{V} \cap \mathcal{K}_A \neq \emptyset \text{ is bounded}, \tag{3.42}$$

$$\mathcal{V} \cap \mathcal{K}_A \neq \emptyset \quad \text{and} \quad \operatorname{bar} \mathcal{V} + \mathcal{K}_A^\circ = \mathbb{S}^n \tag{3.43}$$

are each equivalent to BPCQ for p, hence imply (3.40)*.*

Proof: a) First, observe that , with $h = \delta_{\mathcal{V}}$, condition i) in Lemma 3.3.12 is exactly (3.38). By the same lemma this is equivalent to

$$\operatorname{hzn} \sigma_{\mathcal{V}} \cap (-\mathcal{K}_A^\circ) = \{0\}.$$

Moreover, as $\sigma_{\mathcal{V}} = \sigma_{\mathcal{V}}^\infty$, we have

$$\operatorname{hzn} \sigma_{\mathcal{V}} = \{V \mid \sigma_{\mathcal{V}}(V) \leq 0\} = \mathcal{V}^-.$$

Invoking [8, Section 3.3, Exercise 16 (a)] implies that

$$\operatorname{hzn} \sigma_{\mathcal{V}} \cap (-\mathcal{K}_A^\circ) = \{0\} \iff \operatorname{cl}(\overline{\operatorname{cone}} \mathcal{V} - \mathcal{K}_A) = \mathbb{S}^n,$$

where the closure in the latter statement can clearly be dropped, e.g. by interpreting [36, Theorem 6.3] accordingly.

b) Use (1.4) to infer that PCQ holds for p if and only if

$$\operatorname{pos}(\mathcal{C}(A, B)) + \operatorname{bar} V = \operatorname{pos}(\mathcal{C}(A, B) + \bar{V}) = \operatorname{span}(\mathcal{C}(A, B) + \operatorname{bar} \mathcal{V}).$$

c) The equivalences of BPCQ, (3.41), and (3.42) are clear. Since $\mathcal{V}^\infty$ and $\operatorname{cl}(\operatorname{bar} \mathcal{V})$ are paired in polarity, see (1.9), [8, Section 3.3, Exercise 16 (a)] implies that

$$\mathcal{V}^\infty \cap \mathcal{K}_A = \{0\} \iff \operatorname{cl}(\operatorname{bar} \mathcal{V} + \mathcal{K}_A^\circ) = \mathbb{S}^n,$$

where the closure in the latter statement can be dropped as in a). This establishes all equivalences. □

The following result provides sufficient conditions for the occurence of $p = p^{**}$ when p is given as in (3.37), i.e. in the case that h is an indicator function.

Corollary 3.5.2 *Let p be given by* (3.37)*. Then $p \in \Gamma_0(\mathbb{R}^{n\times m})$ (i.e. $p = p^{**}$) under any of the conditions in* (3.38)-(3.43)*. Under condition* (3.38)-(3.39) *it is also finite-valued.*

Proof: Follows from Lemma 3.5.1 and Theorem 3.3.5 c) and Theorem 3.3.14, respectively. □

We treat the case $A = 0$ and $B = 0$ separately as we will use it in Section 3.5.2.

Corollary 3.5.3 *Let p be given as in (3.37) and assume that $A = 0$ and $B = 0$ and such that $\mathcal{V} \cap \mathbb{S}^n_+$ is nonempty. Then we have*

$$PCQ \iff \mathbb{S}^n_+ + \operatorname{bar} \mathcal{V} = \mathbb{S}^n \iff BPCQ.$$

Moreover, $p \in \Gamma_0(\mathbb{R}^{n\times m})$, i.e. $p = p^{**}$ under any of following conditions:

i) $\mathcal{V} \cap \mathbb{S}^n_{++} \neq \emptyset$ (CCQ);

ii) $\mathcal{V} \cap \mathbb{S}^n_+ \neq \emptyset$ is bounded (or equivalently $\mathbb{S}^n_+ + \operatorname{bar} \mathcal{V} = \mathbb{S}^n$) ((B/S)PCQ).

Under condition i) p is also finite-valued.

Proof: For the first statement notice that $\mathcal{C}(0,0) = \mathbb{S}^n_- = \mathcal{K}_0^\circ$ and invoke Lemma 3.5.1. The rest follows from Corollary 3.5.2 and Lemma 3.5.1. □

To compute the conjugate p^*, instead of using Theorem 3.3.5, a direct derivation relying on [10, Theorem 3.2] yields a powerful result.

Theorem 3.5.4 (Infimal projection with an indicator function) *Let p be given by (3.37). Then its conjugate $p^* : \mathbb{R}^{n\times m} \to \overline{\mathbb{R}}$ is given by*

$$p^*(Y) \;=\; \frac{1}{2}\sigma_{\mathcal{V}\cap\mathcal{K}_A}\left(YY^T\right) + \delta_{\{Y \mid AY=B\}}(Y)\,.$$

In particular, for $A = 0$ and $B = 0$ we obtain

$$p^*(Y) = \frac{1}{2}\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}\left(YY^T\right).$$

Proof: By (3.4), we have

$$\begin{aligned} p^*(Y) &= \sup_X \left[\langle X, Y\rangle - \inf_V \sigma_{\mathcal{D}(A,B)}(X,V) + \delta_{\mathcal{V}}(V)\right] \\ &= \sup_V \sup_X \left[\langle X, Y\rangle - \sigma_{\mathcal{D}(A,B)}(X,V) - \delta_{\mathcal{V}}(V)\right] \\ &= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} \; \sup_{\operatorname{rge}\binom{X}{B}\subset\operatorname{rge} M(V)} \operatorname{tr}\left(-\frac{1}{2}\begin{pmatrix}X\\B\end{pmatrix}^T M(V)^\dagger \begin{pmatrix}X\\B\end{pmatrix} + Y^T X\right) \end{aligned}$$

for $Y \in \mathbb{R}^{n\times m}$. Since $\operatorname{rge}\binom{X}{B} \subset \operatorname{rge} M(V)$, we can make the substitution $M(V)\binom{U}{W} = \binom{X}{B}$, to obtain

$$\begin{aligned}
p^*(Y) &= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} \sup_{\substack{U,W\\ AU=B}} \operatorname{tr}\left(-\frac{1}{2}\begin{pmatrix}U\\W\end{pmatrix}^T M(V)\begin{pmatrix}U\\W\end{pmatrix} + Y^T(VU + A^TW)\right)\\
&= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} -\sum_{i=1}^m \inf_{\substack{u_i,w_i\\ Au_i=b_i}} \left(\frac{1}{2}\begin{pmatrix}u_i\\w_i\end{pmatrix}^T M(V)\begin{pmatrix}u_i\\w_i\end{pmatrix} - y_i^TVu_i - w_i^TAy_i\right)\\
&= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} -\sum_{i=1}^m \inf_{\substack{u_i,w_i\\ Au_i=b_i}} \left(\frac{1}{2}u_i^TVu_i - \langle Vy_i, u_i\rangle + \langle w_i, b_i - Ay_i\rangle\right)\\
&= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} -\sum_{i=1}^m \left[\inf_{Au_i=b_i}\left(\frac{1}{2}u_i^TVu_i - \langle Vy_i, u_i\rangle\right) + \inf_{w_i}\left(\langle w_i, b_i - Ay_i\rangle\right)\right]\\
&= \delta_{\{Z \mid AZ=B\}}(Y) + \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} -\sum_{i=1}^m \inf_{Au_i=b_i}\left(\frac{1}{2}u_i^TVu_i - \langle Vy_i, u_i\rangle\right),
\end{aligned}$$

where the final equality follows since $\delta_{\{y \mid b_i - Ay_i\}}(y_i) = \sup_{w_i}\langle w_i,\, b_i - Ay_i\rangle$ $(i = 1,\ldots,m)$. By hypothesis $\operatorname{rge} B \subset \operatorname{rge} A$, and so, by [10, Theorem 3.2]

$$-\frac{1}{2}\begin{pmatrix}Vy_i\\b_i\end{pmatrix}^T M(V)^\dagger\begin{pmatrix}Vy_i\\b_i\end{pmatrix} = \inf_{Au_i=b_i}\left(\frac{1}{2}u_i^TVu_i - \langle Vy_i, u_i\rangle\right) \quad (i = 1,\ldots,m),$$

Therefore, when $AY = B$, we have

$$\begin{aligned}
p^*(Y) &= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} -\sum_{i=1}^m -\frac{1}{2}\begin{pmatrix}Vy_i\\b_i\end{pmatrix}^T M(V)^\dagger\begin{pmatrix}Vy_i\\b_i\end{pmatrix} \qquad \begin{pmatrix}\text{where } Ay_i = b_i \text{ so}\\ \binom{Vy_i}{b_i} = M(V)\binom{y_i}{0}\end{pmatrix}\\
&= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} \frac{1}{2}\sum_{i=1}^m \left(M(V)\begin{pmatrix}y_i\\0\end{pmatrix}\right)^T M(V)^\dagger\left(M(V)\begin{pmatrix}y_i\\0\end{pmatrix}\right)^T\\
&= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} \frac{1}{2}\sum_{i=1}^m \begin{pmatrix}y_i\\0\end{pmatrix}^T M(V)\begin{pmatrix}y_i\\0\end{pmatrix}^T\\
&= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} \frac{1}{2}\sum_{i=1}^m y_i^TVy_i\\
&= \sup_{V\in\mathcal{V}\cap\mathcal{K}_A} \frac{1}{2}\operatorname{tr}(Y^TVY),
\end{aligned}$$

which proves the general expression for p^*. The case $A = 0, B = 0$ follows readily. □

We now study the subdifferential of p given by (3.37).

Corollary 3.5.5 *Let p be given by (3.37). If $\mathcal{V} \cap \operatorname{int} \mathcal{K}_A \neq \emptyset$ (CCQ) then*

$$\partial p(\bar{X}) = \underset{Y}{\operatorname{argmax}}\{\langle \bar{X}, Y \rangle - \inf_{(Y,T)\in\Omega(A,B)} \sigma_{\mathcal{V}}(-T)\}$$

is nonempty and compact for all $\bar{X} \in \operatorname{dom} p$. If, in addition, $\operatorname{pos} \mathcal{C}(A,B) + \operatorname{bar} \mathcal{V} = \operatorname{span}(\mathcal{C}(A,B) + \operatorname{bar} \mathcal{V})$ (PCQ), then

$$\partial p(\bar{X}) = \left\{\bar{Y} \;\middle|\; \exists \bar{V}, \bar{T} : \; -\bar{T} \in N_{\mathcal{V}}(\bar{V}), \; (\bar{Y}, \bar{T}) \in \partial \sigma_{\Omega(A,B)}(\bar{X}, \bar{V}) \right\}$$

is nonempty and compact for all $\bar{X} \in \mathbb{R}^{n\times m}$.

Proof: Follows readily from Proposition 3.3.16 in combination with Lemma 3.5.1. □

3.5.1 $B = 0$ and $0 \in \mathcal{V}$.

We now consider the important special case of p given by (3.37) where $0 \in \mathcal{V}$ and $B = 0$. In this case p turns out to be a squared gauge function, see Corollary 3.5.9. We start with a technical lemma.

Lemma 3.5.6 *Let $C, K \subset \mathbb{E}$ be nonempty, convex with K being a cone. Then $(C + K)^\circ = C^\circ \cap K^\circ$. If $C + K$ is closed with $0 \in C$, then $(C^\circ \cap K^\circ)^\circ = C + K$. In particular, the set $C + K$ is closed if C and K are closed and $K \cap (-C^\infty) = \{0\}$.*

Proof: Clearly, $C^\circ \cap K^\circ \subset (C + K)^\circ$. Conversely, if $z \in (C + K)^\circ$, then $\langle z, x + ty\rangle \leq 1$ for all $x \in C$, $y \in K$, and $t > 0$. Multiplying this inequality by t^{-1} and letting $t \to \infty$, we see that $z \in K^\circ$. By letting $t \downarrow 0$, we see that $z \in C^\circ$.

Now assume that $C + K$ is closed with $0 \in C$. Then $C + K$ is closed and convex with $0 \in C + K$. Hence, by [36, Theorem 14.5], $C + K = (C + K)^{\circ\circ} = (C^\circ \cap K^\circ)^\circ$.

The final statement of the lemma follows from [36, Corollary 9.1.1]. □

The first main result in this section is concerned with a representation of the conjugate p^* under the standing assumptions.

Corollary 3.5.7 (The gauge case I) *Let p be given by (3.37) with $0 \in \mathcal{V}$ and $B = 0$ and let P be the orthogonal projection onto* $\ker A$. *Moreover, let*

$$\mathcal{S} := \{W \in \mathbb{S}^n \mid \operatorname{rge} W \subset \ker A\} = \{W \in \mathbb{S}^n \mid W = PWP\}.^4$$

Then the following hold:

a) We have

$$p^*(Y) = \frac{1}{2}\sigma_{(\mathcal{V}\cap\mathcal{K}_A)+\mathcal{S}^\perp}\left(YY^T\right) = \frac{1}{2}\gamma_{(\mathcal{V}\cap\mathcal{K}_A)^\circ\cap\mathcal{S}}\left(YY^T\right)$$

where $\mathcal{S}^\perp = \{V \in \mathbb{S}^n \mid PVP = 0\}$. In particular, p^* is positively homogeneous of degree 2.

b) If $\mathcal{V}^\circ + \mathcal{K}_A^\circ$ is closed (e.g. when $\mathcal{K}_A^\circ \cap -(\operatorname{cone}\mathcal{V})^\circ = \{0\}$) then

$$p^*(Y) = \frac{1}{2}\gamma_{(\mathcal{V}^\circ\cap\mathcal{S})+\mathcal{K}_A^\circ}\left(YY^T\right), \tag{3.44}$$

where $\operatorname{dom} p^* = \left\{Y \;\middle|\; YY^T \in \operatorname{cone}\mathcal{V}^\circ \cap \mathcal{S} + \mathcal{K}_A^\circ\right\}$.

Proof: a) We have

$$\begin{aligned}
p^*(Y) =& \frac{1}{2}\sigma_{\mathcal{V}\cap\mathcal{K}_A}\left(YY^T\right) + \delta_{\{Y \mid AY=0\}}\\
=& \frac{1}{2}\sigma_{\mathcal{V}\cap\mathcal{K}_A}\left(YY^T\right) + \frac{1}{2}\delta_{\mathcal{S}}\left(YY^T\right)\\
=& \frac{1}{2}\sigma_{\mathcal{V}\cap\mathcal{K}_A}\left(YY^T\right) + \frac{1}{2}\sigma_{\mathcal{S}^\perp}\left(YY^T\right)\\
=& \frac{1}{2}\sigma_{(\mathcal{V}\cap\mathcal{K}_A)+\mathcal{S}^\perp}\left(YY^T\right)\\
=& \frac{1}{2}\gamma_{(\mathcal{V}\cap\mathcal{K}_A)^\circ\cap\mathcal{S}}\left(YY^T\right).
\end{aligned}$$

Here the first equality uses Theorem 3.5.4, the second equality follows from the fact that $\operatorname{rge} Y = \operatorname{rge} YY^T$, the third can be seen from [37, Example 7.4], the fourth uses (1.8), and the final equivalence follows from [36, Theorem 14.5] and Lemma 3.5.6.

[4] Here we consider $\mathcal{S} = \mathbb{S}^n \cap \operatorname{Ker}_n A$ as a subset in the space $\mathbb{S}^n$.

b) If $\mathcal{V}^\circ + \mathcal{K}_A^\circ$ is closed, then Lemma 3.5.6 also tells us that $(\mathcal{V} \cap \mathcal{K}_A)^\circ = \mathcal{V}^\circ + \mathcal{K}_A^\circ$. Since $\mathcal{K}_A^\circ \subset \mathcal{S}$, see Lemma 3.2.1 b), we have

$$(\mathcal{V}^\circ + \mathcal{K}_A^\circ) \cap \mathcal{S} = (\mathcal{V}^\circ \cap \mathcal{S}) + \mathcal{K}_A^\circ$$

which, using a), gives the first equivalence in (3.44). □

Our final goal is to show that p, under the standing assumption in this section, is a squared gauge. To this end, the next result is key.

Lemma 3.5.8 *Let $0 \in C \subset \mathcal{E}$ be closed and convex and define $q : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ through $q(x) := \frac{1}{2}\gamma_C^2(x)$. Then $q^* = \frac{1}{2}\gamma_{C^\circ}^2$.*

Proof: Apply [37, Proposition 11.21] with $\theta = \frac{1}{2}(\cdot)^2$. □

We are now in a position to prove the last result of this section announced earlier. Here we denote by $\mathbb{B}_F$ the (closed) unit ball in the Frobenius norm.

Corollary 3.5.9 (The gauge case II) *Let p be as in Theorem 3.5.4 with $0 \in \mathcal{V}$ and $B = 0$. For $P \in \mathbb{R}^{n\times n}$ the orthogonal projector on* $\ker A$, *define the (closed, convex) sets*

$$\mathcal{V}_A^{1/2} := \left\{L \in \mathbb{R}^{n\times n} \;\middle|\; LL^T \in P(\mathcal{V} \cap \mathcal{K}_A)P\right\}, \quad \mathcal{F} := \left\{LZ \;\middle|\; L \in \mathcal{V}_A^{1/2},\ Z \in \mathbb{B}_F\right\},$$

and the subspace $\mathcal{U} := \mathrm{Ker}_m A$.[5] *Then*

$$p = \frac{1}{2}\gamma^2_{\mathcal{F}+\mathcal{U}^\perp} \quad \textit{and} \quad p^* = \frac{1}{2}\gamma^2_{\mathcal{F}^\circ \cap \mathcal{U}}.$$

In particular, for $A = 0$ and $\mathcal{F} := \left\{LZ \;\middle|\; LL^T \in \mathcal{V} \cap \mathbb{S}^n_+,\ Z \in \mathbb{B}_F\right\}$ we obtain

$$p = \frac{1}{2}\gamma^2_{\mathcal{F}} \quad \textit{and} \quad p^* = \frac{1}{2}\gamma^2_{\mathcal{F}^\circ}.$$

Proof: For all $Y \in \mathbb{R}^{n\times m}$, by Theorem 3.5.4 and the definition of $\mathcal{U}$, we have

$$p^*(Y) = \frac{1}{2}\sigma_{\mathcal{V}\cap\mathcal{K}_A}(YY^T) + \delta_{\mathcal{U}}(Y) = \frac{1}{2}\sup_{V \in \mathcal{V}\cap\mathcal{K}_A} \left\langle PVP,\, YY^T\right\rangle + \delta_{\mathcal{U}}(Y).$$

---

[5] Hence $\mathcal{U}^\perp = \mathrm{Rge}_m A^T$.

In turn, by the definitions of $\mathcal{V}_A^{1/2}$ and the Frobenius norm, the latter equals

$$\frac{1}{2}\sup_{L\in\mathcal{V}_A^{1/2}}\left\langle LL^T,\, YY^T\right\rangle+\delta_{\mathcal{U}}(Y)=\frac{1}{2}\sup_{L\in\mathcal{V}_A^{1/2}}\|L^TY\|_F^2+\delta_{\mathcal{U}}(Y).$$

On the other hand, by the monotonicity and continuity of $t\in\mathbb{R}_+\mapsto t^2$ as well as the self-duality of the Frobenius norm, we find that the latter can be written as

$$\frac{1}{2}\left[\sup_{L\in\mathcal{V}_A^{1/2}}\|L^TY\|_F\right]^2+\delta_{\mathcal{U}}(Y)=\frac{1}{2}\left[\sup_{(Z,L)\in\mathbb{B}_F\times\mathcal{V}_A^{1/2}}\left\langle L^TY,\, Z\right\rangle\right]^2+\delta_{\mathcal{U}}(Y).$$

This, however, using the definition of $\mathcal{F}$ and the convention $(+\infty)^2=+\infty$, we can rewrite as

$$\frac{1}{2}\sigma_{\mathcal{F}}(Y)^2+\delta_{\mathcal{U}}(Y)=\frac{1}{2}\left[\sigma_{\mathcal{F}}(Y)+\delta_{\mathcal{U}}(Y)\right]^2.$$

All in all, using the latter, [37, Example 11.4], (1.8), and [37, Example 11.19] and the polar cone calculus from, e.g., [8, p. 70], we conclude that

$$p^*(Y)=\frac{1}{2}\left[\sigma_{\mathcal{F}}(Y)+\delta_{\mathcal{U}}(Y)\right]^2=\frac{1}{2}\left[\sigma_{\mathcal{F}}(Y)+\sigma_{\mathcal{U}^\perp}(Y)\right]^2=\frac{1}{2}\sigma^2_{\mathcal{F}+\mathcal{U}^\perp}(Y)=\frac{1}{2}\gamma^2_{\mathcal{F}^\circ\cap\mathcal{U}}(Y).$$

This proves the representation for p^*; the one for p then follows from Lemma 3.5.8.

□

3.5.2 Variational Gram Functions

Given a closed, convex set $\mathcal{V}\subset\mathbb{S}^n$ we define

$$\Omega_{\mathcal{V}}:\mathbb{R}^{n\times m}\to\overline{\mathbb{R}},\quad \Omega_{\mathcal{V}}(Y):=\frac{1}{2}\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}(YY^T). \tag{3.45}$$

These kinds of functions are called *variational Gram function (VGF)* and have received some attention lately in the machine learning community due to their orthogonality promoting properties when used as penalty functions, cf. [30].

Note that our definition explicitly intersects $\mathcal{V}$ with the positive semidefinite cone $\mathbb{S}^n_+$ while in the analysis in [30] a standing assumption is that $\Omega_{\mathcal{V}}=\Omega_{\mathcal{V}\cap\mathbb{S}^n}$. These (equivalent) conventions guarantee that $\Omega_{\mathcal{V}}$ is convex. We also scale by $\frac{1}{2}$ to have more elegant formulas.

Our first result follows readily from our above analysis and refines [30, Proposition 4] about the conjugate of a VGF.

Proposition 3.5.10 (Conjugate of VGFs and VGFs as Squared Gauges) *Let* $\Omega_{\mathcal{V}}$ *be given by* (3.45). *Under either of the following assumptions*

i) $\mathcal{V} \cap \mathbb{S}^n_{++} \neq \emptyset$,

ii) $\mathcal{V} \cap \mathbb{S}^n_{+} \neq \emptyset$ *bounded (or equivalently* $\mathbb{S}^n_+ + \operatorname{bar} \mathcal{V} = \mathbb{S}^n$*),*

we have

$$\Omega^*_{\mathcal{V}}(X) = \inf_V \sigma_\Omega(X,V) + \delta_{\mathcal{V}}(V) = \frac{1}{2} \inf_{\substack{V \in \mathcal{V} \cap \mathbb{S}^n_+ : \\ \operatorname{rge} X \subset \operatorname{rge} V}} \operatorname{tr}\left(X^T V^\dagger X\right) \quad (X \in \mathbb{R}^{n\times m}).$$

Under i), $\Omega^*_{\mathcal{V}}$ *is finite-valued, and under ii),* $\Omega_{\mathcal{V}}$ *is finite-valued. In addition, if* $0 \in \mathcal{V}$ *we also have*

$$\Omega_{\mathcal{V}} = \frac{1}{2}\gamma^2_{\mathcal{F}^\circ} \quad \textit{and} \quad \Omega^*_{\mathcal{V}} = \frac{1}{2}\gamma^2_{\mathcal{F}}$$

with $\mathcal{F} = \left\{ LZ \;\middle|\; LL^T \in \mathcal{V} \cap \mathbb{S}^n_+,\ Z \in \mathbb{B}_F \right\}$.

Proof: Using Theorem 3.5.4, Corollary 3.5.3 and the function p occuring there, we have $\Omega^*_{\mathcal{V}} = p^{**} = p$. The rest is clear from the definition of p and the matrix-fractional function as well as the respective results from Section 3.5, in particular Corollary 3.5.9 for the last statement. □

Next we are interested in the subdifferential of a VGF in the sense of (3.45). Although, by our definition, a VGF is always convex, we take the *convex-composite* perspective, see e.g. [11], since essentially a VGF is simply the composition of a closed, proper, convex function $\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}$ and a nonlinear map $H : Y \mapsto YY^T$. It turns out, that the *basic constraint qualification* for $\Omega_{\mathcal{V}} = \frac{1}{2}\sigma_{\mathcal{V}\cap\mathbb{S}^n_+} \circ H$, which reads

$$N_{\operatorname{dom} \sigma_{\mathcal{V}\cap\mathbb{S}^n_+}}(\bar{Y}\bar{Y}^T) \cap (\operatorname{Ker}_n \bar{Y}^T) = \{0\} \quad (\bar{Y} \in \operatorname{dom} \Omega_{\mathcal{V}}), \tag{3.46}$$

and which is essential for full subdifferential calculus of convex-composites, is intimately linked with condition ii) in Corollary 3.5.3.

Lemma 3.5.11 (BCQ for VGF) *Let $\Omega_{\mathcal{V}}$ be given by* (3.45) *and assume that $\mathbb{S}^n_+ \cap \mathcal{V} \neq \emptyset$. Then the following are equivalent:*

i) There exists $\bar{Y} \in \operatorname{dom}\Omega_{\mathcal{V}}$ such that (3.46) *holds;*

ii) $\mathcal{V}^\infty \cap \mathbb{S}^n_+ = \{0\}$ (or equivalently $\mathcal{V} \cap \mathbb{S}^n_+$ is bounded);

iii) (3.46) *holds at every $\bar{Y} \in \operatorname{dom}\Omega_{\mathcal{V}}$.*

Proof: 'i)⇒ii)': Assume ii) were violated, i.e. there exists $0 \neq W \in (\mathcal{V} \cap \mathbb{S}^n_+)^\infty = \mathcal{V}^\infty \cap \mathbb{S}^n_+$. Moreover, by assumption there exists $\bar{V} \in \mathbb{S}^n_+ \cap \mathcal{V}$. By the properties of the horizon cone of closed, convex sets, see (1.5), we have

$$V_t := \bar{V} + tW \in \mathcal{V} \cap \mathbb{S}^n_+ \quad (t > 0). \tag{3.47}$$

Now, take any $\bar{Y} \in \operatorname{dom}\Omega_{\mathcal{V}}$. Then, for all $t > 0$, we have

$$\begin{array}{rcl} +\infty & > & \Omega_{\mathcal{V}}(\bar{Y}) \\ & = & \sup\limits_{V \in \mathbb{S}^n_+ \cap \mathcal{V}} \left\langle V, \bar{Y}\bar{Y}^T \right\rangle \\ & \geq & \left\langle V_t, \bar{Y}\bar{Y}^T \right\rangle \\ & \geq & t\left\langle W, \bar{Y}\bar{Y}^T \right\rangle. \end{array}$$

Since $W \succeq 0$, we have $\left\langle \bar{Y}\bar{Y}^T, W \right\rangle = \operatorname{tr}(\bar{Y}^T W \bar{Y}) \geq 0$. In view of the above chain of inequalities this implies $\left\langle W, \bar{Y}\bar{Y}^T \right\rangle = 0$ and as $W,\ \bar{Y}\bar{Y}^T \succeq 0$ this gives $W\bar{Y}\bar{Y}^T = 0$. Since $\operatorname{rge}\bar{Y} = \operatorname{rge}\bar{Y}\bar{Y}^T$ this implies $W\bar{Y} = 0$ or, equivalently, $\bar{Y}^T W = 0$. Therefore, we have $0 \neq W \in (\mathcal{V} \cap \mathbb{S}^n_+)^\infty \cap (\operatorname{Ker}_n \bar{Y}^T)$. Now, observe that $N_{\operatorname{dom}\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}}(Z) = (\mathcal{V} \cap \mathbb{S}^n_+)^\infty$ for any $Z \in \operatorname{dom}\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}$, see e.g. [37]. This shows that (3.46) is violated at $\bar{Y}$. Since $\bar{Y} \in \operatorname{dom}\Omega_{\mathcal{V}}$ was chosen arbitrarily, this establishes the desired implication.

'ii)⇒iii)': If $\mathcal{V} \cap \mathbb{S}^n_+$ is bounded, then $\operatorname{dom}\sigma_{\mathcal{V}\cap\mathbb{S}^n_+} = \mathbb{S}^n$, and hence $N_{\operatorname{dom}\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}}(\bar{Y}\bar{Y}^T) = \mathbb{S}^n$ for every $\bar{Y} \in \operatorname{dom}\Omega_{\mathcal{V}}$, which gives the desired implication.

'iii)⇒i)': Obvious. □

We now derive the formula for the subdifferential of the VGF from (3.45).

Proposition 3.5.12 *Let* $\Omega_{\mathcal{V}}$ *be given by* (3.45)*. Then*

$$\partial\Omega_{\mathcal{V}}(\bar{Y}) \supset \left\{\bar{V}\bar{Y} \;\middle|\; \bar{V} \in \mathcal{V} \cap \mathbb{S}^n_+ :\ \left\langle \bar{V},\, \bar{Y}\bar{Y}^T \right\rangle = \Omega_{\mathcal{V}}(\bar{Y})\right\} \quad (\bar{Y} \in \operatorname{dom}\Omega_{\mathcal{V}}).$$

If $\mathbb{S}^n_+ \cap \mathcal{V}$ *is nonempty and bounded, equality holds and* $\operatorname{dom}\Omega_{\mathcal{V}} = \mathbb{R}^{n\times m}$*.*

Proof: Combine Lemma 3.5.11 with [37, Theorem 10.6], [37, Corollary 8.25] and the fact that for $H : Y \to YY^T$ we have $\nabla H(Y)^*V = 2VY$ for all $(Y, V) \in \mathbb{E}$. □

We next consider an example.

Example 3.5.13 (Failure of subdifferential calculus for VGF) *Let* $\mathcal{V} := \operatorname{pos}\{I\} \subset \mathbb{S}^n$*, put* $m := 1$ *and let* $H : Y \mapsto YY^T$*. Then clearly condition i) in Proposition 3.5.10 holds, but condition ii) and hence the BCQ* (3.46) *fails. We have*

$$\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}(W) = \sup_{\alpha\geq 0} \alpha \operatorname{tr}(W) = \delta_{\{U\in\mathbb{S}^n \,|\, \operatorname{tr}(U)\leq 0\}}(W) \quad (W \in \mathbb{S}^n). \tag{3.48}$$

Hence, we obtain $\operatorname{dom}\Omega_{\mathcal{V}} = \{0\}$ *and* $\nabla H(0)^*\partial\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}(0) = \{0\}$*. On the other hand, we have* $\Omega_{\mathcal{V}} = \frac{1}{2}\sigma_{\mathcal{V}\cap\mathbb{S}^n_+} \circ H = \delta_{\{0\}}$*. Therefore, we have*

$$\partial\Omega_{\mathcal{V}}(0) = N_{\{0\}}(0) = \mathbb{R}^{n\times m} \supsetneq \{0\} = \nabla H(0)^*\partial\sigma_{\mathcal{V}\cap\mathbb{S}^n_+}(0).$$

Example 3.5.13 establishes various things: First, it shows that condition i) in Proposition 3.5.10 does not yield equality in the subdifferential formula for VGFs. It also illustrates that equality in the subdifferential formula may fail tremendously in the absence of BCQ, even for a convex-composite which is, in fact, convex.

Much effort is made in [30] to compute the conjugate of a (convex) VGF, cf. [30, Proposition 7] and its proof. A slightly refined version of the latter result follows readily from our analysis.

Proposition 3.5.14 (Subdifferential of $\Omega^*_{\mathcal{V}}$) *Let* $\Omega_{\mathcal{V}}$ *be given by* (3.45) *and assume that that* $0 \in \operatorname{ri}(\mathcal{C} + \operatorname{bar}\mathcal{V})$*. Under either of the following assumptions*

i) $\mathcal{V} \cap \mathbb{S}^n_{++} \neq \emptyset$*,* *ii)* $\mathcal{V}\cap\mathbb{S}^n_+\neq\emptyset$ *and* $\mathcal{V}^\infty\cap\mathbb{S}^n_+=\{0\}$ *(or equivalently* $\mathcal{V}\cap\mathbb{S}^n_+\neq\emptyset$ *bounded),*

for any $\bar{X}\in\mathbb{R}^{n\times m}$ *where* $\bar{X}$ *is finite we have*

$$\partial\Omega^*_{\mathcal{V}}(\bar{X})=\left\{\bar{Y}\;\middle|\;\begin{array}{l}\exists\bar{V}\in\mathcal{V}\cap\mathbb{S}^n_+:\operatorname{rge}\bar{X}\subset\operatorname{rge}\bar{V},\\ \Omega^*_{\mathcal{V}}(\bar{X})=\frac{1}{2}\operatorname{tr}\left(\bar{X}^T\bar{V}^\dagger\bar{X}\right)=\langle\bar{X},\,\bar{Y}\rangle-\Omega_{\mathcal{V}}(\bar{Y})\end{array}\right\}$$

with $\operatorname{dom}\partial\Omega^*_{\mathcal{V}}=\operatorname{dom}\Omega^*_{\mathcal{V}}$*.*

Proof: Using Corollary 3.5.3 and the function p occuring there, we have $\Omega^*_{\mathcal{V}}=p$ and $\Omega_{\mathcal{V}}=p^*$ under either i) or ii). The subdifferential formula follows then from Proposition 3.3.16 (see in particular the third identity in c)).

The fact that $\operatorname{dom}\partial\Omega^*_{\mathcal{V}}=\operatorname{dom}\Omega^*_{\mathcal{V}}$ is due to the fact that the latter is a subspace, hence relatively open, cf. Lemma 3.3.1 c). □

3.5.3 VGFs and squared Ky Fan norms

For $p\geq 1$, $1\leq k\leq\min\{m,n\}$, the *Ky Fan (p,k)-norm* [28, Ex. 3.4.3] of a matrix $X\in\mathbb{R}^{n\times m}$ is defined as

$$\|X\|_{p,k}=\left(\sum_{i=1}^k\sigma_i^p\right)^{1/p},$$

where σ_i are the singular values of X sorted in nonincreasing order. In particular, the $(p,\min\{m,n\})$-norm is the Schatten-p norm and the $(1,k)$-norm is the standard Ky Fan k-norm, see [28]. For $1\leq p\leq\infty$, denote the closed unit ball for $\|\cdot\|_{p,k}$ by $\mathbb{B}_{p,k}:=\{X\mid\|X\|_{p,k}\leq 1\}$. For $1\leq p\leq\infty$, define $s:=p/2$. Then, for $2\leq p\leq\infty$, we have

$$\begin{aligned}\frac{1}{2}\|X\|^2_{p,k}&=\frac{1}{2}\left[\sum_{i=1}^k(\sigma_i^2)^s\right]^{1/s}\\&=\frac{1}{2}\|XX^T\|_{s,k}=\frac{1}{2}\sigma_{\mathbb{B}^\circ_{s,k}}(XX^T)=\frac{1}{2}\sigma_{\mathbb{B}^\circ_{s,k}\cap\mathbb{S}^n_+}(XX^T)\\&=\frac{1}{2}\Omega_{\mathbb{B}^\circ_{s,k}}(X),\end{aligned}$$

where the first equality follows from the definition of s, the second from the definition of the singular values, the third from properties of gauges and their polars, the fourth from the equivalence $\langle V, XX^T \rangle = \sum_{j=1}^m x_j^T V x_j$ with the x_j's the columns of X, and the final from (3.45). For the Schatten norms, where $k = \min\{n, m\}$ we have $\mathbb{B}_{s,k}^\circ = \mathbb{B}_{\hat{s},k}$, where $\hat{s}$ satisfies $\frac{1}{s} + \frac{1}{\hat{s}} = 1$, see [27]. For other values of k, the representation of $\mathbb{B}_{s,k}^\circ$ can be significantly more complicated, e.g. see [18].

3.6 Final remarks

In this chapter we studied partial infimal projections of the generalized matrix-fractional function with a closed, proper, convex function $h : \mathbb{S}^n \to \overline{\mathbb{R}}$. Sufficient conditions for closedness and properness as well as representations of both the conjugate and the subdifferential of the infimal projections are given, along with the essential constraint qualifications. Particular emphasis was given in the instances where the function h is a support or an indicator function of a closed, convex set in $\mathbb{S}^n$. As a special case of support functions, infimal projections with suitable linear functionals yielded smoothing variational representations for the family of scaled nuclear norms. In the indicator case, it was shown that, under appropriate assumptions, the infimal projection is positively homogeneous of degree two, in fact, a squared gauge. Moreover, in a special case, it was proven that the conjugate of the infimal projection coincides with a variational Gram function (VGF) of the underlying set. Thus we were able to easily establish a variational calculus for VGFs as a consequence of our more general analysis. In addition, we made a connection with Ky Fan norms.

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