Branko Grünbaum

Preface to "Notes on Arrangements"

In the Spring Quarter of 1974 I gave a course on "Special Topics in Geometry" at the University of Washington. The guiding idea of the course was to present a detailed picture of several topics involving points and lines in the real projective plane, and some analogous results in the real projective space. The material I intended to describe was not available in any published form. Therefore I prepared handouts which were distributed to the participants as the quarter went on, and several copies were also sent to mathematicians that I hoped will be interested in the topics discussed.

The notes are reproduced by digitally scanning the only copy of the notes that is still in my possession. Originally, I typed the pages and drew the diagrams on (purple) ditto masters, supplemented for some of the diagrams by ditto-masters in other colors to illustrate the constructions. It is rather remarkable that after more than 35 years, and with the purple ditto reproductions something completely unknown to today's younger generation, it turned out to be possible to scan and digitize the complete set of the notes. The readability of the text is no worse than it was originally, and although some of the colors did not scan perfectly, with a modicum of good-will all diagrams are intelligible.

The main reason for making these notes now available in digital form is that they have been mentioned in a number of publications. As explained below, the notes have led to the development of several of the topics discussed in them and many of the problems and conjectures mentioned are still open.

The first section of the notes is self-explanatory. It was necessary since most of the students in the class have never been exposed to projective geometry of any kind.

The second section reprises and presents in some detail several aspects of the topic of arrangements of lines in the plane, that has been considered in some of my earlier publications. Particular attention is given to *simplicial arrangements*, that have appeared in various contexts – mostly due to their extremal properties; see, for example, Erdös and Purdy [6], and Artés *et al*. [1]. Another part of the attraction of simplicial arrangements lies in the mystery of the *sporadic* ones among them (see Conjecture 2.1). A recent publication [11] presents an updated listing of all such arrangements, together with a new way of looking at the kinship among them. This leads to a negative solution of Conjecture 2.3: Counterexamples are arrangements listed in [11] as $\mathcal{A}(16,5)$, $\mathcal{A}(17,8)$, and several others. The other conjectures of this section are still open; also, the open-ended Exercises 4 and 5 are still without a satisfactory answer. The claim in the notes that there are 91 sporadic simplicial arrangements needs to be corrected; only 90 are known (see [11]).

A different source of interest in simplicial arrangements is due to their relation to the *free arrangements* considered by Orlik and Terao [14] – although the precise relationship is not clear. On the other hand, the connection between simplicial arrangements and *simple zonohedra* is well known (Coxeter [3]), as is their relevance to *partial cubes* (see Eppstein [5]). Very recently, it has been found that simplicial arrangements have connections with certain Weyl grupoids (see Cuntz and Heckenberger [4]).

In Section 3 we come to the main innovation of the notes – the consideration of simplicial arrangements in three and higher dimensions. While some of the results have been briefly mentioned, without proofs, in [7], here are presented justifications of these and other claims. Much of the material of Section 3 has been used as the basis of [12]. In particular, the corrected catalogue of simplicial 3-arrangements that appears as an Appendix to Section 3 was included in [12], together with one additional arrangement not in the Appendix. The presentation in the Notes, and later in [12], prompted several publications of J. E. Wetzel and coauthors, see [16] for details and references. The research of Wetzel and coauthors led to the discovery of four simplicial 3-arrangements beyond the ones in [12], and also to a number of results about simplicial arrangements in higher dimensions that solved some of the problems raised in the Notes.

The section on Euclidean arrangements surveys the results known at the time, corrects some of them, and raises a number of open problems.

Similarly, the section on arrangements of colored lines presents the known results, introduces a number of new concepts, and raises several problems and conjectures. Some of the material was presented in the abstract [8], and published in [9]. Many new developments on this and related topics are reported in [15].

The final section of the Notes deals with arrangements in which there are "omittable lines" – that is, lines such that their omission does not lead to the loss of any vertices. The topic was started by Koutsky and Polak [13], and the Notes contain strengthenings of their results as well as new material. An exposition of this topic in [10], based on the Notes, led to renewed interest that resulted in a new point of view (*aggregates of lines)* and to far-reaching generalizations, see [2]. Very recently, I have started studying the analogous problem in 3-space, concerning *omittable planes*.

It is my hope that making the Notes readily available after such a long time will contribute to the interest in the topics presented, and in the open problems and conjectures described in the text.

Seattle, April 2010

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LN-29

NOTES ON ARRANGEMENTS

LN 29

University of Washington
Math 553A Spring 1974

Branko Grünbaum:

NOTES ON ARRANGEMENTS

1. Introduction: the projective plane.

The aim of these notes is to present some of the known results and open problems concerning various kinds of arrangements. Without sticking to formalities we shall mainly stress the 2-dimensional case, only briefly indicating changes that occur in higher dimensions.

We shall be interested in Euclidean as well as in projective arrangements, and we find it best to consider them together; more detailed reasons for this attitude will be given later. A definition of arrangements will also be delayed till the next section, since we shall devote the present one to a few words about the real projective plane P^2. The reader wishing to supplement the brief remarks that follow may choose from many excellent texts that discuss the projective plane from the axiomatic, geometric, and algebraic points of view. Our approach is informal, intuitive and pragmatic; we shall assume that the reader is well acquainted with the Euclidean plane E^2 and the Euclidean d-space E^d.

For simplicity of expression - and for occasional use in proofs - we shall interpret the real projective plane P^2 in any of the following - mutually equivalent - ways:

Model (i). The plane P^2 consists of the Euclidean plane E^2 augmented by one "ideal" "point at infinity" for each family of mutually parallel lines, and by a "line at infinity" formed by all the "points at infinity". This is historically the first model of the projective plane; once the terminology is assimilated the model is very convenient. Most of our graphical illustrations will be in this model.

Model (ii). The points of P^2 are represented by ordered triplets $x = (x_0, x_1, x_2)$ of real numbers, not all equal to 0, two triplets x and $y = (y_0, y_1, y_2)$ representing the same point if and only if there exists a real $\lambda \neq 0$ such that $x_j = \lambda y_j$ for all j. A line is represented by a triplet $a = (a_0, a_1, a_2) \neq (0,0,0)$, with proportional triplets representing the same line. A line a and a point x are incident if and only if $\langle a, x \rangle = a_0 x_0 + a_1 x_1 + a_2 x_2 = 0$. The "points at infinity" in model (i) correspond to those triplets x for which $x_0 = 0$, the "line at infinity" to the triplet $(1,0,0)$.

Model (iii). The points of P^2 are all pairs of antipodal (opposite) points on a (unit) sphere S^2 in Euclidean 3-space E^3, while lines of P^2 correspond to great circles on S^2. Related to this is model (iv), in which points of P^2 are represented by lines through the origin O in E^3, and lines of P^2 are represented by planes through O in E^3.

Model (v). P^2 is represented in a closed circular disc D in the Euclidean plane E^2. The points of P^2 are: the points of the interior of D, and antipodal pairs of points of bd D. The lines of P^2 are represented by: the diameters of D, semi-ellipses having diameters of D as major axes, and the boundary bd D of D.

Model (vi). As in model (v), P^2 is represented in a closed circular disc D, and its points are the points of int D and the antipodal pairs of bd D. The lines of P^2 are: The diameters of D, the boundary of D, and circular arcs in D connecting antipodal points on bd D.

A discussion of model (v) may be found in Gans [1954], [1958], [1969]. Model (vi) was brought to my attention by G. C. Shephard in 1969. Both (v) and (vi) may be obtained from the "lower hemisphere" of model (iii), the first by parallel projection, the second by stereographic projection. Additional models (less easy to describe) may be obtained by projections from other points. Restricting in models (v) and (vi) the attention to the interior of the disc D, two models of the Euclidean plane E^2 are obtained.

* * *

One of the advantages of P^2 over E^2 is the existence of a duality in P^2, that is a one-to-one mapping φ from the points and lines of P^2 to the lines and points of P^2, such that incidences of points and lines are preserved. A mapping φ is particularly easy to describe in models (ii), (iii) and (iv). In the first, the image $\varphi(x)$ of a point x is the line with triplet x, and vice versa. In the second, the image $\varphi(x)$ of a point x of P, i.e. of an antipodal pair of points on the sphere S^2, is the great circle perpendicular to the diameter determined by the antipodal pair of points, and conversely. In model (iv), mutually orthogonal lines and planes through O in E^3 correspond to mutually dual points and lines of P^2.

It should be stressed that there are infinitely many different dualities on P^2, and that there is no "natural", privileged duality. However, in a specific model (such as (ii) or (iii)) of P^2 it may happen that one duality is singled out as more "natural" or geometrically meaningful than others.

* * *

One of the reasons for the choice of the real projective plane as the medium in which most of our discussions will take place is the validity of the following statements:

If a set of straight lines $L_1, \ldots, L_n$ is given in P^2 such that no point of P^2 belongs to all of them, then the complement of the union $L_1 \cup \ldots \cup L_n$ consists of a finite number of open connected sets $F_1, \ldots, F_k$. For each F_j there exists a straight line L which misses the closure of F_j; if such an L is chosen as the "line at infinity" in model (1) of P^2 then the closure of F_j is a convex polygon.

Detailed proofs of those properties of P^2 may be found in Veblen-Young [1918, Chapter 9]; a simple and direct approach may be found also in Carver [1941].

* * *

All the above generalizes readily to the real projective space P^d of dimension d. The duality in P^d interchanges j-subspaces with (d-j-1)-subspaces, for $j = 0, 1, \ldots, d-1$.

Exercises.

1. Prove that the six models of P^2 described above are actually equivalent to each other.

2. Formulate the analogue for P^d of the models (i) to (vi).

3. Consider the projections from the center O onto S^2 of the different Platonic solids placed so that their centroids coincide with O . Using model (iii) of P^2 the projections of the cube, octahedron, dodecahedron and icosahedron give rise to certain tessellations of P^2 . Describe those tessellations in the various models of P^2 . What about the tetrahedron, or about the Archimedean solids ?

4. Describe a duality in each of the models (i) , (v) and (vi), and a duality different from those discussed above in each of the other three models.

5. Prove the main property of the stereographic projection: The image of every circle in S^2 that misses the north pole N is a circle in the Euclidean plane tangent at the south pole, while circles through N (deleted at N) are mapped anto lines in the plane.

6. Many statements of Euclidean geometry have analogues in projective geometry obtainable (in model (i)) by replacing "parallel lines" by "lines intersecting at a point at infinity". By such a "projectivization" of a well-known fact about Euclidean triangles the situation shown in Figure 1.1 is obtained. There is a triangle formed by lines A, B, C, a line marked ∞ , and lines marked P, Q, R. The three lines (marked S, T, U) that are determined by the vertices of the triangle and the intersections of A, B, C with P, Q, R meet at one point. From which Euclidean fact does this follow ?

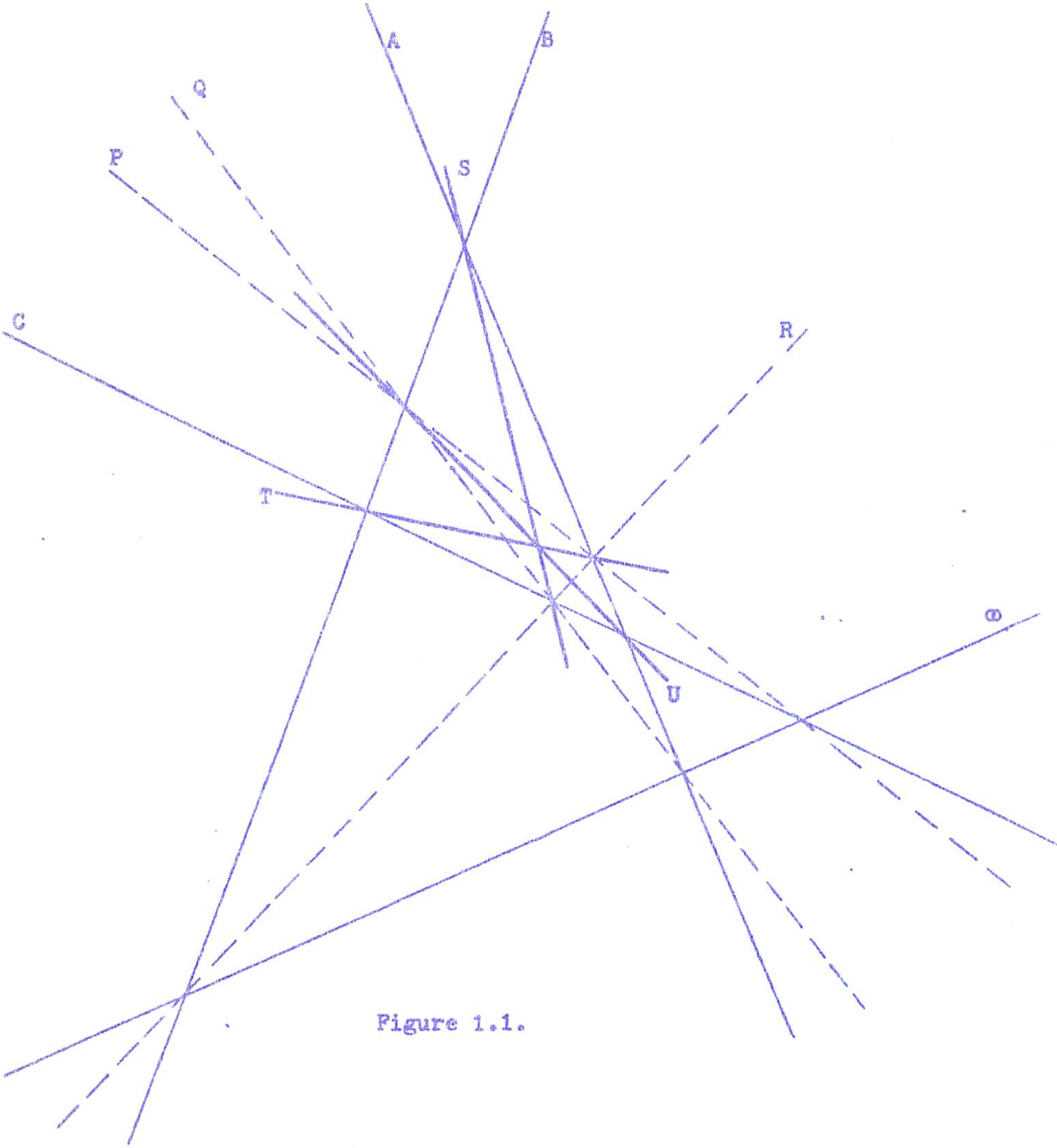

Figure 1.1.

7. Can you formulate the result dealt with in question 6 in any of the other models of P^2 ?

8. Prove the following theorem of G. Ewald (private communication): A finite family $\mathcal{C}$ of circles in the Euclidean plane may be obtained as the stereographic projection of a suitable family of great circles on a suitable sphere if and only if the following conditions are satisfied:

(i) Each two circles in $\mathcal{C}$ intersect at precisely two points;

(ii) For each three circles C, C', C'' in $\mathcal{C}$, the two points of $C \cap C'$ either coincide with the two points of $C \cap C''$, or else separate them on C.

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9. Can you generalize the result of question 8 to higher dimensions ?

<u>References</u>

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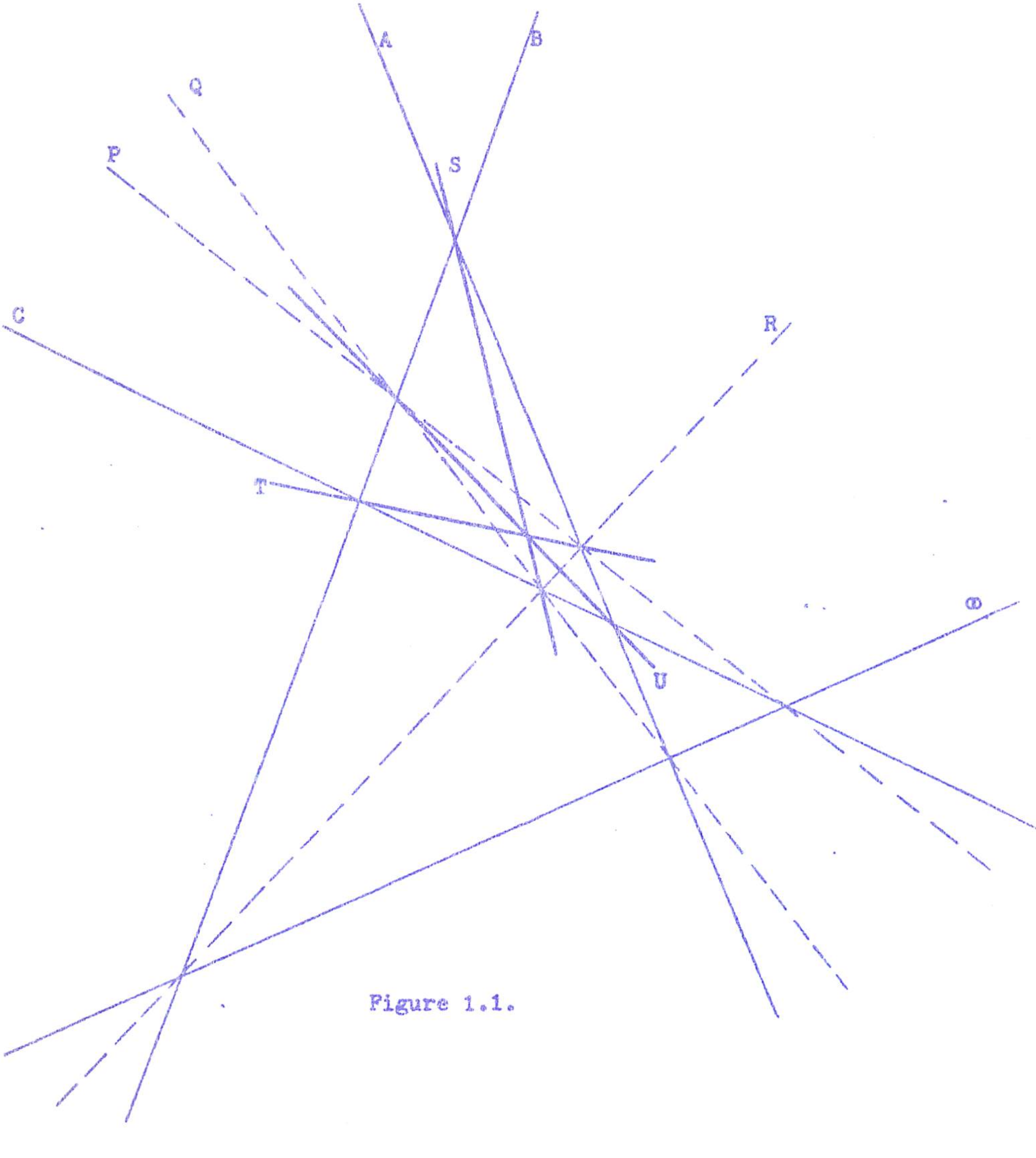

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2. Arrangements of lines in the projective plane.

An **arrangement of lines** $\mathcal{A}$ in the projective plane P^2 is any finite family $\{L_1, \ldots, L_n\}$ (where $n \geq 1$) of lines in P^2, considered with the structure introduced into P^2 by the lines L_j : the points of intersection of the lines (called **vertices** of $\mathcal{A}$), the segments of the L_j's determined by the vertices of $\mathcal{A}$ (we shall call them the **edges** of $\mathcal{A}$), and the connected polygonal regions which form the complement of $L_1 \cup \ldots \cup L_n$: the closure of each such region is called a **face** of $\mathcal{A}$. Thus, each arrangement of lines determines a cell-complex in P^2 and we shall say that two arrangements are **isomorphic** provided the cell-complexes generated by them in the plane are isomorphic.

If all the lines of an arrangement $\mathcal{A}$ in P^2 pass through one point the arrangement is called **trivial**. Although trivial arrangements are in certain respects exceptional and have to be excluded from certain considerations, we shall not follow the frequent convention to deal only with non-trivial arrangements. It is obvious that two trivial arrangements are isomorphic if and only if they contain the same number of lines.

One of the most natural questions about arrangements is the **enumeration problem**: Determine the number $c(n)$ of distinct isomorphism classes of arrangements of n lines in P^2.

The total knowledge available at present on the enumeration problem is contained in:

Theorem 2.1. $c(1) = 1$, $c(2) = 1$, $c(3) = 2$, $c(4) = 3$, $c(5) = 5$, $c(6) = 18$.

The proof of theorem 2.1 is by exhaustive inductive construction, followed by elimination of the arrangements obtained in duplicate. Representatives of each type are shown in Figure 2.1. (Our values

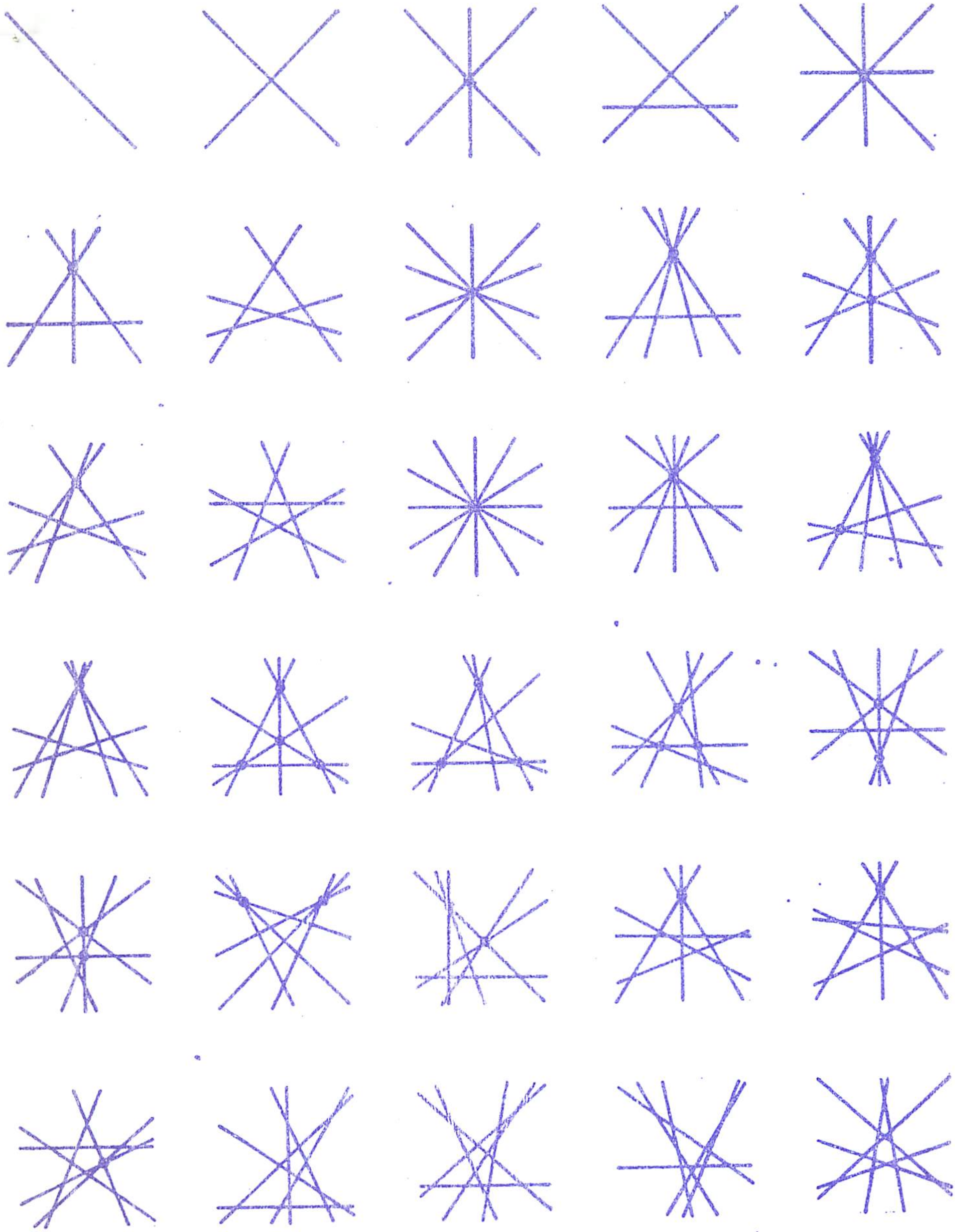


Figure 2.1. The different isomorphism types of arrangements of at most 6 lines. Black circles indicate vertices belonging to 3 or more lines.

for $c(n)$ differ from those found in many other places because of our including the trivial arrangements that are excluded by most other authors.)

It would be very interesting to obtain at least some information on the behaviour of $c(n)$ for large values of n; however, it appears that it is very hard to establish any non-trivial estimates. Part of the reasons for that may possibly be found in the following observations.

Instead of considering arrangements in the real projective plane we could have modified our definitions to fit arrangements of lines in any projective plane $P^2(F)$ over an ordered field F. (In model (ii) of Section 1, consider triplets (x_0, x_1, x_2) of elements of F.) The polygonal Jordan curve theorem, needed to define the cell complex determined by a family of lines in $P^2(F)$, holds in those projective planes, and the definitions of isomorphism, etc. remain unchanged. However, the number of distinct isomorphism types of arrangements of n lines in $P^2(F)$ may depend on F, at least for certain n and F. For example, it is not hard to verify that the arrangement of 9 lines shown in Figure 2.2 exists in $P^2(F)$ if and only if F contains a subfield isomorphic to $Q(5^{\frac{1}{2}})$, the extension of the field of rational numbers by $\sqrt{5}$. Therefore $c(9)$ is strictly greater than the number of isomorphism types of arrangements of 9 lines in (for example) the projective plane over the rationals. Actually (compare Exercises 1,2,3) the number of different isomorphism types depends very strongly on the properties of real algebraic closure of F - which appear to be hard to take into account in enumeration procedures.

In order to appreciate another difficulty in enumeration, we consider the question of how could one construct inductively all the

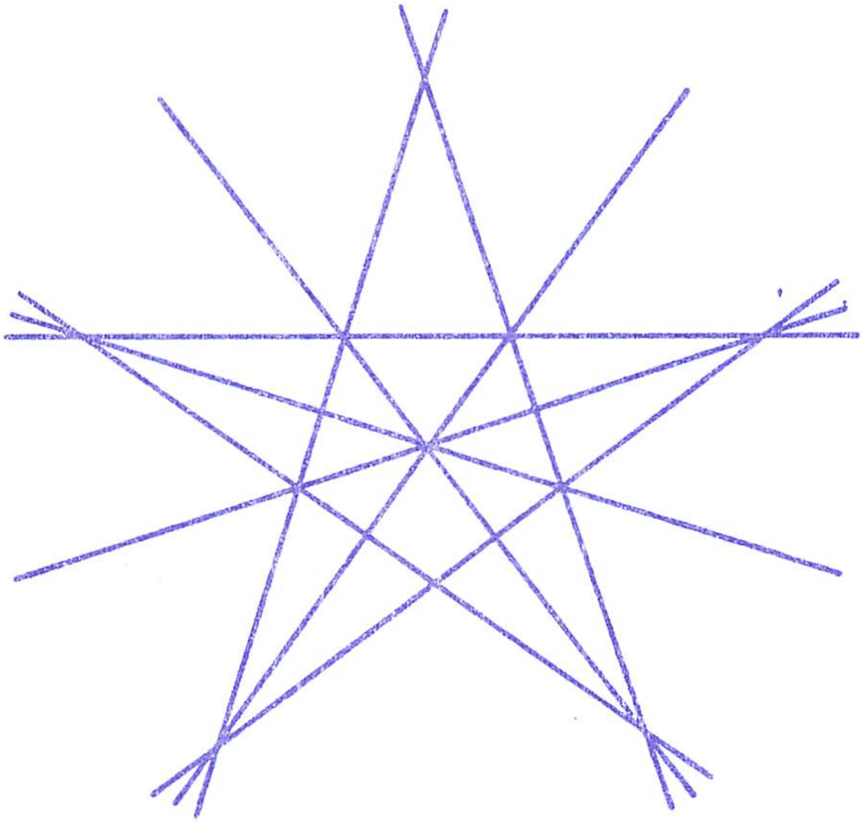
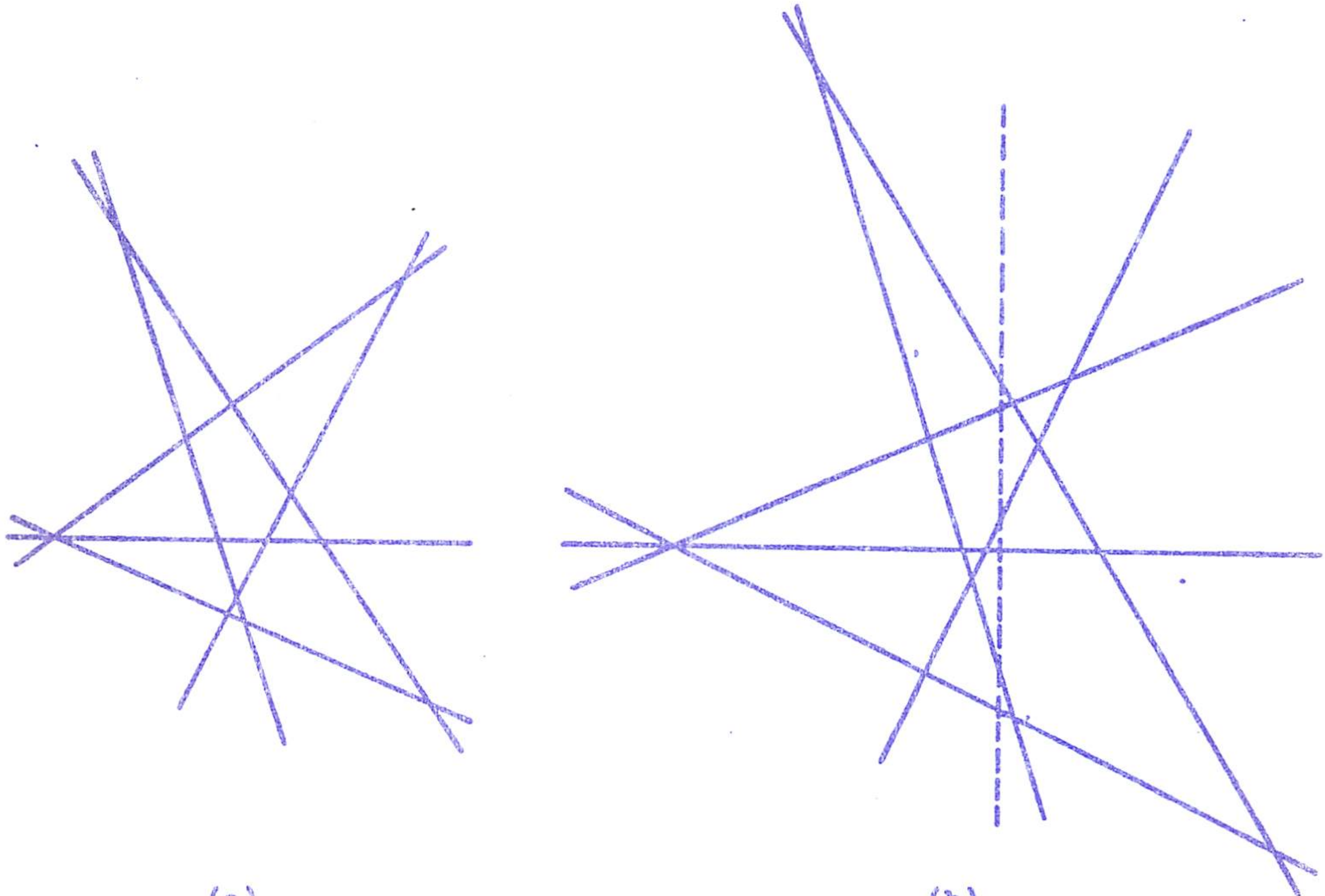


Figure 2.2.

types of arrangements of $n+1$ lines in P^2 if one knows all the types of arrangements of n lines. Since the deletion of any line from an arrangement of $n+1$ lines yields an arrangement with n lines, the most simpleminded idea is to add - in all possible ways - an additional line to a representative of each type of arrangements of n lines. However, experimentation with this idea quickly shows that the outcome may depend on which particular representative of an isomorphism type of n lines one starts with. For example, the two arrangements of 6 lines shown by solid lines in Figure 2.3 are clearly isomorphic; however, the arrangement of 7 lines obtained in Figure 2.3(b) by the addition of the dashed line is not obtainable from the representative of the same type in Figure 2.3(a). In other words, the inductive construction may require different representatives of the same isomorphism class - but it is not known ahead of time which representatives.

One attempt to overcome this difficulty is as follows: Taking one fixed representative of an isomorphism type of arrangements of n lines, we add the (n+1)st line L_{n+1} "schematically" - namely we do not insist on the straightness of L_{n+1} but only that it cross each of the other lines precisely once. (We shall later discuss in more detail such "pseudolines" and arrangements formed by them.) This would enable one to find all the different isomorphism types, and then one would only need to go and choose convenient representatives of the appropriate types so that L_{n+1} may be drawn as a straight line. However, this program is not satisfactory because in some cases it is not possible to realize by straight lines in P^2 the "arrangement" obtained. The smallest known (and probably the smallest possible) examples of that situation occurs for $n = 8$; one is shown in



(a) (b)

Figure 2.3.

Figure 2.4, where the "pseudoline" L_9 is indicated by the dashed line. This L_9 can not be drawn as a straight line for any representative of the isomorphism class indicated by the 8 solid lines in Figure 2.4, since - by the well known Pappus theorem - the line determined by the points A and B necessarily passes through C. (It should be noted that this difficulty does not prevent the inductive proof of Theorem 2.1 simply because for each "pseudoline" it is possible to find - in the cases considered in the theorem - a suitable arrangement that allows it to be drawn as a straight line. Probably the determination of $c(7)$ and $c(8)$ could be made in the same way, but the difficulty would - by the above - appear in connection with $c(9)$.)

In contrast to those negative aspects it should be stressed that the function $c(n)$ is nevertheless algorithmically computable; in other words, the successive values of $c(n)$ could be computed one by one by a suitable Turing machine. This assertion can be established (similarly to the proof given for the analogous statement about polytopes, see Grünbaum [1967, Section 5.5]) using Tarski's [1951] decidability theorem for elementary algebra (see, for example, Robinson [1963, Theorem 4.2.28], Cohen [1967]). However, it is not known whether the function enumerating the different isomorphism types of arrangements of n lines in $P^2(F)$ is algorithmically computable when F is the field of rationals, or any ordered field that does not contain a subfield isomorphic to the real algebraic numbers.

* * *

An arrangement $\mathcal{A}$ of n lines in P^2 is called simple provided no vertex of $\mathcal{A}$ belongs to 3 or more lines. Thus simpleness is a property of the isomorphism types, and one may formulate the enumeration problem for simple arrangements : Determine the number $c^s(n)$ of distinct isomorphism types of simple arrangements of n lines in P^2. We have:

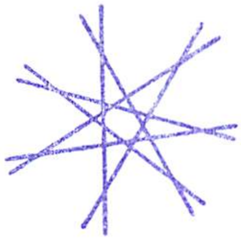
Theorem 2.2. $c^s(1) = 1$, $c^s(2) = 1$, $c^s(3) = 1$, $c^s(4) = 1$, $c^s(5) = 1$, $c^s(6) = 4$, $c^s(7) = 11$, and $c^s(8) = 135$.

The values given in Theorem 2.2 were determined, essentially by variants of the method explained above, for $n \leq 7$ by White [1932], Cummings [1932], [1933], and R. Klee [1938]. Halsey [1972] established $c^s(8) \leq 135$, while Canham [1972] proved $c^s(8) = 135$. Representatives of the 11 types of simple arrangements of 7 lines are shown in Fig.2.5; the only set of drawings of the 135 simple arrangements of 8 lines known to me is in Canham [1972].

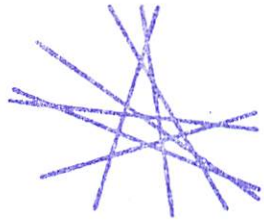
* * *

Another very interesting kind of arrangements are the simplicial ones: An arrangement $\mathcal{A}$ is simplicial provided each face of $\mathcal{A}$ is a triangle. For each $n \geq 3$ there exists at least one isomorphism type of simplicial arrangements of n lines, the near-pencil $\mathcal{A}_0(n)$, which consists of n-1 lines forming a trivial arrangement and another line that does not pass through the point common to the others. Two other infinite families of simplicial arrangements may be described as follows: The arrangement $\mathcal{A}_1(2k)$ is formed, for $k \geq 3$, by the k lines determined by the edges of a regular k-gon in the Euclidean plane, and the k axes of symmetry of that k-gon; the arrangement $\mathcal{A}_1(4m+1)$ is obtained, for $m \geq 2$, from the arrangement $\mathcal{A}_1(4m)$ by the addition of the line at infinity.

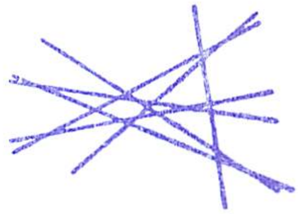
Concerning the number $c^{\Delta}(n)$ of distinct isomorphism types



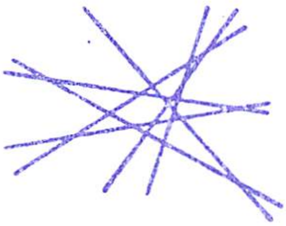
(1)



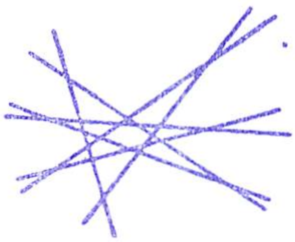
(2)



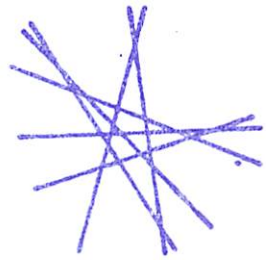
(3)



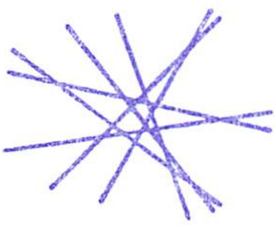
(4)



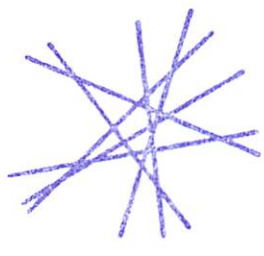
(5)



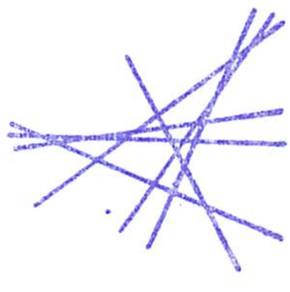
(6)



(7)

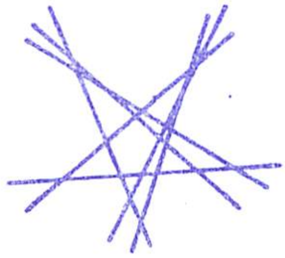


(8)



(9)

(10)



(11)

Figure 2.5.

of simplicial arrangements of n lines we have:

Theorem 2.3. $c^{\mathcal{A}}(3) = c^{\mathcal{A}}(4) = c^{\mathcal{A}}(5) = 1$, $c^{\mathcal{A}}(6) = c^{\mathcal{A}}(7) = c^{\mathcal{A}}(8) = c^{\mathcal{A}}(9) = 2$, $c^{\mathcal{A}}(10) = 4$, $c^{\mathcal{A}}(11) = 2$, $c^{\mathcal{A}}(12) = 4$, $c^{\mathcal{A}}(13) = c^{\mathcal{A}}(14) = 5$, $c^{\mathcal{A}}(15) = 6$.

Representatives of all the isomorphism types of simplicial arrangements of at most 15 lines are shown in Figure 2.6.

As we shall see later, simplicial arrangements are the solutions of several extremal problems. It is therefore rather remarkable that the following conjecture seems to be valid:

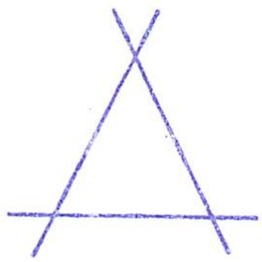
Conjecture 2.1. Except for the infinite families $\mathcal{A}_0(n)$ $(n \geq 3)$, $\mathcal{A}_1(2k)$ $(k \geq 3)$, and $\mathcal{A}_1(4m+1)$ $(m \geq 2)$ there exist only finitely many isomorphism classes of simplicial arrangements of n lines, all with $n \leq 37$.

A total of 91 types of simplicial arrangements not in the three infinite families is known; 90 of them are listed and illustrated in Grünbaum [1971], an additional type in Grünbaum [1972, Fig. 2.3].

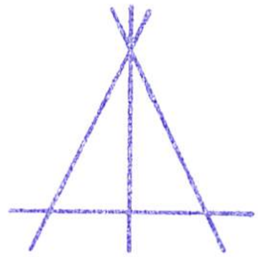
The simplicial arrangements were introduced in Melchior [1940] – a rather naive but still remarkable paper, that appears to have been forgotten (along with simplicial arrangements) for almost 30 years; a version of Conjecture 2.1 was formulated by Melchior, and we shall have occasion to mention him several more times.

* * *

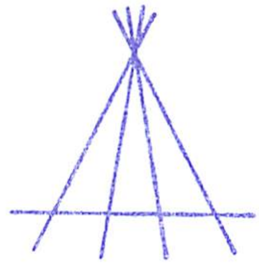
A projective transformation is a mapping of P^2 onto itself that preserves collinearity. In model (ii) of Section 1, each projective transformation may be described by a (non-singular) linear transformation of the vector space of triplets (x_0, x_1, x_2) onto itself. Two arrangements of lines in P^2 are projectively equivalent provided there exists a projective transformation of P^2



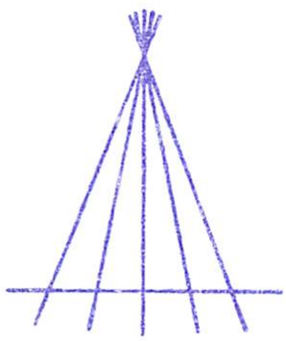
$A_o(3)$



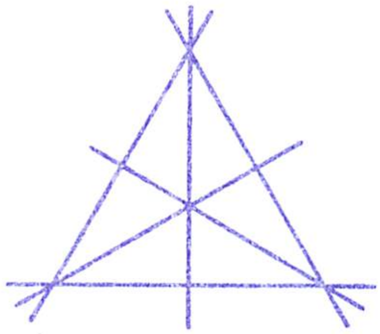
$A_o(4)$



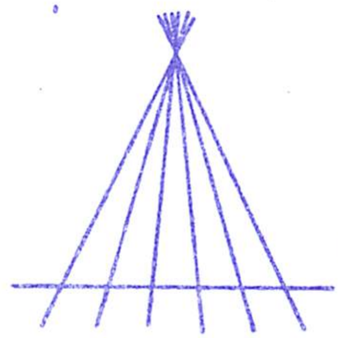
$A_o(5)$



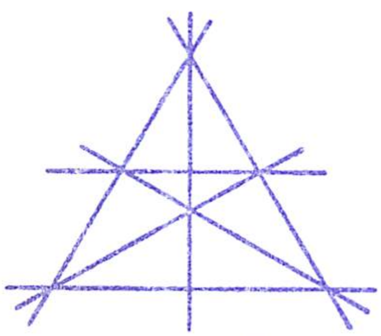
$A_o(6)$



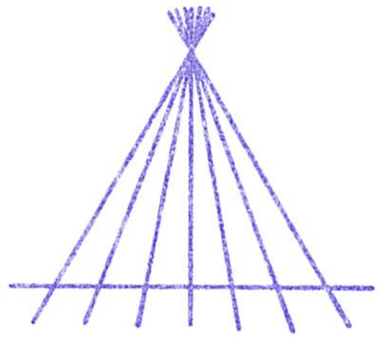
$A_1(6)$



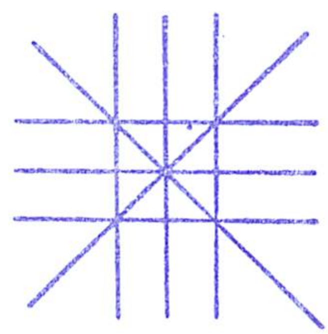
$A_o(7)$



$A_1(7)$



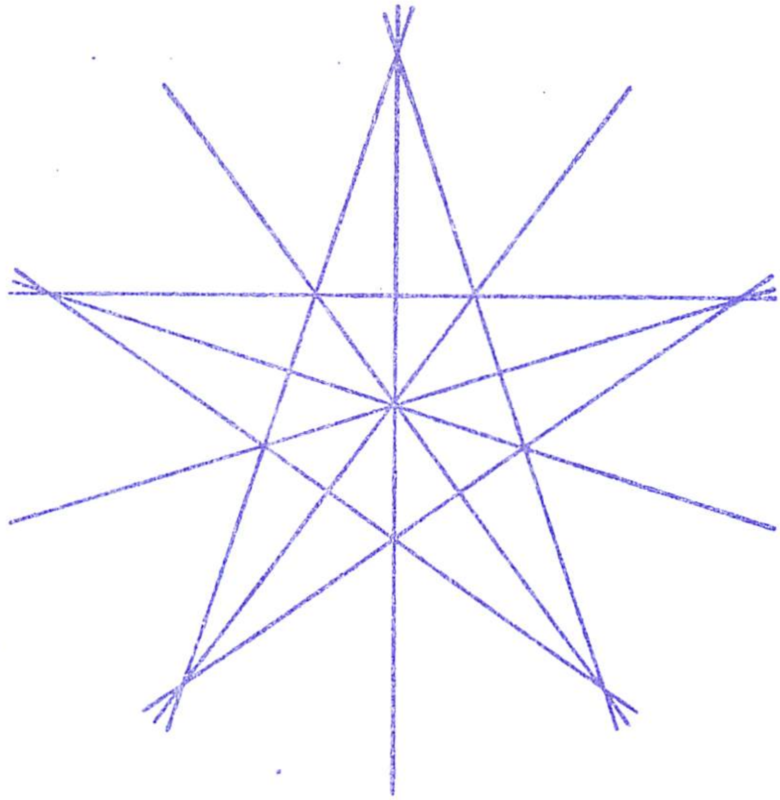
$A_o(8)$



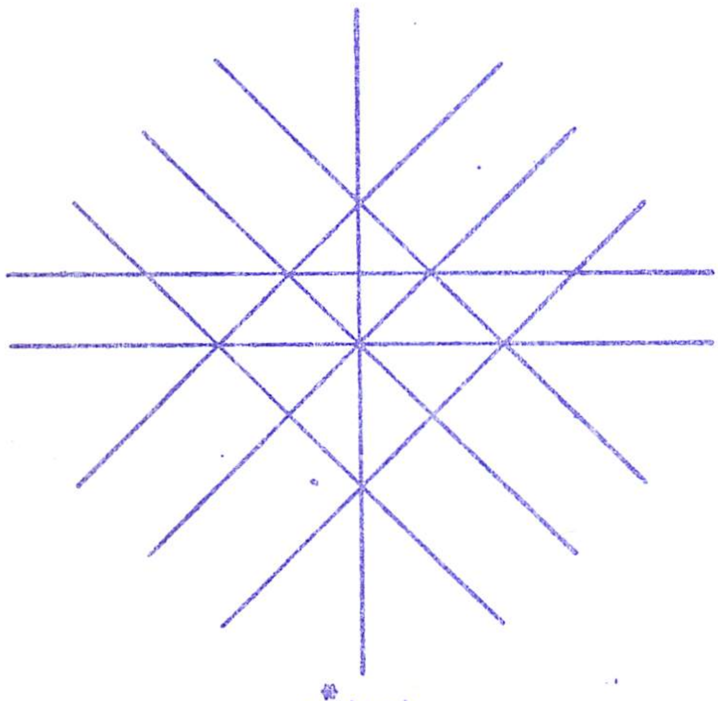
$A_1(8)$, $A_1^*(9)$

Figure 2.6. Simplicial arrangements with at most 15 lines.

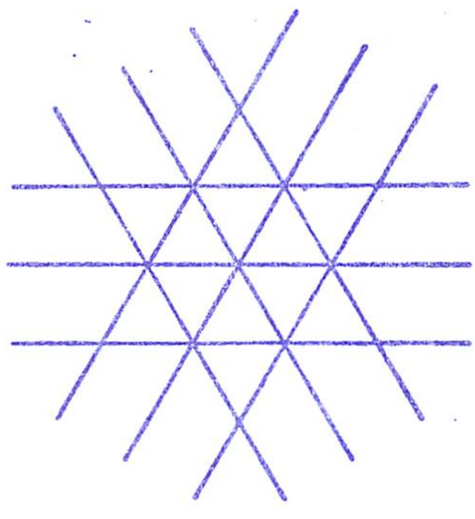
The asterisk * indicates that the line at infinity belongs to the arrangement. The near-pencils $N(n) = A_n(n)$ not shown for $n \geq 9$.



$A_1(10)$

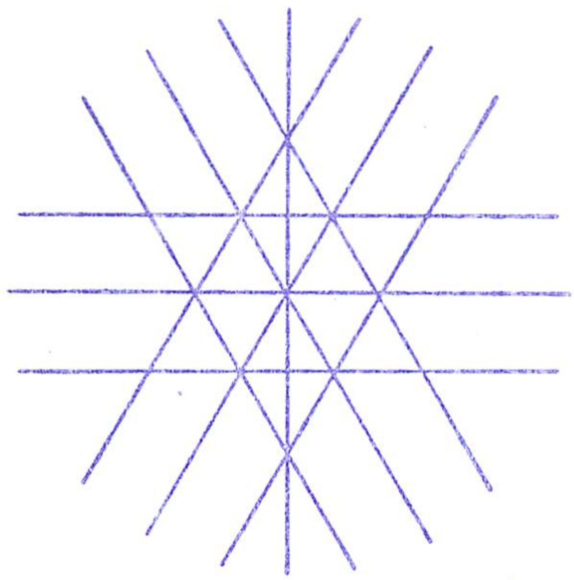


$A_2^*(10)$

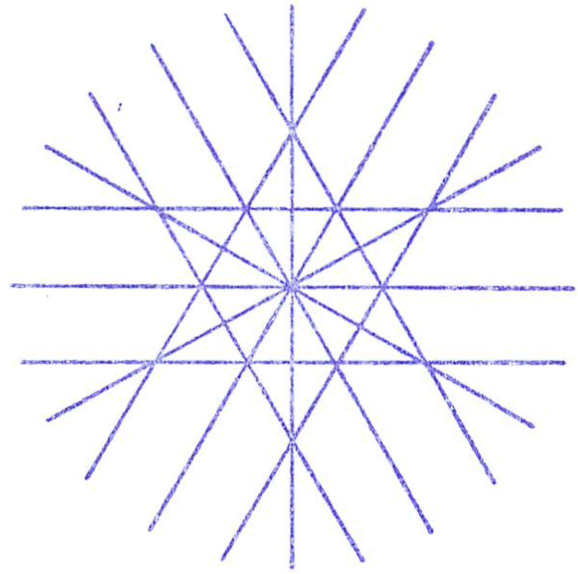


$A_3^*(10)$

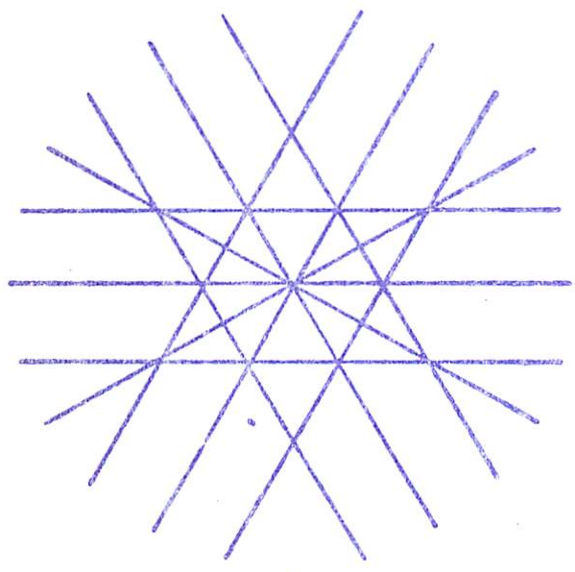
Figure 2.6. (Continued)



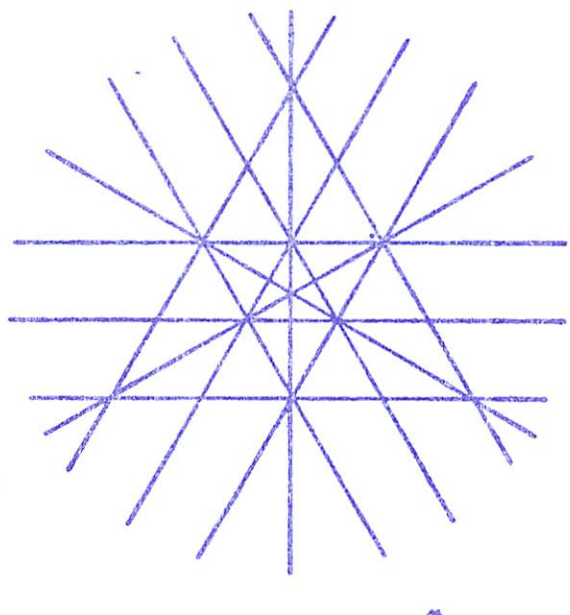
$A_1^*(11)$



$A_1(12)$ $A_1^*(13)$

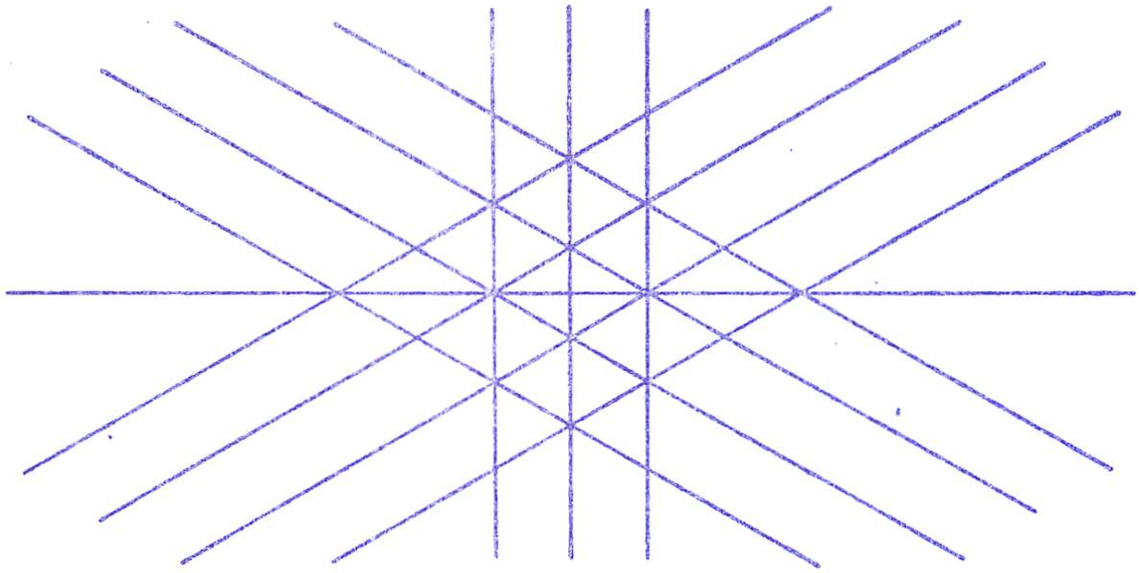


$A_2^*(12)$

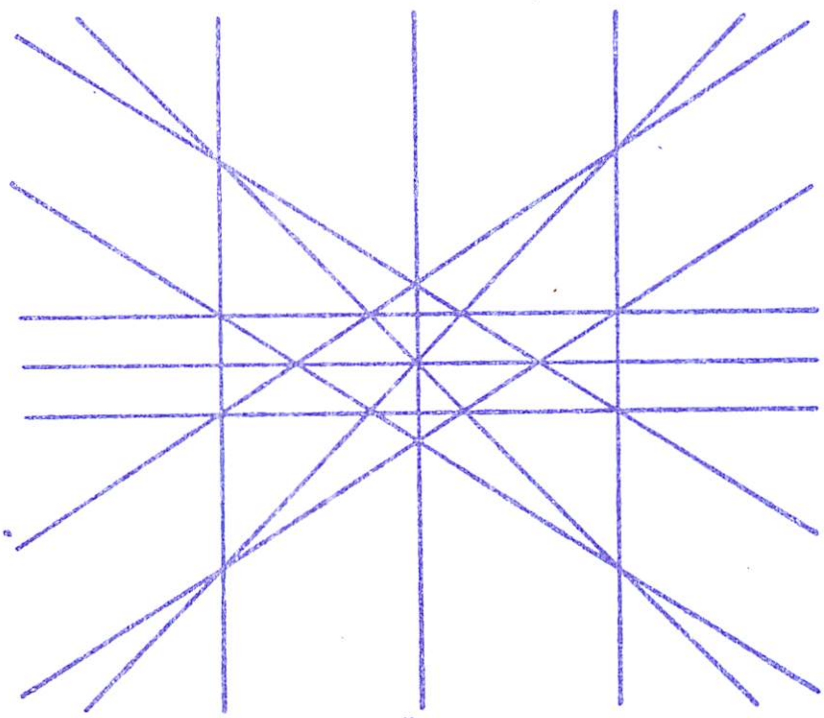


$A_3(12)$ $A_2^*(13)$

Figure 2.6. (Continued)

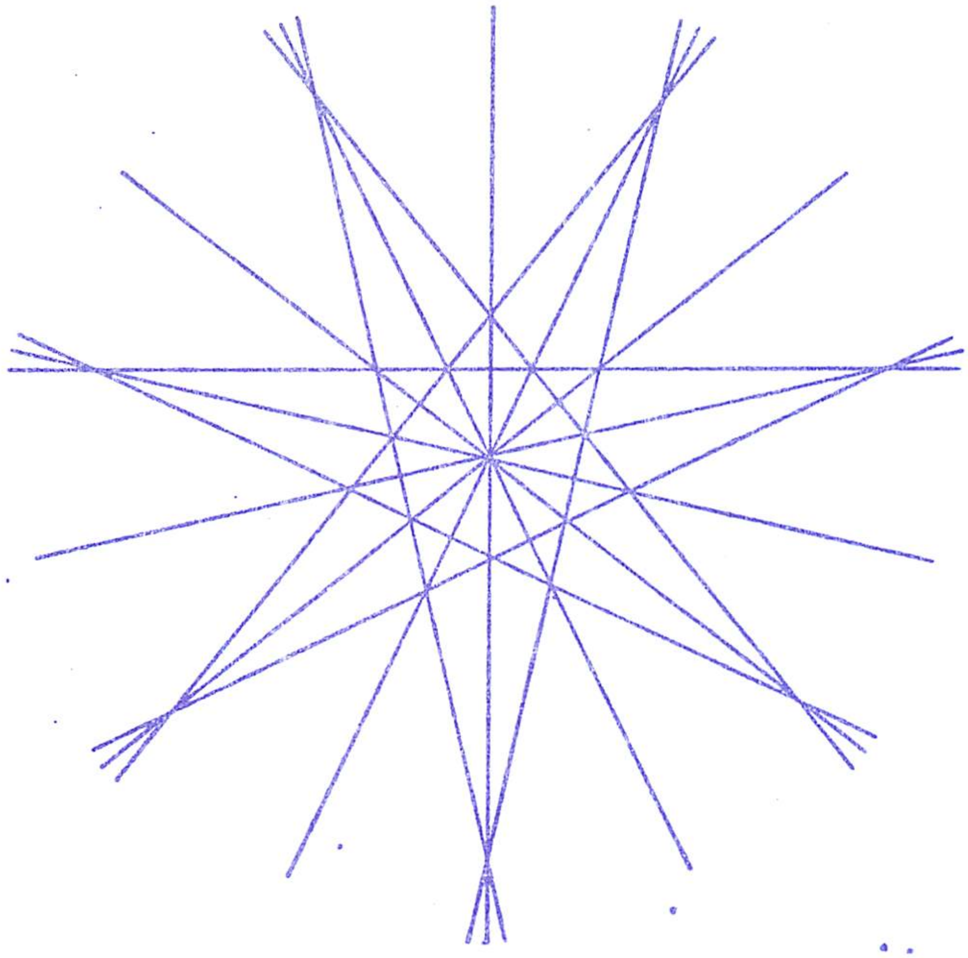


$A_3^*(13)$

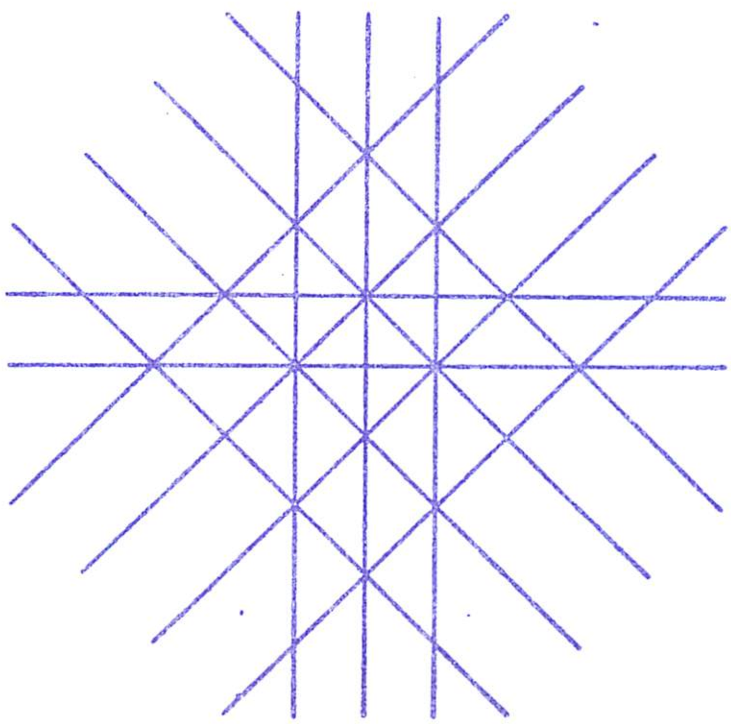


$A_4^*(13)$

Figure 2.6. (Continued)

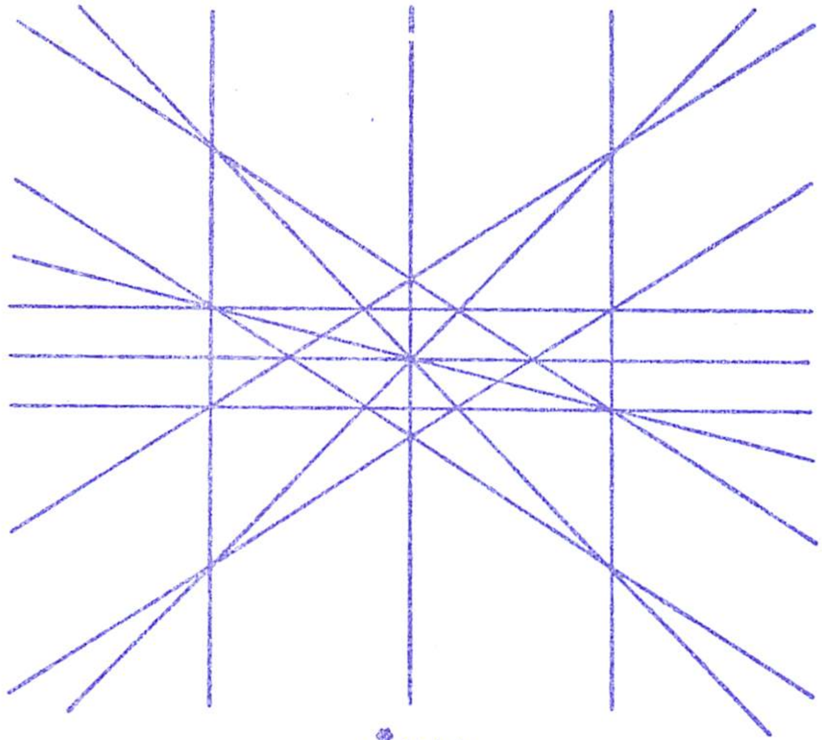


$A_1(14)$

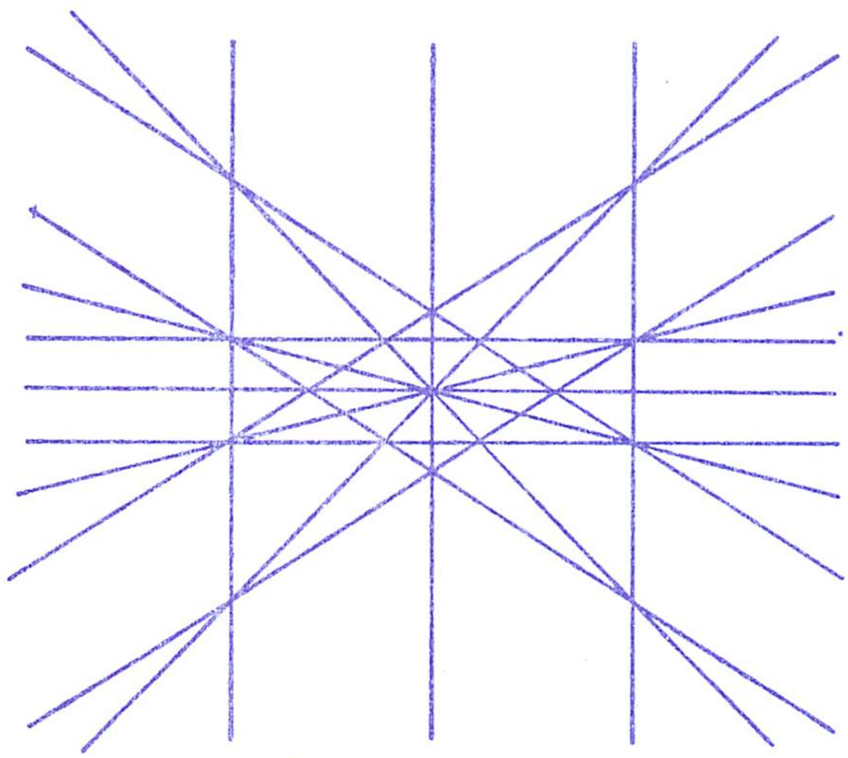


$A_2^*(14)$

Figure 2.6. (Continued)

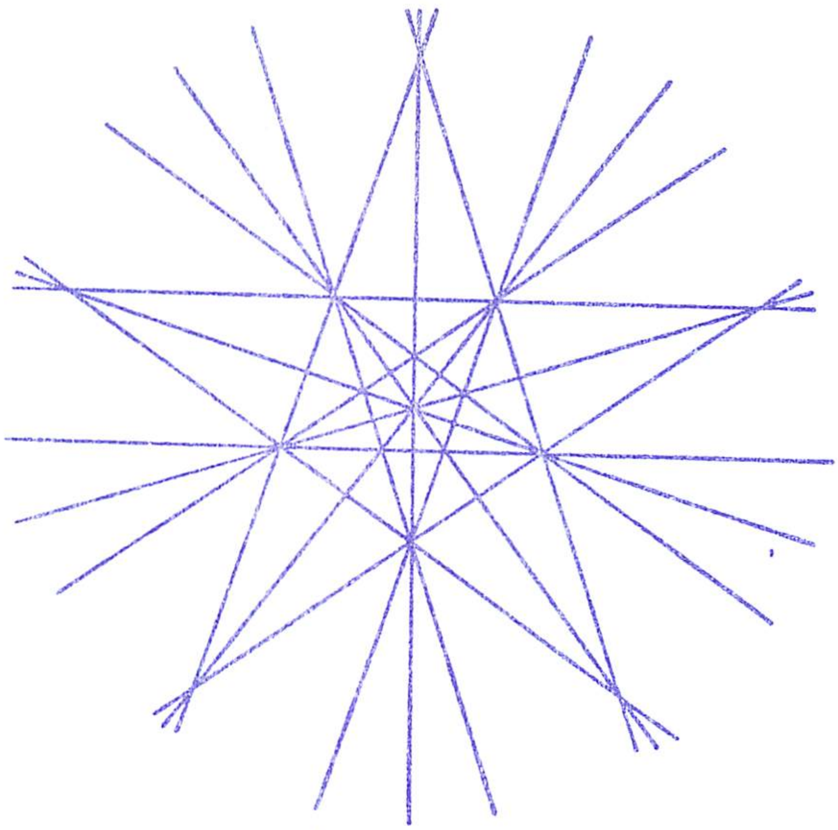


$A_3^*(14)$

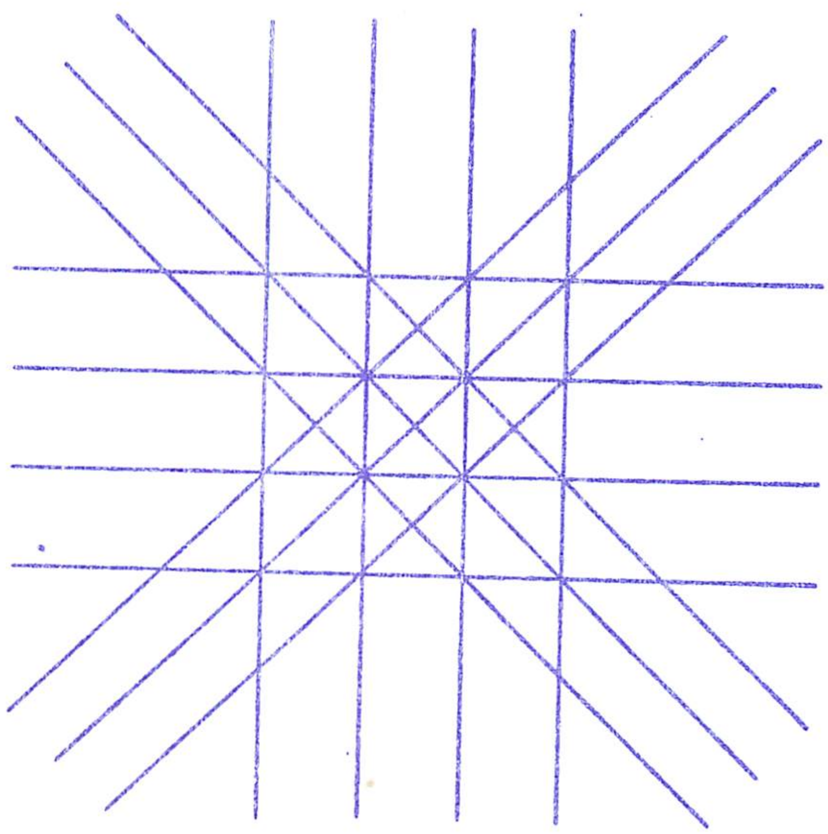


$A_4(14)$ $A_4^*(15)$

Figure 2.6.(Continued)

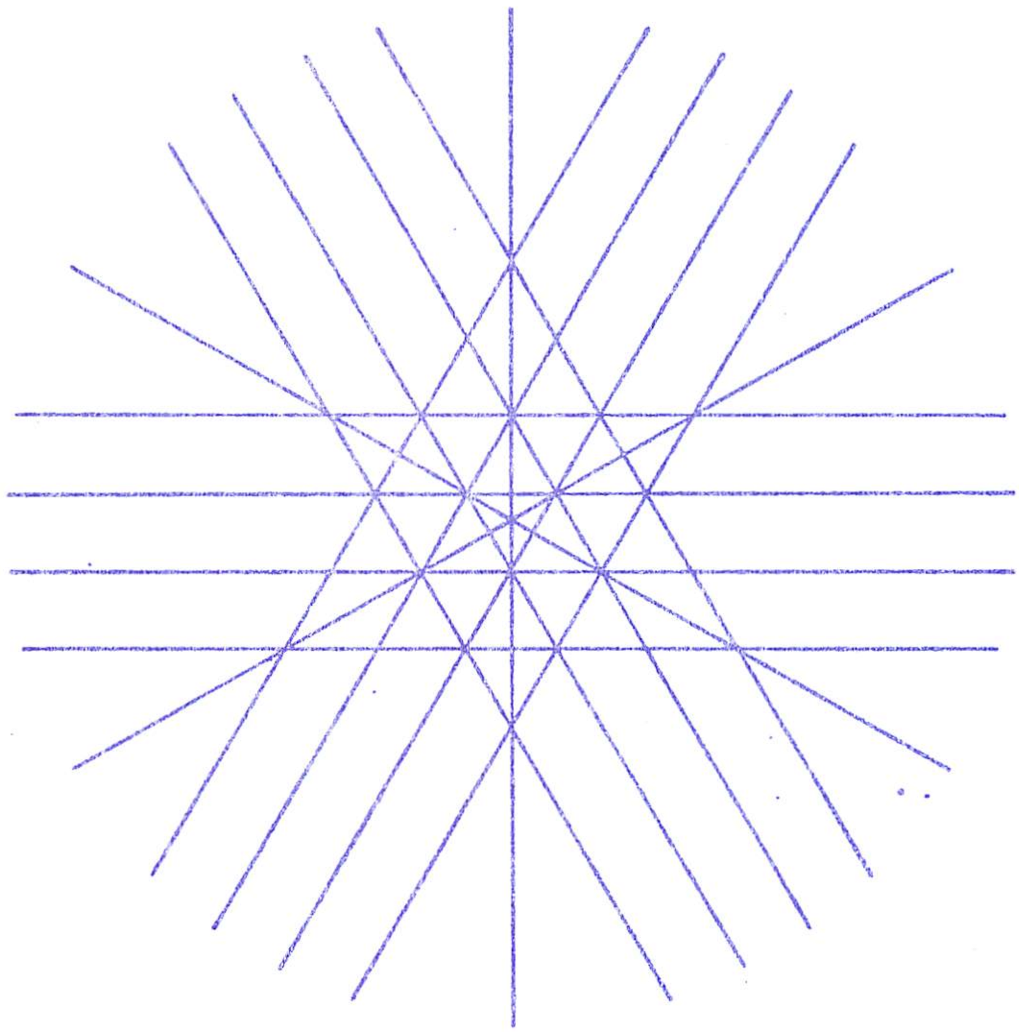


$A_1(15)$



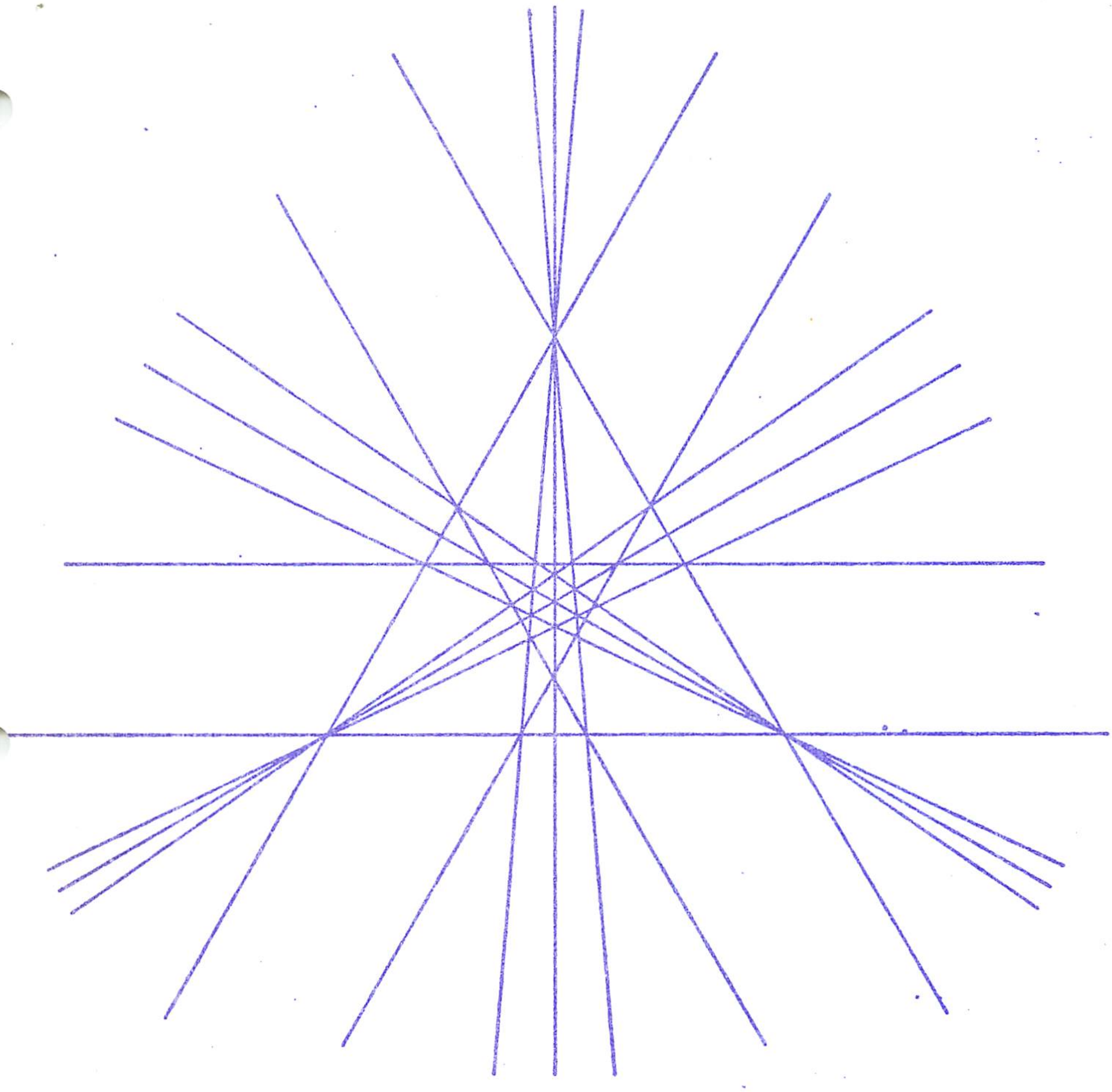
$A_2^*(15)$

Figure 2.6. (Continued)



$A_3(15)$

Figure 2.6. (Continued)



$A_5(15)$

Figure 2.6 (continued).

that maps one arrangement onto the other. An arrangement $\mathcal{A}$ of lines in P^2 is said to be of a <u>projectively unique type</u> if every arrangement isomorphic to $\mathcal{A}$ is projectively equivalent to $\mathcal{A}$.

It is easily checked that all arrangements of at most 4 lines are of projectively unique types, except the trivial arrangement of 4 lines. All simple arrangements of 5 or more lines, and all near-pencils of at least 5 lines, are not of projectively unique types.

<u>Conjecture 2.2</u>. Except for near-pencils with $n \geq 5$, all simplicial arrangements are of projectively unique types.

The validity of Conjecture 2.2 has been established for all known simplicial arrangements.

<u>Conjecture 2.3</u>. Every arrangement of $n \geq 6$ lines that is of a projectively unique type may be obtained from a simplicial arrangement by successively adding lines determined by already present vertices.

* * *

Let isomorphic arrangements of lines $\mathcal{A}(0)$ and $\mathcal{A}(1)$ be called <u>isotopic</u> provided there exist arrangements $\mathcal{A}(t)$ of lines, for $0 \leq t \leq 1$, that depend continuously on t, such that each $\mathcal{A}(t)$ is isomorphic with $\mathcal{A}(0)$. It is clear that isotopy is, formally, a finer equivalence relation than isomorphism. However, we have:

<u>Conjecture 2.4</u>. Every two isomorphic arrangements of lines in P^2 are isotopic.

The validity of Conjecture 2.4 for arrangements of at most 6 lines may be established by an examination of the possible types of arrangements (see Figure 2.1). Similarly, it is not hard (but tedious) to establish its validity for all types of simple arrangements of at most 8 lines, and for all known types of simplicial arrangements.

Exercises.

1. Find an arrangement of lines that exists in a projective plane $P^2(F)$ over an ordered field F if and only if F contains a subfield isomorphic to $Q(2^{\frac{1}{2}})$, the extension of the rationals by $\sqrt{2}$.

2. Prove that $P^2(F)$ contains an isomorphic copy of every arrangement possible in P^2 if and only if F contains a subfield isomorphic to the field A of real algebraic numbers. (Compare Grünbaum [1967, Section 5.5].)

3. Let A again denote the field of all real algebraic numbers. Prove that $P^2(A)$ contains isomorphic copies of all arrangements of lines possible in any $P^2(F)$, where F is an ordered field, Archimedean or not. (For the technique, see Lindström [1971].)

4. It would be interesting to find an arrangement $\mathcal{A}$ of n lines with the property that in order to obtain all the arrangements of $n + 1$ lines that are obtainable from arrangements isomorphic to $\mathcal{A}$ one has to use several different arrangements. (In other words, the example in Figure 2.3 shows that for certain isomorphism classes some representatives are not "suitable" for all extensions; are there isomorphism classes in which no representative is "suitable" for all extensions ?)

5. Can you characterize those types for which every representative is "suitable" for all extensions ? Certainly all arrangements of projectively unique types have this property, but many other arrangements have it as well.

6. Prove that every two simple arrangements of n lines are transformable into each other by a finite sequence of steps, each of which is either an isotopy, or else a "switching" of a triangle (as indicated in Figure 2.7). (Ringel [1957].)

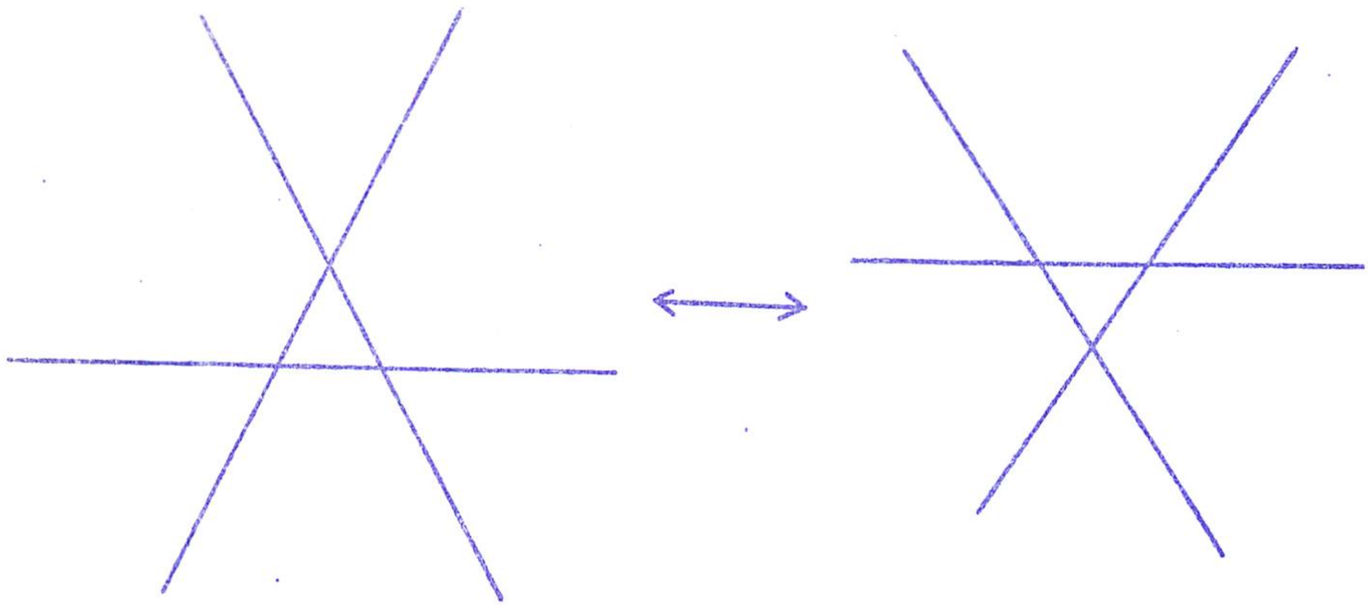


Figure 2.7. The "switching" of a triangle.

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3. d-Arrangements.

The d-dimensional real projective space P^d may be defined and visualized for $d \geq 3$ by analogues of the methods discussed in Setion 1 for P^2. Some familiarity with P^d and its simplest properties will be assumed in the sequel.

A d-arrangement, or arrangement of hyperplanes in P^d, is any finite family $\mathcal{A}$ of (d-1)-dimensional hyperplanes, together with the cell-decomposition of the space P^d determined by the hyperplanes. A d-arrangement $\mathcal{A}$ is trivial if the intersection of all the hyperplanes in $\mathcal{A}$ is non-empty. The number of hyperplanes in an arrangement $\mathcal{A}$ is denoted by $n(\mathcal{A})$.

Two d-arrangements are called isomorphic provided the cell complexes determined by them are isomorphic. The number of k-faces (that is, k-dimensional cells) of an arrangement $\mathcal{A}$ shall be denoted by $f_k(\mathcal{A})$. The d-faces of a d-arrangement are frequently called facets. The f-vector of a d-arrangement $\mathcal{A}$ is $f(\mathcal{A}) = (f_0(\mathcal{A}), f_1(\mathcal{A}),\ldots,f_d(\mathcal{A}))$.

A d-arrangement $\mathcal{A}$ is called simple provided each vertex of $\mathcal{A}$ belongs to precisely d hyperplanes of $\mathcal{A}$. It follows that in each simple d-arrangement every k-face is contained in precisely d-k hyperplanes of $\mathcal{A}$.

A d-arrangement $\mathcal{A}$ is called simplicial provided all faces of $\mathcal{A}$ are simplices of the appropriate dimensions.

In analogy to the definitions in the case of arrangements of lines, we shall denote by $\bar{c}(n,d)$, $c(n,d)$, $c^s(n,d)$ and $c^4(n,d)$ the numbers of distinct isomorphism types of d-arrangements of n hyperplanes, counting either all types, or all non-trivial, or all simple, or all simplicial ones. For $d \geq 3$ very little is known about these numbers; we have:

Theorem 3.1. (i) $c(d,d) = 0$, $\bar{c}(1,1) = 1$.

(ii) $c(n,1) = \bar{c}(n,1) = c^s(n,1) = c^\Delta(n,1) = 1$ for $n \geq 2$.

(iii) $\bar{c}(n,d) = c(n,d) + \bar{c}(n,d-1)$.

(iv) $c(d+1,d) = c^s(d+1,d) = c^\Delta(d+1,d) = 1$.

(v) $c(d+2,d) = d$.

(vi) $c^s(d+1,d) = c^s(d+2,d) = c^s(d+3,d) = 1$.

(vii) $c^s(7,3) = 11$, $c^s(8,4) = 135$.

(viii) $c^\Delta(d+1,d) = c^\Delta(d+2,d) = 1$.

(ix) $c^\Delta(4,1) = c^\Delta(5,2) = 1$, $c^\Delta(d+3,d) = 2$ for $d \geq 3$.

Proof. Parts (i) to (v) and (viii) are trivial, as are the first two assertions in (vi) and in (ix). Part (vii) results from a theorem of McMullen [1971] and theorem 2.2 of the preceding section, although already White [1939] had conjectured that $c^s(7,3) = 11$. McMullen [1971] established also the third part of (vi), and computed $c(d+3,d)$ for all d (though the formula seems to be marred by misprints). We shall prove $c^\Delta(d+3,d) = 2$ for $d \geq 3$ a little later, after describing a method of construction of simplicial arrangements.

If $\mathcal{A}'$ is a d'-arrangement and $\mathcal{A}''$ is a d"-arrangement, the join $\mathcal{A} = \mathcal{A}' \vee \mathcal{A}''$ is a d-arrangement (where $d = d'+d''+1$) defined as follows: In P^d we take skew subspaces $P^{d'}$ and $P^{d''}$, that contain $\mathcal{A}'$ and $\mathcal{A}''$. The (d-1)-hyperplanes that form $\mathcal{A}$ are precisely those spanned by the (d'-1)-hyperplanes of $\mathcal{A}'$ with $P^{d''}$, and by the (d"-1)-hyperplanes of $\mathcal{A}''$ with $P^{d'}$. It may easily be verified that $n(\mathcal{A}) = n(\mathcal{A}') + n(\mathcal{A}'')$, and

$$f_k(\mathcal{A}) = f_k(\mathcal{A}') + f_k(\mathcal{A}'') + 2 \sum_{\substack{i+j+1=k \\ i,j \geq 0}} f_i(\mathcal{A}')f_j(\mathcal{A}'') \quad \text{for} \quad 0 \leq k \leq d ,$$

and that $\mathcal{A}' \vee \mathcal{A}''$ is simplicial if and only if $\mathcal{A}'$ and $\mathcal{A}''$ are simplicial. Moreover, the join operation is associative and commutative.

The join construction applies even in case $d'' = 0$, when the (unique) 0-arrangement $A^0(0)$ has $n(A^0(0)) = 0$, $f_0(A^0(0)) = 1$, and is simplicial.

We shall use the symbol $A^d(n)$ to denote a simplicial d-arrangement of n hyperplanes; subscripts will be used to distinguish between non-isomorphic arrangements with the same d and n. Thus, the 1-arrangement $A^1(n)$ is determined by n points in P^1 and satisfies $n(A^1(n)) = f_0(A^1(n)) = f_1(A^1(n)) = n$. The 2-arrangement $A_0^2(n+1) = A^1(n) \vee A^0(0)$ is easily seen to be the near-pencil which in Section 2 was denoted by $A_0(n+1)$.

It is easy to verify by induction on d that for $d \geq 2$ the only type of simplicial d-arrangement with d+2 hyperplanes is $A^d(d+2) = A^1(3) \vee \underbrace{A^0(0) \vee \dots \vee A^0(0)}_{d-1 \text{ terms}}$, while $A^d(d+1) = \underbrace{A^0(0) \vee \dots \vee A^0(0)}_{d+1 \text{ terms}}$.

For each $d \geq 3$ we have the following two types of simplicial d-arrangements of d+3 hyperplanes:

$$A_1^d(d+3) = A^1(4) \vee \underbrace{A^0(0) \vee \dots \vee A^0(0)}_{d-2 \text{ terms}}$$

and

$$A_2^d(d+3) = A^1(3) \vee A^1(3) \vee \underbrace{A^0(0) \vee \dots \vee A^0(0)}_{d-3 \text{ terms}} .$$

It is easily verified that $A_1^d(d+3)$ and $A_2^d(d+3)$ are not isomorphic.

To see that those are the only possible types of simplicial d-arrangements with d+3 hyperplanes we use induction on d.

For $d = 3$, a given simplicial 3-arrangement $A^3(6)$ either contains a vertex that belongs to 5 planes, or not. In the former case $A^3(6)$ is the join of $A^0(0)$ with the only type $A^2(5) = A^1(4) \vee A^0(0)$ so $A^3(6)$ is $A_1^3(6)$. In the other case the simplicial 2-arrangement induced in each plane must be $A^2(4)$, one line of which is the trace of two planes of $A^3(6)$; since that line can not pass through the vertex determined by two other lines, it follows that the six planes of $A^3(6)$ form two pencils of 3 planes each, so that $A^3(6)$ is $A^1(3) \vee A^1(3)$

For $d \geq 4$, any given $A^d(d+3)$ must contain a vertex that belongs to $d+2$ hyperplanes. Indeed, if this were not so, then the $(d-1)$-arrangements induced in each of the hyperplanes of $A^d(d+3)$ by the other hyperplanes would have to be either of type $A^{d-1}(d)$ or of type $A^{d-1}(d+1)$, with some $(d-2)$-flat the trace of (at least) two hyperplanes of $A^d(d+3)$. In the first case some vertex of $A^{d-1}(d)$ would be incident with all but one hyperplane of $A^d(d+3)$, while in the second case the same conclusion results since because of the simpliciality of $A^d(d+3)$ the "double" $(d-2)$-flat can not be the exceptional flat of the near-pencil $A^{d-1}(d+1)$. But if some vertex of $A^3(d+3)$ belongs to $d+2$ hyperplanes, then $A^d(d+3) = A^0(0)$ $A^{d-1}(d+2)$, and the assertion (ix) of theorem 3.1 follows.

The last part of Theorem 3.1 may probably be generalized as follows:

<u>Conjecture 3.1</u>. For each $m \geq 1$ there exists a constant γ_m such that $c^{\Delta}(d+m,d) \leq \gamma_m$ for all d.

It is even possible that one may take $\gamma_m = c^{\Delta}(2m,m)$.

* * *

Besides the (extensive) collection of simplicial 2-arrangements discussed in Section 2 (and in the catalogue of Grünbaum [1971]), and the "trivial" simplicial arrangements $A^0(0)$ and $A^1(n)$ for $n \geq 1$, we have seen just one method of generating simplicial arrangements: the joining of lower-dimensional simplicial arrangements. Two other methods are known, and we shall now describe them.

The first of those methods is a generalization to higher dimensions of the formation (as described on p. 2.5) of the 2-arrangements $A_1^2(2k)$ and $A_1^2(4m+1)$. Starting from a regular polytope C in E^d (see Coxeter [1948] for material on regular polytopes), a simplicial d-arrangement is obtained by taking the hyperplanes generated by the facets of C together with (all, or a suitable subset of) the hyperplanes invariant under some of the symmetries of C, and with the possible inclusion of the hyperplane at infinity. However, as we shall see below, the indication in Grünbaum [1971] that the inclusion of all the hyperplanes of symmetry always leads to a simplicial arrangement is not true.

We shall first describe in some detail the application of this method to the cases of d-simplices and d-cubes.

We shall denote by $A^{d,\Delta}$ the d-arrangement of $\binom{d+2}{2}$ hyperplanes obtained in the following manner: $d+1$ hyperplanes are determined by the facets of a regular d-simplex $T^d \subset E^d$, while $\binom{d+1}{2}$ hyperplanes are hyperplanes of symmetry of T^d, each determined by $d-1$ vertices of T^d and the midpoint of the edge connecting the remaining two vertices. Thus $A^{2,\Delta}$ is the arrangement denoted by $A_1(6)$ in Figure 2.6.

<u>Theorem 3.2</u>. The d-arrangement $A^{d,\Delta}$ is simplicial for each $d \geq 2$.

<u>Proof</u>. Since the assertion is obviously true for $d = 2$, we may proceed inductively as follows. Each of the (d-1)-arrangements induced in the hyperplanes of $A^{d,\Delta}$ by the other hyperplanes of $A^{d,\Delta}$ is isomorphic to $A^{d-1,\Delta}$, and is therefore simplicial. Each facet of $A^{d,\Delta}$ that is contained in T^d is a simplex having one vertex at the centroid of T^d, while its remaining vertices are just the vertices of one of the facets of $A^{d-1,\Delta}$. Each of the facets of

$A^{d,\Delta}$ outside T^d is the convex hull that crosses the hyperplane at infinity of a j_1-simplex and a j_2-simplex induced by $A^{d,\Delta}$ on the faces of dimensions j_1 and j_2 of T^d, where $j_1+j_2+1 = d$. It is not hard to see that $f_d(A^{d,\Delta}) = (d+2)!/2$, and $f_{d-1}(A^{d,\Delta}) = (d+2)!(d+1)/4$. (Note that the value of f_{d-1} can be computed in two ways: By observing that it is $\binom{d+2}{2} f_{d-1}(A^{d-1,\Delta})$, and also by noting that $2f_{d-1} = (d+1)f_d$; the agreement confirms the simplicial character of $A^{d,\Delta}$.)

For $0 \le k \le d+1$ we shall denote by $A^{d,\Box;k}$ the d-arrangement of $2d + 2\binom{d}{2} + k$ hyperplanes obtained in the following way: Starting from the d-cube $C^d = \{(x_1,\dots,x_d) \in E^d \mid |x_i| \le 1 \text{ for } 1 \le i \le d\}$ we take the $2d$ hyperplanes determined by the facets of C^d together with the $2\binom{d}{2}$ hyperplanes of symmetry of C^d determined by equations of the type $x_i = \pm x_j$ for $1 \le i < j \le d$, and with the k midplanes of C^d, each of which is determined by one of the equations $x_i = 0$, for $i = 1,2,\dots,k$ if $k \le d$, and with all those and the hyperplane at infinity if $k = d+1$.

<u>Theorem 3.3</u>. The d-arrangement $A^{d,\Box;k}$ is simplicial for all $d \ge 1$ and for all k, $0 \le k \le d+1$.

The proof can be accomplished in analogy to the proof of theorem 3.2, observing that the arrangements induced in the hyperplanes of $A^{d,\Box;k}$ are all of types $A^{d-1,\Box;j}$ for various values of j. For data on arrangements $A^{3,\Box;k}$ see the Appendix.

It is not known what other simplicial d-arrangements may be obtained by systematic procedures from regular simplices and cubes, utilizing their various symmetries. Information on the known possibilities of this kind for $d = 3$ is collected in the Appendix.

Similarly, it is not known what simplicial arrangements may be derived from the d-crosspolytopes. Again, the results available in the case of the octahedron (that is, d=3) are presented in the Appendix.

Applied to the regular dodecahedron, the method leads to at least two simplicial 3-arrangements: one with 27 planes (12 planes determined by the faces of the dodecahedron, and 15 planes of mirror-symmetry of the dodecahedron) and one with 28 planes (obtained from the previous one by adjoining the plane at infinity). For details see the Appendix.

In contrast to that we have:

<u>Theorem 3.4</u>. The 3-arrangements $\mathcal{A}^3(35)$ and $\mathcal{A}^3(36)$ of 35 or 36 planes, formed by the 20 planes spanned by the facets of a regular icosahedron and its 15 planes of mirror-symmetry, with the plane at infinity in case of $\mathcal{A}^3(36)$, are not simplicial.

<u>Proof</u>. The 2-arrangements induced on each of the planes of $\mathcal{A}^3(35)$ or $\mathcal{A}^3(36)$ by the other planes are easily seen to be the following ones; since in b) non-simplicial 2-arrangements are obtained, the theorem is established.

a) In each of the facet-planes, the arrangement obtained is as shown in Figure 3.1; it is a simplicial arrangement of 22 lines, denoted $A_3(22)$ in the catalogue of Grünbaum [1971].

b) In each of the planes of symmetry, the induced 2-arrangement is the non-simplicial one shown in Figure 3.2.

c) In case of $\mathcal{A}^3(36)$, the arrangement induced in the plane at infinity is shown in Figure 3.3; it is the simplicial arrangement of 25 lines denoted $A_3(25)$ in Grünbaum [1971].

The second method for generating simplicial d-arrangements starts from (all, or some suitable subsets of) the d-hyperplanes

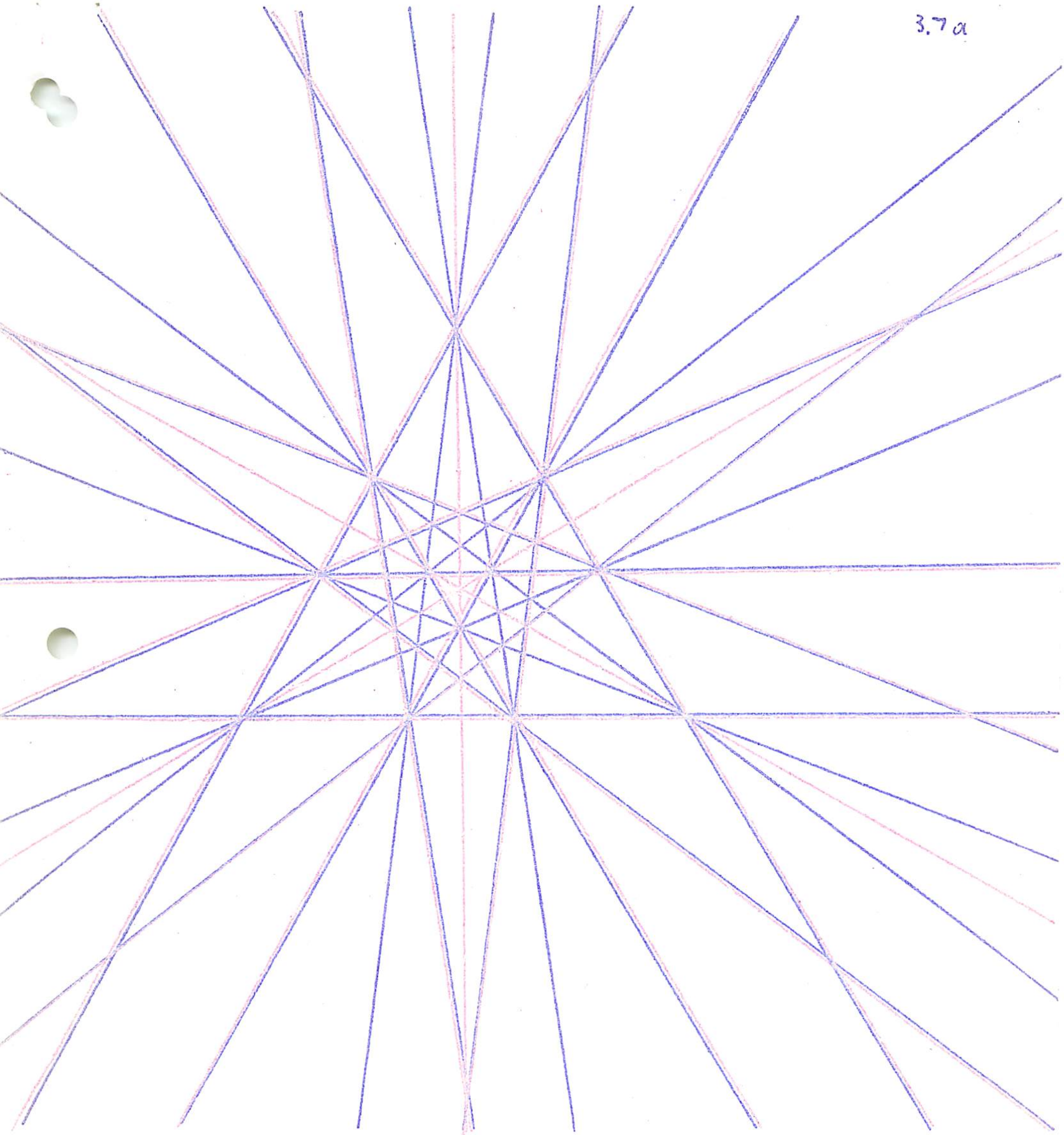


Figure 3.1. The simplicial 2-arrangement $A_3^*(22)$ induced in each facet-plane of the regular icosahedron by the other facet-planes (purple lines and line at infinity) and the planes of symmetry (red lines). Closely spaced lines actually coincide.

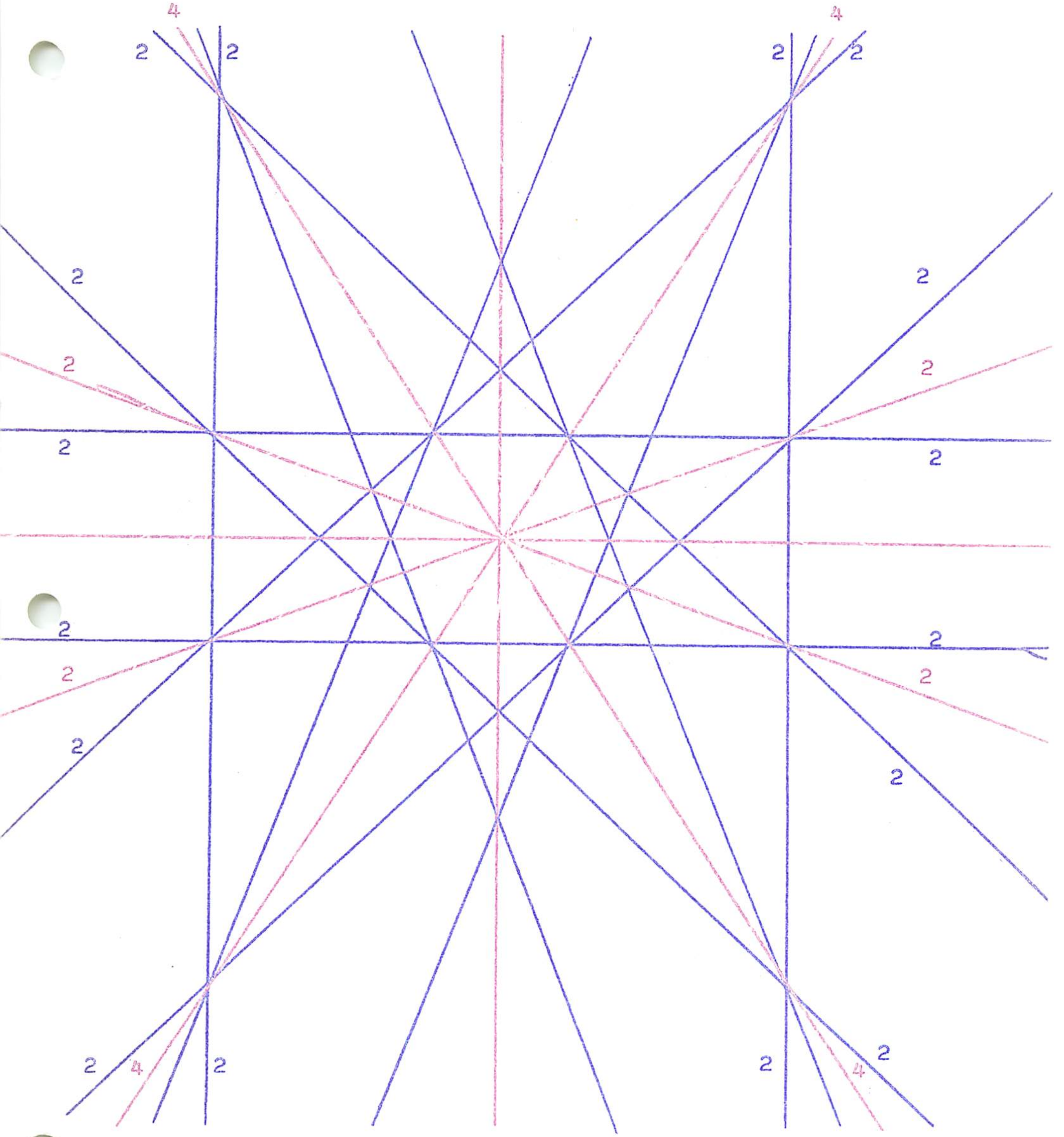


Figure 3.2. The 2-arrangement of 18 lines induced in each symmetry plane of the regular icosahedron by the 20 facet planes (purple lines) and the other 14 symmetry planes. The numbers indicate how many of the planes have coinciding traces. In case of the 3-arrangement $\mathcal{A}^3(36)$, the line at infinity should be adjoined to yield an $\mathcal{A}^2(19)$.

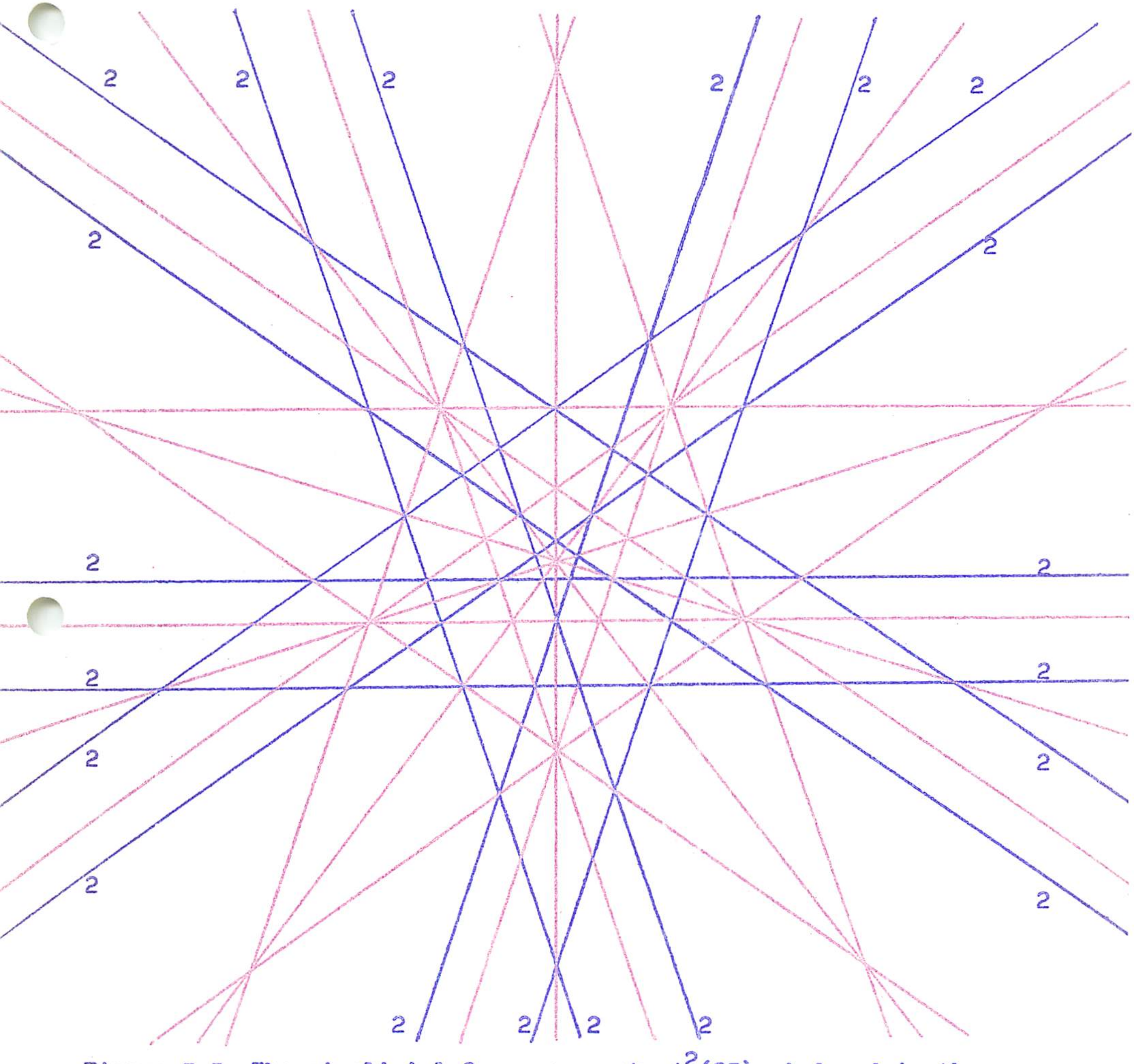


Figure 3.3. The simplicial 2-arrangement $A_3^2(25)$ induced in the plane at infinity by the 20 facet-planes of the regular icosahedron (purple lines) and the 15 planes of symmetry (red lines). The numbers indicate how many planes have coinciding traces.

naturally associated with symmetries of a regular (d+1)-polytope. The d-arrangement is obtainable by intersecting this family of d-hyperplanes with the hyperplane at infinity of P^{d+1}. It is easily checked that if all the hyperplanes of mirror-symmetry of a regular (d+1)-polytope are taken, the resulting d-arrangement is simplicial. However, simplicial arrangements arise from many other sets of symmetry hyperplanes as well.

The simplicial 2-arrangements obtainable by this method from the tetrahedron, the cube (or octahedron), and the dodecahedron (or icosahedron) are shown in Figures 3.4, 3.5, and 3.6.

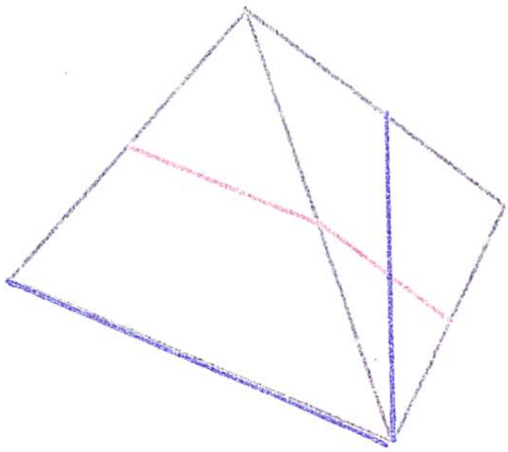
For each $d \geq 2$, the d-arrangement obtained by intersecting with the hyperplane at infinity all the hyperplanes of mirror symmetry of the (d+1)-simplex T^{d+1} is isomorphic with $A^{d,\Delta}$.

The 12 3-flats bisecting the main body diagonals of the 24-cell in E^4 intersect the hyperplane at infinity in a 3-arrangement isomorphic to the one designated $A_1^3(12)$ in the Appendix; an isomorphic 3-arrangement results from the intersection of the hyperplane at infinity by the 12 3-flats of mirror-symmetry of the 24-cell that are parallel to a pair of facets of the 24-cell. The two sets of 12 3-flats taken together induce in the hyperplane at infinity the arrangement $A_1^3(24)$.

Actually, there are two other, rather obvious, ways of obtaining simplicial arrangements from known ones:

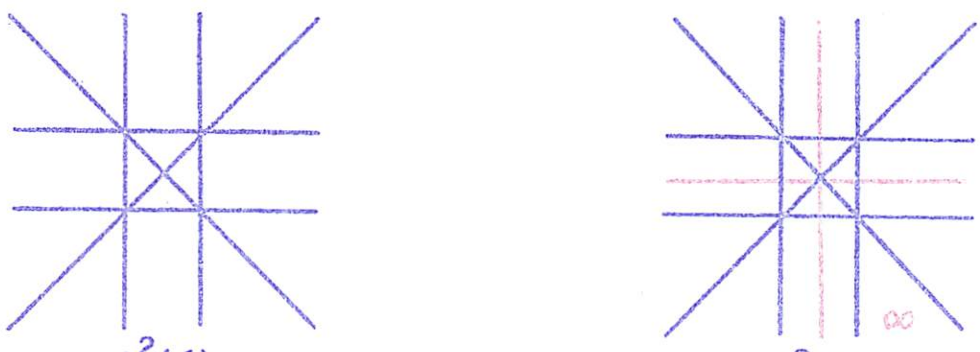
(i) The (d-1)-arrangement induced in one hyperplane of a simplicial d-arrangement by the other hyperplanes is itself simplicial;

(ii) In many cases a simplicial d-arrangement may be obtained



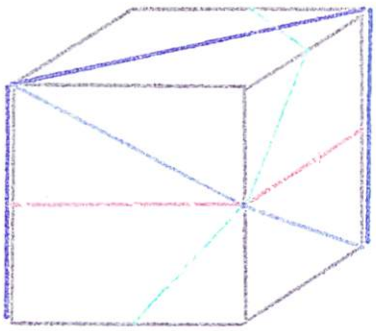
Purple: 6 planes of mirror-symmetry.

Red: 3 midplanes between skew edges.



$A_1^2(6)$ $A_1^2(9)$

Figure 3.4. Simplicial 2-arrangements obtained by intersecting the hyperplane at infinity with symmetry planes of the regular tetrahedron. ∞ indicates the line at infinity.

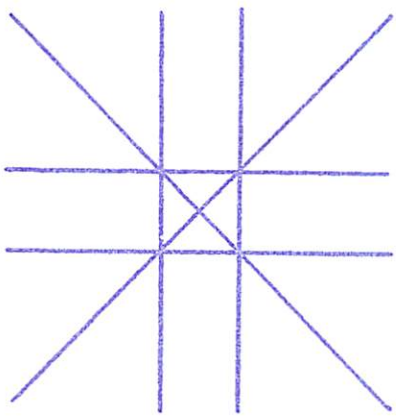


Purple: 6 planes of mirror-symmetry

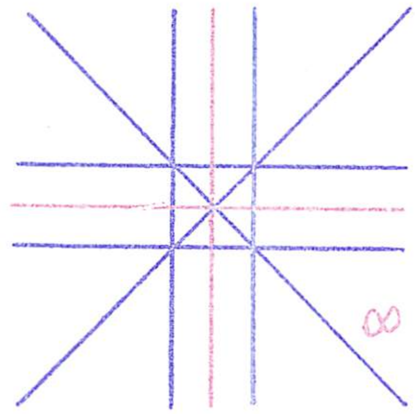
Red: 3 midplanes (mirror symmetry)

Green: 4 bisectors of body diagonals

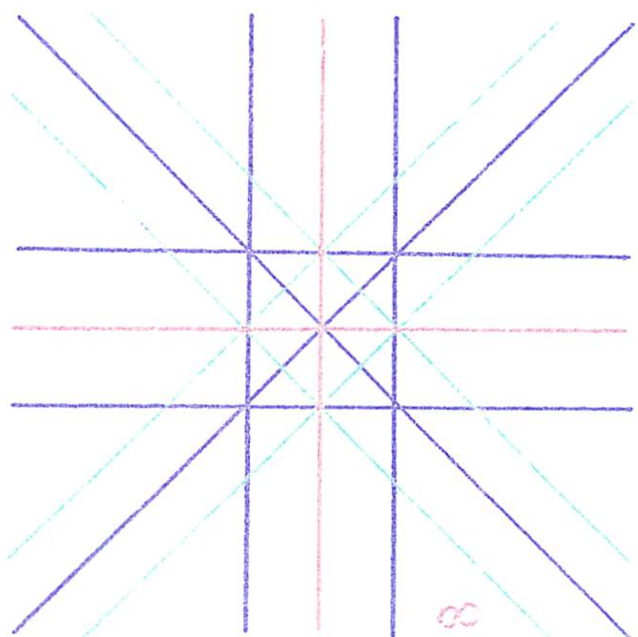
Blue: 12 bisectors.



$A_1^2(6)$

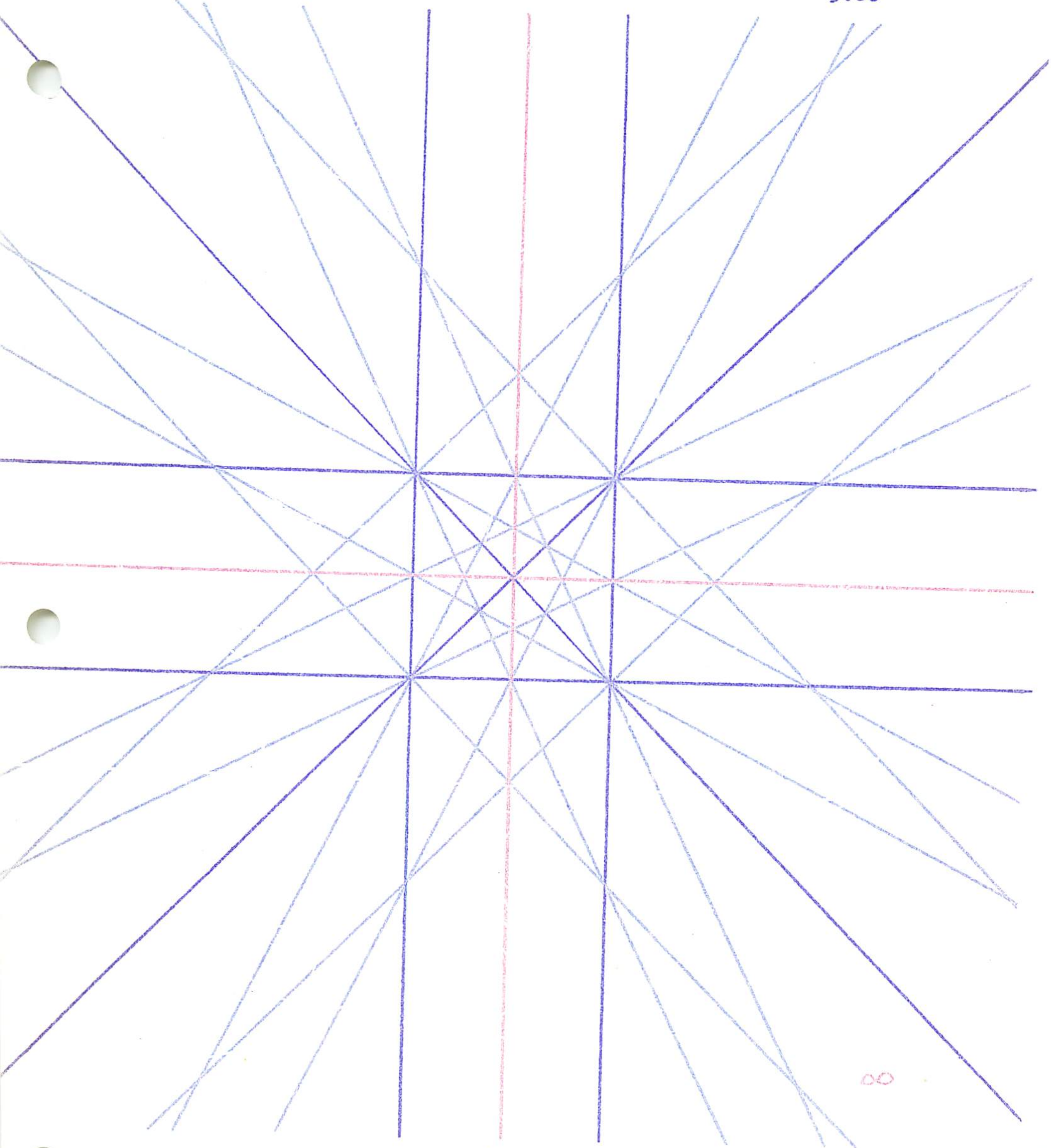


$A_1^2(9)$



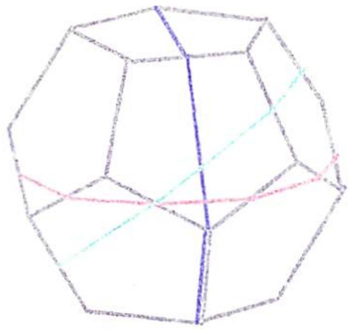
$A_2^2(13)$

Figure 3.5. (first part). Simplicial 2-arrangements obtained by intersecting the hyperplane at infinity with symmetry planes of the regular cube.



$A_3^2(21)$

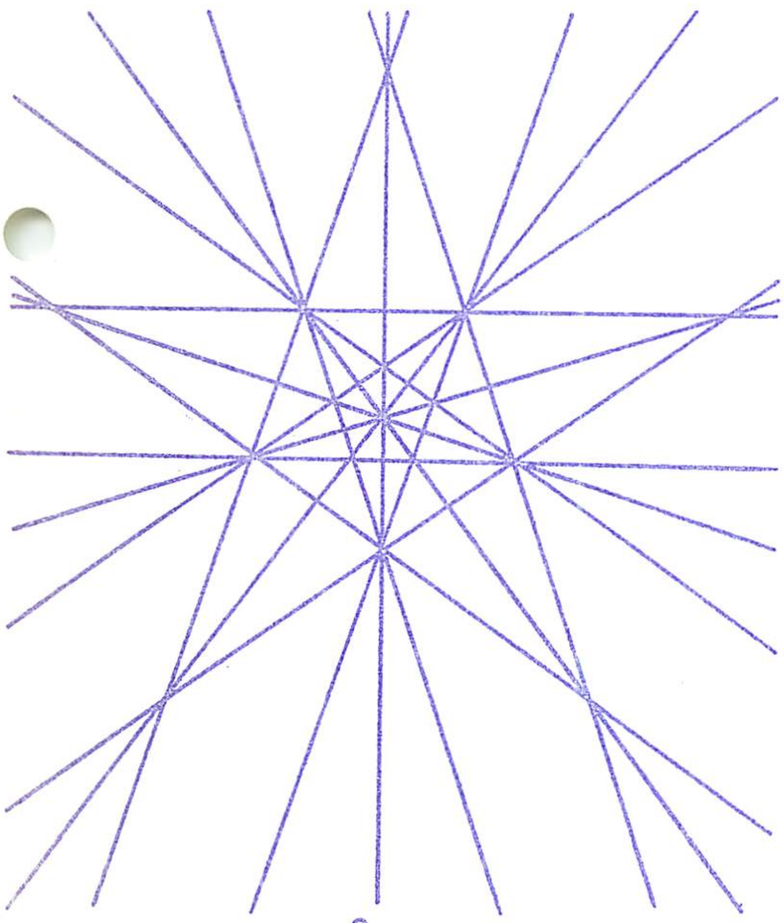
Figure 3.5 (second part).



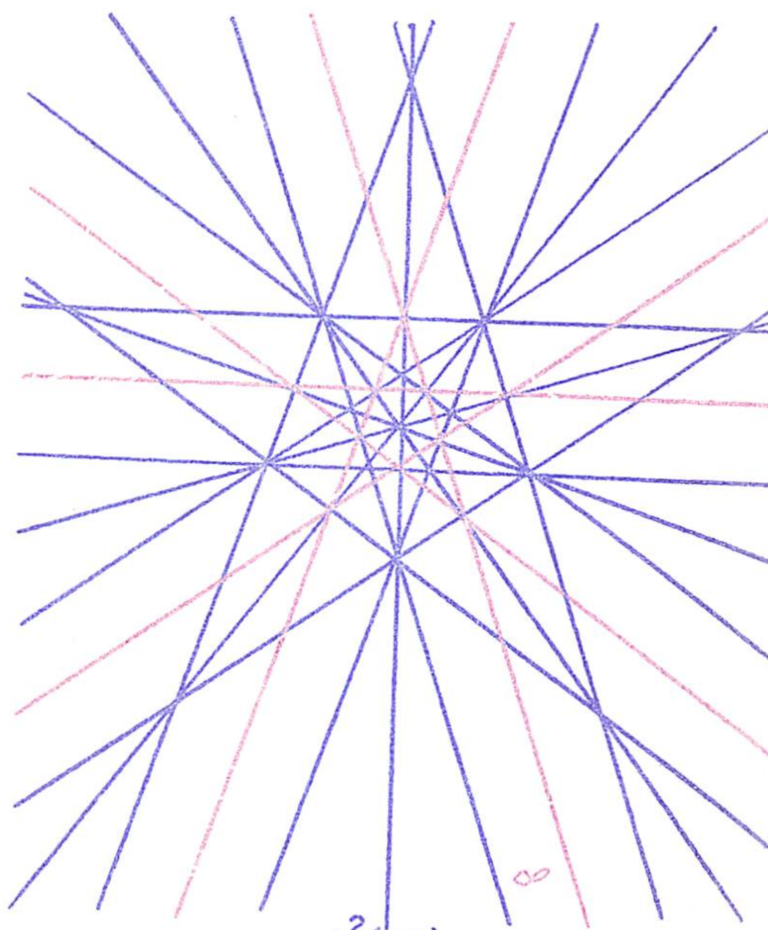
Purple: 15 planes of mirror-symmetry

Red: 6 midplanes of parallel facets

Green: 10 bisectors of main diagonals.

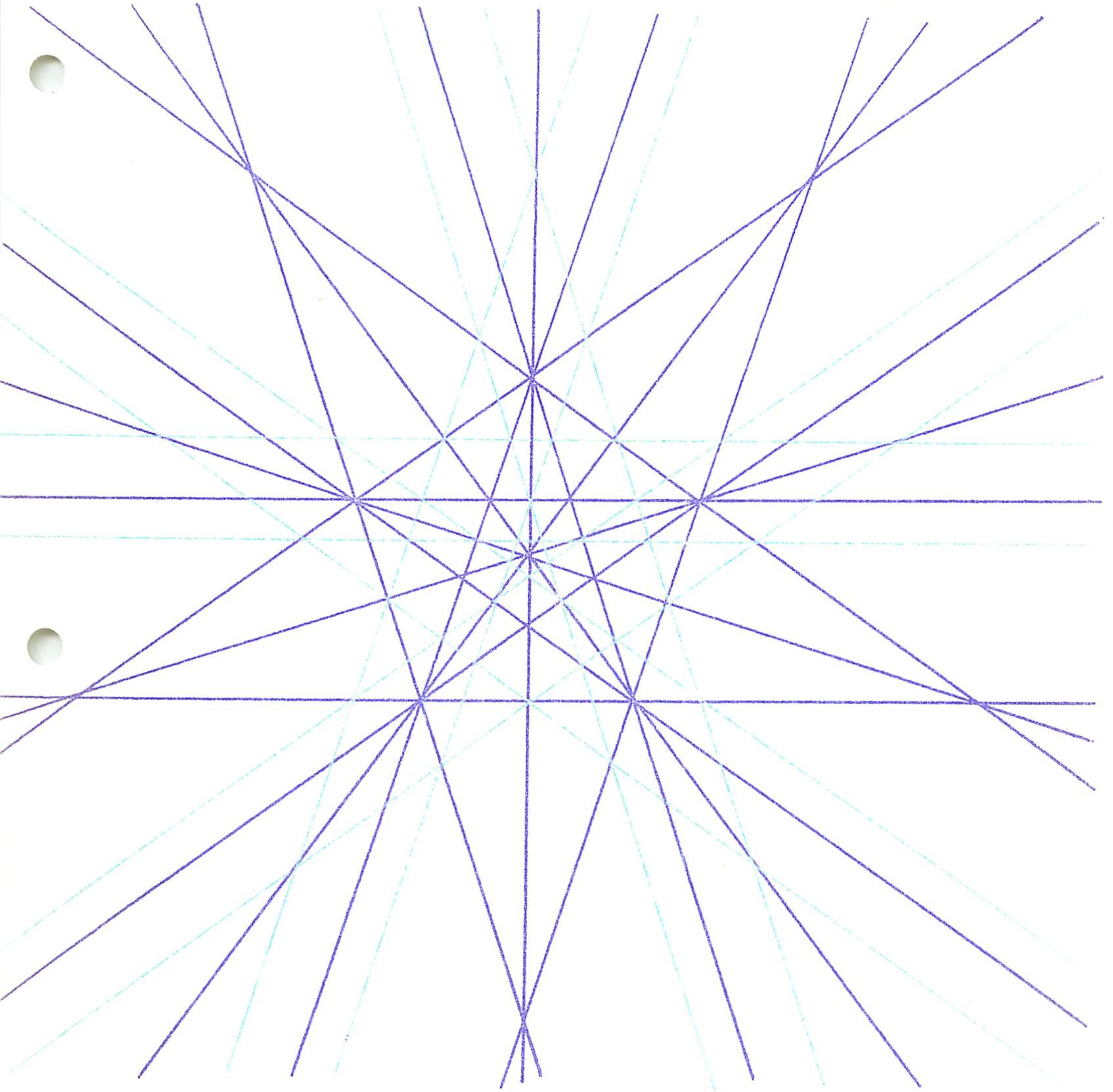


$A_1^2(15)$



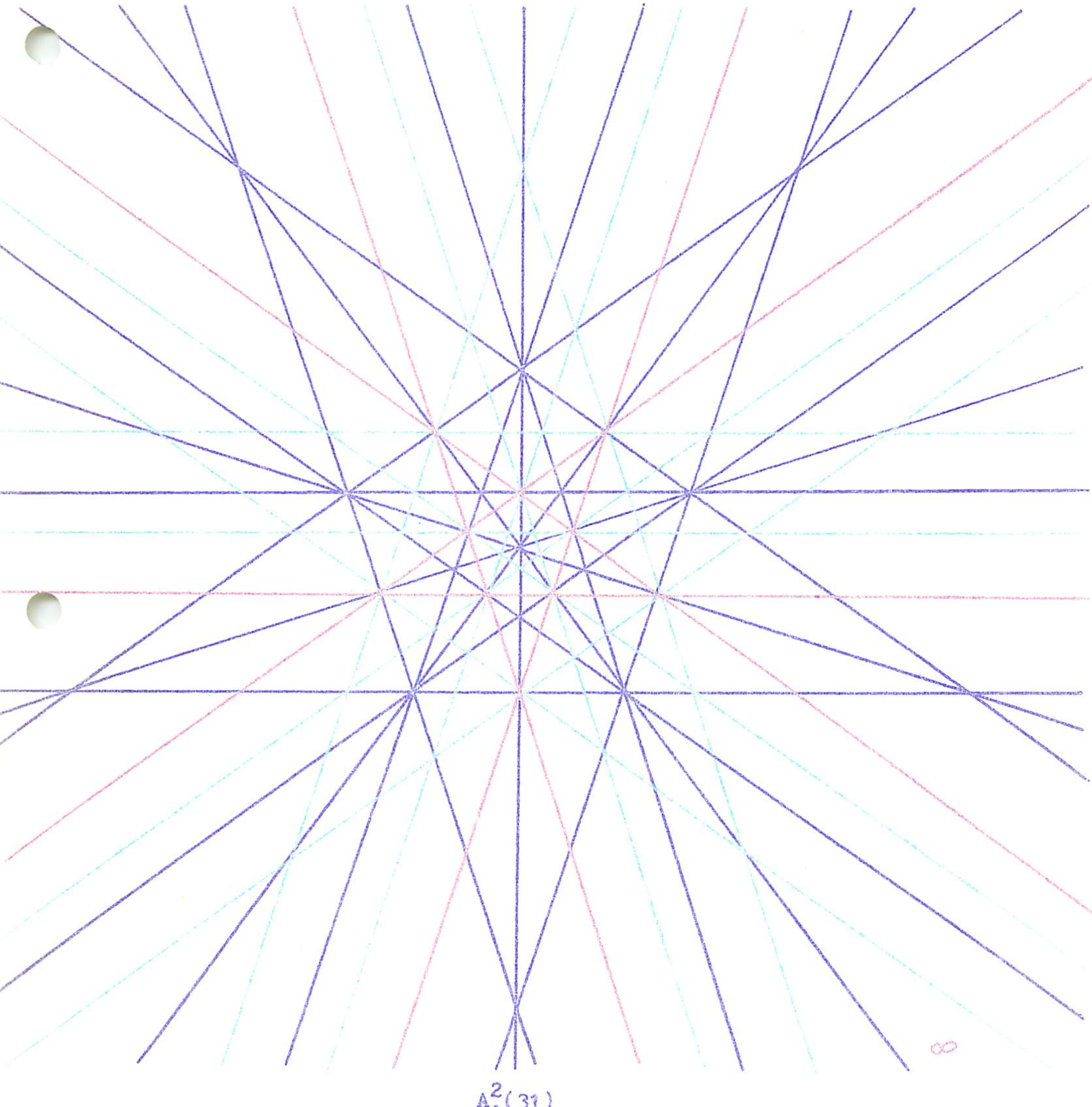
$A_2^2(21)$

Figure 3.6 (first part). Simplicial 2-arrangements obtained by intersecting the hyperplane at infinity with symmetry planes of the regular dodecahedron.



$A_3^2(25)$

Figure 3.6 (second part).



$A_1^2(31)$

Figure 3.6 (third part).

from another d-arrangement by deleting one or several of its hyperplanes.

A large part of the known simplicial 2-arrangements may be obtained by combinations of the methods discussed above, although there appear several "unsystematic" ones that do not seem obtainable in this way.

Exercises.

1. Show that an arrangement isomorphic to $A^{d,\Delta}$ is generated by all the hyperplanes determined by any $d+2$ points in "general position" in P^d.

2. Present a detailed proof of Theorem 3.3.

3. Show that if the hyperplane at infinity is adjoined to $A^{d,\square;k}$, where $0 \leq k \leq d$, the arrangement obtained is isomorphic to $A^{d,\square;k+1}$.

4. Determine the f-vectors, and the numbers of flats of various dimensions, for the arrangements $A^{d,\Delta}$ and $A^{d,\square;k}$.

5. Find examples of 3-arrangements that are not simplicial although all their 2-faces are triangles.

6. Show that the family of all 25 symmetry planes of the 3-cube considered in Figure 3.5 does not intersect the plane at infinity in a simplicial 2-arrangement.

7. Prove the assertion made on p. 3.8 that the arrangement $A^{d,\Delta}$ may be obtained by intersecting the hyperplane at infinity with the d-flats of mirror symmetry of the regular (d+1)-simplex.

8. Investigate the arrangements obtainable by the various constructions discussed on pages 3.5 to 3.8, starting from the d-simplex, the d-cube, the d-cross-polytopes, the 24-cell, the 120-cell, the 600-cell.

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CATALOGUE OF SIMPLICIAL 3-ARRANGEMENTS:

Corrections and additions

1. Page C.6 should be discarded, since the arrangement denoted $A_2^3(13)$ is nor simplicial; even the traces in the 6 planes of mirror symmetry are not simplicial - as is visible in the illustration (wrongly denoted $A_1^2(7)$).

2. On page C.11 , lines 2 and -1, instead of $A_1^2(10)$ should be $A_2^2(10)$.

3. Insert page C-10a, in which a new simplicial 3-arrangement $A_1^3(17)$ is described.

4. Add pages C-20 to C-25, in which a new simplicial 3-arrangement $A_1^3(30)$ is described.

Seattle, June 1974

Arrangement: $A_1^3(17)$

17 planes: 3 $A_1^2(8)$, 8 $A_2^2(10)$, 6 $A_3^2(10)$

$f = (53, 293, 480, 240)$

$t_3 = 9$, $t_4 = 23$, $t_6 = 14$, $t_8 = 6$, $t_9 = 1$

$\omega = 32$

$h = 65$; $h_2 = 34$, $h_3 = 28$, $h_4 = 3$.

Constructions:

(i) From $A_1^3(18)$ by deleting the plane at infinity.

(ii) From the regular octahedron: 8 facet planes
6 planes of mirror symmetry
3 midplanes between opposite vertices.

Appendix to Section 3:

CATALOGUE OF SIMPLICIAL 3-ARRANGEMENTS

We have seen that the joins of lower-dimensional simplicial arrangements yield simplicial arrangements. In case of 3-arrangements it is clearly possible to join either one 1-arrangement with another 1-arrangement, or else a simplicial 2-arrangement with a point (that is, the 0-arrangement $A^0(0)$). The following catalogue deals only with simplicial 3-arrangements that are not obtainable as joins of lower-dimensional ones.

At present, 13 such simplicial arrangements have been investigated, although it is clear that additional ones exist. (For example, one derived from the intersections of the 3-flats of mirror-symmetry of the regular 120-cell (or of the 600-cell) with the hyperplane at infinity of E^4.)

For each of the 13 arrangements described below, the catalogue contains the following data:

1. The symbol of the arrangement.

2. A description of the 2-arrangements induced in each plane by the other planes. This comprizes the symbol of the 2-arrangement, and a diagram showing the 2-arrangement and relating it to the construction of the 3-arrangement.

3. The f-vector of the arrangement, where f_k is the number of k-faces.

4. The numbers t_k of vertices each of which is incident with precisely k planes.

5. ω, the number of ordinary vertices of the arrangement.

6. h, the number of lines of the arrangement, and the numbers h_k of lines contained in precisely k planes.

7. A list of all known methods of generating the arrangement, and alternate designations used elsewhere.

In the tracings of the 2-arrangements induced in the planes by the other planes, closely spaced lines indicate planes with coinciding traces. ∞ indicates the line at infinity.

Arrangement: $A_1^3(10)$

10 planes: 10 $A_1^2(6)$

$f = (15, 75, 120, 60)$

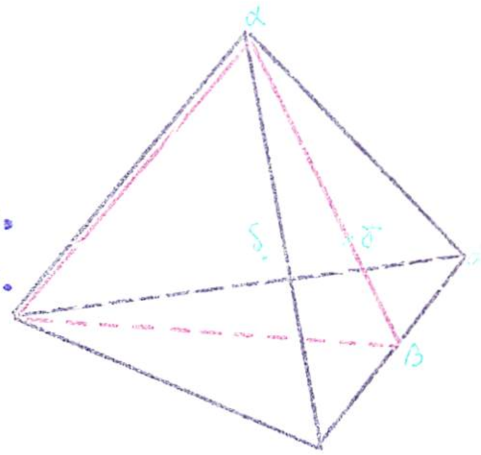
$t_4 = 10$ $t_6 = 5$

$\omega = 10$

$h = 25$; $h_2 = 15$, $h_3 = 10$

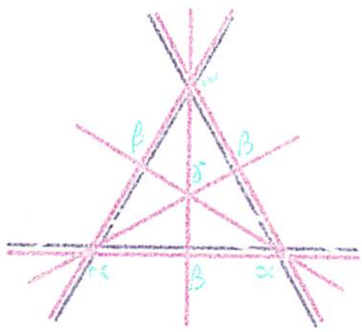
Constructions:

(i) 4 facet-planes of the regular tetrahedron, together with 6 planes of mirror symmetry.

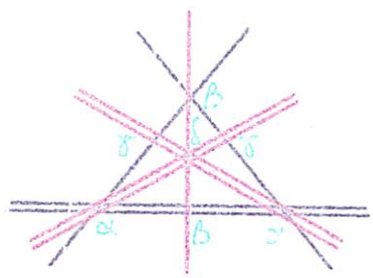


Traces:

4 facet-planes $A_1^2(6)$



6 planes of symmetry $A_1^2(6)$



(ii) $A^{3,\Delta}$.

(iii) The ten planes determined by any 5 points in "general position" in the projective 3-space.

(iv) Intersection of the 3-flats of mirror symmetry of the regular 4-simplex with the hyperplane at infinity.

Arrangement: $A_1^3(12)$

12 planes: 12 $A_1^2(7)$

$f = (24, 120, 192, 96)$

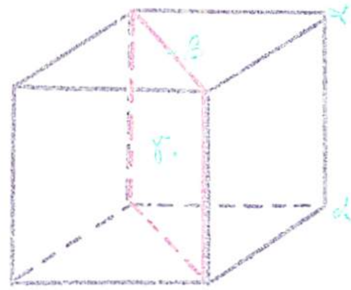
$t_3 = 12$ $\quad$ $t_6 = 12$

$\omega = 12$

$h = 34$; $h_2 = 18$, $h_3 = 16$

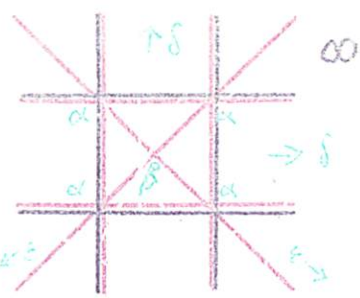
Constructions:

(i) 6 facet-planes of the regular cube and 6 planes of mirror symmetry

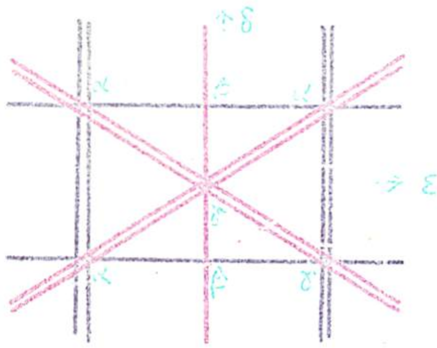


Traces:

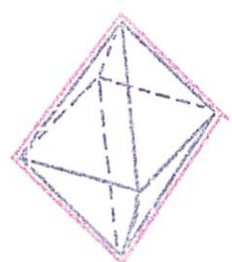
6 facet-planes $A_1^2(7)$



6 planes of symmetry $A_1^2(7)$

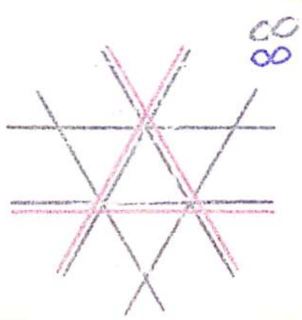


(ii) 8 facet-planes of regular octahedron, 3 planes of mirror symmetry, and 1 plane at infinity.

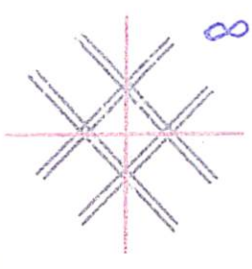


Traces:

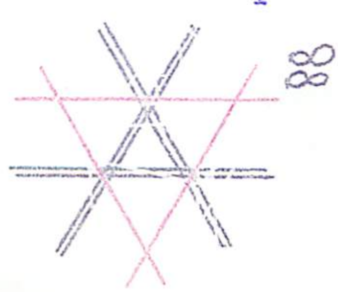
8 facet-planes $A_1^2(7)$



3 planes of symmetry $A_1^2(7)$



1 plane at infinity $A_1^2(7)$



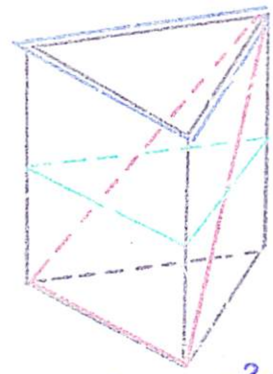
Arrangement: $A_1^3(12)$ - continued

(iii) 3 planes of side facets

2 base planes

1 midplane between bases

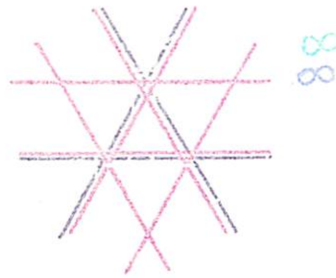
6 cutting planes



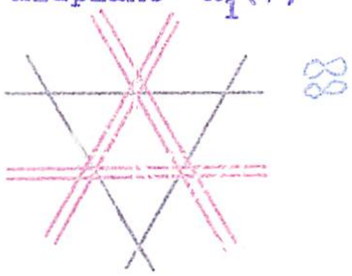
Traces:

3 side planes $A_1^2(7)$

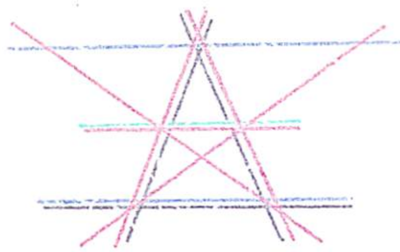
2 base planes $A_1^2(7)$



1 midplane $A_1^2(7)$



6 cutting planes $A_1^2(7)$



(iv) Intersection of the 12 perpendicular bisectors of the main body diagonals of the regular 24-cell with the hyperplane at infinity.

(v) Intersection of the 12 mid-3-flats parallel to pairs of opposite facets of the regular 24-cell with the hyperplane at infinity.

(vi) A^3, □: 0.

Arrangement: $A_1^3(13)$

13 planes: 6 $A_1^2(7)$, 6 $A_1^2(8)$, 1 $A_1^2(9)$

$f = (28, 148, 240, 120)$

$t_3 = 6$, $t_4 = 10$, $t_6 = 9$, $t_7 = 3$

$\omega = 16$

$h = 40$; $h_2 = 21$, $h_3 = 19$

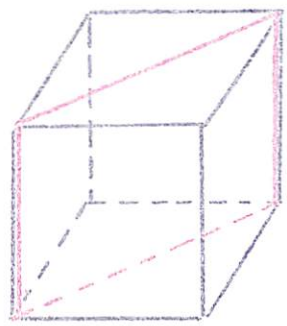
Constructions:

(i) 6 facet-planes of the regular cube

6 planes of symmetry

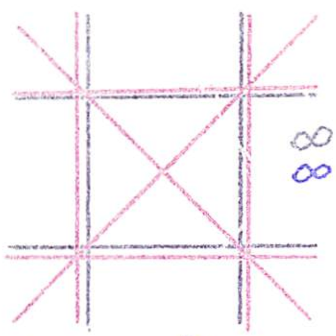
1 plane at infinity

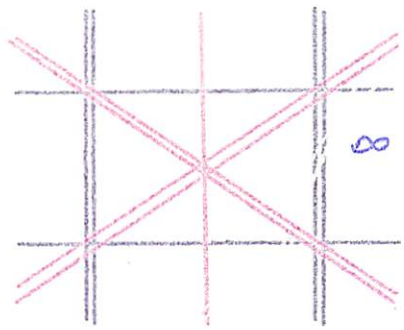
($A_1^3(12)$ with plane at infinity)

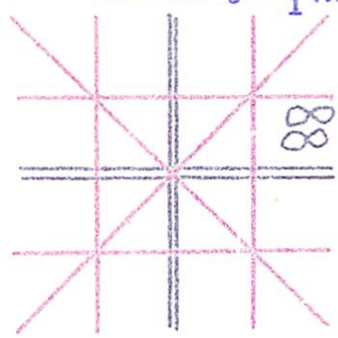


Traces:

6 facet-planes $A_1^2(7)$ 6 planes of symmetry $A_1^2(8)$ 1 plane at infinity $A_1^2(9)$







(ii) From $A_1^3(12)$ by adjoining one midplane parallel to a pair of opposite facets.

(iii) From $A_2^3(15)$ by deleting the two bases.

(iv) $A^{3,\square;1}$.

Arrangement: $A_2^3(13)$

13 planes: 6 $A_1^2(7)$, 3 $A_1^2(8)$, 4 $A_1^2(9)$

$f = (33, 159, 252, 126)$

$t_3 = 12$, $t_4 = 10$, $t_6 = 10$, $t_9 = 1$

$\omega = 22$

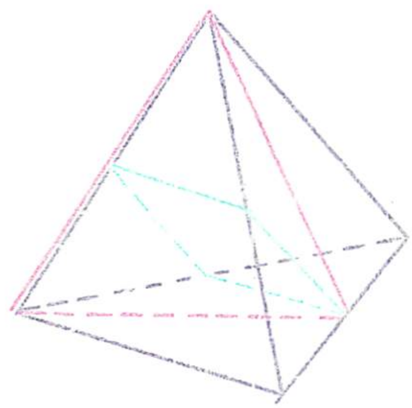
$h = 43$; $h_2 = 30$, $h_3 = 10$, $h_4 = 3$

Constructions:

(i) 4 facet-planes of the tetrahedron

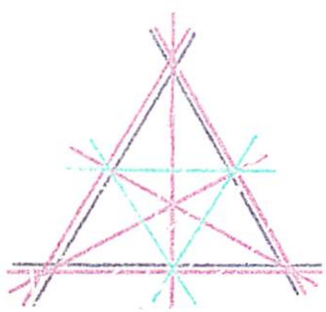
6 planes of mirror-symmetry

3 midplanes between skew edges

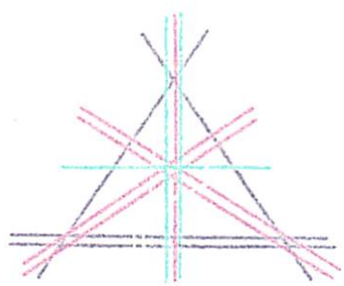


Traces:

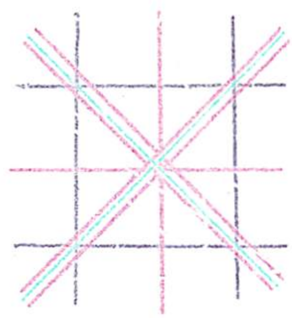
4 facet-planes $A_1^2(9)$



6 planes of symmetry $A_1^2(7)$



3 midplanes $A_1^2(8)$



Arrangement: $A_1^3(14)$

14 planes: 2 $A_1^2(7)$, 8 $A_1^2(8)$, 4 $A_1^2(9)$

$f = (32, 176, 288, 144)$

$t_3 = 2$, $t_4 = 16$, $t_5 = 2$, $t_6 = 8$, $t_7 = 2$, $t_8 = 2$

$\omega = 20$

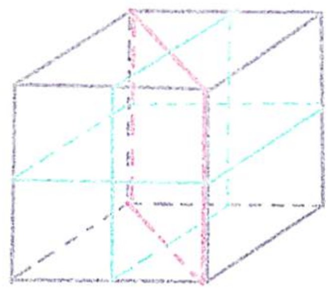
$h = 46$; $h_2 = 25$, $h_3 = 20$, $h_4 = 1$

Constructions:

(i) 6 facet-planes of the regular cube

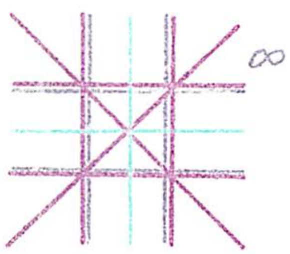
6 planes of mirror-symmetry

2 midplanes between parallel facets

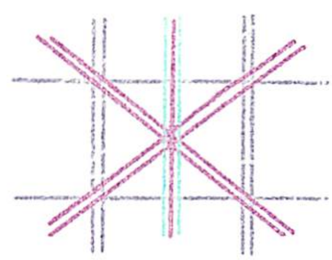


Traces:

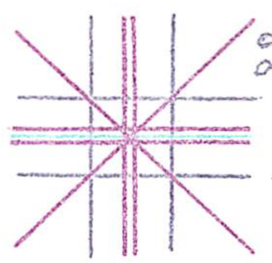
2 facet planes $A_1^2(9)$



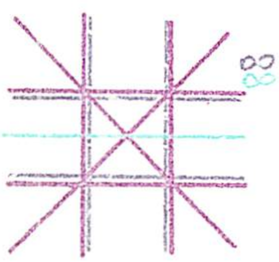
2 planes of symmetry $A_1^2(7)$



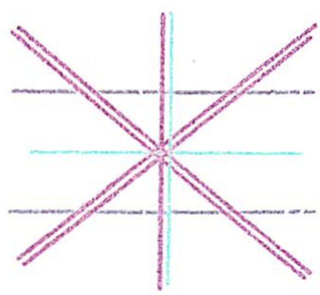
2 midplanes $A_1^2(9)$



4 facet-planes $A_1^2(8)$



4 planes of symmetry $A_1^2(8)$



(ii) From $A_1^3(13)$ by adding a midplane parallel to facets

(iii) $A^{3,\square;2}$.

Arrangement: $A_1^3(15)$

15 planes: 6 $A_1^2(8)$, 9 $(A_1^2(9)$

$f = (36, 204, 336, 168)$

$t_4 = 18$, $t_5 = 6$, $t_6 = 8$, $t_8 = 3$, $t_9 = 1$

$w = 24$

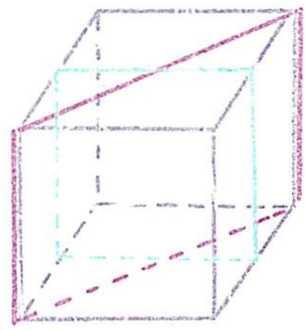
$h = 52$; $h_2 = 30$, $h_3 = 19$, $h_4 = 3$

Constructions:

(i) 6 facet-planes of the regular cube

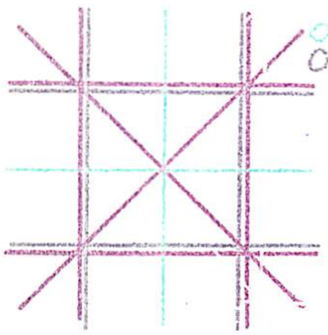
6 planes of mirror-symmetry

3 midplanes between parallel facets

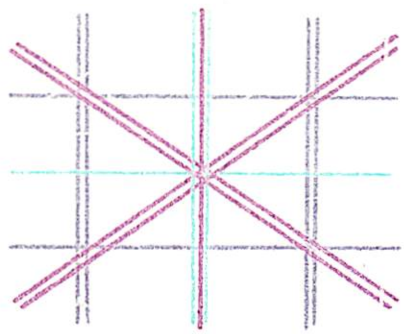


Traces:

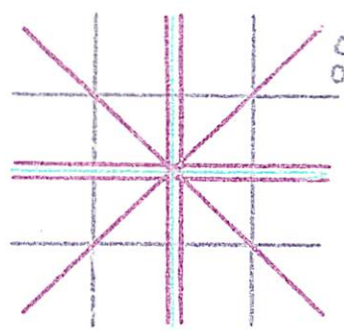
6 facet-planes $A_1^2(9)$



6 planes of symmetry $A_1^2(8)$



3 midplanes $A_1^2(9)$



(ii) From $A_1^3(14)$ by adding either the third midplane, or the plane at infinity; A^{3}, □:3 .

Arrangement: $A_2^3(15)$

15 planes: 9 $A_1^2(8)$, 6 $A_2^2(10)$

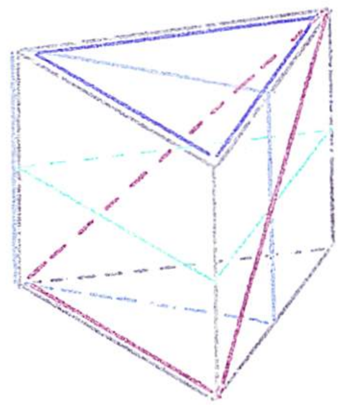
$f = (39, 219, 360, 180)$

$t_4 = 24$, t $t_6 = 6$, $t_7 = 9$

$\omega = 24$

$h = 53$; $h_2 = 27$, $h_3 = 26$

Constructions:



(i) 3 side-facet planes of regular 3-sided prism

(ii) 2 base planes

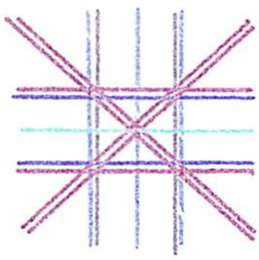
(iii) 1 midplane between bases

(iv) 3 planes of mirror-symmetry

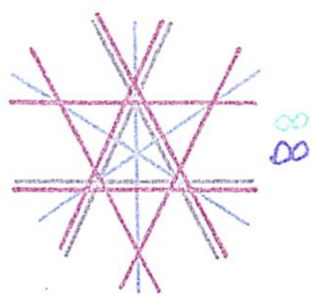
(v) 6 cutting planes

Traces:

3 side facets $A_1^2(8)$

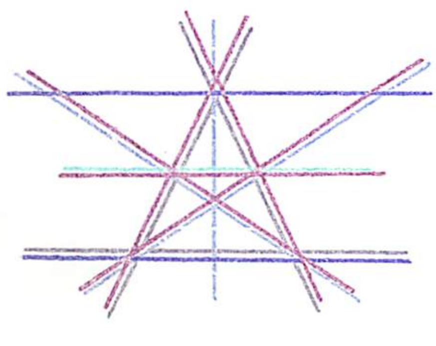


2 base planes $A_2^2(10)$

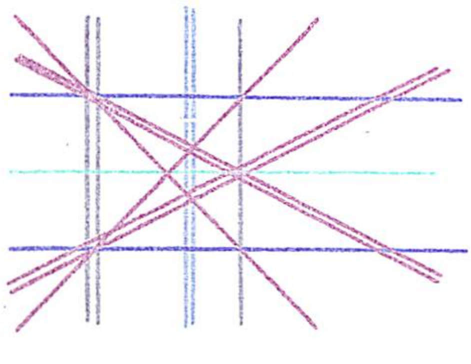


1 midplane $A_2^2(10)$

6 cutting planes $A_1^2(8)$



3 planes of mirror-symmetry $A_2^2(10)$



Arrangement: $A_1^3(16)$

16 planes: 16 $A_1^2(9)$

$f = (40, 232, 384, 192)$

$t_4 = 16, \quad t_5 = 12, \quad t_6 = 8, \qquad t_9 = 4$

$\omega = 28$

$h = 58; \quad h_2 = 36, \quad h_3 = 16, \quad h_4 = 6$

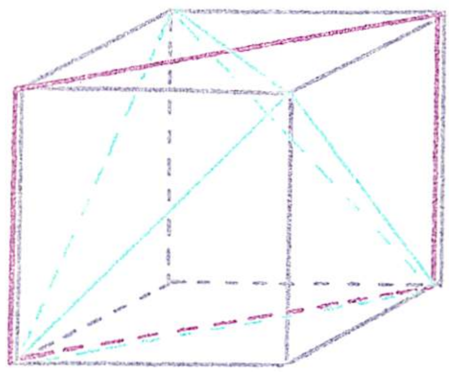
Constructions:

(i) Adding the plane at infinity to $A_1^3(15)$; $A^3, \square; 4$.

(ii) 6 facet-planes of the regular cube

6 planes of mirror-symmetry

4 skew planes (facet-planes of inscribed regular tetrahedron)



Arrangement: $A_1^3(18)$

18 planes: 3 $A_1^2(9)$, 8 $A_1^2(10)$, 6 $A_1^2(11)$, 1 $A_2^2(13)$

$f = (60, 348, 576, 288)$

$t_4 = 36$, $t_5 = 3$, $t_6 = 8$, $t_7 = 6$, $t_8 = 6$, $t_9 = 1$

$\omega = 39$

$h = 74$, $h_2 = 39$, $h_3 = 32$, $h_4 = 3$

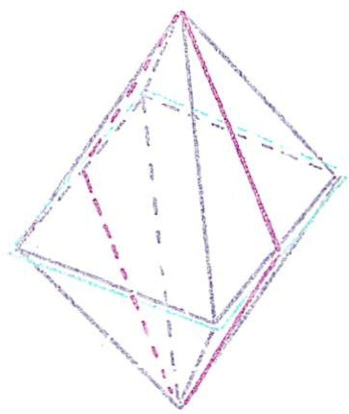
Constructions:

(i) 8 facet-planes of the regular octahedron

6 planes of mirror-symmetry

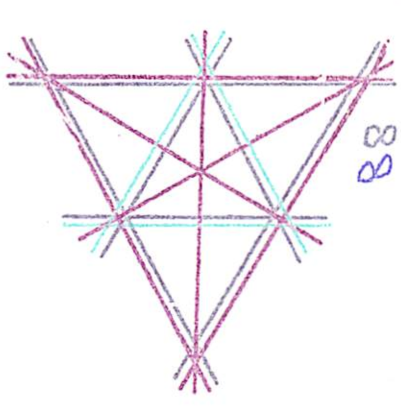
3 midplanes between opposite vertices

1 plane at infinity

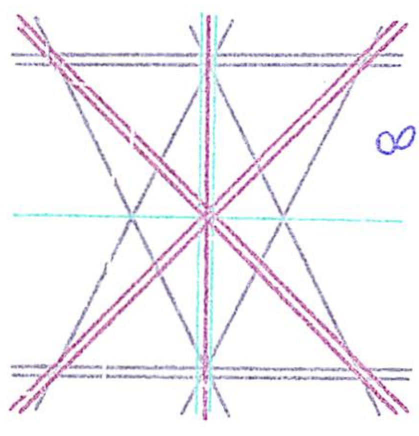


Traces:

8 facet-planes $A_1^2(10)$

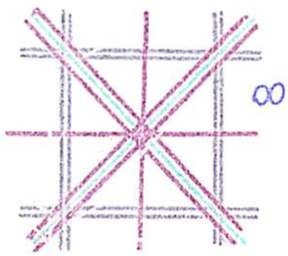


6 planes of symmetry $A_1^2(11)$

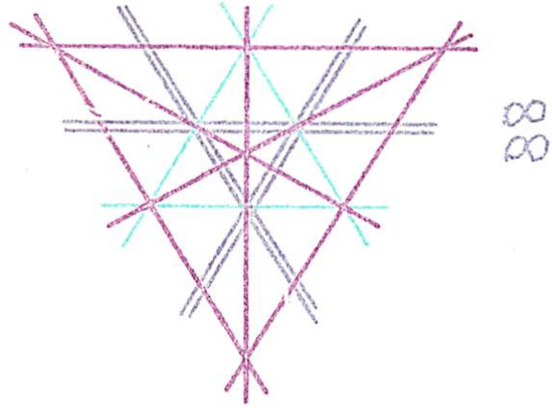


Traces (continued):

3 midplanes $A_1^2(9)$



1 plane at infinity $A_2^2(13)$



Arrangement: $A_1^3(24)$

24 planes: 24 $A_2^2(13)$

$f = (120, 696, 1152, 576)$

$t_4 = 96$, $t_9 = 24$

$\omega = 96$

$h = 122$; $h_2 = 72$, $h_3 = 32$, $h_4 = 18$

Constructions:

(i) Intersecting the 12 3-flats that perpendicularly bisect the main body-diagonals of the 24-cell, and the 12 midplanes between parallel facets, with the hyperplane at infinity of E^4.

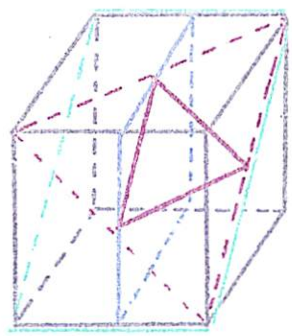
(ii) 6 facet-planes of the regular cube

6 planes of mirror-symmetry

3 midplanes between parallel facets

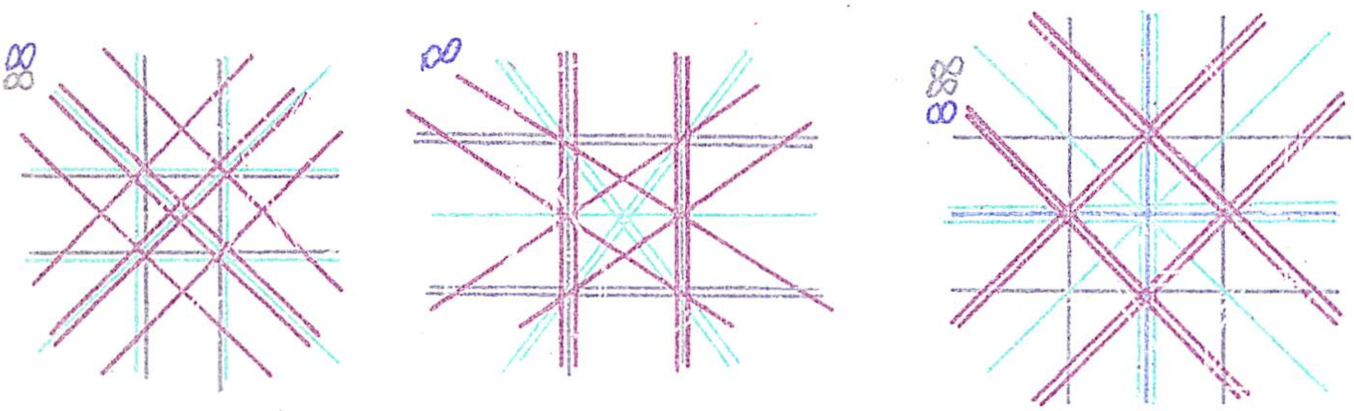
8 skew planes (bounding a regular octahedron)

1 plane at infinity



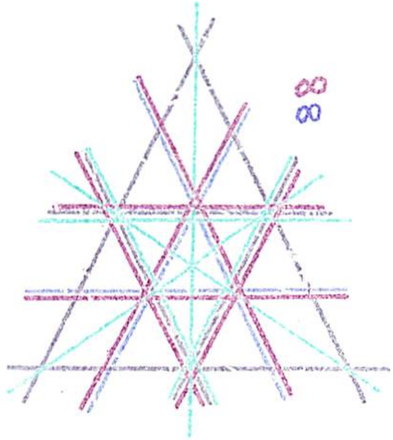
Traces:

6 facet-planes $A_2^2(13)$ 6 planes of symmetry $A_2^2(13)$ 3 midplanes $A_2^2(13)$

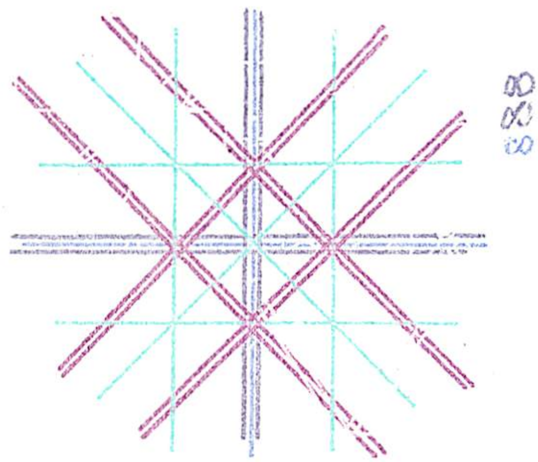


Traces (continued):

8 skew planes $A_2^2(13)$



1 plane at infinity $A_2^2(13)$



Arrangement: $A_1^3(27)$

27 planes: 12 $A_4^2(16)$, 15 $A_4^2(14)$

$f = (170, 1010, 1680, 840)$

$t_3 = 30$, $t_4 = 60$, $t_6 = 67$, $t_{10} = 12$, $t_{15} = 1$

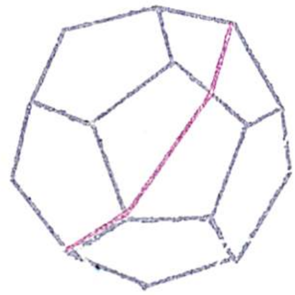
$\omega = 102$

$h = 157$; $h_2 = 81$, $h_3 = 70$, $h_5 = 6$

Constructions:

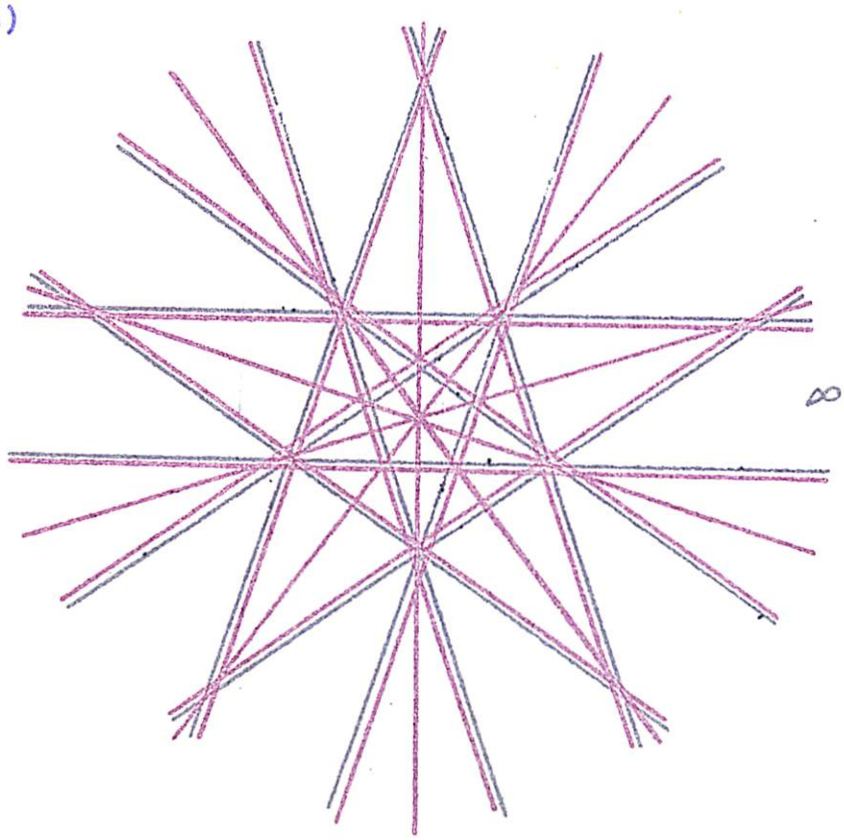
(i) 12 facet-p'anes of the regular dodecahedron

15 planes of mirror-symmetry



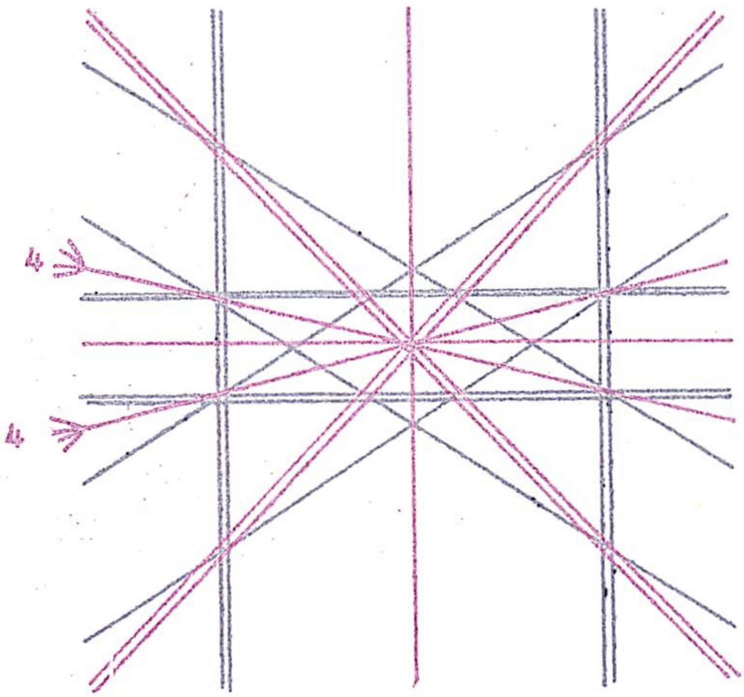
Traces:

12 facet-planes $A_4^2(16)$



Traces (continued):

15 planes of symmetry $A_4^2(14)$



Arrangement: $A_1^3(28)$

28 planes: 12 $A_4^2(16)$, 15 $A_4^2(15)$, 1 $A_2^2(21)$

$f = (186, 1146, 1920, 960)$

$t_4 = 100$, $t_6 = 58$, $t_7 = 15$, $t_{10} = 12$, $t_{15} = 1$

$\omega = 118$

$h = 172$; $h_2 = 90$, $h_3 = 76$, $h_5 = 6$

Constructions:

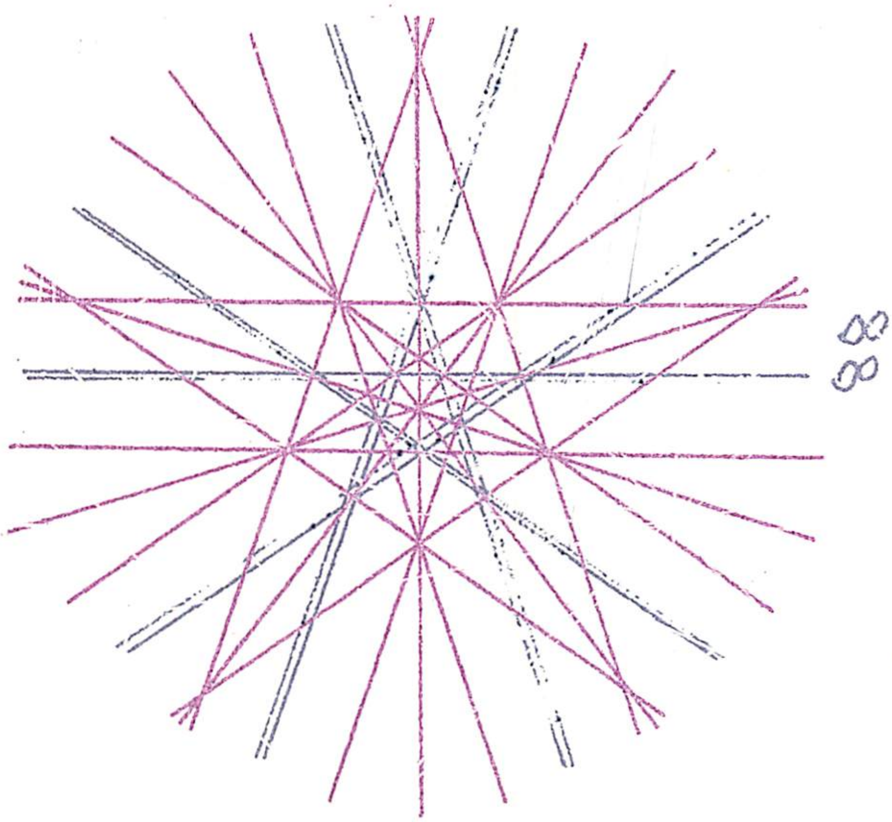
(i) The plane at infinity added to $A_1^3(27)$

(that is: 12 facet planes of the regular dodecahedron

15 planes of mirror symmetry

1 plane at infinity.)

Traces: 1 plane at infinity $A_2^2(21)$



Arrangement: $A_2^3(28)$

28 planes: 4 $A_1^2(13)$, 4 $A_2^2(13)$, 8 $A_3^2(16)$, 6 $A_2^2(17)$, 6 $A_4^2(17)$

$f = (194, 1154, 1920, 960)$

$t_3 = 24$, $t_4 = 84$, $t_5 = 18$, $t_6 = 40$, $t_8 = 18$, $t_9 = 3$,
$t_{11} = 6$, $t_{13} = 1$

$\omega = 126$

$h = 170$; $h_2 = 90$, $h_3 = 64$, $h_4 = 16$

Constructions:

(i) 6 planes of square facets of the regular cuboctahedron

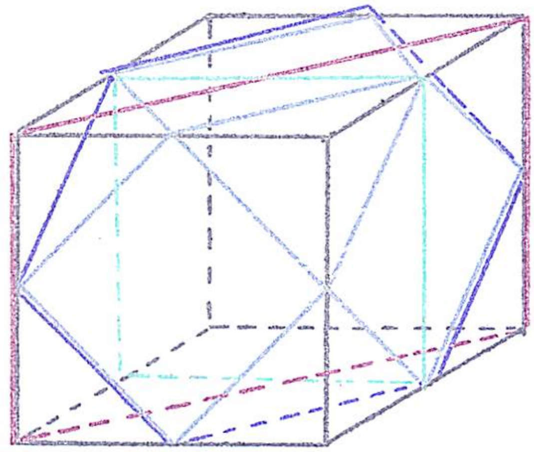
8 planes of triangular facets

6 planes of mirror-symmetry

3 midplanes

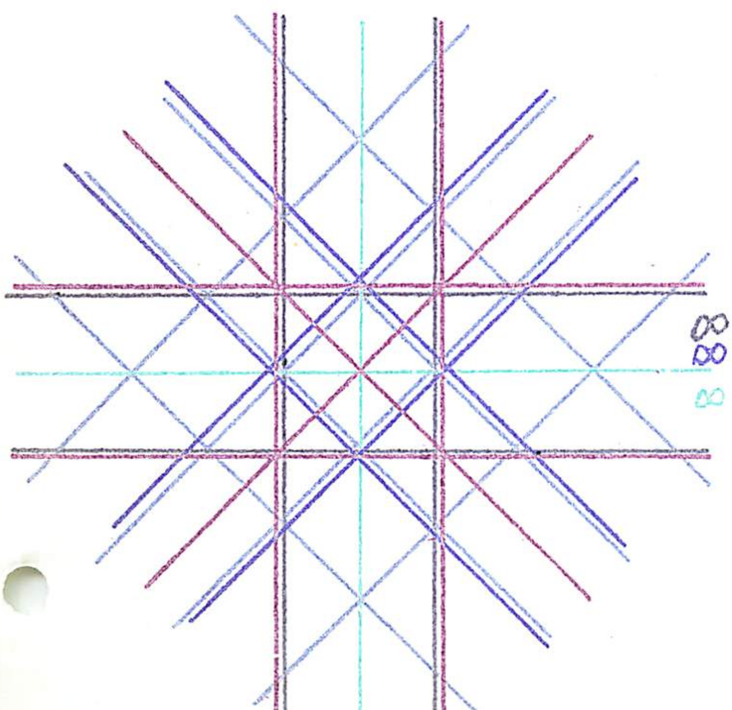
4 bisecting planes

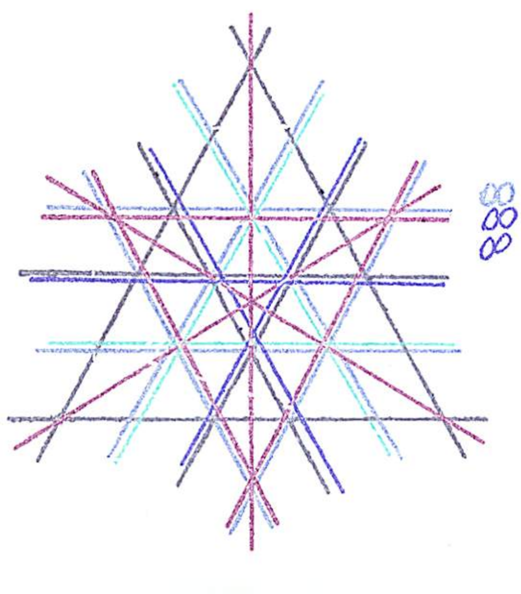
1 plane at infinity



Traces:

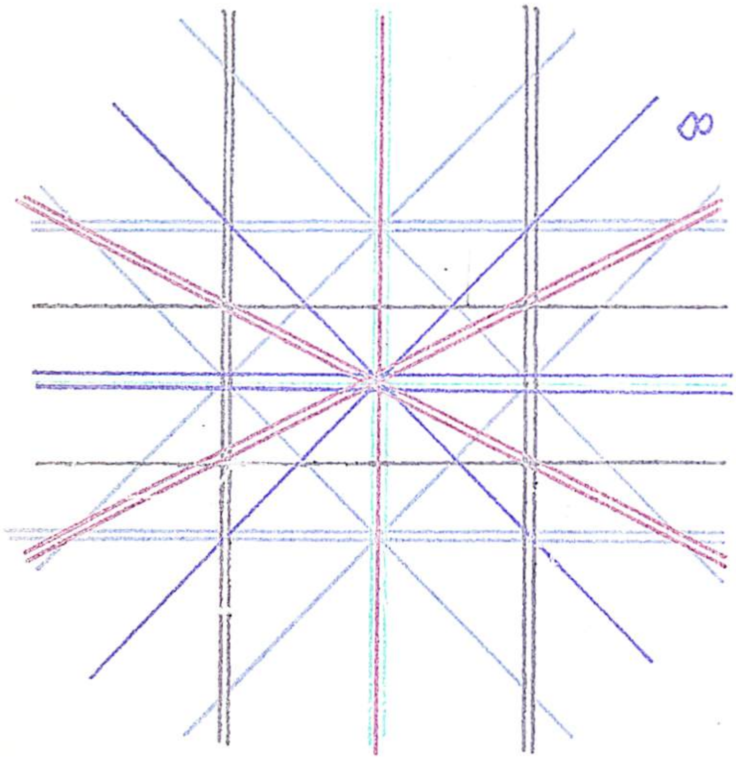
6 planes of square facets $A_4^2(17)$ 8 planes of triangles $A_3^2(16)$



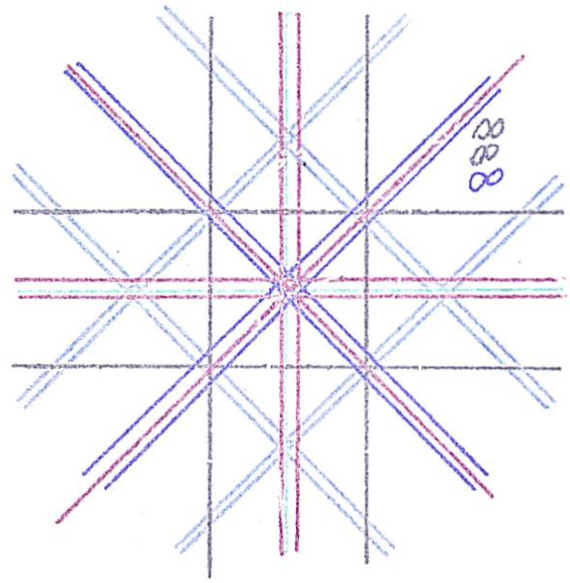


Traces (continued):

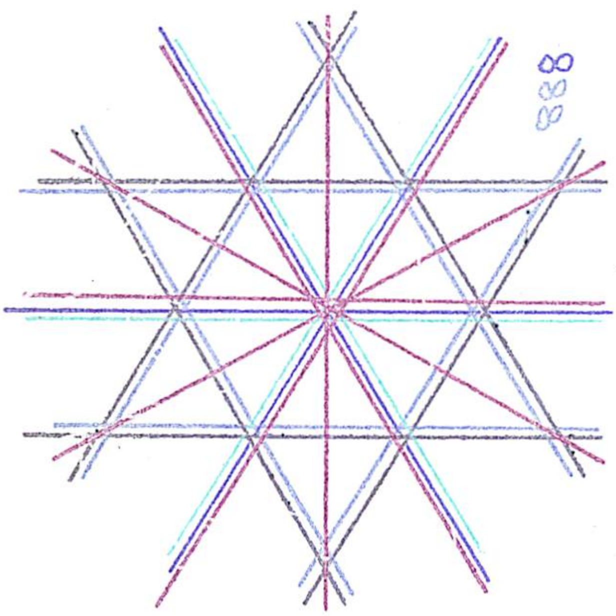
6 planes of symmetry $A_2^2(17)$



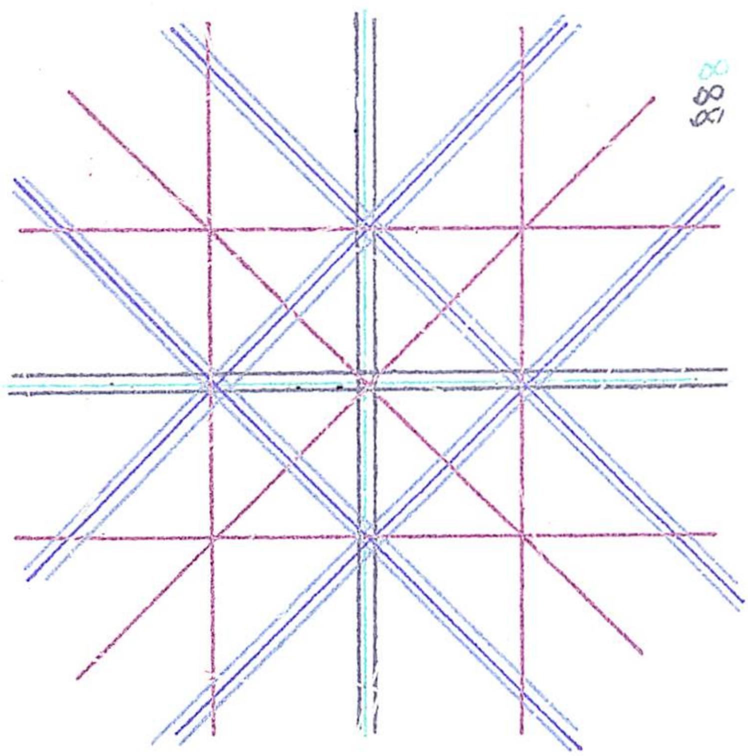
3 midplanes $A_2^2(13)$



4 bisecting planes $A_1^2(13)$



1 plane at infinity $A_2^2(13)$



Arrangement: $A_1^3(30)$

30 planes: 6 $A_1^2(13)$, 18 $A_4^2(17)$, 6 $A_1^2(19)$

$f = (\ 228,\ 1380,\ 2304,\ 1152\ )$

$t_4 = 144$, $t_6 = 36$, $t_7 = 24$, $t_8 = 18$, $t_{13} = 6$

$\omega = 150$

$h = 194$; $h_2 = 99$, $h_3 = 84$, $h_4 = 9$, $h_6 = 2$

Constructions:

(i) 2 base-facet planes of a six-sided prism

6 side-facet planes

3 planes of symmetry

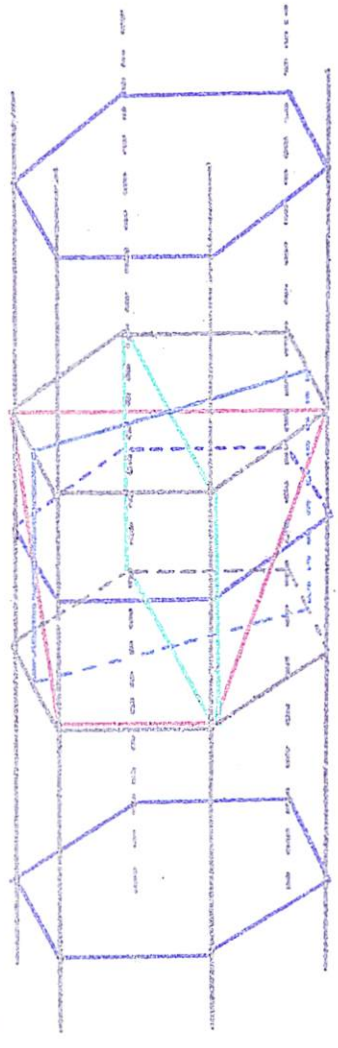
3 planes of symmetry

1 midplane

2 parallel planes

12 skew planes

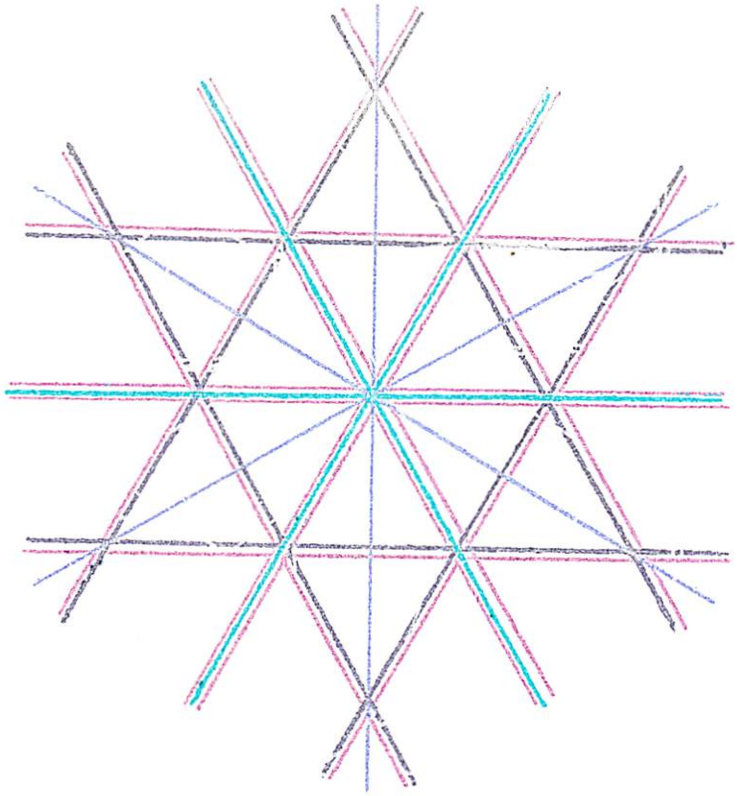
1 plane at infinity



Traces:

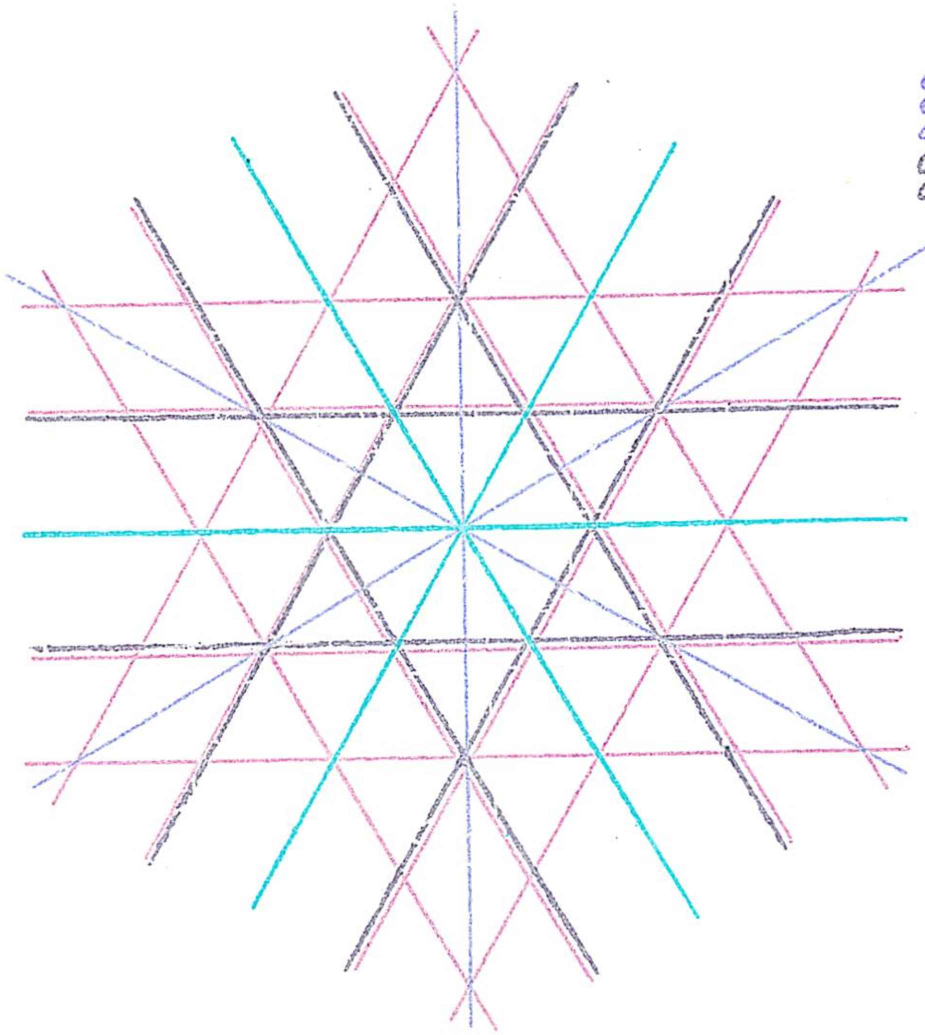
2 base-facet planes

$A_1^2(13)$

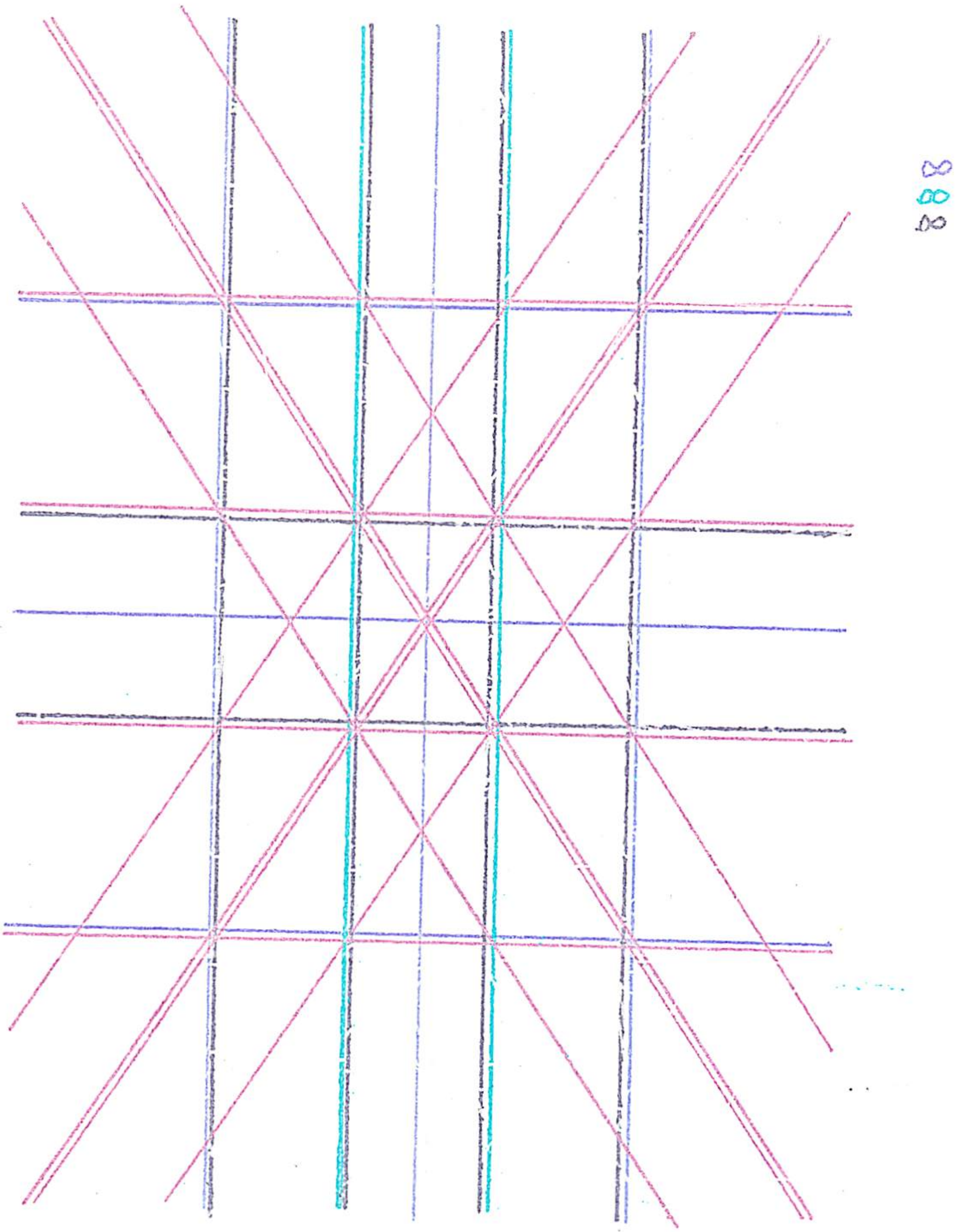


2 parallel planes

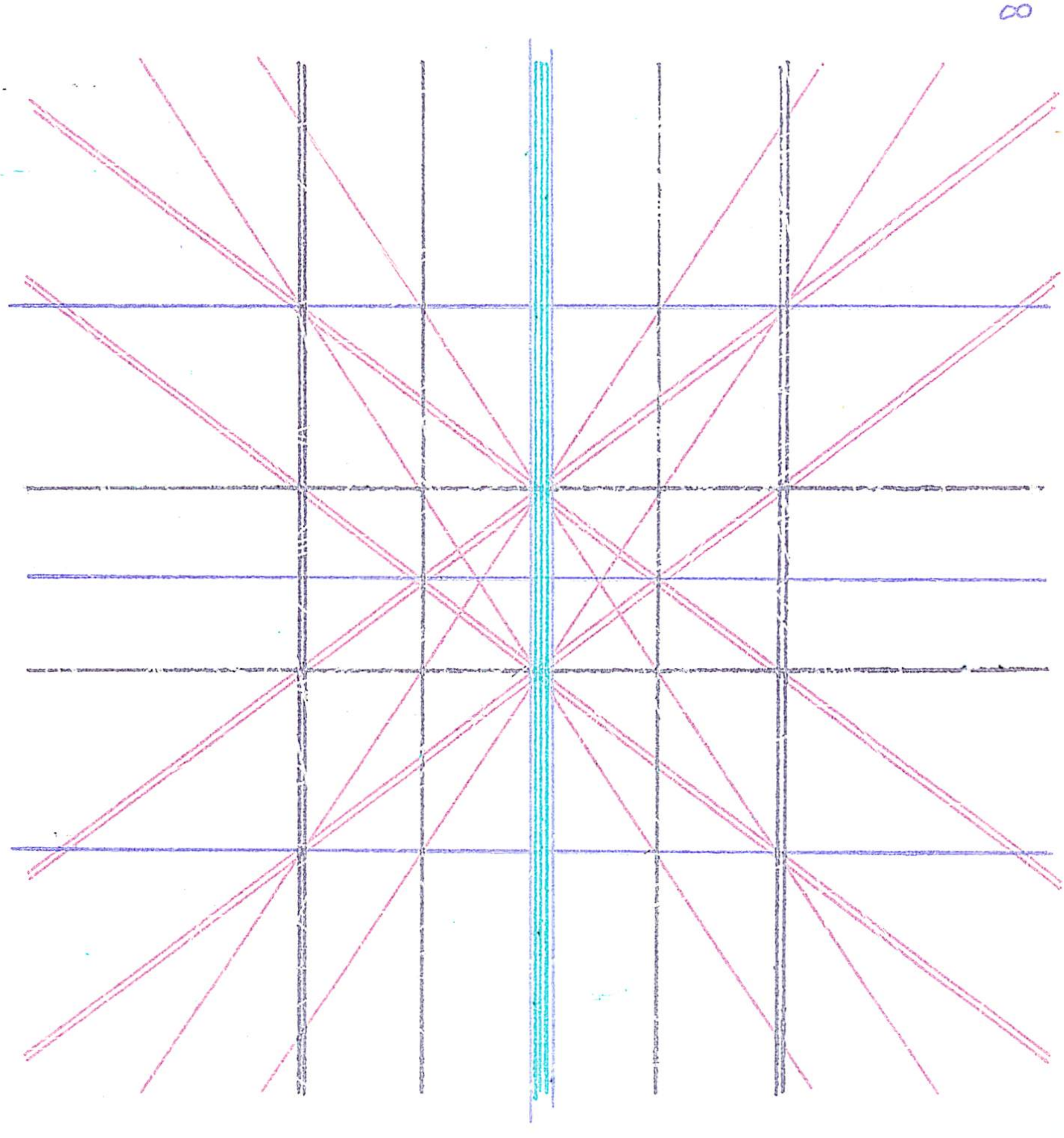
$A_1^2(19)$



6 side-facet planes $A_4^2(17)$

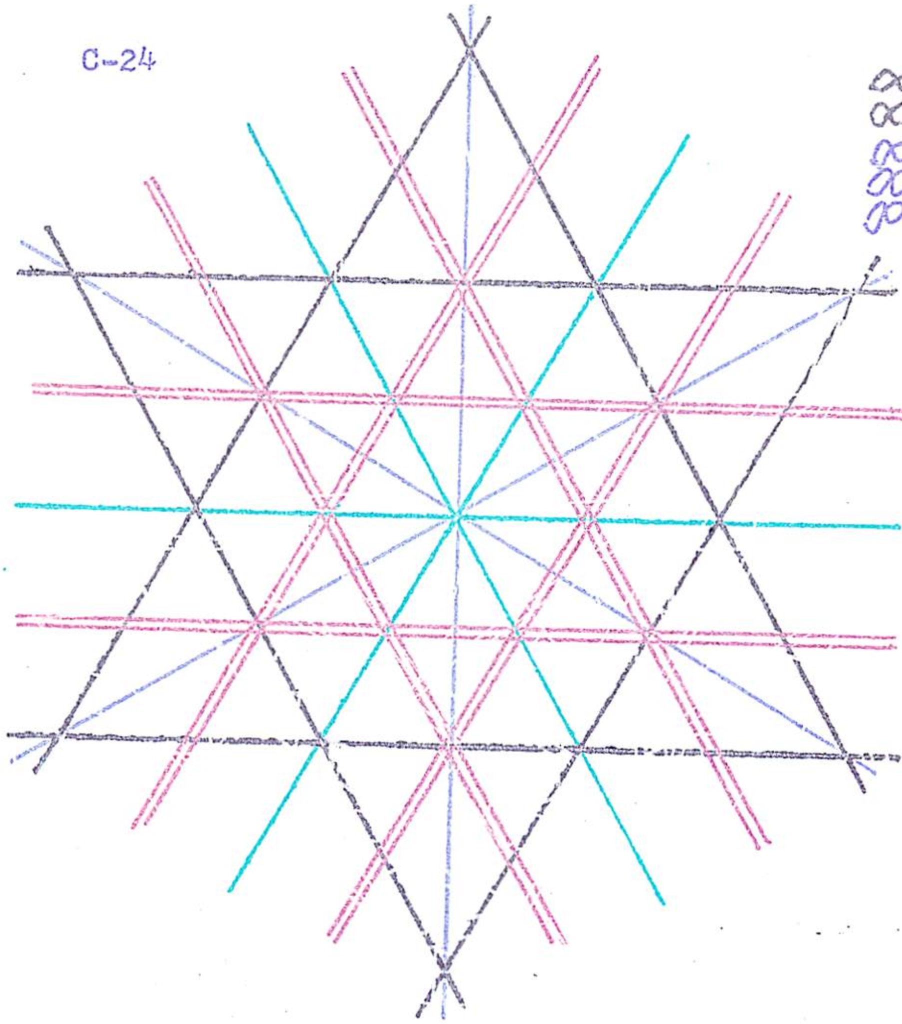


3 planes of symmetry $A_1^2(19)$

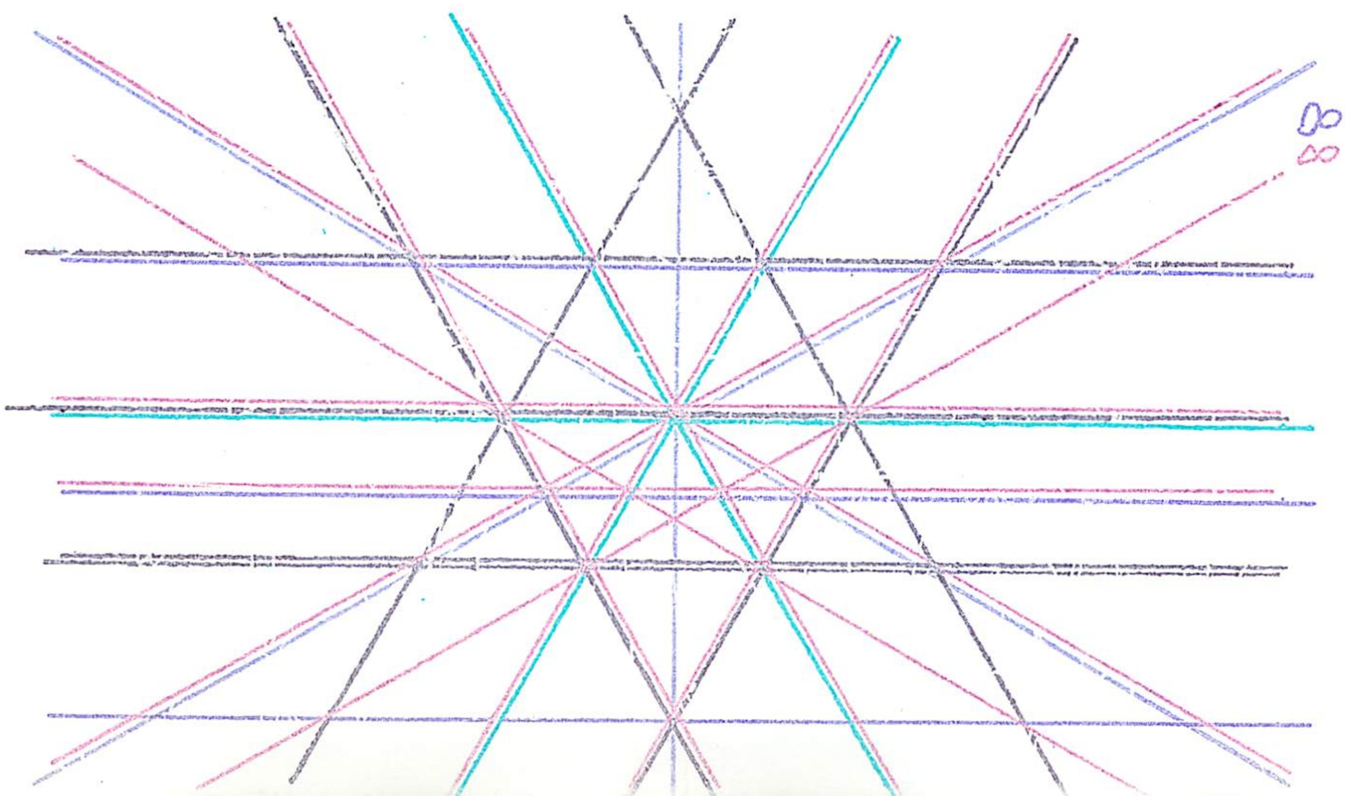


C-24

1 midplane $A_1^2(19)$

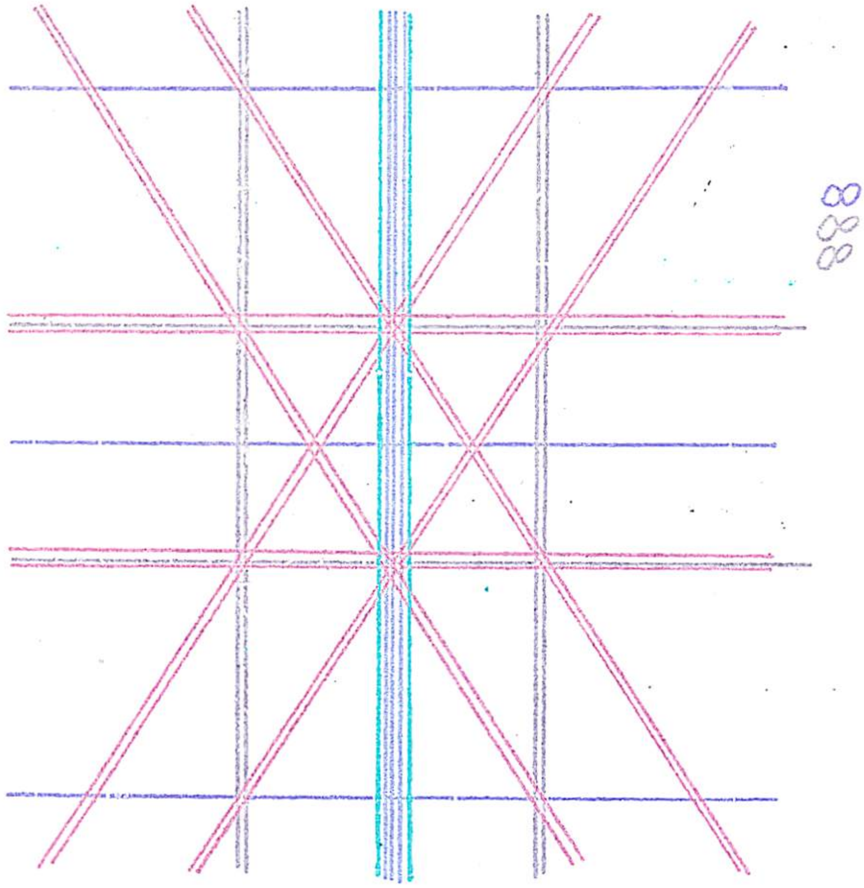


12 skew planes $A_4^2(17)$



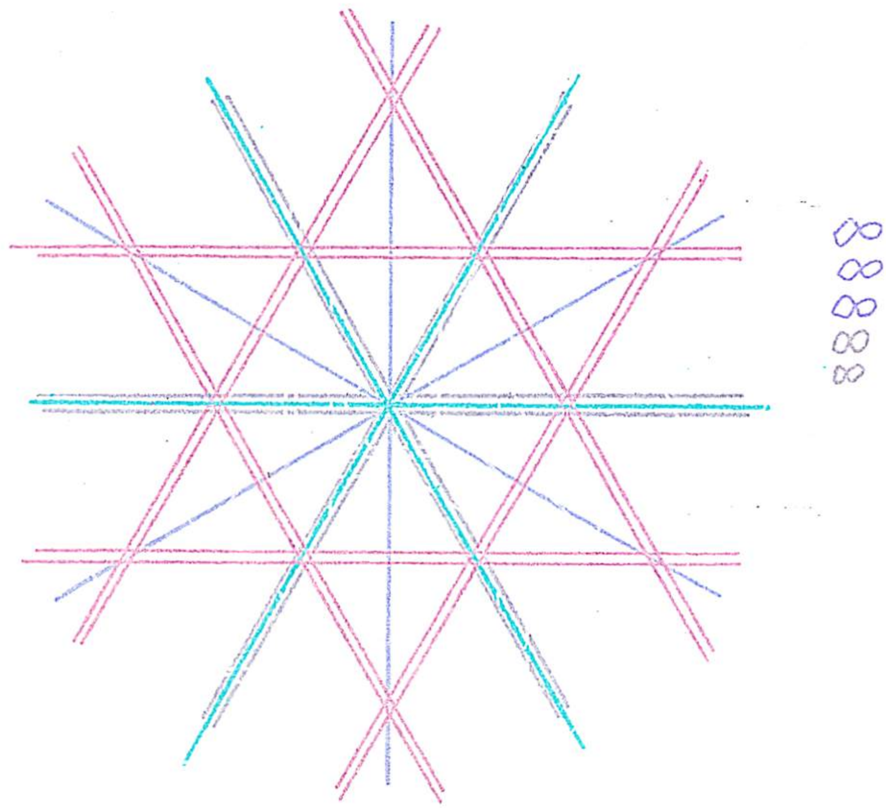
3 planes of symmetry

$A_1^2(13)$



1 plane at infinity

$A_1^2(13)$



Euclidean arrangements

Although this concept (and the dual one) were among the first notions related to arrangements that were considered (see Steiner [1826], as well as the long list of other references given in Grünbaum [1967, p. 391]), the first general and satisfactory definition seems to be that given in Canham [1972]. Following Canham, we shall say that a non-trivial Euclidean d-arrangement (or arrangement of hyperplanes in Euclidean d-space E^d) is the cell-decomposition of E^d determined by any family of n hyperplanes such that no point of E^d belongs to all the hyperplanes, and no line is parallel to all of them. A Euclidean d-arrangement is called simple if no point of E^d belongs to more than d hyperplanes, and no line is parallel to d hyperplanes.

Two Euclidean d-arrangements are weakly isomorphic if the cell complexes determined by them are isomorphic under a homeomorphism φ of E^d onto itself. Two Euclidean d-arrangements are isomorphic if they are weakly isomorphic under an orientation-preserving homeomorphism φ of E^d onto itself.

The completion of E^d to P^d by the addition of the hyperplane at infinity clearly associates with each Euclidean d-arrangement a d-arrangement in P^d, obtained by taking the completions in P^d of the hyperplanes of the Euclidean arrangement, together with the hyperplane at infinity. Conversely, from each d-arrangement $\mathcal{A}$ in P^d each choice of one hyperplane of $\mathcal{A}$ as the "hyperplane at infinity" determines a Euclidean d-arrangement.

This observation, and his enumeration and construction of all types of simple arrangements of at most 8 lines in P^2, enabled Canham [1972] to determine the following values for the numbers of Euclidean 2-arrangements with at most 7 lines. In the formulation of Canham's result we shall denote by $\hat{c}_e^s(n,d)$ and $c_e^s(n,d)$ the number of distinct weakly isomorphic, or distinct isomorphic, types of simple Euclidean d-arrangements of n hyperplanes.

Theorem 1. The numbers of distinct types of simple Euclidean arrangements of at most 7 lines are:
$\hat{c}_e^s(3,2) = c_e^s(3,2) = 1$; $\hat{c}_e^s(4,2) = c_e^s(4,2) = 1$; $\hat{c}_e^s(5,2) = 6$, $c_e^s(5,2) = 7$; $\hat{c}_e^s(6,2) = 43$, $c_e^s(6,2) = 79$; $\hat{c}_e^s(7,2) = 922$, $c_e^s(7,2) = 1765$.

Actually, all the numbers quoted in the theorem were obtained already by R. Klee [1938], who asserted also that $c_e^s(8,2) = 77,064$ (along with some other related numbers). However, the value Klee assigns to $c_e^s(8,2)$ is certainly wrong, although it possibly expresses the number of non-isomorphic types of simple Euclidean arrangements of pseudolines. The reason for this (and for my crediting the result of the theorem to Canham) is that Klee's method is appropriate for the counting of types of pseudolines, but can not distinguish the stretchable ones among them. Klee is aware of the danger, but by mistaken arguments (on pp. 12, 13) convinces himself that all the arrangements he is obtaining are stretchable. By Canham's result, this stretchability actually occurs for $n = 7$, but examples derivable from the non-stretchable simple arrangement of 9 pseudolines in P^2 (Ringel [1956], Grünbaum [1972, p.42]) show that the stretchability fails for $n = 8$.

The <u>finite part</u> of a Euclidean arrangement consists of all the bounded cells of the arrangement; it has been the object of several investigations about which we shall report below. If the Euclidean arrangement is viewed as a projective arrangement from which one hyperplane was deleted as the "hyperplane at infinity", the "finite part" of the Euclidean arrangement becomes the complex consisting of all the (closed) cells of the projective arrangement that miss the "hyperplane at infinity". This provides a possibility of treating different questions on Euclidean arrangements in a projective formulation; often this variant is actually a strengthening of the purely projective version established earlier. The following examples should clarify this relatioship; many other interesting problems of this type remain to be investigated.

Consider first the following analogue of the well-known Sylvester problem:

<u>Theorem 2</u>. Each non-trivial Euclidean 2-arrangement has a simple vertex (that is, a vertex that belongs to precisely two lines of the arrangement).

The equivalent statement in the projective setting is the following one, which is clearly stronger than just an affirmative solution of Sylvester's problem:

<u>Theorem 2*</u>. If all the simple vertices of a non-trivial arrangement $\mathcal{A}$ in P^2 are on one line L of $\mathcal{A}$, then $\mathcal{A}$ is a near-pencil and L is the exceptional line of $\mathcal{A}$.

Pavlick [1973] mentions Theorem 2, and claims that it had been proved by Melchior [1940]. This in incorrect, since Melchior proved only that each non-trivial projective arrangement has simple vertices,- that is, gave a solution to Sylvester's problem. Although there are several simple and straightforward solutions of Sylvester's problem in the projective plane (based either on Euler's formula, or on other elementary considerations), none seems to be adaptable to a proof of Theorem 2 or 2*. The only proof of those I know uses the much stronger result of Kelly-Moser [1958] to the effect that each non-trivial projective arrangement $\mathcal{A}$ with $n(\mathcal{A})$ lines has at least $t_2(\mathcal{A}) \geq$ $\geq 3n(\mathcal{A})/7$ simple vertices.

Proof of Theorem 2*. Assume all simple vertices of the arrangement $\mathcal{A}$ are on the line L ; if the arrangement $\mathcal{A}'$ obtained from $\mathcal{A}$ by omitting the line L is trivial, then $\mathcal{A}$ is a near-pencil and L its exceptional line, as claimed. Hence we need consider only the case in which $\mathcal{A}'$ is non-trivial. All the $t_2(\mathcal{A}') \geq 3n(\mathcal{A}')/7 =$ $= 3(n(\mathcal{A}) - 1)/7$ simple vertices of $\mathcal{A}'$ must correspond to vertices of $\mathcal{A}$ incident with precisely 3 lines, one of which is L. Therefore L must be intersected by at least $\frac{3}{7}n(\mathcal{A}) + 2.\frac{3}{7}(n(\mathcal{A}) - 1) = 9n(\mathcal{A})/7 - 6/7$ lines of $\mathcal{A}$, which is impossible since $\mathcal{A}$ contains only $n(\mathcal{A}) - 1$ lines different from L.

Assuming Theorem 2, Pavlick [1973] gives the easy proof (by induction on n) of the following result:

Theorem 3. If a Euclidean 2-arrangement of n lines has at least 2 vertices then it has at least $n-1$ vertices.

An equivalent projective formulation is:

Theorem 3*. If $\mathcal{A}$ is a non-trivial arrangement of n lines in P^2 and if L is any line of $\mathcal{A}$, then either $\mathcal{A}$ is a near-pencil and L its exceptional line, or else $\mathcal{A}$ has at least $n-2$ vertices not on L.

One of the earliest results on Euclidean arrangements is the following (Roberts [1888]):

Theorem 4. In each simple Euclidean 2-arrangement of n lines, the finite part contains at least $n-2$ triangles.

Actually, Roberts' proof of Theorem 4 is invalid, and no proof of the assertion has ever been published. In his forthcoming thesis, Shannon [1975] establishes it in the following vastly generalised form:

Theorem 5. If $\mathcal{A}$ is an arrangement of n hyperplanes in P^d such that no point of P^d belongs to $n-2$ or more of the hyperplanes, and if H is one of the hyperplanes of $\mathcal{A}$, then among the facets of $\mathcal{A}$ there are at least $n-(d+1)$ d-simplices each of which has no $(d-1)$-face in H.

Exercises and problems.

1. The enumerations in Klee [1938] were based on the following construction: If in each 2-face of a simple Euclidean arrangement of n lines or pseudolines we mark a "node", and if we connect "nodes" in faces that have a common (bounded or unbounded) edge by an "arc", we obtain a planar map which is isomorphic (as a planar map) to a "Klee diagram", that is, a subdivision of the regular (2n)-gon into rhombi. Conversely, each such subdivision is a Klee diagram and corresponds to (a family of mutually isomorphic) simple arrangements of n pseudolines. The example in Figure 1 should help clarify the correspondence; in its first part "nodes" are represented by open circles, "arcs" by dashed lines. (Prove the validity of the above statement about Klee diagrams; probably the simplest way is by induction on n of the more general construction, in which the regular (2n)-gon is replaced by one with only a center of symmetry.) What misled Klee (at least psychologically) is the curious fact that even the Klee diagrams of non-stretchable arrangements of pseudolines may be represented by (rectilinear) rhombi.

A non-stretchable Euclidean arrangement of 8 pseudolines, obtained from the non-stretchable simple arrangement of 9 pseudolines in the projective plane described by Ringel [1956] and Grünbaum [1972, Fig. 3.3] by taking one of them as the "line at infinity", has the Klee diagram shown in Figure 2. (Check that assertion; the nearly-vertical line in Figure 3.3 of Grunbaum [1972] was taken as the line at infinity in the preparation of Figure 2.) The construction of Klee diagrams of Euclidean arrangements may be compared to the correspondence between projective arrangements and zonotopes (Coxeter [1962], McMullen [1971], Grünbaum [1971].)

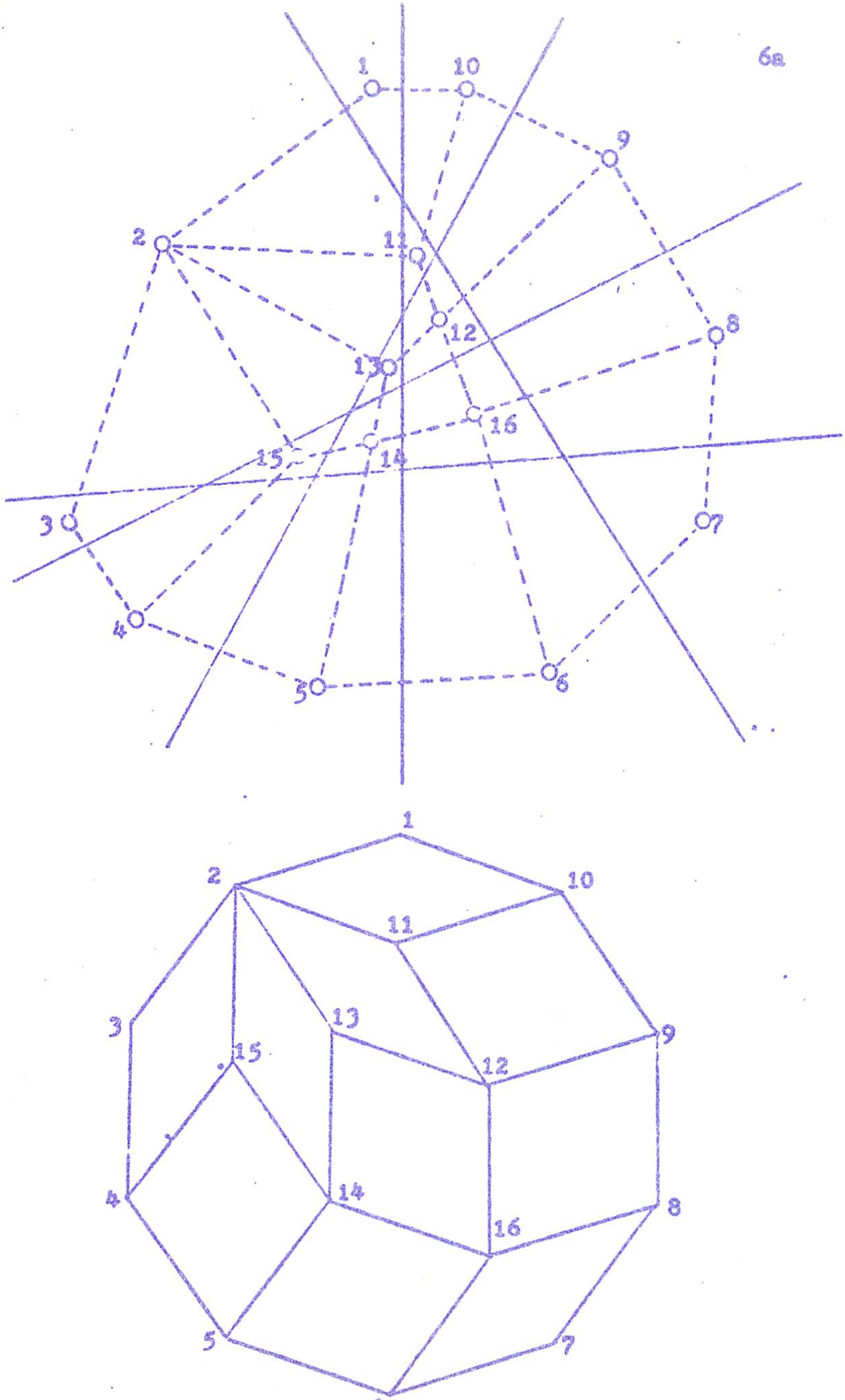


Figure 1.

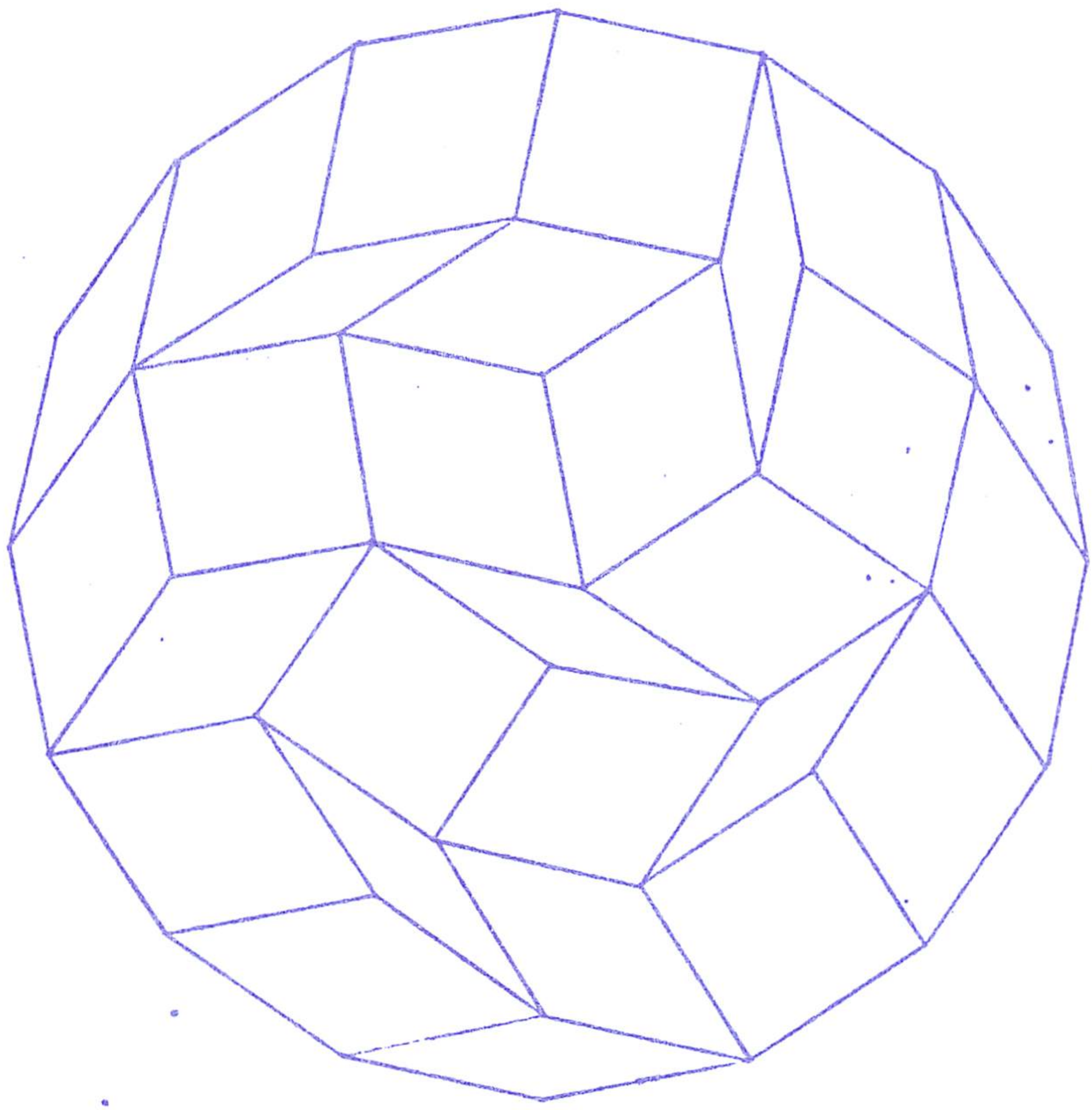


Figure 2.

2. Generalize the construction of Klee diagrams to not necessarily simple Euclidean arrangements of pseudolines.

3. Consider analogues of Klee diagrams for Euclidean arrangements in 3-dimensional space.

4. The proof of theorem 2* given above may easily be modified so as to show that in every non-trivial 2-arrangement $\mathcal{A}$ that is not a near-pencil, and for each line L in $\mathcal{A}$, there are at least $2n(\mathcal{A})/21$ simple vertices of $\mathcal{A}$ that are not on L.

5. Prove that theorem 2* remains valid even if it is not assumed that the line L that contains all the simple vertices of $\mathcal{A}$ is a line of $\mathcal{A}$.

6. Prove theorem 2* for arrangements of pseudolines.

7. Provide examples establishing that no analogue of theorem 2* is valid for d-arrangements in P^d dor $d \geq 3$.

8. No analogue of theorem 3 holds for arrangements of planes in E^3 (or in higher dimensions). For each $n \geq 4$ and each k, $2 \leq k \leq n$, find an arrangement of n planes in E^3 that has precisely k vertices.

9. As a complement of theorem 3, Pavlick [1973] formulates the following conjecture (verified by him for n = 4,5,6,7): For each n and for each choice of p such that either p = 0 or p = 1 or $n-1 \leq p \leq \binom{n}{2}$, there exists a Euclidean 2-arrangement with n lines and precisely p vertices.

Prove that this conjecture is true for n = 8 and n = 9, and that it is false for each $n \geq 10$.

10. A problem somewhat complementary to theorem 4 has recently been proposed by K. Fujimura (see M. Gardner [1972]). In our terminology. Fujimura's problem is: What is the maximal number of triangles possible in the finite part of a Euclidean arrangement of n lines ? As maximal known number of such triangles Gardner mentions:

Number of lines:	3	4	5	6	7	8	9
Maximal known number of triangles:	1	2	5	7	11	15	21 .

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Arrangements of colored lines.

This is a very interesting topic with many novel types of problems, in which relatively little has been published so far although many people appear to have thought at least fleetingly about some of the questions.

Let $\mathcal{A}$ be an arrangement of n lines in P^2 ; we say that $\mathcal{A}$ is k-colored if the lines of $\mathcal{A}$ are grouped in k sets, the "colors" of $\mathcal{A}$. We call a vertex of $\mathcal{A}$ monochromatic if all the lines of $\mathcal{A}$ incident with it have the same color.

The first problem we shall discuss originated with R. L. Graham; it concerns the existence of monochromatic vertices in 2-colored arangements. An affirmative answer was announced by Motzkin [1967], and a proof was published by Chakerian [1970]. More precisely:

Theorem 1. If a non-trivial arrangement $\mathcal{A}$ in P^2 is colored by two colors then $\mathcal{A}$ contains at least one monochromatic vertex.

Proof. (From Chakerian [1970]). Consider the arrangement $\mathcal{A}$ in the model of P^2 on the 2-sphere. Then the assertion is obviously a special case of the following result, which is known as Cauchy's lemma; a variant of it was first established by Cauchy [1813] in his proof of the famous "rigidity theorem" for convex 3-polytopes. (It is ironic that Cauchy's version was actually insufficient for the needs of his proof, although it would have been adequate for the proof of Theorem 1.)

Cauchy's lemma. There is no possibility to color each edge of a planar graph G (without loops or multiple edges) by one of two colors so that at each vertex of G, in the cyclic sequence of edges around the vertex there be at least 4 changes of color.

Proof. Assume such a coloring of a graph G possible. Let v, e, f, c be the numbers of vertices, edges, faces (countries) and connected components of G, and let p_k be the number of k-gonal faces of G. Then $2e = \sum k p_k$, $f = \sum p_k$, and - by Euler's relation -, $v = 1 + c + e - f$. The total number m of changes of color around all vertices satisfies, on the one hand, $m \geq 4v$. On the other hand, since m is also the total number of changes of color around the faces of G, and since around a k-gonal face there may be at most $2[k/2]$ changes of color, we have $m \leq \sum_{k \geq 3} 2[k/2]p_k$. Combining those inequalities with the previous equations we obtain $4 + 4c + 2p_5 + 2p_6 + 4p_7 + 2p_8 + 6p_9 + \ldots \leq 0$, which is clearly impossible. ∎

A different proof of Theorem 1 was privately communicated to me in 1971 by G. D. Chakerian, who credited it to Sherman K. Stein; it is as follows:

Call "good configuration" any near-pencil formed by 4 lines, two of each color. Each "good configuration" contains two triangles with a common edge so that the disposition of colors is as indicated (for the shaded triangles) in the schemes of Figure 1. (Actually, each "good configuration" contains two such pairs.) Call the union of these two triangles a "characteristic triangle" of the "good configuration". Clearly, if $\mathcal{A}$ were a non-trivial arrangement colored by 2 colors, without any monochromatic vertices, then $\mathcal{A}$ would contain some "good configurations". Let us choose one such, the "characteristic triangle" of which is minimal in the sense that it does not properly contain the "characteristic triangle" of any other "good configuration". Without loss of generality we

Figure 1.

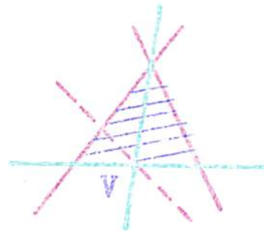


Figure 2.

Figure 3. Colorings of near-pencils $A_0(n)$.

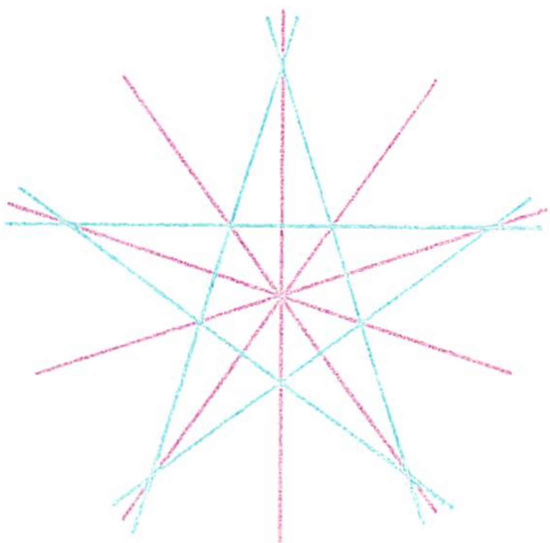


Figure 4. $A_1(10)$, typical for $A_1(4k+2)$.

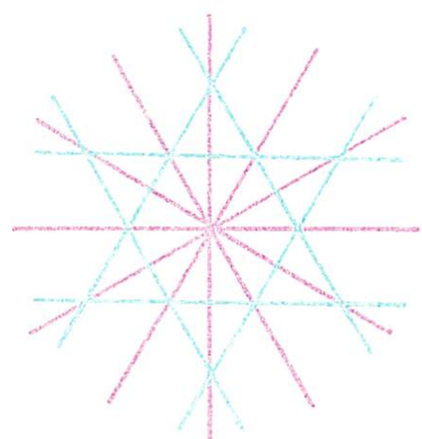


Figure 5. $A_1(12)$, typical for $A_1(4k)$. With (green) line at infinity: $A_1(13)$, typical for $A_1(4k+1)$.

may assume that the situation is as indicated by the solid lines in Figure 2, the minimal "characteristic triangle" being again shaded. Now, if the vertex V were not monochromatic, there obviously would exist a "good configuration" (of the other color-scheme) with apex at V, the "characteristic triangle" of which would be contained in one of the parts forming the supposedly minimal one we started with. The contradiction reached completes the proof of Theorem 1. ▥

Although both proofs of Theorem 1 have great similarities with proofs of Sylvester's problem about simple points, there are significant and interesting differences. First, as mentioned by Motzkin [1967], a "combined theorem" that would assert the existence of a simple monochromatic vertex does not hold. Also, the number of monochromatic vertices does not have to increase with the number of lines in the arrangement. Indeed, in each of the 3 known infinite families of simplicial arrangements (Grünbaum [1971], [1972]) it is possible to color the lines by 2 colors so that there is only one monochromatic vertex, and it is incident with as many lines as desired. The examples in Figures 3, 4, and 5 should be sufficient to explain the idea.

For an arrangement $\mathcal{A}$ colored by k colors $1,2,\ldots,k$, let $n_i = n_i(\mathcal{A})$ denote the number of lines of color i, and let $s_i = s_i(\mathcal{A})$ denote the number of monochromatic vertices of $\mathcal{A}$ all lines through which have color i. A k-colored arrangement $\mathcal{A}$ is called biased provided $s_i(\mathcal{A}) = 0$ for all $i \geq 2$. In a 2-colored biased arrangement we call $\delta(\mathcal{A}) = n_2(\mathcal{A}) - n_1(\mathcal{A})$ the chromatic deficit of $\mathcal{A}$.

Correction

In Conjecture 1 the words "that is not a near-pencil" should be replaced by "for which the lines of color 2 do not form a trivial arrangement (pencil)".

Analogous corrections should be made in Conjecture 2, and in Remark 4 (on page 8).

Motzkin [1967] mentions that he is aware of 10 infinite families of 2-colored biased arrangements $\mathcal{A}$ with $0 \leq \delta(\mathcal{A}) \leq 1$, and that the greatest chromatic deficit $\delta(\mathcal{A}) = 4$ known to him occurs in the arrangement of Figure 6. (Color 1 = red, color 2 = green). We supplement Motzkin's findings by the following:

Theorem 2. There exist at least two infinite families of 2-colored biased arrangements with $\delta = 4$.

(Obviously, strictly speaking the only formal assertion of this theorem is that there are infinitely many examples; however, as seen in the proof, they form "families" in a natural way.)

Proof. The first such family has $n_1 = 6k$, $n_2 = 6k + 4$, for $k = 1,2,\dots$. Its formation is best seen from Figure 7, in which the cases $k = 1,2,3$ are illustrated (in each case, the line at infinity belongs to the arrangement and has color 2 = green). The second family is illustrated (by cases $k = 1,2,3$) in Figures 7a and 8. Each arrangement in it may be interpreted as obtained from an arrangement in the first family by deleting certain lines.

To complement Theorem 2 we make

Conjecture 1. For each 2-colored biased arrangement (that is not a near-pencil) the chromatic deficit is at most 4; $\delta(\mathcal{A}) \leq 4$.

2-colored arrangements with a single monochromatic vertex appear to have many special properties. We have:

Theorem 3. (Shannon [1974]) If $\mathcal{A}$ is a 2-colored biased arrangement with $s_1(\mathcal{A}) = 1$, then all the lines of $\mathcal{A}$ that have color 1 are incident with that vertex.

Conjecture 2. If $\mathcal{A}$ is a 2-colored biased arrangement (not a near-pencil) with $s_1(\mathcal{A}) = 1$ then $\delta(\mathcal{A}) \leq 1$.

The arrangements $A_1(4k+1)$ in Figure 5 show that equality in this conjecture holds for infinitely many arrangements. Actually,

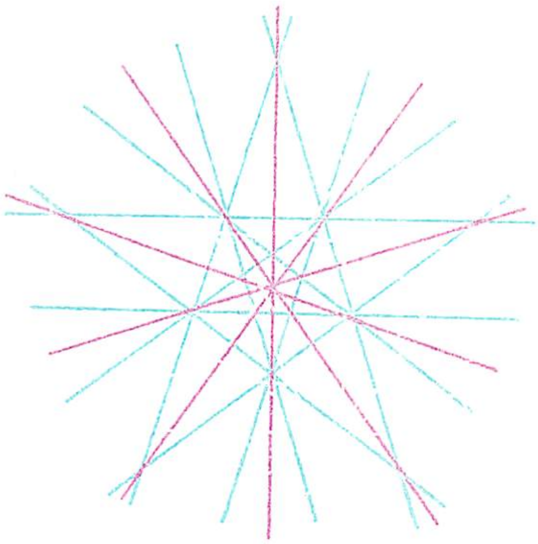


Figure 6. $A_4(16)$, with red line at infinity.

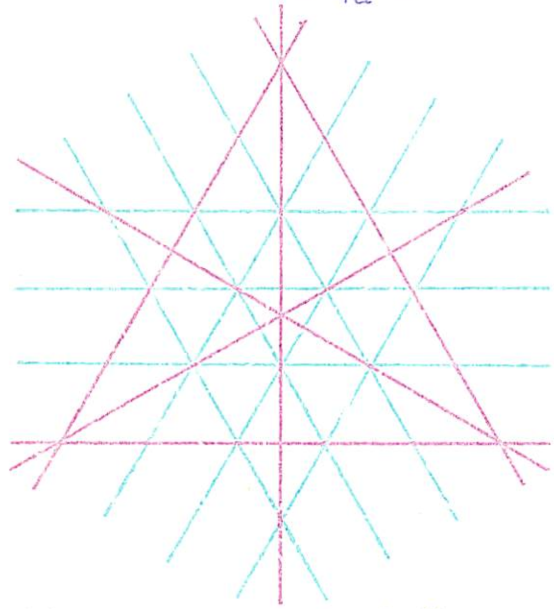


Figure 7a. $A_3(16)$
6 red lines, 10 green, with green line at infinity.

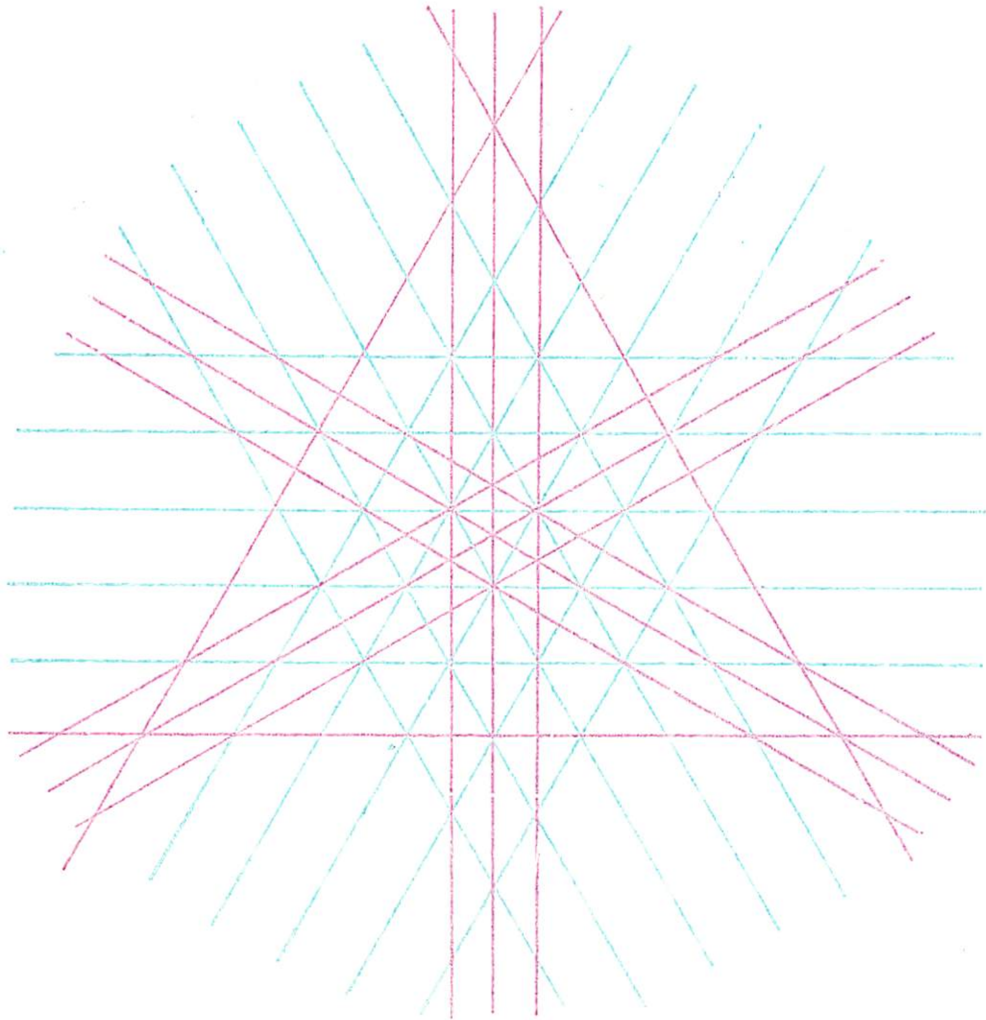


Figure 7b.
12 red lines
16 green,
with ∞
green.

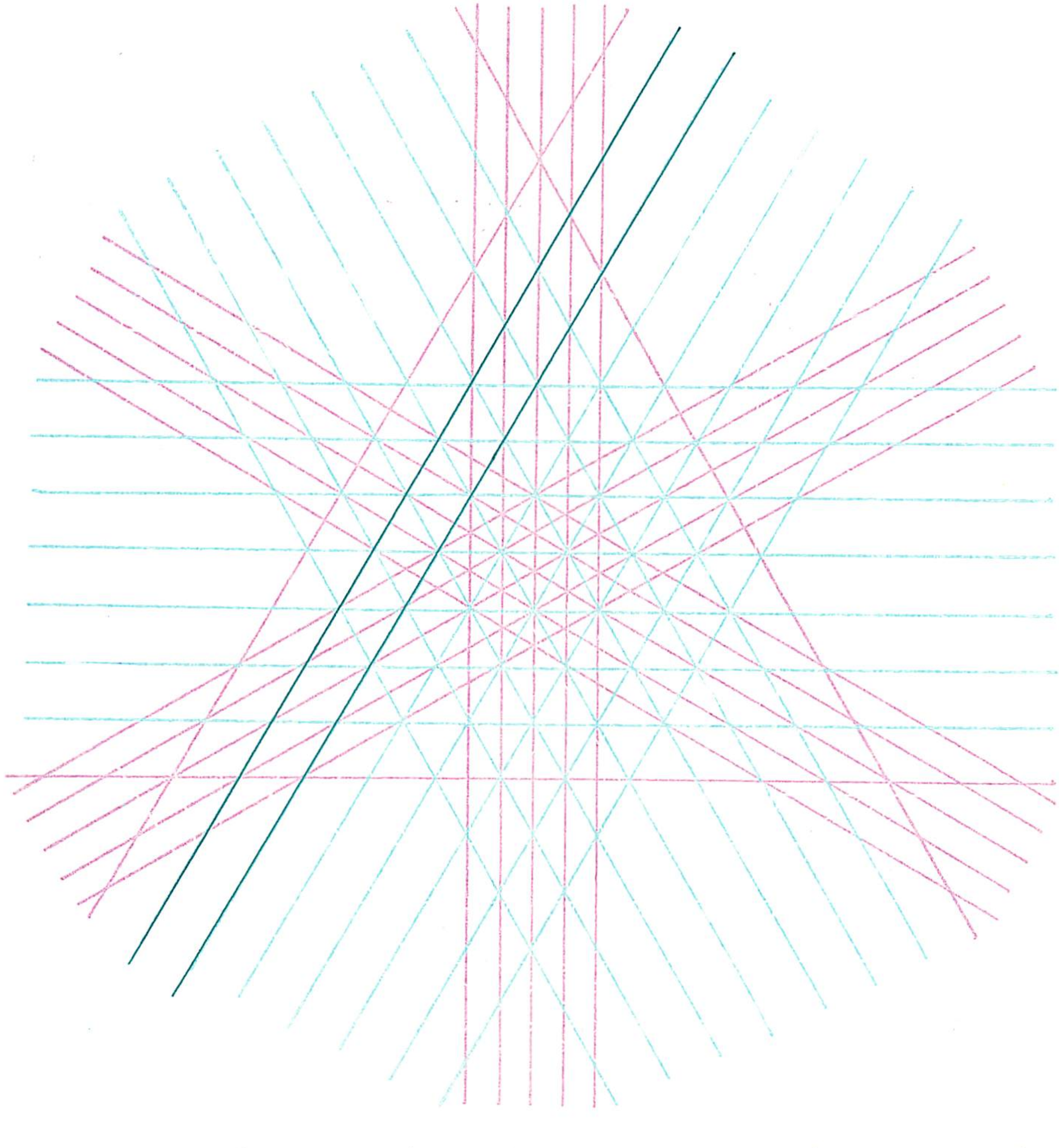


Figure 7c. 18 red lines, 22 green ones (including ∞).

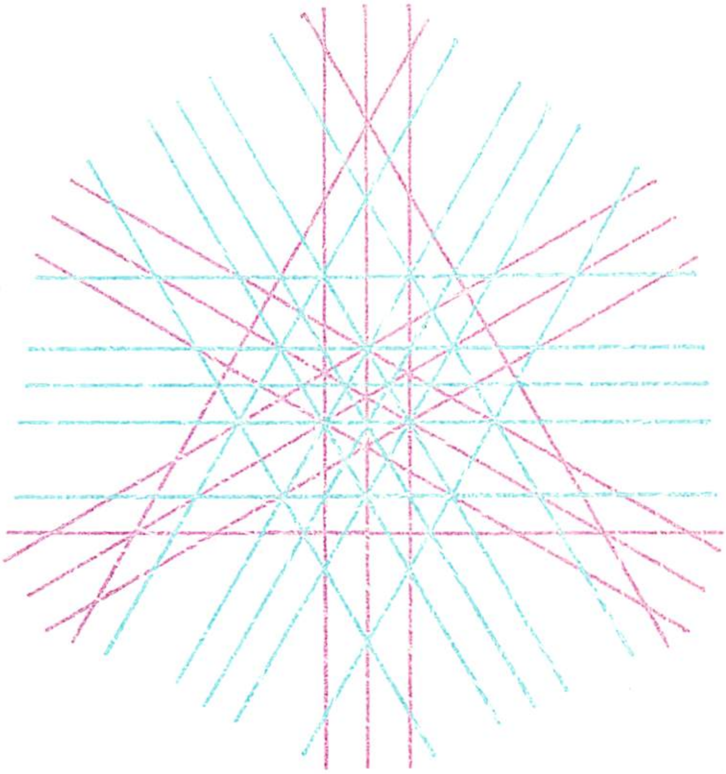


Figure 8a.
12 red lines
16 green (including ∞)

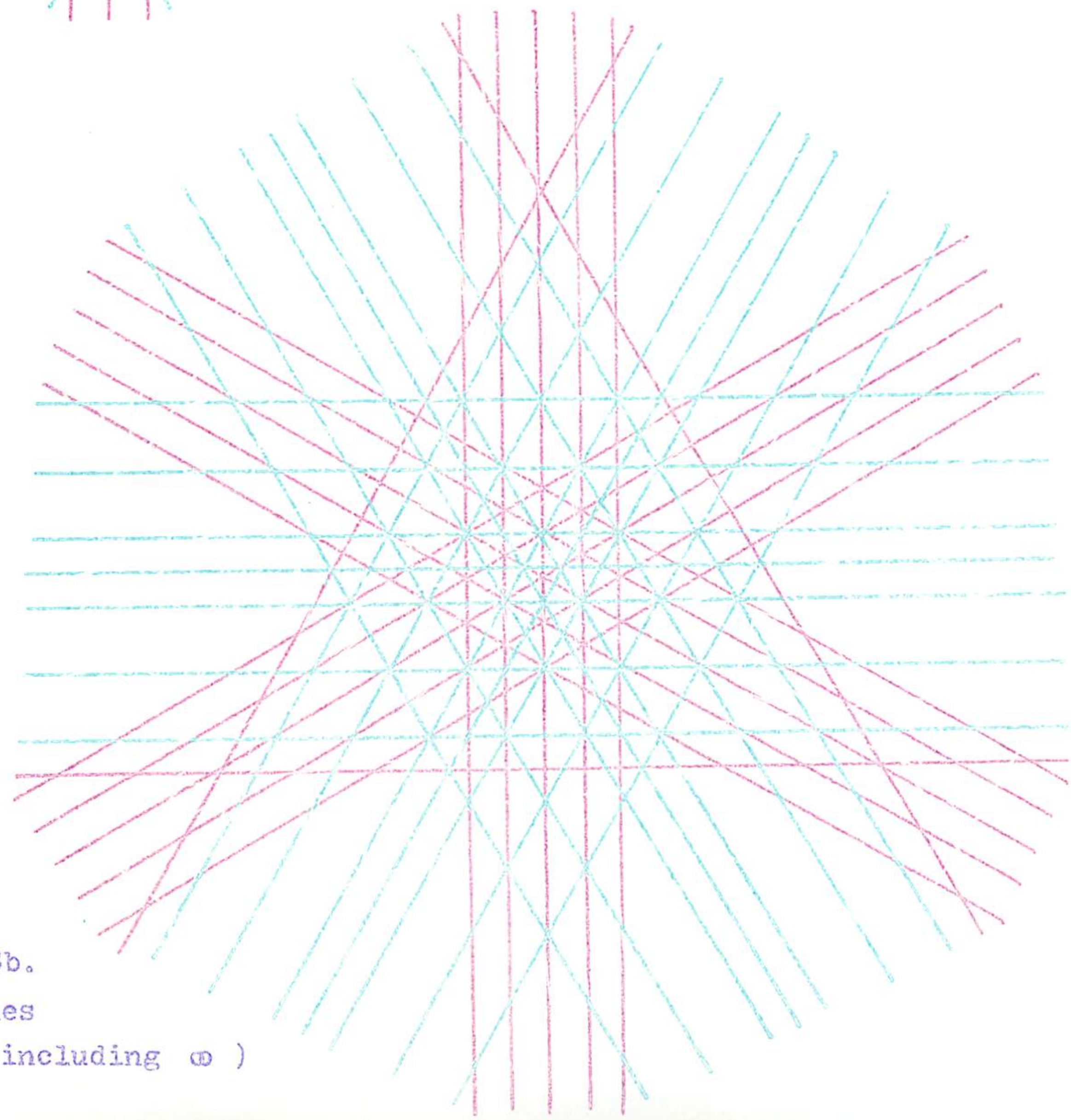


Figure 8b.
18 red lines
22 green (including ∞)

there are many other 2-colored biased arrangements $\mathcal{A}$ with $s_1(\mathcal{A}) = 1 = \delta(\mathcal{A})$.

Turning to a different type of questions, we shall say that a k-coloring of an arrangement $\mathcal{A}$ is nice provided $k \geq 2$ and no vertex of $\mathcal{A}$ is bichromatic (that is, incident with lines of just two different colors). The results of Grünbaum [1956], Edelstein [1957], Herzog-Kelly [1960], Edelstein-Herzog-Kelly [1963], Edelstein -Kelly [1966] may be reformulated (by duality) to assert that among certain families of arrangements of infinitely many lines only the trivial arrangements are nicely colorable (by any number of colors). However, as noted in several of those papers, there exist non-trivial nicely colored (finite) arrangements. The (dual of the) first example (Herzog-Kelly [1960]) is shown in Figure 9b; it may be generalized to the nicely 3-colorable arrangements of 3k lines, $k \geq 2$, obtained by extending the sides of a regular (2k)-gon, and taking also its k longest diagonals (Edelstein-Kelly [1966]);] see Figure 9 for k = 2, 3,4,5 . The number of colors used in nicely colored arrangements may be arbitrarily large, as is shown by the following result (in which the notation of Grünbaum [1971] is used):

Theorem 4. All arrangements $A_1(4k+2)$ and $A_1(8k+4)$, where $k \geq 1$, have nice (2k+1)-colorings.

Proof. The method of coloring is rather clearly indicated in Figure 10 and 11, which illustrate the cases k = 1, 2 .

It may be observed that many other simplicial arrangements have nice colorings. For example, $A_4(13)$ has a nice 7-coloring, $A_1(15)$ has a nice 5-coloring, while $A_5(15)$ has nice colorings with 6, 7, 8, or 9 colors (see Figure 12).

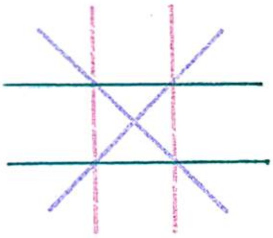


Figure 9a

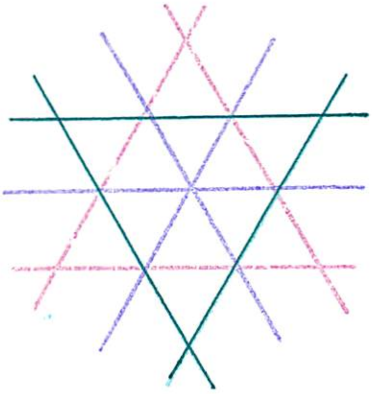


Figure 9 b

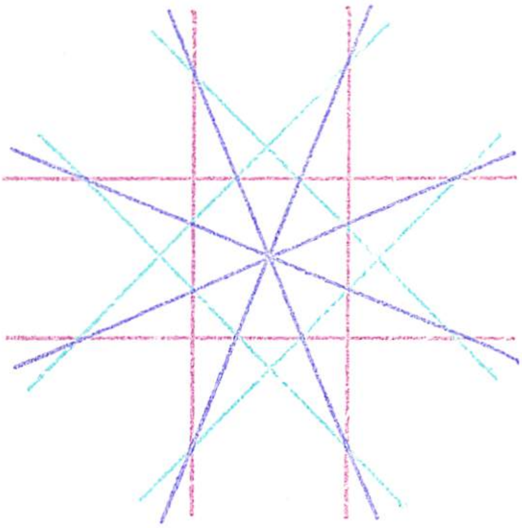


Figure 9c

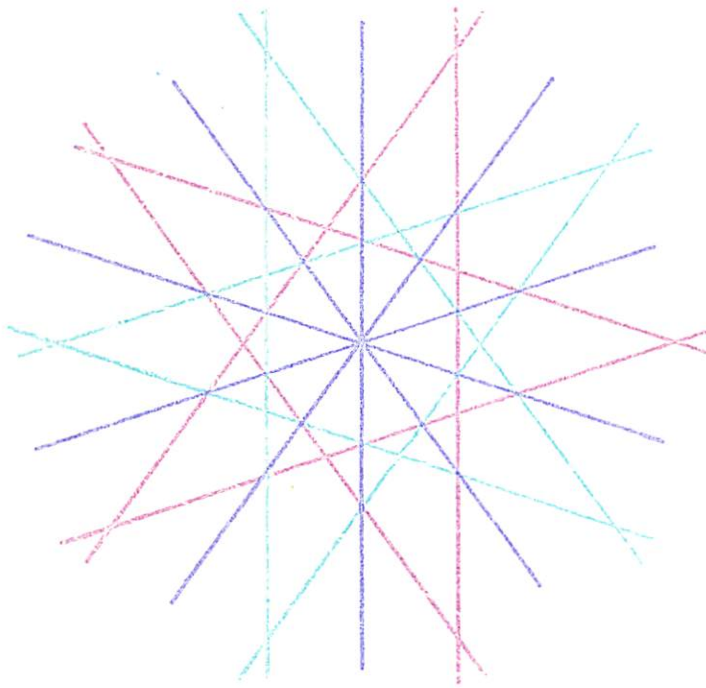


Figure 9d

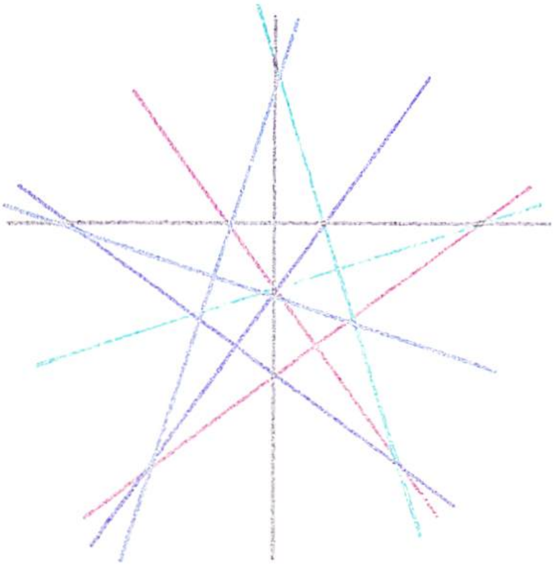
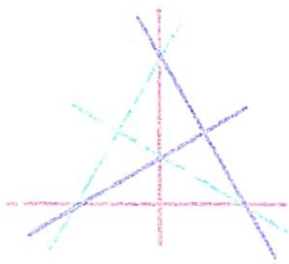
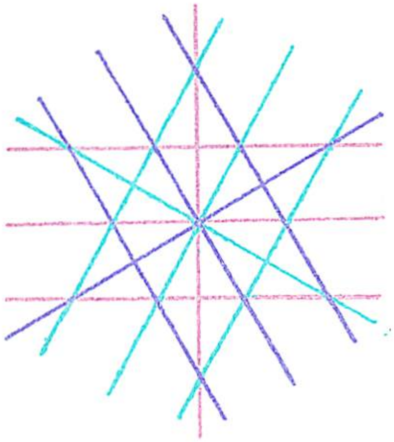


Figure 10





$A_1(12)$

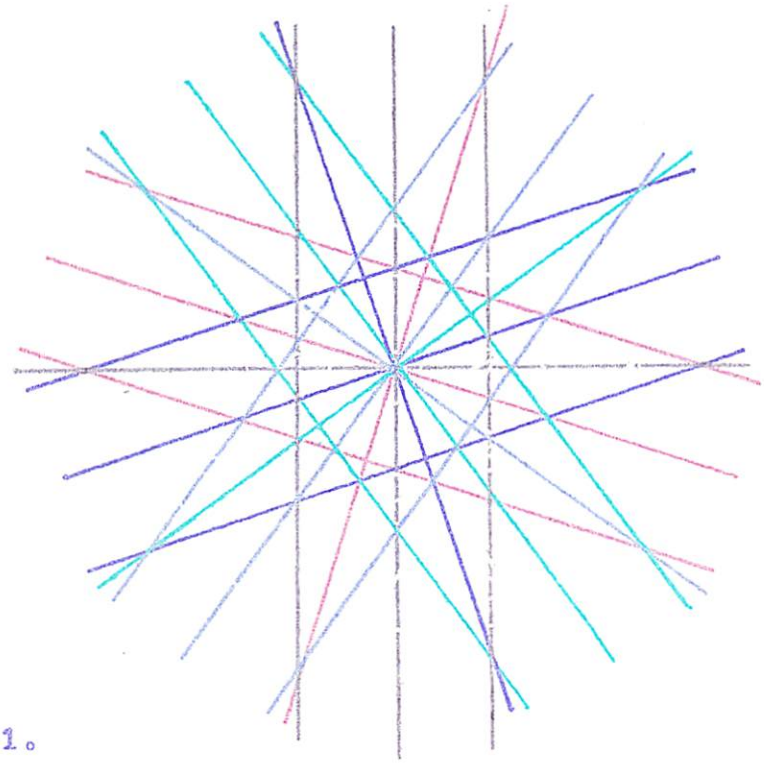


Figure 11.

$A_1(20)$

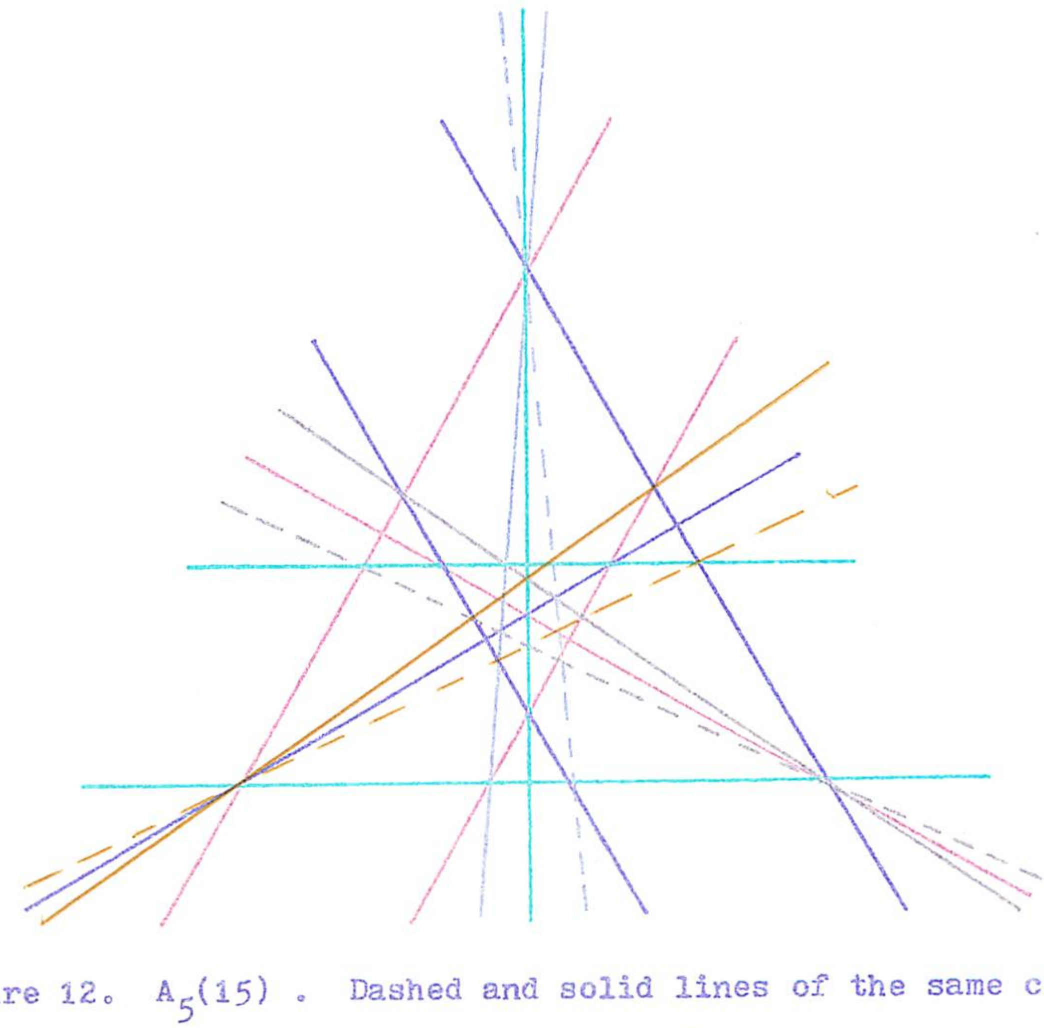


Figure 12. $A_5(15)$. Dashed and solid lines of the same color may be assigned different colors as well.

In all known examples of nicely colored arrangements each color is assigned to relatively few lines. We have two conjectures:

Conjecture 3. If an arrangement is nicely k-colored with $k \geq 5$, then some color is assigned to 4 or fewer lines.

Conjecture 4. There exists an absolute constant c such that for each nice k-coloring of an arrangement with $k \geq 4$, each color is assigned to at most c lines.

The example of a nicely 5-colored arrangement of 15 lines shown in Figure 13 proves that if c exists then $c \geq 7$.

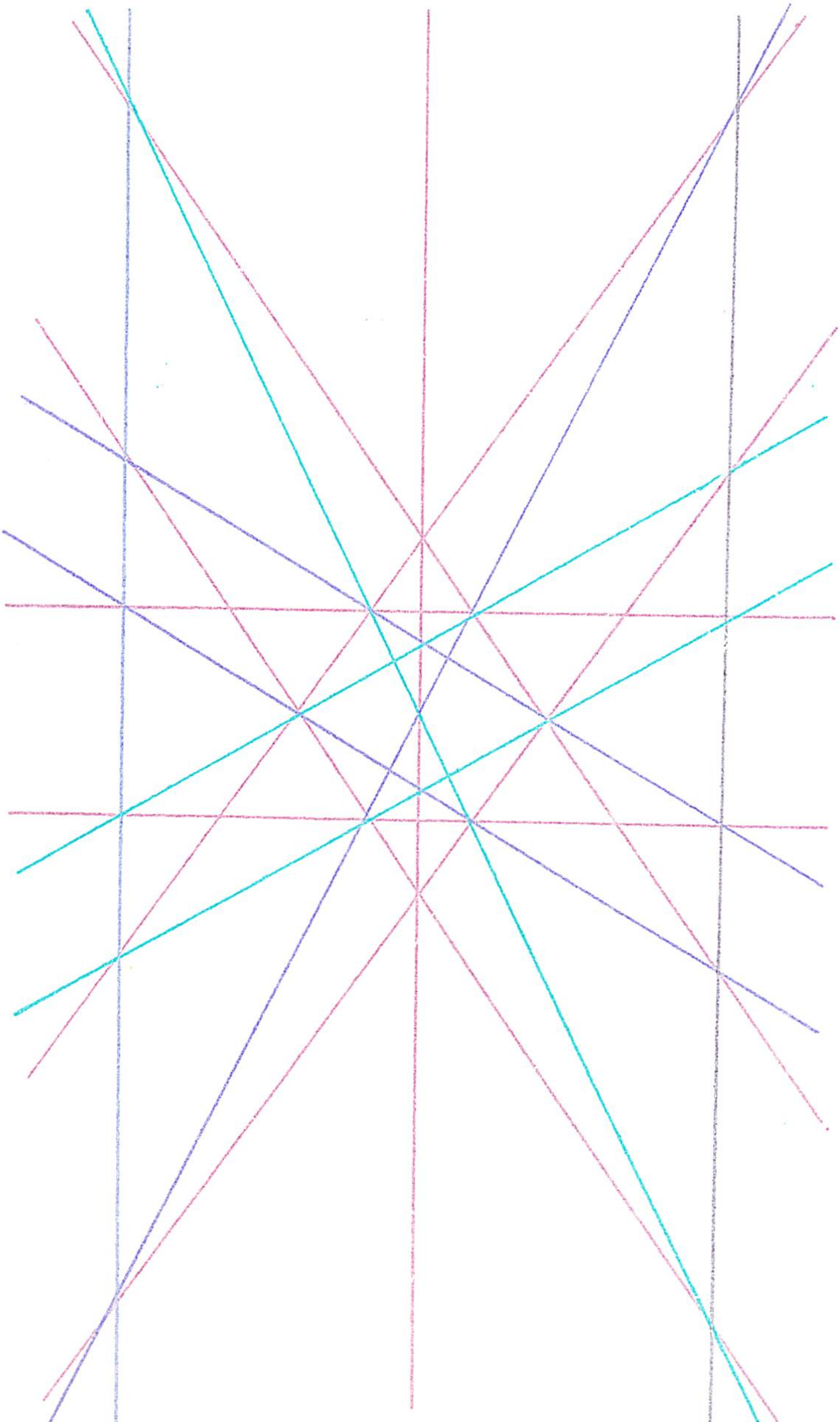


Figure 13.

Remarks and problems.

1. From biased arrangements with a single monochromatic vertex (such as those in Figures 4 and 5) it is easy to obtain examples of non-trivial arrangements colored by 3 colors (each assigned to arbitrarily large numbers of lines) that have no monochromatic vertices (Motzkin [1967]).

2. Generalizing Theorem 1 to d-arrangements, Shannon [1974] has proved the following result:

If the hyperplanes of a d-arrangement $\mathcal{A}$ in projective d-space are colored by at most d colors, there exists in $\mathcal{A}$ a monochromatic (d-2)-flat (that is, a (d-2)-flat that is determined by the hyperplanes of $\mathcal{A}$, and is such that all hyperplanes of $\mathcal{A}$ that contain it have the same color).

3. Already for d=3 it is easy to find 2-colored d-arrangements with no monochromatic vertices. For example (in the notation of Section 3 of these Notes) it is possible to 2-color the planes in all arrangements $A^1(p) \vee A^1(q)$, with $p, q \geq 2$, so that no vertex be monochromatic, and the same is possible in every arrangement $A^0(0) \vee \mathcal{A}$, where $\mathcal{A}$ is a biased 2-colored arrangement of lines in P^2. But there are examples with this property that are not joins of lower-dimensional arrangements. For example, if 5 of the planes of symmetry in $A_1^3(10)$ are given one color, while the sixth plane of symmetry and the 4 other planes have another color, there is no monochromatic vertex. Similarly for $A_1^3(15)$ (with 6 planes of symmetry of one color, 6 facet-planes and 3 mid-planes of the other), $A_1^3(16)$, $A_1^3(18)$ (6 planes of symmetry have one color while 8 facet-planes, 3 midplanes and plane at infinity have the other), $A_1^3(27)$,

$A_1^3(28)$, and $A_2^3(28)$ (6 planes of symmetry and 3 midplanes one color, the other 19 planes the other). It would be interesting to know whether every 3-arrangement without simple vertices has this property.

4. Theorem 1 (and both proofs given for it) generalize to arrangements of pseudolines in P^2 . However, Conjecture 1 does not hold for such arrangements. Indeed, in Figures 14 and 15 we show two 2-colored biased arrangements of 28 pseudolines each, such that the chromatic deficit of each equals 6 . We conjecture that the chromatic deficit of every 2-colored biased arrangement of pseudolines (other than a near-pencil) is at most 6 .

5. Theorem 1 and Chakerian's [1970] proof of it apply as well to the case of digon-free arrangements of curves. (Concerning such arrangements, and the terminology we use, see Grünbaum [1972].) However, if digons are permitted, Theorem 1 need not hold. In Figure 16 we show two non-trivial examples of 2-colored arrangements of circles without monochromatic vertices.

6. Mediating Shannon's theorem 3 it is easy to see that Conjecture 2 is equivalent with the following conjecture of Shannon:

If n points of the Euclidean plane are not collinear they determine at least n-1 different slopes (directions) .

7. As a counterpart to the existence of nicely colored arrangements, Herzog-Kelly [1960] prove that there is no non-trivial arrangement colored with $k \geq 2$ colors that lacks both bichromatic and trichromatic vertices.

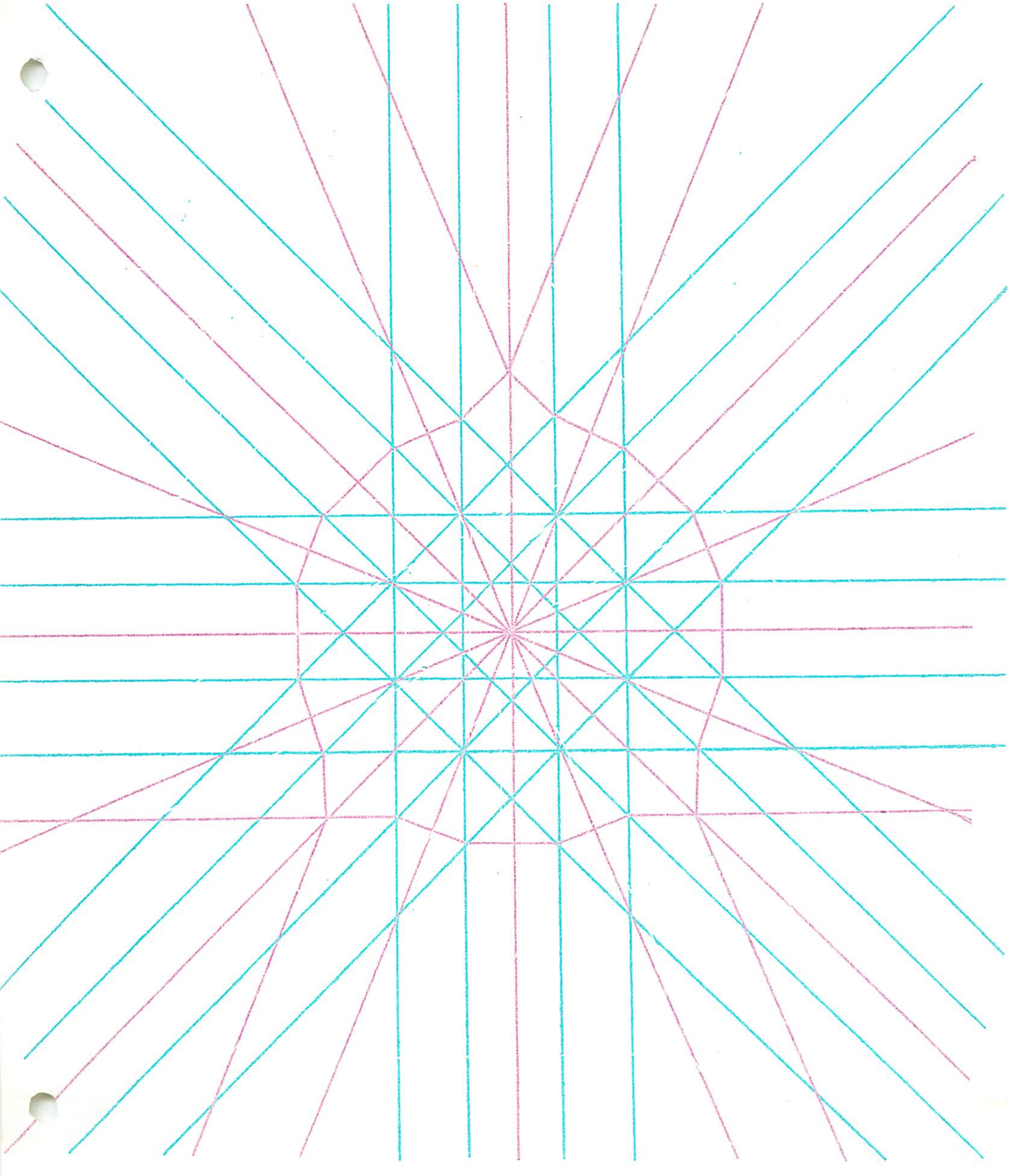


Figure 14. 11 red pseudolines, 17 green ones (including ∞) .

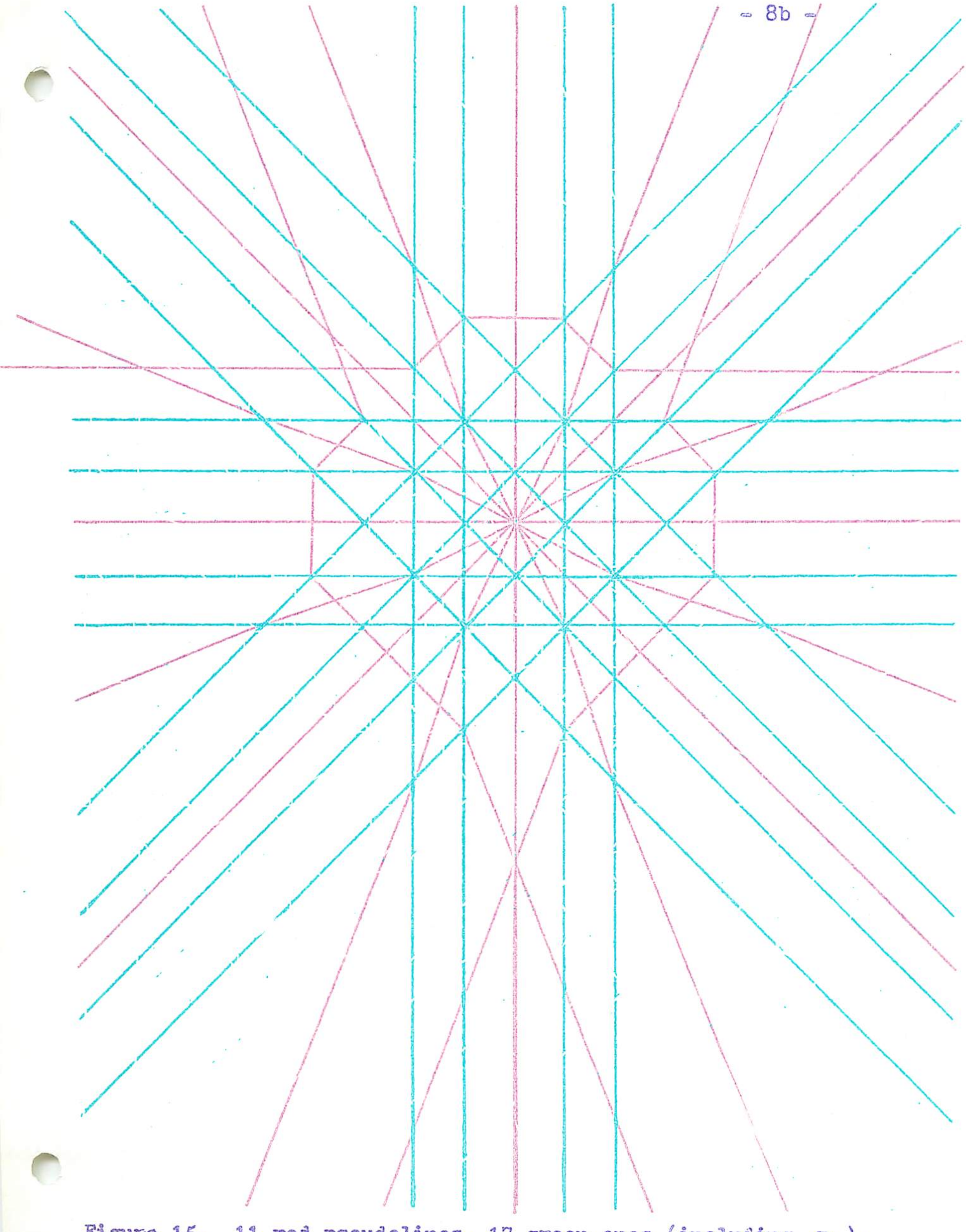


Figure 15. 11 red pseudolines, 17 green ones (including ∞).

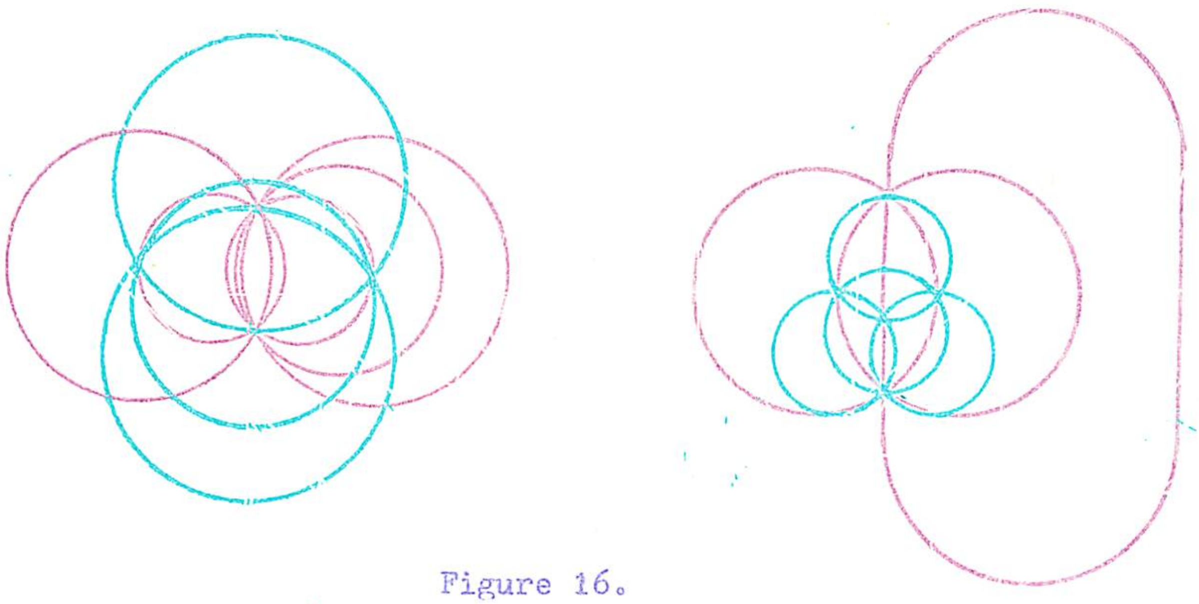
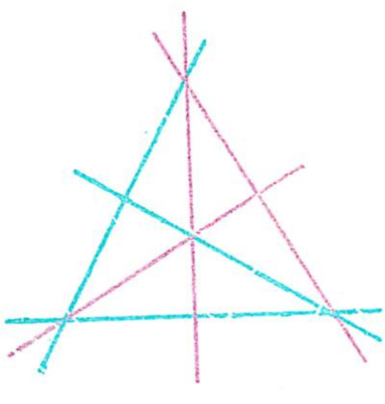
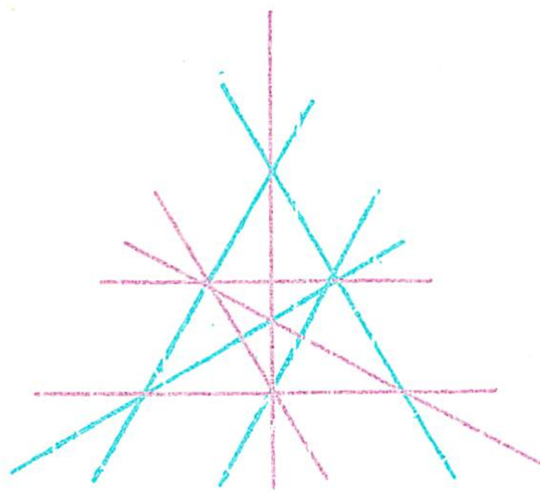


Figure 16.



n = 6   m = 2



n = 9   m = 3

Figure 17. Properly 2-colored arrangements.

8. A k-coloring of a d-arrangement $\mathcal{A}$ is <u>nice</u> provided each (d-2)-flat of $\mathcal{A}$ is either monochromatic, or else is contained in hyperplanes of at least 3 colors. It was observed by Edelstein-Kelly [1966] that the 3-arrangement $A_1^3(12)$ (formed by the 6 facet-planes of a cube, and its 6 planes of mirror-symmetry through pairs of opposite edges) has a nice 3-coloring (each color is assigned to two parallel facet-planes and the two symmetry planes perpendicular to them). Remarkably, no other (non-trivial) nicely colored 3-arrangement is known. Moreover, Edelstein-Kelly [1966] prove: For $d \geq 4$ there exist no non-trivial nicely colored d-arrangements.

9. In the dual formulation, the notion of nice k-coloring may be phrased as follows: A set $S \subset P^d$ is k-<u>nice</u> provided each partition of S into k disjoint non-empty sets $S_1,\dots,S_k$ has the property that any line meeting two distinct S_i's meets at least one more S_i. The papers mentioned on page 5 deal mainly with the following types of problems related to k-niceness: Prove, under suitable conditions on the nature of the pairwise disjoint sets $K_1,\dots K_k \subset P^d$, that if every line meeting two eof the K_i's meets at least a third, then $\bigcup_i K_i$ is contained in a line. We shall say that such families have the <u>Sylvester property</u>, since the original problem of Sylvester was to show that families in which each K_i is a single point have that property. The result of Edelstein-Kelly [1966] mentioned in 8. above establishes the Sylvester property of families of finite sets such that the affine hull of $\bigcup_i K_i$ has dimension at least 4. Edelstein-Herzog-Kelly [1963] prove the Sylvester property for families of compact sets K_i such that $\bigcup_i K_i$ is infinite.

10. An arrangement $\mathcal{A}$ of lines in P^2 is properly k-colored provided each line of $\mathcal{A}$ is assigned to one of k colors so that every monochromatic vertex is incident with two lines only. It is easily checked that the arrangement $A_1(10)$ (see Figure 4) is not properly 2-colorable. Although we have not established the existence of a k such that every arrangement is properly k-colorable, no examples are known that need 4 colors for a proper coloring.

Conjecture. Every arrangement of lines in P^2 is properly 3-colorable.

Another open problem is whether in properly 2-colored arrangements of n lines (other than near-pencils) the number m of monochromatic vertices is necessarily large for large n. Experimental evidence seems to point that way. The largest n for which a properly 2-colored arrangement of n lines with $m = 2$ is known is $n = 6$; for $m = 3$ the largest known n is 9 (see Figure 17).

11. In a recent letter, M. A. Rabin informed me that he had heard of Graham's problem in 1966, and had at time found a proof of Theorem 1. His paper with T. S. Motzkin, mentioned in Chakerian [1970], will soon be submitted for publication; it will contain (among others) proofs of the statements in Motzkin [1967].

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Omittable lines.

Following Koutský-Polák [1960] (who dealt with the dual situation) we shall call a line L of an arrangement $\mathcal{A}$ of lines in P^2 omittable provided each vertex of $\mathcal{A}$ is also a vertex of the arrangement $\mathcal{A} \setminus \{L\}$; in other words, L is omittable provided no vertex of $\mathcal{A}$ that lies on L is simple. Let $s(\mathcal{A},V)$ denote the number of omittable lines of $\mathcal{A}$ that pass through the vertex A of $\mathcal{A}$.

Koutský-Polák [1960] proved the following result:

Theorem 1. $s(\mathcal{A},V) \leq n(\mathcal{A})/3$.

Proof. Consider the dual situation, in which points P_i are given, and all lines determined by them form the arrangement. Let the collinear omittable points $P_1,\ldots,P_s$ be on the line at infinity. Let K be the convex hull of the remaining points $P_{s+1},\ldots,P_n$; K is clearly 2-dimensional. Each of the two support lines of K through the point P_i (for each $i = 1,2,\ldots,s$) contains at least two of the points P_j , $s < j \leq n$; hence K has at least $2s$ vertices, $n \geq 3s$, and the theorem is established.

The estimate in Theorem 1 is best possible, as is shown by the examples in which $n = 3s$, and the lines of $\mathcal{A}$ are : (i) the $2s$ extended sides of a regular $(2s)$-gon centered at V ; (ii) the s lines of symmetry of the $(2s)$-gon that pass through pairs of diametral vertices.

Another result of Koutský-Polák [1960] is:

Theorem 2. Given k lines $L_1,\ldots,L_k$ passing through a common vertex V , there exists an arrangement $\mathcal{A}$ such that the lines $L_1,\ldots,L_k$ are the only omittable lines of $\mathcal{A}$.

Proof. We shall prove this also in the dual formulation; the cases $k = 1$ and $k = 2$ are trivial. If $k = 3$, let the collinear points P_1, P_2, P_3 (which are to become omittable) be at the line at infinity. We take points V_1, V_2 collinear with P_1, then consider translates of them, V_3, V_4, in direction of P_2, then translates of these four in the direction of P_3. If there are more points (directions), we keep repeating the process, arriving at a set consisting of at most 2^k additional points and having the given k points omittable. Careful choices of the translations used prevent the introduction of other omittable points. (See Fig. 1 for $k = 3$.)

Let $\mathring{\mathcal{A}}$ denote the arrangement generated by the omittable lines of $\mathcal{A}$, let $s(\mathcal{A})$ denote the number of lines in $\mathring{\mathcal{A}}$, and let $s(n) = \max\{s(\mathcal{A}) \mid n(\mathcal{A}) = n\}$. From the example following Theorem 1 (and easy variants) it is obvious that $s(n) \geq [n/3]$; if the (2s)-gon used in the construction has odd s, by adjoining the line at infinity we see that $s(6k+4) \geq 2k+2$ (which is strictly greater than $[(6k+4)/3]$).

The only other cases in which a value of $s(n)$ greater than $[n/3]$ is known are listed in Table 1. Concerning the behaviour for large n we venture

Conjecture. $\lim_{n \to \infty} s(n)/n = 1/3$.

Other related problems are: Given an arrangement $\mathcal{A}_0$ of lines $L_1, \ldots, L_k$, does there exist an arrangement $\mathcal{A}$ that extends $\mathcal{A}_0$ for which the lines $L_1, \ldots, L_k$ are omittable (that is, $\mathcal{A}_0 \subset \mathring{\mathcal{A}}$). An analogous problem arises if it is required that the lines of $\mathcal{A}_0$

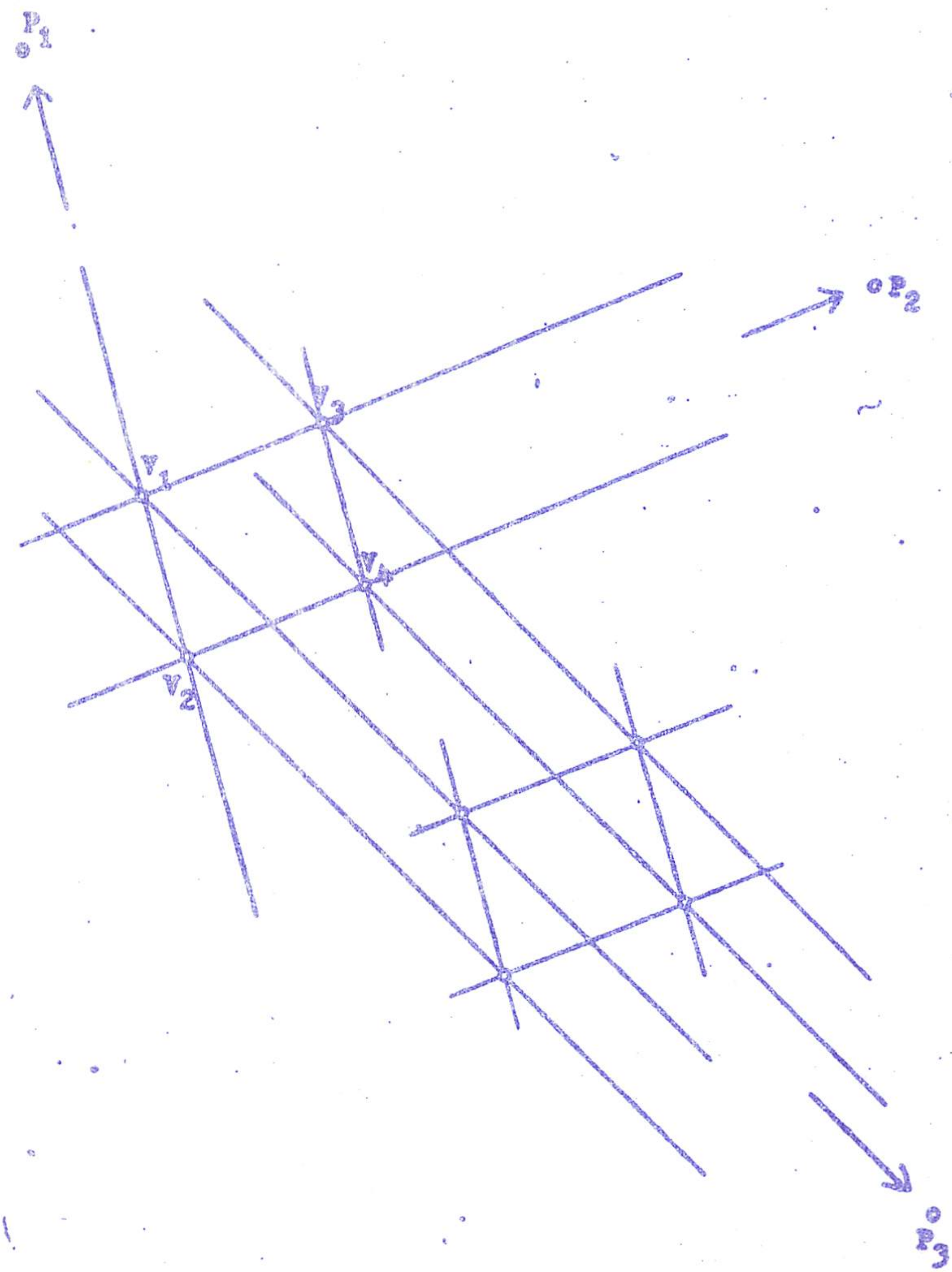

Figure 1.

be precisely the omittable lines of $\mathcal{A}$ (that is, $\mathcal{A}_0 = \mathring{\mathcal{A}}$); we shall call this the strict problem.

It is not hard to see that the answer is affirmative (even for the strict problem) whenever $k \leq 4$. Also, a modification of the construction used in the proof of Theorem 2 shows that the answer is affirmative (even for the strict problem) whenever $\mathcal{A}_0$ is a near-pencil. The only other cases in which an answer is known are those noted in Table 1, or derivable from them.

Conjecture. There exists an $\bar{n}$ such that if $n(\mathring{\mathcal{A}}) \geq \bar{n}$, where $\mathring{\mathcal{A}}$ is the set of omittable lines of some arrangement $\mathcal{A}$, then $\mathring{\mathcal{A}}$ is either trivial, or a near-pencil.

n	s(n) ≥	Example	$\mathcal{A}$ for that example
10	4	* $A_3(10)$	$A_0(4)$
15	6	$A_5(15)$	No 3 lines concurrent
16	6	*	$A_0(6)$
21	9	$A_3(21)$	$A_1(9)$
22	8	*	$A_0(8)$
25	10	$B_{11}(25)$	$A_1(10)$
28	10	*	$A_0(10)$
34	10	*	$A_0(12)$
37	13	$A_3(37)$	$A_2(13)$
40	14	*	$A_0(14)$
40	16	$B_2(40)$	8 sides of regular octagon and 8 lines through its center
46	16	*	$A_0(16)$

Table 1. * indicates arrangement obtained by the construction mentioned on page 2. $A_j(n)$ refers to the simplicial arrangements listed in Grünbaum [1971], and in part also in these notes. $B_j(n)$ refers to the simplicial arrangements of pseudolines shown in Figures 2 and 3.

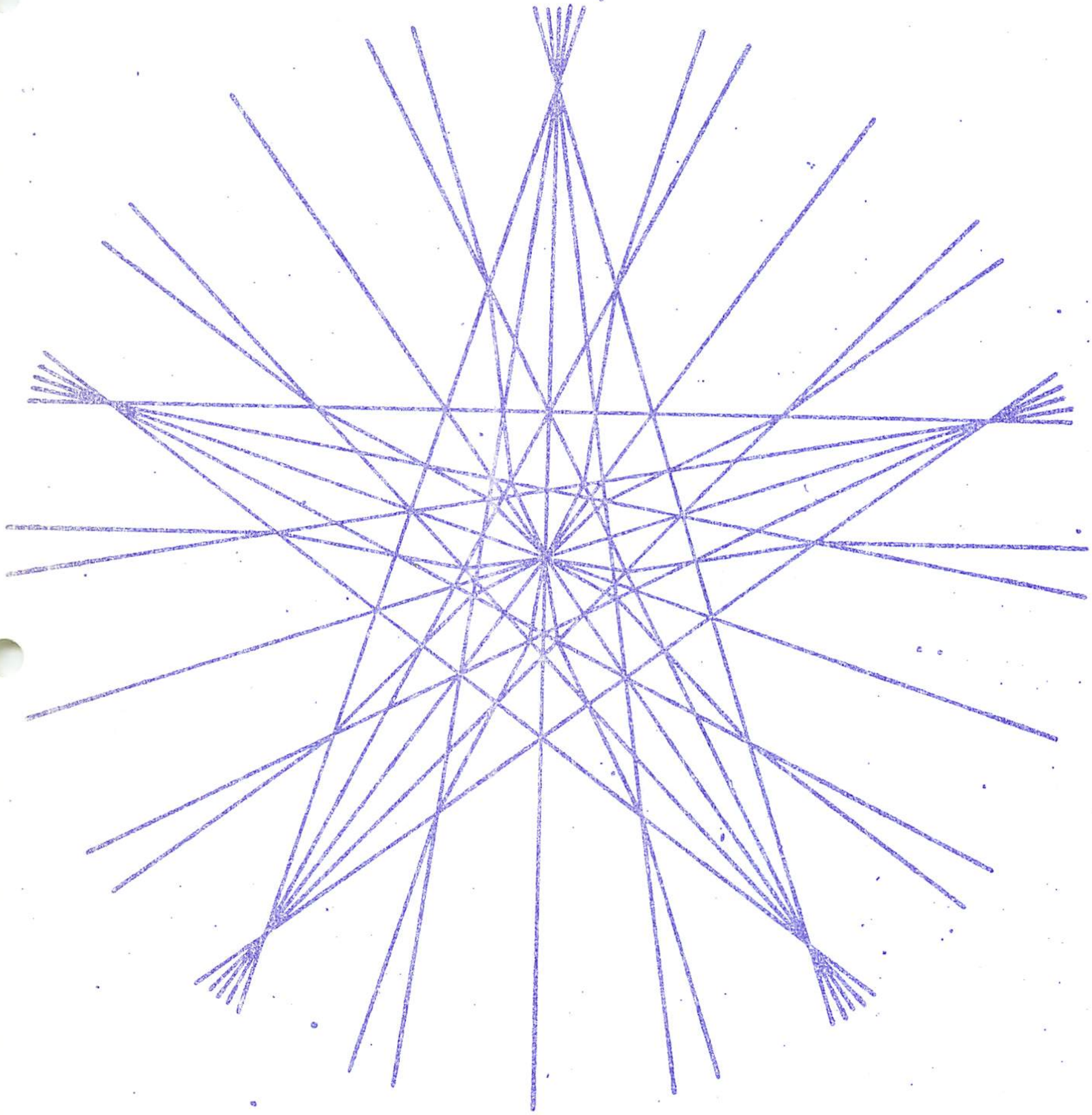

Figure 2. $B_{11}(25)$, a simplicial arrangement of 25 pseudolines.

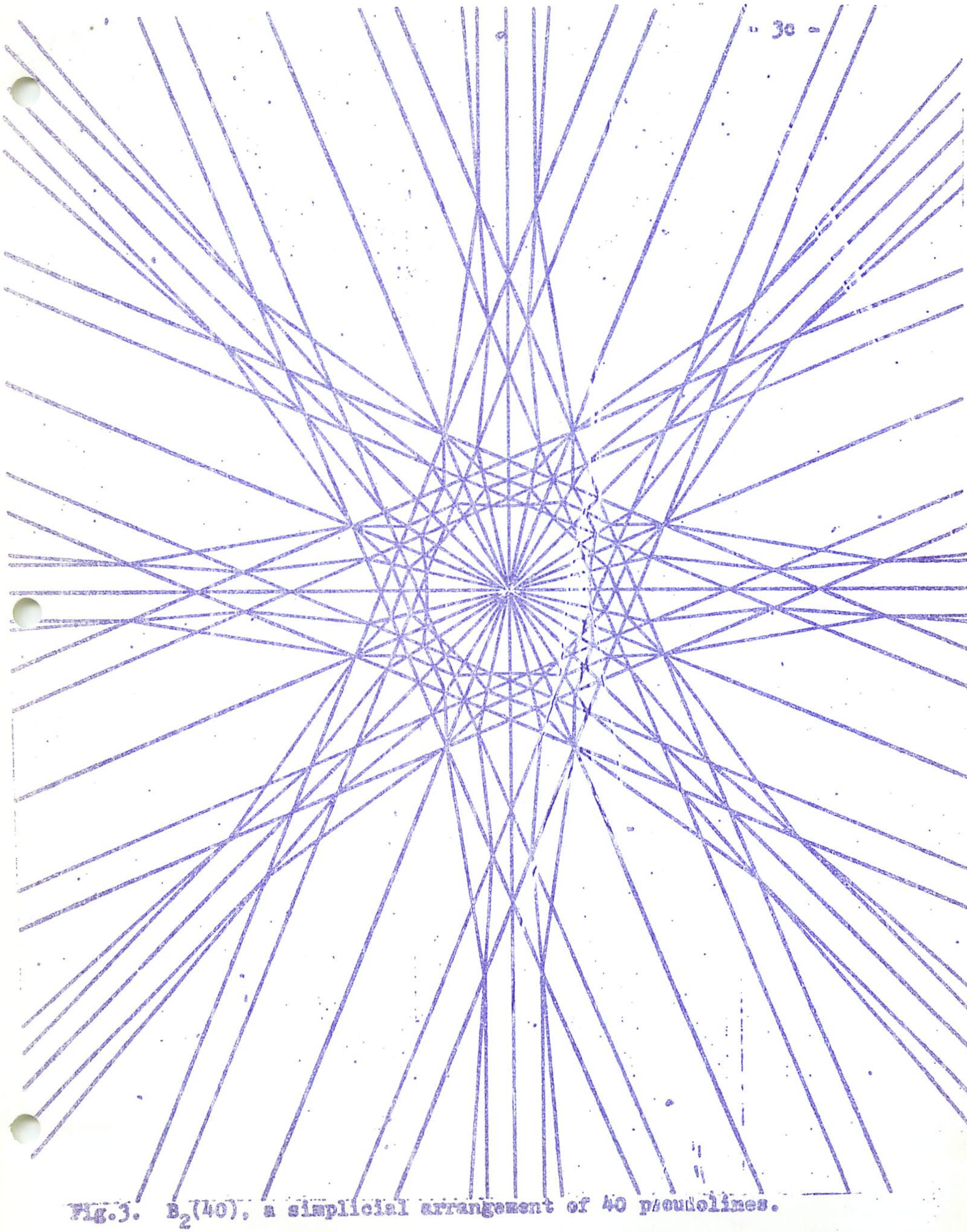

Fig.3. $B_2(40)$, a simplicial arrangement of 40 pseudolines.

Exercises, problems.

1. Provide a detailed proof of the assertion made on page 3 that the strict extension problem has an affirmative solution whenever $\mathcal{A}_0$ is a near-pencil. (In Figure 4 one way of showing it is indicated in case $\mathcal{A}_0 = A_0(4)$; again the dual formulation of the problem is employed.)

\# 2. Determine whether the 5 extended sides of a regular pentagon can be (the only) omittable lines of some arrangement.

\# 3. Show that the construction described in the proof of Theorem 2 is "essentially" best possible in the following sense: There exists a constant $c > 0$ such that for each k one can find k concurrent lines $L_1, \ldots, L_k$ with the property that any arrangement $\mathcal{A}$ for which each L_i is omittable satisfies $m(\mathcal{A}) > c\,2^k$.

\# indicates unsolved problem.

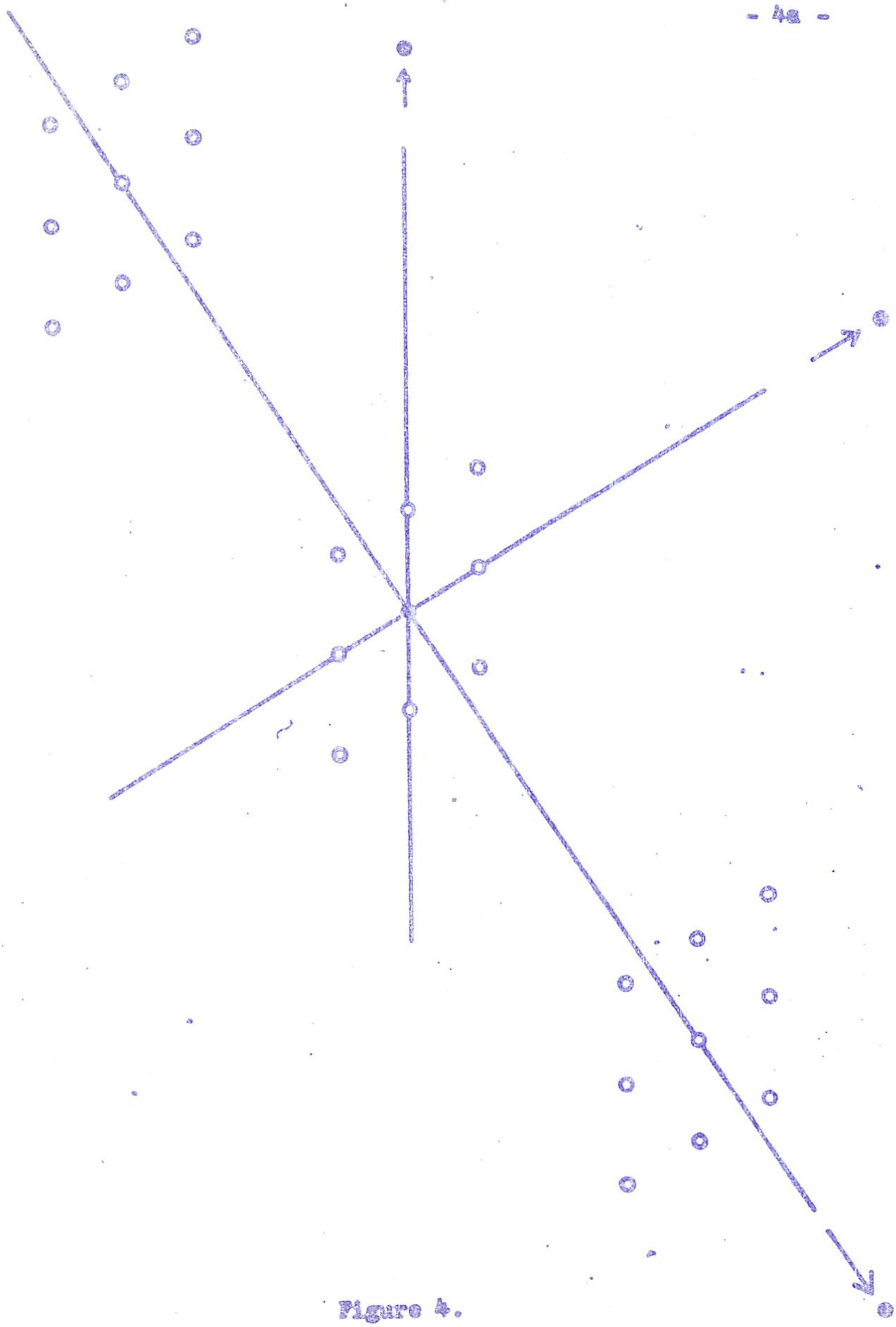

Figure 4.

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