Toward the compactification of the stack of Lie(G)-forms using perfect complexes

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Abstract

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To establish geometric properties of an algebraic stack, one can find a compactification. This method has been successfully employed to find irreducible components for example of the moduli stack of curves [**DM69**], vector bundles on a surface [**O'G96**], and Azumaya algebras on a surface [**Lie09**].

The latter two are moduli stacks of torsors, but these two classify locally free sheaves with possibly additional algebraic structure. For an arbitrary algebraic group G, one can study the stack of Lie(G)-forms, and try to find a compactification via degenerating the underlying locally free sheaves to perfect complexes.

In order to avoid having to truncate the stack, we define a Lie ∞-operad, and Lie algebra objects in the additive symmetric monoidal ∞-category of perfect complexes. In the special case Lie(G) $\cong \mathfrak{sl}_n$, to get a candidate for a compactification as its essential image, we construct a functor mapping a perfect totally supported sheaf of rank n to the Lie algebra object of the traceless part of its derived endomorphism complex, in the setting of higher algebra.

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CHAPTER 1

Introduction

1.1. Compactification of stacks of torsors

Let $X \xrightarrow{f} S$ be a a morphism of schemes. An X-stack $\mathscr{X}$ is a *trivial gerbe*, if it classifies objects locally isomorphic to a given object. More formally, $\mathscr{X}$ is a trivial gerbe, if it satisfies the following criteria.

(1) Its sheafification $\tau_{\leq 0}\mathscr{X}$ is the trivial sheaf h_X.
(2) There exists a global section $X \xrightarrow{s} \mathscr{X}$.

Example 1.1.1. The stack $\mathrm{LF}_X(n)$ of locally free sheaves of rank n on X is a trivial gerbe with global section $s = \mathscr{O}_X^{\oplus n}$.

Example 1.1.2. The stack $\mathrm{Az}_X(n)$ of Azumaya algebras of rank n on X is a trivial gerbe with global section $s = \underline{\mathrm{End}}(\mathscr{O}_X^{\oplus n})$.

Example 1.1.3. Let G be an algebraic group on X. The stack $\mathrm{TF}_X(\mathrm{Lie}(G))$ of $\mathrm{Lie}(G)$-forms is a trivial gerbe with global section the $\mathscr{O}_X$-Lie algebra $s = \mathrm{Lie}(G)$.

Example 1.1.4. Let G be a sheaf of groups on X. The stack of G-torsors $\mathrm{Tors}(G/X)$ is a trivial gerbe with global section $s = G_G$.

Stacks of torsors give canonical representing elements for equivalence classes of trivial gerbes:

Proposition 1.1.5. [**Gir71**, III, Corollaire 2.2.6]

Let $\mathscr{X}$ be a trivial gerbe on X, and let G denote the automorphism sheaf Aut(s). *Then the morphism of stacks*

$$\mathscr{X} \xrightarrow{x \mapsto \mathrm{Isom}(s(p(x)),x)} \mathrm{Tors}(G/X)$$

is an equivalence.

Corollary 1.1.5.1. *Via the identification of the automorphism group of the trivial sections, we get equivalences for the stacks given in Examples 1.1.1 and 1.1.2:*

$$\mathrm{LF}_X(n) \simeq \mathrm{Tors}(\mathrm{GL}_n/X), \quad \mathrm{Az}_X(n) \simeq \mathrm{Tors}(\mathrm{PGL}_n/X).$$

Note that all of Examples 1.1.1, 1.1.2 and 1.1.3 classify locally free sheaves with additional algebraic structure. The method employed to get a compactification which identifies irreducible components is to relax the locally free condition. It has already been successfully carried out in the first two cases, provided that f is a surface over $\mathbf{C}$ resp. a separably closed field:

Theorem 1.1.6. [**O'G96**, Theorem D]

Let (X, H) be a smooth projective polarized complex surface. For $\xi = (r, L, c_2)$ let $\mathcal{M}_\xi$ denote the stack of families of semistable, torsion-free sheaves F with rank r, determinant line bundle L, and second Chern class c_2. Then there exists a positive integer $\Delta(r, X, H)$ such that if

$$\Delta_\xi = c_2 - \frac{r-1}{2r}c_1(L)^2 > \Delta(r, X, H),$$

then the following statements hold.

1. *The stack $\mathcal{M}_\xi$ is irreducible and proper.*
2. *It is the closure of the open substack of μ-stable vector bundles.*

Theorem 1.1.7. [**Lie09**, Theorem 6.3.1]

Let (X, H) be a smooth projective polarized surface over a separably closed field. Let $\mathscr{P}^{\mathscr{O}}_{X/S}(n)$ denote the stack of totally pure S-flat sheaves of rank n on X with trivialized determinant. Let $\mathrm{gAz}_X(n)$ denote the category fibered in groupoids of algebra objects in the monoidal category $(\mathbf{D}(U), \otimes^{\mathbf{L}})$, which are locally quasi-isomorphic to $\mathbb{R}\mathrm{End}(F)$ for some $F \in \mathscr{P}^{\mathscr{O}}_{X/S}(n)$. Then the following assertions hold.

1. *The category fibered in groupoids $\mathrm{gAz}_X(n)$ is a stack.*
2. *The morphism*

$$\mathscr{P}^{\mathscr{O}}_X(n) \xrightarrow{p:F \mapsto \mathbb{R}\mathrm{End}(F)} \mathrm{gAz}_X(n)$$

   is a $\boldsymbol{\mu}_n$-gerbe, where the action of $\boldsymbol{\mu}_n$ is given by the $\mathscr{O}_X$-module structure.

Let $c > 0$ and let $\mathscr{X} \xrightarrow{\pi} X$ be a $\boldsymbol{\mu}_n$-gerbe. Let $\mathrm{gAz}_{X/S}(n)$ denote the pushforward. Let $\mathrm{gAz}_{\mathscr{X}/S}(n, c)^s \subset \mathrm{gAz}_{X/S}(n)$ denote the substack containing those sections $U \xrightarrow{s} \mathrm{gAz}_{X/S}(n)$ which fit into a 2-Cartesian *diagram of the form.*

$$\begin{tikzcd}
\mathscr{X}_U \arrow[r, "F"] \arrow[d, "\pi"'] & \mathscr{P}^{\mathscr{O}}_{X/S}(n) \arrow[d, "p"] \\
X_U \arrow[r, "s"] & \mathrm{gAz}_X(n)
\end{tikzcd}$$

such that moreover F is stable, and $\deg c_2(F) = c$. *Then there exists $D > 0$ such that if $c > D$, then the following statements are true.*

(1) The stack $\mathrm{gAz}_{\mathscr{X}/S}(n,c)^s$ *is proper and irreducible.*

(2) It is the closure of the open substack of stable Azumaya algebras A with $\deg c_2(A) = c$.

1.2. The need for sheaves of ∞-groupoids

One of the reasons the result for Azumaya algebras was restricted to the surface case, is that this ensures the category fibered in groupoids $\mathrm{gAz}_X(n)$ is a 1-stack. This is not true in general, which can be indicated by the following argument. Suppose that we have a covering $\{U_i \to U\}$, and complexes of quasi-coherent $\mathscr{O}_{U_i}$-modules E_i. If we tried gluing these in the derived category, we would provide quasi-isomorphisms on the restrictions $E_i \xrightarrow{\phi_{ij}} E_j$ such that they satisfy the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$. But as we have learnt from stack theory, the witnesses of equalities of equivalence classes need to be retained, because they need to satisfy coherence criteria. Let us suppose that we have homotopies ϕ_{ijk} from ϕ_{ik} to $\phi_{jk} \circ \phi_{ij}$. These can be pictured as commutative triangles

$$\begin{tikzcd}
E_i \arrow[rr, "\phi_{ik}"] \arrow[dr, "\phi_{ij}"'] & \Downarrow \phi_{ijk} & E_k \\
& E_j. \arrow[ur, "\phi_{jk}"'] &
\end{tikzcd}$$

Following the pasting diagram

$$\begin{tikzcd}
 & & E_k & \\
 & E_j & \Uparrow\phi_{jk\ell} & \\
E_i & & & E_\ell \\
 & \Downarrow\phi_{ijk} & E_k & \\
 & E_j & &
\arrow[from=1-3, to=3-4, "\phi_{k\ell}"]
\arrow[from=2-2, to=1-3, "\phi_{jk}"]
\arrow[from=2-2, to=3-4, "\phi_{j\ell}" description]
\arrow[from=3-1, to=2-2, "\phi_{ij}"]
\arrow[from=2-2, to=3-2, phantom, "\Uparrow\phi_{ij\ell}"]
\arrow[from=3-1, to=3-4, "\phi_{i\ell}" description]
\arrow[from=3-1, to=4-3, "\phi_{ik}" description]
\arrow[from=3-3, to=4-3, phantom, "\Downarrow\phi_{ik\ell}"]
\arrow[from=3-1, to=5-2, "\phi_{ij}"']
\arrow[from=4-3, to=3-4, "\phi_{k\ell}"']
\arrow[from=5-2, to=4-3, "\phi_{jk}"']
\end{tikzcd}$$

we get two homotopies

$$\begin{tikzcd}
\phi_{i\ell} \arrow[r, Rightarrow, bend left, "(\phi_{jk\ell}\cdot\phi_{ij})\circ\phi_{ij\ell}"] \arrow[r, Rightarrow, bend right, "(\phi_{k\ell}\cdot\phi_{ijk})\circ\phi_{jk\ell}"'] & \phi_{k\ell}\circ\phi_{jk}\circ\phi_{ij},
\end{tikzcd}$$

which need to satisfy the cocycle condition given by a 2-homotopy between them.

This shows that to avoid having to restrict to situations where all the negative ext groups of the studied complexes vanish, we need to keep track of all the higher coherence data. One solution is to formulate the moduli problem with sheaves of ∞-groupoids. In order to formulate functor of points-style arguments about these, we need an $(\infty, 1)$-category of them. A very intuitive model using quasi-categories, which includes a topos theory adapted to this situation has been developed in [**Lur09**].

1.3. Lie algebras in additive symmetric monoidal ∞-categories

The compactifications evoked in Theorems 1.1.6 and 1.1.7 were not constructed for the stacks of torsors, but the equivalent stacks classifying locally free sheaves with algebraic structure. For an arbitrary algebraic group G, we have as algebraic structure that of Lie(G)-forms. That is, via the adjoint representation $G \circlearrowright \mathrm{Lie}(G)$, we can twist the Lie algebra Lie(G) by G-torsors, thus getting a functor

$$\mathrm{Tors}(G/X) \to \mathrm{LAlg}(X),$$

which maps into the substack of Lie(G)-forms. We want to compactify the stack of Lie(G)-forms considering Lie algebras of perfect complexes. We constructed an additive ∞-operad $\mathrm{Lie}^{\otimes}$, and defined Lie algebras in an additive symmetric monoidal ∞-category $\mathscr{C}^{\otimes}$ as functors $\mathrm{Lie}^{\otimes} \to \mathscr{C}^{\otimes}$ compatible both with the additive and operadic structures. This required adjoining finite $N(\mathrm{Fin}_*)$-coproducts to the ∞-category of operators constructed in [**Lur14**, Construction 2.1.1.7].

In the case $G = \mathrm{SL}_n$, we could construct a functor $\mathscr{T}_X^{\mathrm{perf}}(n) \to \mathrm{LAlg}\,\mathrm{QC}(X)$ mapping a perfect totally supported sheaf of rank n to the Lie algebra on the traceless part of its derived endomorphism complex. This lets us define the stack of generalized $\mathfrak{sl}_n$-forms as the essential image of this functor.

Overview

In Chapter 2, we summarize the theory of ∞-categories and ∞-topoi, as developed in [**Lur09**]. In Chapter 3, we summarize parts of the theory of ∞-operads developed in [**Lur14**], and describe the construction of the symmetric monoidal ∞-category of perfect complexes, following [**Pan11**]. In Chapter 4, we construct a Lie ∞-operad, and the functor $F \mapsto \mathfrak{sl}(F)$.

CHAPTER 2

A summary of higher topos theory

2.1. Simplicial sets and geometric realizations

In this chapter, we summarize the theory of ∞-categories modelled by quasi-categories, developed in [**Lur09**].

NOTATION 2.1.1. The essential image of the fully faithful functor Poset $\to$ Cat consists of small categories where each hom set has at most one element. We will identify the posets with their images.

Let $\Delta \subseteq$ Poset denote the full subcategory with objects the finite nonempty ordinals

$$[0] = \{0\}, [1] = \{0, 1\}, \ldots$$

For any category C, we can represent functors $[n] \to C$ as composable chains in C:

$$C_0 \xrightarrow{f_0} C_1 \to \cdots \to C_{n-1} \xrightarrow{f_{n-1}} C_n.$$

DEFINITION 2.1.2. The *category of simplicial sets* Set_Δ is the presheaf category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. For a simplicial set X, its *n-simplices* are the objects of the set $X_n =_{\mathrm{df}} X([n])$. For $n \geq 0$ and $0 \leq j \leq n$, we have the following maps.

(1) The *j-th face map* $X_n \xrightarrow{d_j} X_{n-1}$ is the image by X of the map $[n-1] \xrightarrow{d^j} [n]$:

$$0 \to 1 \to \cdots \to (j-1) \to (j+1) \to \cdots \to n.$$

(2) The *j-th degeneracy map* $X_n \xrightarrow{s_j} X_{n+1}$ is the image by X of the map $[n+1] \xrightarrow{s^j} [n]$:

$$0 \to 1 \to \cdots \to (j-1) \to j \to j \to (j+1) \to \cdots \to n.$$

NOTATION 2.1.3. For each $n \geq 0$, we will denote the representable presheaf $\Delta^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_\Delta(-,[n])} \mathrm{Set}$ by Δ^n. These are called the *standard simplicial sets*. Via Yoneda's lemma, we will refer to n-simplices of a simplicial set X using arrows $\Delta^n \to X$. For example, the edge $f \in X_1$ with $d_0(f) = y$ and $d_1(f) = x$ is going to be denoted by $\Delta^1 \xrightarrow{x \xrightarrow{f} y} X$. We will also denote Δ^0 by $*$.

Remark 2.1.3.1. The simplicial set $*$ is a final object of the category of simplicial sets Set_Δ.

Notation 2.1.4. For any finite nonempty linearly ordered set I of cardinality $n+1$, we write $\Delta^I = \Delta^n$. For any inclusion of a nonempty subset $J \subseteq I$, we get a map $\Delta^J \to \Delta^I$, which we will refer to as *canonical*. In general, all morphisms of finite nonempty linearly ordered sets $I \to I'$ give a map $\Delta^I \to \Delta^{I'}$.

Remark 2.1.4.1. The restriction map $\mathrm{Fun}(\mathrm{FinOrd}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Set}_\Delta$ is an isomorphism.

Definition 2.1.5. Let $\mathcal{C} \xrightarrow{F} \mathcal{E}$ and $\mathcal{C} \xrightarrow{K} \mathcal{D}$ be functors between categories. A pair (Lan_K, ξ) fitting into the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\
\mathcal{D} \arrow[ur, "\mathrm{Lan}_K F"'] \arrow[phantom, u, shift left=4, "\Downarrow_\xi"] &
\end{tikzcd}$$

is a *left Kan extension of F along K*, if for every pair (G, ζ) of a functor $\mathcal{D} \xrightarrow{G} \mathcal{E}$ and a natural transformation $F \xrightarrow{\zeta} G \circ K$, there exists a unique natural transformation $\mathrm{Lan}_K F \xrightarrow{\epsilon} G$ such that $\zeta = \epsilon \cdot \xi$.

Proposition 2.1.6. [**Rie14**, Construction 1.5.1]

Let C be a cocomplete, locally small category, and let $\Delta \xrightarrow{C^\bullet} C$ be a cosimplicial object of C. If $(||_{C^\bullet}, \xi)$ denotes the left Kan extension of $C^\bullet$ along the Yoneda embedding:

$$\begin{tikzcd}
\Delta \arrow[r, "C^\bullet"] \arrow[d, "h_\bullet"'] & C, \\
\mathrm{Set}_\Delta \arrow[ur, "||_{C^\bullet}"'] \arrow[phantom, u, shift left=4, "\Downarrow_\xi"] &
\end{tikzcd}$$

then the following assertions hold.

(1) The natural transformation ξ is an isomorphism.

(2) The functor $||_{C^\bullet}$ *admits a right adjoint:*

$$\begin{tikzcd}
\mathrm{Set}_\Delta \arrow[r, bend left, "||_{C^\bullet}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\mathrm{Sing}_{C^\bullet} : C \mapsto (\mathrm{Hom}_{\mathcal{C}}(C^n, C))"]
\end{tikzcd}$$

Corollary 2.1.6.1. *Since the right adjoint* $\mathrm{Sing}_{C^\bullet}$ *only depends on the cosimplicial object* $C^\bullet$*, every left adjoint* $\mathrm{Set}_\Delta \xrightarrow{F} \mathcal{C}$ *is determined by* $F|\Delta$ *up to unique isomorphism. In particular, if* $\mathcal{C}$ *is locally presentable, then every colimit-preserving functor* F *is determined by* $F|\Delta$ *up to unique isomorphism.*

Definition 2.1.7. The functor $\mathrm{Set}_\Delta \xrightarrow{||_{C^\bullet}} \mathcal{C}$ is called a *geometric realization functor*. If we don't specify $\mathcal{C}$ and $C^\bullet$, we mean the cosimplicial topological space mapping the finite ordinal $[n]$ to the standard n-simplex

$$|\Delta^n| = \{(x_0, \dots, x_n) \in (\mathbf{R}_{\geq 0})^n : x_0 + \cdots + x_n \leq 1\}.$$

Here, the target category is usually the full subcategory CGHaus ⊆ Top of compactly generated Hausdorff spaces. In this case, we write $|X|$ as the geometric realization of a simplicial set X.

Definition 2.1.8. Let C be an object of a category $\mathcal{C}$, and K a set. The *copower of* C *by* K, denoted by $K \otimes C$, is the coproduct $C^{\sqcup K}$. We say that the category $\mathcal{C}$ is *copowered over* Set, if there exists a functor

$$\mathrm{Set} \times \mathcal{C} \xrightarrow{(K,C) \mapsto K \otimes C} \mathcal{C},$$

which moreover sends an arrow $(K, C) \xrightarrow{(\phi, f)} (K', D)$ to the arrow $K \otimes C \to K' \otimes D$ determined by the collection of composite arrows

$$\left\{ C \xrightarrow{f} D \xrightarrow{i_{\phi(v)}} K' \otimes D : v \in K \right\}.$$

Definition 2.1.9. Let $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{H} \mathcal{E}$ be a functor. The *coend of* H, denoted by $\int^C H$, is a coequalizer of the form

$$\coprod_{C \xrightarrow{f} D} H(C, D) \underset{H(f,\mathrm{id})}{\overset{H(\mathrm{id},f)}{\rightrightarrows}} \coprod_{C} H(C, C) \longrightarrow \int^C H.$$

Proposition 2.1.10. **[Rie14**, Theorem 1.2.1]

Let $\mathcal{C}$ *be a small category,* $\mathcal{D}$ *a locally small category, and* $\mathcal{E}$ *a cocomplete category. Then there is a left Kan extension* $\mathcal{D} \xrightarrow{\mathrm{Lan}_K F} \mathcal{E}$ *of a functor* $\mathcal{C} \xrightarrow{F} \mathcal{E}$ *along a functor* $\mathcal{C} \xrightarrow{K} \mathcal{D}$*, which satisfies, for all objects*

$D \in \mathcal{D}$,

$$\mathrm{Lan}_K F(D) = \int^{C \in C} \mathcal{D}(KC, D) \cdot FC.$$

Corollary 2.1.10.1. *In case K is the Yoneda embedding $\Delta \to \mathrm{Set}_\Delta$, and F is the inclusion of the subcategory of standard simplices with standard maps $\Delta \to$ CGHaus, we get that up to a unique natural isomorphism, for any simplicial set X,*

$$|X| = \int^{[n] \in \Delta} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X) \cdot |\Delta^n|.$$

In other words, the geometrical realization $|X|$ is the colimit of the forgetful functor $\Delta \downarrow X \to \Delta$, where $\Delta \downarrow X$ is the full subcategory of the overcategory $(\mathrm{Set}_\Delta)_{/X}$ on the standard simplicial sets Δ^n.

Definition 2.1.11. An n-simplex $\Delta^n \xrightarrow{\sigma} X$ of a simplicial set is called *nondegenerate*, if it is not in the image of any degeneracy map.

The boundary simplicial set $\partial\Delta^n$ is the largest sub-simplicial set of Δ^n, which does not contain $\Delta^n \xrightarrow{\mathrm{id}_n} \Delta^n$. It has all the other nondegenerate simplices of Δ^n.

For $0 \leq i \leq n$, the *i-th horn* Λ^n_i is the largest sub-simplicial set of Δ^n, which does not contain the face $\Delta^{n-1} \xrightarrow{h_{d^i}} \Delta^n$. It has all the other nondenerate m-simplices of Δ^n, for $m < n$.

Definition 2.1.12. Let C be a category, and let S and T be two collections of morphisms. We say that S *has the left lifting property with respect to* T, and that T *has the right lifting property with respect to* S, if all commutative diagrams in C of solid arrows

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

such that $f \in S$ and $g \in T$ can be filled in with a dashed arrow keeping it commutative. The collection of morphisms having the right lifting property with respect to S is denoted by $S^\perp$, and the collection of morphisms having the left lifting property with respect to T is denoted by ${}^\perp T$.

Definition 2.1.13. A morphism of simplicial sets $X \xrightarrow{f} Y$ is called a *Kan fibration*, if it has the right lifting property with respect to the inclusions $\Lambda^n_i \to \Delta^n$:

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

A simplicial set X is called a *Kan complex*, if the unique map $X \to *$ is a Kan fibration.

A morphism of simplicial sets $X \xrightarrow{f} Y$ is called a *weak homotopical equivalence*, if the geometric realization

$$|X| \xrightarrow{|f|} |Y|$$

is a weak homotopical equivalence.

Definition 2.1.14. The category of simplicial sets can be endowed with a cartesian closed structure. In this context, we will denote it by **S**. We define the direct product of the simplicial sets $\Delta^{\mathrm{op}} \underset{Y}{\overset{X}{\rightrightarrows}} \mathrm{Set}$ by the universal property:

$$\mathrm{Hom}(\Delta^n, X \times Y) = \mathrm{Hom}(\Delta^n, X) \times \mathrm{Hom}(\Delta^n, Y),$$

and the mapping complex $\mathrm{Map}(X, Y)$ by the adjointness property:

$$\mathrm{Hom}(\Delta^n, \mathrm{Map}(X, Y)) = \mathrm{Hom}(\Delta^n \times X, Y).$$

A *simplicial category* is an **S**-enriched category, and their category is denoted by Cat_Δ.

Theorem 2.1.15. [**Hov99**, Theorem 3.6.5, Proposition 4.2.8]

There exists a cartesian model structure on **S** *with*

(1) monomorphisms as cofibration,

(2) weak homotopy equivalences as weak equivalences, and

(3) Kan fibrations as fibrations.

Moreover, a morphism of simplicial sets $X \xrightarrow{f} Y$ *is a trivial fibration precisely when it has the right lifting property with respect to all inclusions* $\partial\Delta^n \to \Delta^n$*:*

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] & Y.
\end{tikzcd}$$

Definition 2.1.16. The model structure described in Theorem 2.1.15 is called the *Kan model structure.*

2.2. ∞-categories and simplicial categories

Definition 2.2.1. Let $\Delta \xrightarrow{C^\bullet} \mathrm{Cat}$ be the cosimplicial object which is the restriction to Δ of the functor Poset → Cat which maps a poset S to the category with object set S and hom sets

$$|\operatorname{Hom}(x, y)| = \begin{cases} 1 & x \leq y \\ 0 & x > y. \end{cases}$$

The corresponding singular complex functor is called the *simplicial nerve functor*, and it's denoted by $\mathrm{Cat} \xrightarrow{N} \mathrm{Set}_\Delta$.

Remark 2.2.1.1. Note that for a category C, the n-simplices of $N(C)$ are the composable chains in C:

$$C_0 \to C_1 \to \cdots \to C_n,$$

the inner face maps are given by composition, the outer face maps by forgetting one of the arrows at the ends, and the degeneracies by inserting identity maps.

Remark 2.2.1.2. Note also that $\mathrm{Cat} \xrightarrow{N} \mathrm{Set}_\Delta$ is fully faithful.

Proposition 2.2.2. [**Lur09**, Proposition 1.1.2.2]

The essential image of the nerve functor $\mathrm{Cat} \xrightarrow{N} \mathrm{Set}_\Delta$ *consists of those simplicial sets* K *for which for all* $0 < i < n$*, every diagram of solid arrows*

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

where the vertical arrow is the inclusion can be extended to a commutative triangle in a unique way.

Definition 2.2.3. A simplicial set C is an ∞*-category (or quasi-category)*, if for all $n \geq 2$ and every $0 < i < n$, every morphism $\Lambda^n_n \to C$ factorizes through the inclusion $\Lambda^n_i \to \Delta^n$:

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

Definition 2.2.4. A morphism $x \xrightarrow{f} y$ in a simplicial category C is called a *homotopy equivalence*, if the induced morphism $\pi_0(x) \xrightarrow{\pi_0(f)} \pi_0(y)$ in the category $\pi_0(C)$ is an isomorphism.

Let $C \xrightarrow{F} \mathcal{D}$ be a morphism of simplicial categories. The map F is called a *DK-equivalence*, if the following conditions hold.

(1) For all $x, y \in C$, it defines a weak homotopy equivalence on the mapping complexes

$$\mathrm{Map}_C(x, y) \xrightarrow{F_{x,y}} \mathrm{Map}_{\mathcal{D}}(x, y).$$

(2) The induced functor $\pi_0 C \xrightarrow{\pi_0 F} \pi_0 \mathcal{D}$ is an equivalence of categories.

The map F is called a *local fibration*, if the following conditions hold.

(1) For all $x, y \in C$, it defines a Kan fibration on the mapping complexes

$$\mathrm{Map}_C(x, y) \xrightarrow{F_{x,y}} \mathrm{Map}_{\mathcal{D}}(x, y).$$

(2) For all objects X of C and homotopy equivalences $x \xrightarrow{f} y$ in $\mathcal{D}$, there exists a homotopy equivalence $X \xrightarrow{\tilde{f}} Y$ in C such that $F(\tilde{f}) = f$.

Theorem 2.2.5. [**Ber07**, Theorem 1.1]

There exists a model structure on the category Cat_Δ *of simplicial categories such that weak equivalences are DK-equivalences and fibrations are local fibrations.*

Definition 2.2.6. The model structure on Cat_Δ given by Theorem 2.2.5 is called the *Bergner model structure*.

Definition 2.2.7. The cosimplicial simplicial category $\Delta \xrightarrow{\mathfrak{C}} \mathrm{Cat}_\Delta$ with $\mathfrak{C}[\Delta^n]$ having [**Lur09**, Definition 1.1.5.1]

- object set $[n]$, and
- for elements $0 \leq i, j \leq n$, mapping complexes

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) = \begin{cases} \emptyset & j < i, \\ N(P_{i,j}) & i \leq j, \end{cases}$$

where

$$P_{i,j} = \{I \subseteq [n] : i, j \in I \text{ and for all } k \in I,\ i \leq k \leq j\},$$

defines a geometric realization adjoint pair by Proposition 2.1.6

$$\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\Delta \arrow[l, bend left, "N"] \end{tikzcd},$$

where the geometric realization functor $\mathfrak{C}$ is called the *categorical realization functor*, and the singular complex functor N is called *simplicial nerve functor*.

A morphism of simplicial sets $X \xrightarrow{f} Y$ is called a *categorical equivalence*, if its categorical realization $\mathfrak{C}[X] \xrightarrow{\mathfrak{C}[f]} \mathfrak{C}[Y]$ is a DK-equivalence.

THEOREM 2.2.8. [**Lur09**, Theorem 2.2.5.1]

There exists a model structure, called the Joyal model structure *on* Set_Δ *such that the cofibrations are monomorphisms, and the weak equivalences are categorical equivalences.*

The adjoint pair $\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\Delta \arrow[l, bend left, "N"] \end{tikzcd}$ *is a Quillen equivalence if we endow* Set_Δ *with the Joyal model structure, and* Cat_Δ *with the Bergner model structure.*

THEOREM 2.2.9. [**Lur09**, Theorem 2.4.6.1]

The fibrant objects in the Joyal model structure on Set_Δ *are precisely the* ∞*-categories.*

COROLLARY 2.2.9.1. *Let* $\mathcal{C}$ *be a simplicial category such that all of its mapping complexes* $\mathrm{Map}_{\mathcal{C}}(C, D)$ *are Kan complexes. Then the simplicial nerve* $N(\mathcal{C})$ *is an* ∞*-category.*

DEFINITION 2.2.10. Let X and Y be two simplicial sets. Their *join* is the simplicial set $X * Y$ with set of n-simplices

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X * Y) = \sqcup_{\substack{[n]=I \sqcup I' \\ I < I'}} X(I) \times X(I').$$

The *left cone* is $X^{\triangleleft} =_{\text{df}} \Delta^0 * X$, and the *right cone* is $X^{\triangleright} =_{\text{df}} X * \Delta^0$.

Proposition 2.2.11. [**Lur09**, §1.2.8]

For all $i, j \geq 0$, we have natural isomorphisms

$$\Delta^i * \Delta^j \to \Delta^{i+j-1}.$$

Definition 2.2.12. Let C be an ∞-category, and $K \xrightarrow{p} C$ a morphism of simplicial sets. The *overcategory of p*, denoted by $C_{/p}$, has as n-simplices the σ in

$$\begin{tikzcd}
\Delta^n * K \arrow[r, "\sigma"] & C \\
K, \arrow[u] \arrow[ur, "p"'] &
\end{tikzcd}$$

and the *undercategory of p*, denoted by $C_{p/}$, has as n-simplices the σ in

$$\begin{tikzcd}
K * \Delta^n \arrow[r, "\sigma"] & C \\
K. \arrow[u] \arrow[ur, "p"'] &
\end{tikzcd}$$

Definition 2.2.13. Let x and y be two objects in an ∞-category C. The *right morphism space* $\mathrm{Hom}^R_C(x, y)$ is the fiber product

$$\begin{tikzcd}
\mathrm{Hom}^R_C(x, y) \arrow[r] \arrow[d] & C_{/y} \arrow[d] \\
\{x\} \arrow[r] & C,
\end{tikzcd}$$

and the *left morphism space* $\mathrm{Hom}^L_C(x, y)$ is the fiber product

$$\begin{tikzcd}
\mathrm{Hom}^L_C(x, y) \arrow[r] \arrow[d] & C_{x/} \arrow[d] \\
\{y\} \arrow[r] & C.
\end{tikzcd}$$

Definition 2.2.14. Let X be an object in an ∞-category C. It is a *final object*, if the canonical map $C_{/X} \to C$ is a trivial Kan fibration. It is an *initial object*, if the canonical map $C_{X/} \to C$ is a trivial Kan fibration.

Definition 2.2.15. Let C be an ∞-category, and $K \xrightarrow{p} C$ a morphism of simplicial sets. A *limit of p* is a final object of the overcategory $C_{/p}$. A *colimit of p* is an initial object of the undercategory $C_{p/}$.

Definition 2.2.16. Let K be a simplicial set and C be an ∞-category. The *∞-category of functors $K \to C$* is the mapping complex $\mathrm{Fun}(K, C) =_{\mathrm{df}} \mathrm{Map}(K, C)$. We denote $\mathscr{O}_C =_{\mathrm{df}} \mathrm{Fun}(\Delta^1, C)$. It has source and target morphisms $\begin{tikzcd} \mathscr{O}_C \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C \end{tikzcd}$.

Proposition 2.2.17. [**Lur09**, Corollary 2.3.2.2]

Let C be a simplicial set. It is an ∞-category precisely when the restriction map

$$\mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$$

is a trivial Kan fibration.

Proposition 2.2.18. [**Lur09**, Corollary 4.2.1.8]

Let x and y be two objects in an ∞-category C. Let $\mathrm{Hom}_C(x, y)$ be defined as the fiber product

$$\begin{tikzcd}
\mathrm{Hom}_C(x,y) \arrow[r] \arrow[d] & \mathscr{O}_C \arrow[d, "{(s,t)}"] \\
\{x\} \times \{y\} \arrow[r] & C \times C.
\end{tikzcd}$$

Since the n-simplices in $\mathrm{Hom}^R_C(x, y)$, $\mathrm{Hom}_C(x, y)$ and $\mathrm{Hom}^L_C(x, y)$ are the σ fitting into the three respective commutative diagrams

$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d, "d^{n+1}"'] & \Delta^0 \arrow[d, "x"] \\
\Delta^{n+1} \arrow[r, "\sigma"] & C \\
\Delta^{\{n+1\}} \arrow[u] \arrow[ur, bend right, "y"'] &
\end{tikzcd}
, \quad
\begin{tikzcd}
\Delta^n \times \{0\} \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "x"] \\
\Delta^n \times \Delta^1 \arrow[r, "\sigma"] & C \\
\Delta^n \times \{1\} \arrow[u] \arrow[r] & \Delta^0 \arrow[u, "y"']
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\Delta^{\{0\}} \arrow[d] \arrow[dr, bend left, "x"] & \\
\Delta^{n+1} \arrow[r, "\sigma"] & C \\
\Delta^n \arrow[u, "d^0"] \arrow[r] & \Delta^0 \arrow[u, "y"']
\end{tikzcd}$$

precomposition with the respective cofiber maps

$$\Delta^n \times \{1\} \to \Delta^n \times \Delta^1 \xrightarrow{q_R} \Delta^{n+1} \textit{ and } \Delta^n \times \{0\} \to \Delta^n \times \Delta^1 \xrightarrow{q_L} \Delta^{n+1}$$

gives two inclusion maps

$$\mathrm{Hom}^R_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}}(x,y) \leftarrow \mathrm{Hom}^L_{\mathcal{C}}(x,y).$$

These two maps are homotopy equivalences of Kan complexes.

Definition 2.2.19. Let X and Y be simplicial sets. Their *alternative join* $X \diamond Y$ is the colimit

$$X \coprod_{X\times Y\times\{0\}} \left(X \times Y \times \Delta^1\right) \coprod_{X\times Y\times\{1\}} Y.$$

2.3. Presheaves in simplicial categories and mapping spaces

Definition 2.3.1. Let $\mathcal{C} \underset{S}{\overset{T}{\rightrightarrows}} s\,\mathrm{Set}$ be two simplicially enriched functors. Via [**Kel05**, §2.2], we can endow $\mathrm{Set}^{\mathcal{C}}_{\Delta}$ with a simplically enriched structure by letting the mapping simplicial set $\mathrm{Map}_{\mathrm{Set}^{\mathcal{C}}_{\Delta}}(T,S)$ have as n-simplices collections of n-simplices of morphisms

$$(C \in \mathcal{C}) \qquad\qquad \Delta^n \xrightarrow{\alpha_C} \mathrm{Map}_{\mathrm{Set}_\Delta}(TC,SC)$$

such that for each $C, D \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
\Delta^n \times \mathcal{C}(C,D) \arrow[r, "\alpha_D\times(T_{CD})"] \arrow[d, "S_{CD}\times^{\mathrm{sh}}\alpha_C"'] & {[TD,SD]\times[TC,TD]} \arrow[d, "\circ"] \\
{[SC,SD]\times[TC,SC]} \arrow[r, "\circ"'] & {[TC,SD]}
\end{tikzcd}$$

is commutative, ie. for all $\sigma \in \mathcal{C}(C,D)_n$, we have

$$\alpha_D \circ (T\sigma) = (S\sigma) \circ \alpha_C.$$

Proposition 2.3.2. [**Lur09**, Proposition A.3.3.2]
Let $\mathcal{C}$ be a small simplicial category. Then the diagram category $\mathrm{Set}^{\mathcal{C}}_{\Delta}$ *has the following model structures.*

1. *The* injective model structure, *with pointwise cofibrations and weak equivalences.*
2. *The* projective model structure, *with pointwise weak equivalences and fibrations.*

Definition 2.3.3. Let S be a simplicial set and $X \xrightarrow{f} Y$ a map of simplicial sets over S.

(1) The map f is a *covariant equivalence*, if the induced map

$$X^{\triangleleft} \coprod_{X} S \to Y^{\triangleleft} \coprod_{Y} S$$

is a categorical equivalence.

(2) The map f is a *contravariant equivalence*, if the induced map

$$X^{\triangleright} \coprod_{X} S \to Y^{\triangleright} \coprod_{Y} S$$

is a categorical equivalence.

Proposition 2.3.4. [**Lur09**, Proposition 2.1.4.7]

Let S be a simplicial set. Then the overcategory $(\mathrm{Set}_\Delta)_{/S}$ has the following model structures.

(1) The covariant model structure*, where cofibrations are monomorphisms and weak equivalences are covariant equivalences.*

(2) The contravariant model structure*, where cofibrations are monomorphisms and weak equivalences are contravariant equivalences.*

Theorem 2.3.5. [**Lur09**, Theorem 2.2.1.2, Proposition 5.1.1.1]

Let S be a simplicial set, C a simplicial category and $\mathfrak{C}[S] \xrightarrow{\phi} C^{\mathrm{op}}$ a functor. For a map of simplicial sets $X \to S$, let

$$\mathcal{M} = \mathfrak{C}[X^{\triangleright}] \coprod_{\mathfrak{C}[X]} C^{\mathrm{op}}.$$

Let $(\mathrm{Set}_\Delta)_{/S}$ be endowed with the contravariant model structure, and Set_Δ^C be endowed with the projective model structure. Then there is a Quillen adjunction

$$\begin{tikzcd}(\mathrm{Set}_\Delta)_{/S} \arrow[r, bend left, "\mathrm{St}_\phi"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\Delta^C \arrow[l, bend left, "\mathrm{Un}_\phi"]\end{tikzcd}$$

satisfying the following conditions.

(1) For a map of simplicial sets $X \to S$, the functor $\mathrm{St}_\phi X$ is defined by

$$C \xrightarrow{C \mapsto \mathrm{Map}_{\mathcal{M}}(C,\infty)} \mathrm{Set}_\Delta .$$

(2) If $\mathfrak{C}[S] \xrightarrow{\phi} C^{\mathrm{op}}$ *is an equivalence, then* $(\mathrm{St}_\phi, \mathrm{Un}_\phi)$ *is a Quillen equivalence. Moreover, the unstraightening functor can be endowed with the structure of a simplicial functor.*

Definition 2.3.6. Let $X \xrightarrow{p} S$ be a morphism of simplicial sets.

(1) If p has the right lifting property with respect to all inclusions $\Lambda^n_i \to \Delta^n$ where $0 < i < n$, then it is called an *inner fibration*.

(2) If p has the right lifting property with respect to all inclusions $\Lambda^n_i \to \Delta^n$ where $0 \leq i < n$, then it is called a *left fibration*.

(3) If p has the right lifting property with respect to all inclusions $\Lambda^n_i \to \Delta^n$ where $0 < i \leq n$, then it is called a *right fibration*.

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

Proposition 2.3.7. [**Lur09**, Corollary 2.2.3.12]

Let $X \xrightarrow{p} S$ *be a morphism of simplicial sets. Then it is a right fibration precisely when it is a contravariant fibrant object of* $(\mathrm{Set}_\Delta)_{/S}$.

Proposition 2.3.8. [**Lur09**, §2.2.2]

If the simplicial set S is a final object $X = \{x\}$ *and* $\phi = \mathrm{id}_{\mathfrak{C}[S^{\mathrm{op}}]}$, *then the straightification functor*

$$\mathrm{Set}_\Delta \xleftarrow{\cong} (\mathrm{Set}_\Delta)_{/X} \xrightarrow{\mathrm{St}_X} \mathrm{Set}_\Delta^{\mathfrak{C}[X^{\mathrm{op}}]} \xrightarrow{\cong} \mathrm{Set}_\Delta$$

is isomorphic to the geometric realization functor $|\,|_{Q^\bullet}$, *where the cosimplicial simplicial set* $Q^\bullet$ *is defined as follows. For each* $n \geq 0$, *let* J^n *be the pushout of simplicial sets*

$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d, "d^{n+1}"'] & \Delta^0 \arrow[d, "x"] \\
\Delta^{n+1} \arrow[r, "\{n+1\}\mapsto y"] & J^n.
\end{tikzcd}$$

Let the simplicial set Q^n *be* $\mathrm{Map}_{\mathfrak{C}[J^n]}(x, y)$.

Corollary 2.3.8.1. [**Lur09**, Proposition 2.2.2.13]

Let X, Y be two objects in a simplicial category C. There is a natural isomorphism of simplicial sets

$$\mathrm{Hom}^R_{N(C)}(X, Y) \to \mathrm{Sing}_{Q^\bullet} \mathrm{Map}_C(X, Y).$$

Proof. Let $n \geq 0$. The n-simplices of the space of right morphisms $\mathrm{Hom}^R_{N(C)}(X, Y)$ are the σ fitting into the commutative diagram

$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d, "d^{n+1}"'] & \Delta^0 \arrow[d, "X"] \\
\Delta^{n+1} \arrow[r, "\sigma"] & N(C) \\
\Delta^{\{n+1\}}. \arrow[u] \arrow[ur, bend right, "Y"']
\end{tikzcd}$$

By construction, these are precisely those maps $J^n \to N(C)$, which map x to X and y to Y. In turn, these are in natural bijection with those maps $\mathfrak{C}[J^n] \to C$ which map x to X and y to Y. As the categorical realization functor $\mathrm{Set}_\Delta \xrightarrow{\mathfrak{C}} \mathrm{Cat}_\Delta$ is colimit-preserving, the object set of J^n is $\{x, y\}$, and by construction we have

$$\mathrm{Map}_{\mathfrak{C}[J^n]}(x, x) = \mathrm{Map}_{\mathfrak{C}[J^n]}(y, y) = *, \ \mathrm{Map}_{\mathfrak{C}[J^n]}(y, x) = \emptyset.$$

Therefore, mapping morphisms $\mathfrak{C}[J^n] \to C$ mapping x to X and y to Y to their action on the mapping complexes

$$Q^n = \mathrm{Map}_{\mathfrak{C}[J^n]}(x, y) \to \mathrm{Map}_C(x, y)$$

is a natural bijection.

□

2.4. Cartesian and coCartesian fibrations, classifying maps

Definition 2.4.1. Let $X \xrightarrow{p} S$ be an inner fibration of simplicial sets, and $x \xrightarrow{f} y$ and edge in X.

(1) The edge f is a *p-Cartesian edge*, if the canonical map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

is a trivial Kan fibration.

(2) The edge f is a *p-coCartesian edge*, if the canonical map

$$X_{f/} \to X_{x/} \times_{S_{p(x)/}} S_{p(f)/}$$

is a trivial Kan fibration.

Definition 2.4.2. Let $X \xrightarrow{p} S$ be an inner fibration of simplicial sets.

(1) The map p is a *Cartesian fibration*, if for all vertices y of X and edges $s \xrightarrow{f} p(y)$ of S, there exists a p-Cartesian edge $x \xrightarrow{\phi} y$ of X such that $p(x) = s$.
(2) The map p is a *coCartesian fibration*, if for all vertices x of X and edges $p(x) \xrightarrow{f} t$ of S, there exists a p-coCartesian edge $x \xrightarrow{\phi} y$ of X such that $p(y) = t$.

Definition 2.4.3. A *marked simplicial set* is a pair $(X, \mathcal{E})$ where X is a simplicial set, and $\mathcal{E}$ is a set of edges of X containing all degenerate edges. A *morphism of marked simplicial sets* $(X, \mathcal{E}) \to (X', \mathcal{E}')$ is a morphism of simplicial sets $X \xrightarrow{f} X'$ such that $f(\mathcal{E}) \subseteq \mathcal{E}'$. The category of marked simplicial sets is denoted by Set_Δ^+.

Notation 2.4.4. Let X be a simplicial set. The marked simplicial set $X^\flat$ has as marked edges the degenerate edges of X. The marked simplicial set $X^\sharp$ has as marked edges all the edges of X. Let S be a simplicial set. The overcategory $\left(\mathrm{Set}_\Delta^+\right)_{/S^\sharp}$ is also denoted by $\left(\mathrm{Set}_\Delta^+\right)_{/S}$. Let $X \xrightarrow{p} S$ be a Cartesian fibration of simplicial sets. The marked simplicial set $X^\natural$ has as marked edges the p-Cartesian edges.

Notation 2.4.5. Let S be a simplicial set, and X, Y objects of $\left(\mathrm{Set}_\Delta^+\right)_{/S}$. Then the simplicial sets $\mathrm{Map}_S^\flat(X, Y)$ and $\mathrm{Map}_S^\sharp(X, Y)$ are defined by requiring bijections natural in K:

$$\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/S}}\left(K, \mathrm{Map}^\flat(X, Y)\right) \cong \mathrm{Hom}_{(\mathrm{Set}_\Delta^+)_{/S}}(K^\flat \times X, Y),$$
$$\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/S}}\left(K, \mathrm{Map}^\sharp(X, Y)\right) \cong \mathrm{Hom}_{(\mathrm{Set}_\Delta^+)_{/S}}(K^\sharp \times X, Y).$$

Proposition 2.4.6. [**Lur09**, Proposition 3.1.3.3]
Let S be a simplicial set and $X \xrightarrow{f} Y$ a morphism in $\left(\mathrm{Set}_\Delta^+\right)_{/S}$. The following are equivalent.

(1) For all Cartesian fibrations $Z \to S$, the induced map

$$\mathrm{Map}_S^\flat(Y, Z^\natural) \to \mathrm{Map}_S^\flat(X, Z^\natural)$$

is an equivalence of ∞-categories.

(2) For all Cartesian fibrations $Z \to S$, the induced map

$$\mathrm{Map}^{\sharp}_{S}(Y, Z^{\flat}) \to \mathrm{Map}^{\sharp}_{S}(X, Z^{\flat})$$

is an equivalence of ∞-categories.

Definition 2.4.7. Let S be a simplicial set. If a morphism $X \xrightarrow{f} Y$ in $\left(\mathrm{Set}^{+}_{\Delta}\right)_{/S}$ satisfies the equivalent conditions of Proposition 2.4.6, then it is called a *Cartesian equivalence*. If $X^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} Y^{\mathrm{op}}$ is a Cartesian equivalence in $\left(\mathrm{Set}^{+}_{\Delta}\right)_{/S^{\mathrm{op}}}$, then f is called a *coCartesian equivalence*.

Proposition 2.4.8. [**Lur09**, Proposition 3.1.3.7]

Let S be a simplicial set. The category $\left(\mathrm{Set}^{+}_{\Delta}\right)_{/S}$ can be endowed with the Cartesian model structure*, which has monomorphisms as cofibrations, and Cartesian equivalences as weak equivalences. The category $\left(\mathrm{Set}^{+}_{\Delta}\right)_{/S}$ can also be endowed with the* coCartesian model structure*, which has monomorphisms as cofibrations, and coCartesian equivalences as weak equivalences.*

Proposition 2.4.9. [**Lur09**, Proposition 3.1.4.1]

Let S be a simplicial set. An object X of $\left(\mathrm{Set}^{+}_{\Delta}\right)_{/S}$ is fibrant in the Cartesian model structure precisely when it is isomorphic to $Y^{\natural}$ for some Cartesian fibration $Y \to S$.

2.4.0.1. *Naturality of the unstraightening functor.*

Notation 2.4.10. Let $T \xrightarrow{f} S$ be a morphism of simplicial sets, and $X \to S^{\sharp}$, $Y \to T^{\sharp}$ morphisms of marked simplicial sets. We let $f^{*}X$ denote the morphism of marked simplicial sets $X \times_{S} T \xrightarrow{\mathrm{pr}_{T}} T$, where an edge is marked precisely when its projection onto X is marked. We let $f_{!}Y$ denote the composite $Y \to T \xrightarrow{f} S$. Note that we get an adjoint pair

$$\begin{tikzcd}\left(\mathrm{Set}^{+}_{\Delta}\right)_{/T} \arrow[r, bend left, "f_{!}"] \arrow[r, phantom, "\perp"] & \left(\mathrm{Set}^{+}_{\Delta}\right)_{/S} \arrow[l, bend left, "f^{*}"]\end{tikzcd} .$$

Let $\mathscr{C} \xrightarrow{\pi} \mathscr{C}'$ be a simplicial functor. We let $\left(\mathrm{Set}^{+}_{\Delta}\right)^{\mathscr{C}'} \xrightarrow{\pi^{*}} \left(\mathrm{Set}^{+}_{\Delta}\right)^{\mathscr{C}}$ denote the map given by precomposition, and $\pi_{!}$ a left adjoint.

Proposition 2.4.11. *Let* $T \xrightarrow{f} S$ *be a morphism of simplicial sets, and* $\mathfrak{C}[S]^{\mathrm{op}} \xrightarrow{F} \mathrm{Cat}_\infty$ *a simplicial functor. Then there is a canonical isomorphism*

$$\mathrm{Un}_S(F) \times_S T \cong \mathrm{Un}_T(f^*F).$$

Proof. Let us denote the simplicial functor $\mathfrak{C}[T] \xrightarrow{\mathfrak{C}[f]} \mathfrak{C}[S]$ by π, and its opposite $\mathfrak{C}[T]^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}} \mathfrak{C}[S]^{\mathrm{op}}$ by ϕ. We claim that both functors

$$\begin{tikzcd} \left(\mathrm{Set}_\Delta^+\right)_{/S} & \left(\mathrm{Set}_\Delta^*\right)^{\mathfrak{C}[S]^{\mathrm{op}}} \arrow[l, shift right, "\mathrm{Un}_T \circ \pi^*"'] \arrow[l, shift left, "f^* \circ \mathrm{Un}_S"] \end{tikzcd}$$

are right adjoints of St_ϕ. Let $X \in \left(\mathrm{Set}_\Delta^+\right)_T$. The claim follows from the chains of canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}(X, \mathrm{Un}_T(f^*F)) &\cong \mathrm{Hom}(\mathrm{St}_T X, \pi^*F)) \\ &\cong \mathrm{Hom}(\pi_! \, \mathrm{St}_T X, F) \\ &\cong \mathrm{Hom}(\mathrm{St}_\phi X, F), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}(X, f^* \, \mathrm{Un}_S F) &\cong \mathrm{Hom}(f_! X, \mathrm{Un}_S F) \\ &\cong \mathrm{Hom}(\mathrm{St}_S(f_! X), F) \\ &\cong \mathrm{Hom}(\mathrm{St}_\phi X, F). \end{aligned}$$

□

2.4.0.2. *Classifying maps.*

Notation 2.4.12. Let $X \xrightarrow{f} S$ be a morphism of simplicial sets, and $\mathfrak{C}[S^{\mathrm{op}}] \xrightarrow{\phi} \mathscr{C}$ be a functor of simplicial categories. Since the straightening

$$\mathscr{C} \xrightarrow{\mathrm{St}_\phi f} \mathrm{Set}_\Delta$$

takes $C \in \mathscr{C}$ to the mapping space $\mathrm{Map}_{\mathscr{C}_X^{\mathrm{op}}}(C, *)$, where the simplicial category $\mathscr{C}_X^{\mathrm{op}}$ is defined as the pushout

$$\begin{tikzcd}
\mathfrak{C}[X] \arrow[r] \arrow[d, "\phi'=\phi^{\mathrm{op}}\circ\mathfrak{C}[f]"'] & \mathfrak{C}[X^{\triangleright}] \arrow[d] \\
\mathscr{C}^{\mathrm{op}} \arrow[r] & \mathscr{C}_X^{\mathrm{op}}
\end{tikzcd}$$

its action on the mapping spaces

$$\mathrm{Map}_{\mathscr{C}}(C, D) \to \mathrm{Map}_{\mathrm{Set}_\Delta}(\mathrm{Map}_{\mathscr{C}_X^{\mathrm{op}}}(C, *), \mathrm{Map}_{\mathscr{C}_X^{\mathrm{op}}}(D, *))$$

is given via the composition map

$$\mathrm{Map}_{\mathscr{C}_X^{\mathrm{op}}}(C, *) \times \mathrm{Map}_{\mathscr{C}_X^{\mathrm{op}}}(D, C) \to \mathrm{Map}_{\mathscr{C}_X^{\mathrm{op}}}(D, *).$$

For an n-simplex σ in $\mathrm{Map}_{\mathscr{C}}(C, D)$, we will denote by $(\mathrm{St}_\phi\, f)(C)_n \xrightarrow{\sigma^*} (\mathrm{St}_\phi\, f)(D)_n$ the map it induces.

Let c be a vertex in X. It induces a map

$$\Delta^1 \xrightarrow{c\star *} X^{\triangleright},$$

which ϕ' takes to the map

$$\mathfrak{C}[\Delta^1] \to \mathscr{C}_X^{\mathrm{op}}.$$

The unique vertex of $\mathrm{Map}_{\mathfrak{C}[\Delta^1]}(0, 1)$ is taken to a vertex of $\mathrm{Map}_{\mathscr{C}_X^{\mathrm{op}}}(C, *) = (\mathrm{St}_\phi\, f)(C)$, which we will denote by $\tilde{c}$.

Let $c \xrightarrow{g} d$ be an edge of X. It induces a map

$$\Delta^2 \xrightarrow{g\star *} X^{\triangleright},$$

which ϕ' takes to the map

$$\mathfrak{C}[\Delta^2] \to \mathscr{C}_X^{\mathrm{op}}.$$

The following are true by construction.

(1) The mapping space $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0, 2)$ is the simplicial nerve of the poset $\{\{0, 2\}, \{0, 1, 2\}\}$.

(2) The composition map

$$\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(1,2) \times \mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0,1) \to \mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0,2)$$

takes the pair of vertices $(\{1,2\},\{0,1\})$ to the vertex $\{0,1,2\}$.

This shows that the map $\mathfrak{C}[\Delta^2] \to \mathscr{C}_X^{\mathrm{op}}$ takes the unique nondegenerate edge of $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0,2)$ to an edge of the mapping space $\mathrm{Map}_{\mathscr{C}_X^{\mathrm{op}}}(C,*) = (\mathrm{St}_\phi\, f)(C)$, which we will denote by $\tilde{g}$ as in the diagram

$$\begin{tikzcd}
C \arrow[rr, "G"] \arrow[dr, bend right=40, "\tilde{c}"', ""{name=U}] \arrow[dr, bend left=20, "G^*\tilde{d}", ""'{name=V}] & & D \arrow[dl, "\tilde{d}"] \\
& * &
\arrow[Rightarrow, from=U, to=V, "\tilde{g}"]
\end{tikzcd}$$

Definition 2.4.13. Let $\mathfrak{C}[S^{\mathrm{op}}] \xrightarrow{\phi} \mathscr{C}$ be a functor of simplicial categories. The *marked straightening functor*

$$\left(\mathrm{Set}_\Delta^+\right)_{/S} \xrightarrow{\mathrm{St}_\phi^+} \left(\mathrm{Set}_\Delta^+\right)^{\mathscr{C}}$$

takes a map $(X,\mathscr{E}) \to S^\sharp$ to the simplicial functor $\mathscr{C} \to \mathrm{Set}_\Delta^+$, which maps $C \in \mathscr{C}$ to the marked simplicial set $(\mathrm{St}_\phi\, X, \mathscr{E}_\phi(C))$, where

$$\mathscr{E}_\phi(C) = \{F^*\tilde{g} : g \in \mathscr{E},\ \tilde{d} \xrightarrow{\tilde{g}} G^*\tilde{e},\ F \in \mathrm{Map}_{\mathscr{C}^{\mathrm{op}}}(C,D)_1\}.$$

Proposition 2.4.14. [**Lur09**, Theorem 3.2.0.1, Lemma 3.2.4.1]

Let $\mathfrak{C}[S^{\mathrm{op}}] \xrightarrow{\phi} \mathscr{C}$ be a functor of simplicial categories. Then the marked straightening functor has a right adjoint, the marked unstraightening functor:

$$\begin{tikzcd}
\left(\mathrm{Set}_\Delta^+\right)_{/S} \arrow[r, bend left=10, "\mathrm{St}_\phi^+"] \arrow[r, phantom, "\perp"] & \left(\mathrm{Set}_\Delta^+\right)^{\mathscr{C}} \arrow[l, bend left=10, "\mathrm{Un}_\phi^+"]
\end{tikzcd}$$

such that the following statements hold.

(1) *Let $\left(\mathrm{Set}_\Delta^+\right)_{/S}$ be endowed with the Cartesian model structure, and $\left(\mathrm{Set}_\Delta^+\right)^{\mathscr{C}}$ be endowed with the projective model structure. Then the adjunction $(\mathrm{St}_\phi^+, \mathrm{Un}_\phi^+)$ is a Quillen adjunction.*

(2) If the functor ϕ is an equivalence, then the Quillen adjunction $(\mathrm{St}^+_\phi, \mathrm{Un}^+_\phi)$ is a Quillen equivalence, and the restriction

$$\left(\left(\mathrm{Set}^+_\Delta\right)_{/S}\right)^\circ \xleftarrow{\mathrm{Un}^+_\phi} \left(\left(\mathrm{Set}^+_\Delta\right)^{\mathscr{C}}\right)^\circ$$

can be given the structure of a simplicially enriched functor.

Definition 2.4.15. Let $\mathrm{Cat}^\Delta_\infty$ denote the full subcategory of (cofibrant) and fibrant objects of the simplicial category Set^+_Δ. The *∞-category of ∞-categories* is $\mathrm{Cat}_\infty = N(\mathrm{Cat}^\Delta_\infty)$.

Let $X \xrightarrow{p} S$ be a Cartesian fibration of simplicial sets. We say that a map of simplicial sets $S \xrightarrow{f} \mathrm{Cat}_\infty$ *classifies* p, if there exists an equivalence of Cartesian fibrations $X \to \mathrm{Un}^+_S f'$, where f' is the composite of the map $\mathfrak{C}[S] \to \mathrm{Cat}^\Delta_\infty$ corresponding to f, and the inclusion $\mathrm{Cat}^\Delta_\infty \to \mathrm{Set}^+_\Delta$.

Remark 2.4.15.1. Since the unit map $X \to \mathrm{Un}^+_S \mathrm{St}^+_S X$ is a weak equivalence and the fibers of a Cartesian fibration are ∞-categories, the straightening $\mathrm{St}^+_S X$ factorizes throught $\mathrm{Cat}^\Delta_\infty$, and thus induces a map $S \to \mathrm{Cat}_\infty$ classifying p. If p is a right fibration, then its fibers are Kan complexes, therefore its straightening induces a map $S \to \mathscr{S}$, of which we will also say that it classifies p.

2.5. ∞-categories of presheaves and representability

Definition 2.5.1. Let S be a simplicial set. The *∞-category of presheaves on S* is the mapping space $\mathrm{Fun}(S^{\mathrm{op}}, \mathscr{S})$. We denote it by $\mathscr{P}(S)$.

Notation 2.5.2. Let S be a simplicial set. We denote by $\mathscr{P}'(S)$ the simplicial nerve of the simplicial category of fibrant and cofibrant objects $(\mathrm{Set}_\Delta)^\circ_{/S}$ with respect to the contravariant model structure. Let $\mathfrak{C}[S]^{\mathrm{op}} \xrightarrow{\phi} \mathscr{C}$ be an equivalence of simplicial categories. The simplicial nerve of the simplicial category of fibrant and cofibrant objects $\left(\mathrm{Set}^{\mathscr{C}}_\Delta\right)^\circ$ with respect to the projective model structure is denoted by $\mathscr{P}''(\phi)$.

Proposition 2.5.3. [**Lur09**, Proposition 5.1.1.1]

Let S be a simplicial set and let $\mathfrak{C}[S]^{\mathrm{op}} \xrightarrow{\phi} \mathscr{C}$ be an equivalence of simplicial categories. Then there exist categorical equivalences

$$\mathscr{P}'(S) \xleftarrow{f} \mathscr{P}''(\phi) \xrightarrow{g} \mathscr{P}(S),$$

which can be described as follows.

(1) Since in the Quillen equivalence

$$\begin{tikzcd}(\mathrm{Set}_\Delta)_{/S} \arrow[r, bend left, "\mathrm{St}_\phi"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\Delta^{\mathscr{C}} \arrow[l, bend left, "\mathrm{Un}_\phi"]\end{tikzcd}$$

the unstraightening functor is simplicial, it induces an equivalence of simplicial categories

$$\left(\mathrm{Set}_\Delta^{\mathscr{C}}\right)^\circ \to (\mathrm{Set}_\Delta)^\circ_{/S}\,,$$

the simplicial nerve of which we denote by f.

(2) We can get a functor

$$N((\mathrm{Set}_\Delta^{\mathscr{C}})^\circ) \to \mathscr{P}(S)$$

acting on simplices as the composite of the following maps

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, N((\mathbf{S}^{\mathscr{C}})^\circ)) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Cat}_{\mathrm{Set}_\Delta}}(\mathfrak{C}[\Delta^n], (\mathrm{Set}_\Delta^{\mathscr{C}})^\circ)$$

via the categorical realization adjunction $(\mathfrak{C}, N)$

$$\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n] \times \mathscr{C}, \mathrm{Set}_\Delta^\circ)$$

by definition of the projective model structure

$$\xrightarrow{\circ(\mathrm{id}\times\phi)} \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n] \times \mathfrak{C}[S^{\mathrm{op}}], \mathrm{Set}_\Delta^\circ)$$
$$\to \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n \times S^{\mathrm{op}}], \mathrm{Set}_\Delta^\circ)$$

via precomposition with the canonical map $\mathfrak{C}[\Delta^n \times S^{\mathrm{op}}] \to \mathfrak{C}[\Delta^n] \times \mathfrak{C}[S^{\mathrm{op}}]$

$$\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times S^{\mathrm{op}}, N(\mathrm{Set}_\Delta^\circ))$$

categorical realization again

$$= \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{Fun}(S^{\mathrm{op}}, \mathscr{S})).$$

PROPOSITION 2.5.4 (Enriched Yoneda lemma). [**Kel05**, §2.4]

Let $\mathbf{S}$ *be a symmetric closed monoidal category,* $\mathscr{C}$ *an* $\mathbf{S}$*-enriched category,* X *an object in* $\mathscr{C}$*, and* $\mathscr{C}^{\mathrm{op}} \to \mathbf{S}$ *an* $\mathbf{S}$*-enriched functor. Then the map*

$$F(X) \to \mathrm{Map}_{\mathbf{S}^{\mathscr{C}^{\mathrm{op}}}}(h_X, F)$$

taking $\mathbb{1}_{\mathbf{S}} \xrightarrow{\eta} F(X)$ *to natural transformation which assigns to* $Y \in \mathscr{C}$ *the composite*

$$\mathrm{Map}_{\mathscr{C}}(Y, X) \xleftarrow{\cong} \mathrm{Map}_{\mathscr{C}}(Y, X) \otimes \mathbb{1}_{\mathbf{S}} \xrightarrow{\mathrm{id} \otimes \eta} \mathrm{Map}_{\mathscr{C}}(Y, X) \otimes F(X) \xrightarrow{F'_{Y,X}} F(Y),$$

where $F'_{Y,X}$ *corresponds to* $\mathrm{Map}_{\mathscr{C}}(Y, X) \xrightarrow{F_{Y,X}} \mathrm{Map}_{\mathbf{S}}(F(X), F(Y))$ *via the tensor-hom adjunction is an isomorphism.*

Corollary 2.5.4.1. *The* enriched Yoneda embedding

$$\mathscr{C} \xrightarrow{h_\bullet} \mathbf{S}^{\mathscr{C}^{\mathrm{op}}}$$

is a fully faithful $\mathbf{S}$*-enriched morphism.*

Definition 2.5.5. Let $\mathbf{S}$ be a symmetric closed monoidal category, $\mathscr{C}$ an $\mathbf{S}$-enriched category, $\mathscr{C}^{\mathrm{op}} \xrightarrow{F} \mathbf{S}$ an $\mathbf{S}$-enriched functor, X an object of $\mathscr{C}$, and $\mathbb{1}_{\mathbf{S}} \xrightarrow{\eta} F(X)$ a map in $\mathbf{S}$. We say that F *is representable*, and that F *is represented by the pair* (X, η), if the enriched Yoneda morphism

$$F(X) \to \mathrm{Map}_{\mathbf{S}^{\mathscr{C}^{\mathrm{op}}}}(h_X, F)$$

takes η to an $\mathbf{S}$-natural isomorphism.

Definition 2.5.6. Let K be a simplicial set, and let $\mathfrak{C}[K^{\mathrm{op}}] \xrightarrow{\phi} \mathscr{C}$ be its fibrant replacement map in the Bergner model structure on Cat_Δ. The composite

$$K \to N\left((\mathrm{Set}_\Delta^{\mathscr{C}})^\circ\right) \to \mathscr{P}(K),$$

where the left-hand map corresponds to the composite

$$\mathfrak{C}[K^{\mathrm{op}}] \xrightarrow{\phi^{\mathrm{op}}} \mathscr{C}^{\mathrm{op}} \xrightarrow{h_\bullet} (\mathrm{Set}_\Delta^{\mathscr{C}})^\circ$$

via the adjunction $(\mathfrak{C}, N)$, and the right-hand map is the map g in Proposition 2.5.3 is called the *Yoneda embedding*.

Definition 2.5.7. Let $\mathscr{C}$ be an ∞-category, and $\mathscr{C}^{\mathrm{op}} \to \mathscr{S}$ a presheaf on it. Suppose that there is a pair (C, η) where $C \in \mathscr{C}$ and $\eta \in \pi_0(F(C))$ such that the $\mathscr{H}$-enriched functor

$$\operatorname{Ho} \mathscr{C}^{\mathrm{op}} \xrightarrow{\operatorname{Ho} F} \mathscr{H}$$

is representable by (C, η). In this case, we say that *F is representable*, and that *F is represented by* (C, η).

Proposition 2.5.8. [**Lur09**, Proposition 4.4.4.5]

Let $\tilde{\mathscr{C}} \xrightarrow{p} \mathscr{C}$ be a right fibration of ∞-categories and $\tilde{C}$ an object of $\tilde{\mathscr{C}}$. Suppose p is classified by $\mathscr{C}^{\mathrm{op}} \xrightarrow{F} \mathscr{S}$. Let $C = p(\tilde{C})$ and let $\eta \in \pi_0(F(C)) \cong \pi_0(\tilde{\mathscr{C}} \times_{\mathscr{C}} \{C\})$ be the connected component containing $\tilde{C}$. Then the following are equivalent.

(1) The presheaf F is represented by the pair (C, η).

(2) The object $\tilde{C} \in \tilde{\mathscr{C}}$ is final.

(3) The inclusion $\{\tilde{C}\} \to \tilde{\mathscr{C}}$ is a contravariant equivalence in $(\mathrm{Set}_\Delta)_{/\mathscr{C}}$.

Definition 2.5.9. Let $\tilde{\mathscr{C}} \xrightarrow{p} \mathscr{C}$ be a right fibration between ∞-categories. If the ∞-category $\tilde{\mathscr{C}}$ has a final object, then we say that *p is a representable right fibration*.

2.6. Adjunction, localization and presentable categories

Definition 2.6.1. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. A *correspondence* is a triple (p, h_0, h_1) of a map of ∞-categories $\mathcal{M} \xrightarrow{p} \Delta^1$ and categorical equivalences $\mathcal{C} \xrightarrow{h_0} p^{-1}\{0\}$, $\mathcal{D} \xrightarrow{h_1} p^{-1}\{1\}$. In case p is a Cartesian fibration, we say that a functor $\mathcal{D} \xrightarrow{g} \mathcal{C}$ is *associated to* $\mathcal{M}$, if there exists a map $\mathcal{D} \times \Delta^1 \xrightarrow{s} \mathcal{M}$ in $(\mathrm{Set}_\Delta)_{/\Delta^1}$ fitting into the commutative diagram with canonical vertical arrows

$$\begin{tikzcd}
\mathcal{D} \times \{0\} \arrow[r, "g"] \arrow[d] & \mathcal{C} \arrow[r, "h_0"] & p^{-1}\{0\} \arrow[d] \\
\mathcal{D} \times \Delta^1 \arrow[rr, "s"] & & \mathcal{M} \\
\mathcal{D} \times \{1\} \arrow[rr, "h_1"] \arrow[u] & & p^{-1}\{1\} \arrow[u]
\end{tikzcd}$$

such that the restriction $s \mid \{x\} \times \Delta^1$ is a p-Cartesian edge for all $x \in \mathcal{D}$. If p is a coCartesian fibration, then a functor $\mathcal{C} \xrightarrow{f} \mathcal{D}$ is *associated to* $\mathcal{M}$, if it is associated to the opposite triple $\left(\mathcal{M}^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} \left(\Delta^1\right)^{\mathrm{op}} \xrightarrow{\cong} \Delta^1, h_1, h_0\right)$.

Proposition 2.6.2. [**Lur09**, Proposition 5.2.1.3]

Let $\mathcal{D} \xrightarrow{g} \mathcal{C}$ be a functor of ∞-categories. Then there exists a correspondence (p, h_0, h_1) such that the functor $\mathcal{M} \xrightarrow{p} \Delta^1$ is a Cartesian fibration, the maps $\mathcal{C} \xrightarrow{h_0} p^{-1}\{0\}$, $\mathcal{D} \xrightarrow{h_1} p^{-1}\{1\}$ are isomorphisms, and g is associated to $\mathcal{M}$.

Proposition 2.6.3. [**Lur09**, Proposition 5.2.1.4]

Let $(\mathcal{M} \xrightarrow{p} \Delta^p\Delta^1, \mathcal{C} \xrightarrow{h_0} p^{-1}\{0\}, \mathcal{D} \xrightarrow{h_1} p^{-1}\{1\})$ be a correspondence between ∞-categories. Suppose the map p is a Cartesian fibration. Then there exists a functor $\mathcal{D} \xrightarrow{g} \mathcal{C}$ associated to $\mathcal{M}$.

Existence of g. Consider the diagram

$$\begin{tikzcd}
\mathcal{D}^\flat \times \{1\} \arrow[r, "h_1"] \arrow[d] & \mathcal{M}^\natural \arrow[d] \\
\mathcal{D}^\flat \times (\Delta^1)^\sharp \arrow[r] \arrow[ur, dashed, "s"] & (\Delta^1)^\sharp.
\end{tikzcd}$$

By [**Lur09**, Proposition 3.1.2.1 and Remark 3.1.1.11] the dashed arrow exists. Consider the map $\mathcal{D} \xrightarrow{s_0 = s|\mathcal{D}\times\{0\}} p^{-1}\{0\}$. Since in the Joyal model structure on simplicial sets, the cofibrations are the monomorphisms, the weak equivalences are the categorical equivalences, and the fibrant objects are precisely the ∞-categories, we can factor s_0 through h_0 to get g:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "h_0"] \\
\mathcal{D} \arrow[r, "s_0"'] \arrow[ur, dashed, "g"] & p^{-1}\{0\}.
\end{tikzcd}$$

□

Definition 2.6.4. Let $\mathcal{M} \xrightarrow{p} \Delta^1$ be part of a correspondence between ∞-categories $\mathcal{C}$ and $\mathcal{D}$. Suppose that p is both a Cartesian fibration and a coCartesian fibration. In this case, we say that p is an *adjunction*. If $\mathcal{C} \xrightarrow{f} \mathcal{D}$ and $\mathcal{D} \xrightarrow{g} \mathcal{C}$ are associated to $\mathcal{M}$, then we say *f is a left adjoint of* g, and *g is a right adjoint of f*.

Definition 2.6.5. Let $\begin{tikzcd}\mathscr{C} \arrow[r, bend left, "f"] & \mathscr{D} \arrow[l, bend left, "g"]\end{tikzcd}$ be a pair of functors between ∞-categories. An edge $\mathrm{id} \xrightarrow{\eta} g \circ f$ in $\mathrm{Fun}(\mathscr{C},\mathscr{C})$ is called a *unit transformation for* (f,g), if for all $(C,D) \in \mathscr{C} \times \mathscr{D}$, the map

$$\mathrm{Map}_{\mathscr{D}}(f(C),D) \xrightarrow{g} \mathrm{Map}_{\mathscr{C}}((g \circ f)(C), g(D)) \xrightarrow{u_C \circ} \mathrm{Map}_{\mathscr{C}}(C, g(D))$$

is an isomorphism in $\mathscr{H}$.

Proposition 2.6.6. [**Lur09**, Proposition 5.2.2.8]

Let $\begin{tikzcd}\mathscr{C} \arrow[r, bend left, "f"] & \mathscr{D} \arrow[l, bend left, "g"]\end{tikzcd}$ *be a pair of functors between ∞-categories. Then the following are equivalent.*

1. *(1) The functor f is a left adjoint of g.*
2. *(2) There exists a unit transformation for the pair (f,g).*

Proposition 2.6.7. [**Lur09**, Proposition 5.2.2.12]

Let $\mathscr{C} \xrightarrow{f} \mathscr{G}$ be a functor between ∞-categories. Then f has a right adjoint precisely when the $\mathscr{H}$-enriched functor $\mathrm{Ho}\,\mathscr{C} \xrightarrow{\mathrm{Ho}\,f} \mathrm{Ho}\,\mathscr{D}$ *has one.*

Proposition 2.6.8. [**Lur09**, Proposition 5.2.4.2]

Let $\mathscr{C} \xrightarrow{F} \mathscr{D}$ be a functor between ∞-categories. Then the following are equivalent.

(1) *The functor F has a left adjoint.*

(2) *For every pullback diagram*

$$\begin{tikzcd}\bar{\mathscr{C}} \arrow[r] \arrow[d, "p'"'] & \bar{\mathscr{D}} \arrow[d, "p"] \\ \mathscr{C} \arrow[r, "F"] & \mathscr{D},\end{tikzcd}$$

if p is a representable right fibration, then so is p'.

Notation 2.6.9. Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories. We will denote by $\mathrm{Fun}^{\mathrm{L}}(\mathscr{C},\mathscr{D}) \subseteq \mathrm{Fun}(\mathscr{C},\mathscr{D})$ the full subcategory on functors which are left adjoints, and by $\mathrm{Fun}^{\mathrm{R}}(\mathscr{C},\mathscr{D}) \subseteq \mathrm{Fun}(\mathscr{C},\mathscr{D})$ the full subcategory on functors which are right adjoints.

Proposition 2.6.10. [**Lur09**, Proposition 5.2.6.2]

Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories. The essential image of the fully faithful composite map

$$\mathrm{Fun}^{\mathrm{R}}(\mathscr{C},\mathscr{D}) \to \mathrm{Fun}(\mathscr{C},\mathscr{D}) \to \mathrm{Fun}(\mathscr{C},\mathscr{P}(\mathscr{D})) \to \mathrm{Fun}(\mathscr{C}\times\mathscr{D}^{\mathrm{op}},\mathscr{S})$$

where the arrows are inclusion, postcomposition with the Yoneda embedding and the isomorphism coming from the product-hom adjunction, consists of precisely those maps $\mathscr{C}\times\mathscr{D}^{\mathrm{op}} \xrightarrow{F} \mathscr{S}$, which satisfy the following.

(1) For each $C \in \mathscr{C}$, the restriction $F \mid \{C\}\times\mathscr{D}^{\mathrm{op}}$ is representable.

(2) For each $D \in \mathscr{D}$, the restriction $F \mid \mathscr{C}\times\{D\}$ is corepresentable.

This essential image is the same as that of the fully faithful composite map

$$\mathrm{Fun}^{\mathrm{L}}(\mathscr{D},\mathscr{C})^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{R}}(\mathscr{D}^{\mathrm{op}},\mathscr{C}^{\mathrm{op}}) \to \mathrm{Fun}(\mathscr{D}^{\mathrm{op}},\mathscr{C}^{\mathrm{op}}) \to \mathrm{Fun}(\mathscr{D}^{\mathrm{op}},\mathscr{P}(\mathscr{C}^{\mathrm{op}})) \to \mathrm{Fun}(\mathscr{C}\times\mathscr{D}^{\mathrm{op}},\mathscr{S}),$$

where the first isomorphism is gotten by definition, and the rest is the same as above. In particular, we get a canonical categorical equivalence between $\mathrm{Fun}^{\mathrm{R}}(\mathscr{C},\mathscr{D})$ *and* $\mathrm{Fun}^{\mathrm{L}}(\mathscr{D},\mathscr{C})^{\mathrm{op}}$.

Definition 2.6.11. A functor $\mathscr{C} \to \mathscr{D}$ between ∞-categories is a *localization functor*, if it admits a fully faithful right adjoint.

Let $\mathscr{C}_0 \subseteq \mathscr{C}$ be a subcategory. If the inclusion $\mathscr{C}_0 \to \mathscr{C}$ has a left adjoint, we say that $\mathscr{C}_0$ is a *reflexive subcategory*.

Definition 2.6.12. Let $\mathscr{C}$ be an ∞-category and κ a regular cardinal. We say that $\mathscr{C}$ *is κ-filtered*, if for every κ-small simplicial set K, every map $K \to \mathscr{C}$ extends to a map $K^{\triangleright} \to \mathscr{C}$. We say that $\mathscr{C}$ *is filtered*, if it is ω-filtered.

Notation 2.6.13. Let $\mathscr{C}$ be an ∞-category and κ a regular cardinal. We will denote by $\mathrm{Ind}_\kappa(\mathscr{C}) \subset \mathscr{P}(\mathscr{C})$ the full subcategory on presheaves $\mathscr{C}^{\mathrm{op}} \to \mathscr{S}$, which classify right fibrations $\tilde{\mathscr{C}} \to \mathscr{C}$ such that the ∞-category $\tilde{\mathscr{C}}$ is κ-filtered. We will let $\mathrm{Ind}(\mathscr{C}) = \mathrm{Ind}_\omega(\mathscr{C})$.

Definition 2.6.14. Let κ be a regular cardinal and $\mathscr{C}$ an ∞-category which admits small κ-filtered colimits. We say that a functor $\mathscr{C} \to \mathscr{D}$ between ∞-categories is *κ-continuous*, if it preserves κ-filtered colimits.

Let $C \in \mathscr{C}$ be an object. We say that C *is κ-compact*, if the functor it corepresents $\mathscr{C} \to \hat{\mathscr{S}}$ is κ-continuous.

Definition 2.6.15. Let $\mathscr{C}$ be an ∞-category. We say that $\mathscr{C}$ is *accessible*, if there exists a regular cardinal κ, a small ∞-category $\mathscr{C}_0$, and a categorical equivalence

$$\mathrm{Ind}_\kappa(\mathscr{C}_0) \to \mathscr{C}.$$

Let $\mathscr{C} \xrightarrow{f} \mathscr{D}$ be a functor between ∞-categories and suppose that $\mathscr{C}$ is accessible. We say that the functor f is *accessible*, if it is κ-continuous for some regular cardinal κ.

We say that $\mathscr{C}$ is *presentable*, if it is accessible, and it admits small colimits.

Definition 2.6.16. Let $\mathscr{C}$ be an ∞-category, and S a collection of morphisms in $\mathscr{C}$. We say that an object $Z \in \mathscr{C}$ is *S-local*, if for all morphisms $X \xrightarrow{s} Y$ in S, the precomposition map

$$\mathrm{Map}_{\mathscr{C}}(Y, Z) \xrightarrow{\circ s} \mathrm{Map}_{\mathscr{C}}(X, Z)$$

is a weak equivalence. We say that a morphism $X \xrightarrow{f} Y$ in $\mathscr{C}$ is an *S-equivalence*, if for all S-local objects Z of $\mathscr{C}$, the precomposition map

$$\mathrm{Map}_{\mathscr{C}}(Y, Z) \xrightarrow{\circ f} \mathrm{Map}_{\mathscr{C}}(X, Z)$$

is a weak equivalence.

Proposition 2.6.17. [**Lur09**, Proposition 5.5.4.2]

Let $\mathscr{C}$ be an ∞-category and $\mathscr{C} \xrightarrow{L} \mathscr{C}$ a localization functor. Let S denote the collection of morphisms $X \xrightarrow{s} Y$ in $\mathscr{C}$ such that the morphism Ls is an equivalence. Then the following assertions hold.

1. *An object $C \in \mathscr{C}$ is S-local if and only if it is in $L\mathscr{C}$.*
2. *Every S-equivalence is in S.*
3. *Suppose moreover that $\mathscr{C}$ is accessible. Then the following conditions are equivalent.*
   - (a) *The ∞-category $L\mathscr{C}$ is accessible.*
   - (b) *The localization functor $\mathscr{C} \xrightarrow{L} \mathscr{C}$ is accessible.*
   - (c) *There exists a small subset $S_0 \subseteq S$, such that every S_0-local object of $\mathscr{C}$ is S-local.*

Theorem 2.6.18. [**Lur09**, Theorem 5.5.1.1]

Let $\mathscr{C}$ be an ∞-category. Then the following are equivalent.

1. *$\mathscr{C}$ is presentable.*

(2) There exists a small ∞-category $\mathscr{D}$ such that $\mathscr{C}$ is an accessible localization of $\mathscr{P}(\mathscr{D})$.

Theorem 2.6.19. [**Lur09**, Proposition 5.5.2.2, Corollary 5.5.2.4, Proposition 5.5.2.7]

Let $\mathscr{C}$ be a presentable ∞-category. Then the following statements are true.

(1) A presheaf $\mathscr{C}^{\mathrm{op}} \xrightarrow{F} \mathscr{S}$ is representable precisely when it preserves small colimits.

(2) The ∞-category $\mathscr{C}$ admits small limits.

(3) A functor $\mathscr{C} \xrightarrow{F} \mathscr{S}$ is corepresentable precisely when it is accessible and it preserves small limits.

Theorem 2.6.20 (Adjoint functor theorem). [**Lur09**, Corollary 5.5.2.9]

Let $\mathscr{C} \xrightarrow{F} \mathscr{D}$ be a functor between presentable ∞-categories. Then the following statements hold.

(1) The functor F admits a right adjoint precisely when it preserves small colimits.

(2) The functor F admits a left adjoint precisely when it is accessible and it preserves small limits.

2.7. n-categories, and truncated objects

Definition 2.7.1. Let $\Delta_n \xrightarrow{i_n} \mathrm{Set}_\Delta$ denote inclusion of the full subcategory on $\{0, \ldots, n\}$. There are adjunctions

$$\begin{tikzcd} \mathrm{PSh}(\Delta_n) \arrow[r, bend left, "i_n^*"] \arrow[r, bend right, "i_n^!"'] & \mathrm{Set}_\Delta, \arrow[l, "i_{n*}" description] \end{tikzcd}$$

and we write

$$\mathrm{sk}_n = i_n^* i_{n*}, \ \mathrm{cosk}_n = i_n^! i_{n*}.$$

We call the sk_n *skeleton maps*, and the cosk_n *coskeleton maps*.

Definition 2.7.2. Let $\mathscr{C}$ be an ∞-category. For $n \geq -1$, we say that $\mathscr{C}$ is an *n-category*, if the following conditions hold.

(1) If two n-simplices $\begin{tikzcd} \Delta^n \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\sigma'"'] & \mathscr{C} \end{tikzcd}$ are homotopic relative to $\partial\Delta^n$, then $\sigma = \sigma'$.

(2) For any $m > n$, if for two m-simplices $\begin{tikzcd} \Delta^m \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\sigma'"'] & \mathscr{C} \end{tikzcd}$, we have $\sigma \mid \partial\Delta^m = \sigma' \mid \partial\Delta^m$, then $\sigma = \sigma'$.

We say that $\mathscr{C}$ is a (-2)-*category*, if it is a final object of Set_Δ.

Proposition 2.7.3. [**Lur09**, Proposition 2.3.4.7]

Let $\mathscr{C}$ be an ∞-category and $n \geq -1$. The following are equivalent.

(1) *$\mathscr{C}$ is an n-category.*

(2) *For every pair of maps* $K \underset{f'}{\overset{f}{\rightrightarrows}} \mathscr{C}$ *, if $f \mid \operatorname{sk}_n K$ and $f' \mid \operatorname{sk}_n K$ are homotopic relative to* $\operatorname{sk}_{n-1} K$, *then we have $f = f'$.*

Proposition 2.7.4. [**Lur09**, Proposition 2.3.4.9]

Let $\mathscr{C}$ be an ∞-category and $n \geq 1$. The following are equivalent.

(1) *$\mathscr{C}$ is an n-category.*

(2) *For every $0 < i < m$, every map $\Lambda^m_i \to \mathscr{C}$ can be* uniquely *extended to an m-simplex.*

$$\begin{tikzcd}
\Lambda^m_i \arrow[r] \arrow[d] & \mathscr{C} \\
\Delta^m \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd}$$

Notation 2.7.5. Let $\mathscr{C}$ be an ∞-category, K a simplicial set, and $n \geq 1$. We will denote by $[K, \mathscr{C}]_n$ the set of those diagrams $\operatorname{sk}_n K \to \mathscr{C}$ which extend to $\operatorname{sk}_{n+1} K$. For a pair of maps $\operatorname{sk}_n K \underset{g}{\overset{f}{\rightrightarrows}} \mathscr{C}$, write $f \sim g$ if they are homotopic relative to $\operatorname{sk}_{n-1} K$.

Proposition 2.7.6. [**Lur09**, Proposition 2.3.4.12]

Let C be an ∞-category and $n \geq 1$.

(1) *The presheaf* $\mathrm{Set}_\Delta^{\mathrm{op}} \xrightarrow{K \mapsto [K,\mathscr{C}]_n/\sim} \mathrm{Set}$ *is representable by a simplicial set $h_n\mathscr{C}$.*

(2) *The simplicial set $h_n\mathscr{C}$ is an n-category.*

(3) *If $\mathscr{C}$ is an n-category, then the natural map $\mathscr{C} \xrightarrow{\theta} h_n\mathscr{C}$ is an isomorphism.*

(4) *For every n-category $\mathscr{D}$, composition with θ induces an isomorphism of simplicial sets*

$$\mathrm{Fun}(h_n\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D}).$$

Definition 2.7.7. Let X be a Kan complex and $k \geq 0$. We say that X is *k-truncated*, if for all $i > k$ and every point $x \in X$, the group $\pi_i(X, x)$ is trivial. We say that X is (-1)-*truncated*, if it is either empty or contractible. We say that X is (-2)-*truncated*, if it is contractible.

Proposition 2.7.8. [**Lur09**, Proposition 2.3.4.18]

Let $\mathscr{C}$ be an ∞-category and let $n \geq -1$. Then the following are equivalent.

(1) *There exists an n-category $\mathscr{D}$ and a categorical equivalence $\mathscr{D} \simeq \mathscr{C}$.*

(2) *For every pair of objects $X, Y \in \mathscr{C}$, the mapping space $\mathrm{Map}_{\mathscr{C}}(X, Y) \in \mathscr{H}$ is $(n-1)$-truncated.*

2.8. Groupoid objects, Čech nerves, effective epimorphisms and essential images

Definition 2.8.1. Let Δ_+ denote the 1-category $\Delta^{\triangleright}$. It is equivalent to the category of possibly empty finite linearly ordered sets. Let $\mathscr{C}$ be an ∞-category. A *groupoid object of $\mathscr{C}$* is a simplicial object $N(\Delta)^{\mathrm{op}} \xrightarrow{U_\bullet} \mathscr{C}$ such that for partitions $[n] = S \cup S'$ such that $S \cap S' = \{s\}$, the diagram

$$\begin{tikzcd} U([n]) \arrow[r] \arrow[d] & U(S) \arrow[d] \\ U(S') \arrow[r] & U(\{s\}) \end{tikzcd}$$

is Cartesian.

A *Čech nerve in $\mathscr{C}$* is an augmented simplicial object $N(\Delta_+)^{\mathrm{op}} \xrightarrow{U_\bullet^+} \mathscr{C}$ such that $U_\bullet^+|N(\Delta)^{\mathrm{op}}$ is an ∞-category, and the diagram

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_0 \arrow[r] & U_{-1} \end{tikzcd}$$

is Cartesian. Let $N(\Delta)^{\mathrm{op}} \xrightarrow{V_\bullet} \mathscr{C}$ be a groupoid object. We say that $U_\bullet$ *is effective*, if it can be extended to a Čech nerve.

A Čech nerve $N(\Delta_+)^{\mathrm{op}} \xrightarrow{U_\bullet^+} \mathscr{C}$ is called a *simplicial resolution*, if it is the colimit of the groupoid object $U_\bullet$. In this case, we say that the morphism $U_\bullet|N(\Delta_+^{\leq 0})^{\mathrm{op}}$ is an *effective epimorphism*.

Proposition 2.8.2. [**Lur09**, Proposition 6.1.2.11]

Let $\mathscr{C}$ be an ∞-category, and let $N(\Delta_+)^{\mathrm{op}} \xrightarrow{U_\bullet^+} \mathscr{C}$ be an augmented simplicial object. Then the following are equivalent.

(1) *$U_\bullet^+$ is a Čech nerve.*

(2) *$U_\bullet^+$ is a right Kan extension of $U_\bullet^+|N(\Delta_+^{\leq 0})^{\mathrm{op}}$.*

Definition 2.8.3. Let $\mathscr{X}$ be an ∞-category. We say that *$\mathscr{X}$ is an ∞-topos*, if if there exists a small ∞-category $\mathscr{C}$, and an accessible left exact localization functor $\mathscr{X} \to \mathscr{P}(\mathscr{C})$.

Theorem 2.8.4. [**Lur09**, Theorem 6.1.0.6]

Let $\mathscr{X}$ be an ∞-category. Then the following are equivalent.

(1) $\mathscr{X}$ is an ∞-topos.

(2) The following conditions hold.

  (a) $\mathscr{X}$ is presentable.

  (b) Colimits in $\mathscr{X}$ are universal.

  (c) Coproducts in $\mathscr{X}$ are disjoint.

  (d) Every groupoid object in $\mathscr{X}$ is effective.

Definition 2.8.5. Let $X \xrightarrow{f} Y$ be a morphism in a presentable category $\mathscr{C}$. We say that f *is n-truncated*, if for all $Z \in \mathscr{C}$, all homotopy fibers of the map of Kan complexes

$$\mathrm{Map}_{\mathscr{C}}(Z, X) \to \mathrm{Map}_{\mathscr{C}}(Z, Y)$$

are n-truncated. We say that f is *0-connective*, if it is an effective epimorphism, and the truncation $\tau_{\leq 0}(f) \in \mathscr{C}_{/Y}$ is a final object.

Theorem 2.8.6. [**Lur09**, Example 5.2.8.16]
Let $\mathscr{X}$ be an ∞-topos, and let $X \xrightarrow{f} Y$ be a morphism in $\mathscr{X}$. Then f factorizes as a composite

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z, \arrow[ur, "h"] & \end{tikzcd}$$

where g is 0-connective, and h is (-1)*-truncated.*

Definition 2.8.7. In the situation of Theorem 2.8.6, we call Z the *essential image of f*.

CHAPTER 3

Constructing the stack of perfect complexes

In this chapter, we will summarize a part of the theory developed in [**Lur14**], and how it is used in [**Pan11**] to construct the ∞-stack of perfect complexes $\operatorname{Perf} X$ on a scheme X. We also explicitly describe how can this stack be given the structure of a closed symmetric monoidal ∞-category, which is going to be used in the next chapter to define ∞-Lie algebras and the functor $E \mapsto \mathfrak{sl}_n(E)$ on $\operatorname{Perf} X$.

3.1. Colored operads and ∞-operads

The framework developed in [**Lur14**] uses the Grothendieck construction to encode higher coherence diagrams for algebraic structures.

Definition 3.1.1. A *colored operad* $\mathscr{O}$ consists of the following.

1. A set of objects, which we will also denote by $\mathscr{O}$.
2. For each finite set of objects $\{X_i\}_{i\in I}$ and another object Y, a set of morphisms $\operatorname{Mul}_{\mathscr{O}}(\{X_i\}_{i\in I}, Y)$.
3. For each morphism of finite sets $I \xrightarrow{\alpha} J$, finite sets of objects $\{X_i\}_{i\in I}$, $\{Y_j\}_{j\in J}$ and another object Z, a composition map
$$\prod_{j\in J} \operatorname{Mul}_{\mathscr{O}}(\{X_i\}_{i\in\alpha^{-1}\{j\}}, Y_j) \times \operatorname{Mul}_{\mathscr{O}}(\{Y_j\}_{j\in J}, Z) \to \operatorname{Mul}_{\mathscr{O}}(\{X_i\}_{i\in I}, Z),$$
which is associative in the appropriate sense.
4. For each object $X \in \mathscr{O}$, an identity morphism $\mathrm{id}_X \in \operatorname{Mul}_{\mathscr{O}}(\{X\}, X)$, which is a left and right unit for composition.

Remark 3.1.1.1. The collections $\{X_i\}_{i\in I}$ are actually finite sequences $(X_i)_{i\in I}$, but I would like to keep the notation in [**Lur14**].

Variation 3.1.2. A *simplicial colored operad* admits the same axioms in the Cartesian category of simplicial sets.

Definition 3.1.3. We define a category Fin_* equivalent to that of pointed finite sets as follows.

1. Its objects are the finite pointed sets $\langle n\rangle = \{*, 1, \dots, n\}$, for $n \geq 0$.
2. Its morphisms are the morphisms of pointed sets.

For each $n \geq 0$, we denote by $\langle n\rangle^\circ$ the subset $\{1, \dots, n\} \subseteq \langle n\rangle$. Let $\langle m\rangle \xrightarrow{f} \langle n\rangle$ be a morphism. It is called *inert*, if for each $i \in \langle n\rangle^\circ$, the preimage $f^{-1}\{i\}$ has a unique element. The morphism f is called *active*, if $f^{-1}\{*\} = \{*\}$. For $1 \leq i \leq n$, we let $\langle n\rangle \xrightarrow{\rho^i} \langle 1\rangle$ denote the morphism such that

$$\rho^i(j) = \begin{cases} 1 & \text{if } i = j, \\ * & \text{otherwise.} \end{cases}$$

Construction 3.1.4. Let $\mathscr{O}$ be a colored operad. We define a category $\mathscr{O}^\otimes$ over Fin_* as follows.

1. The objects of $\mathscr{O}^\otimes$ over $\langle n\rangle$ for $n \geq 0$ are the finite collections of objects $\{X_i\}_{1\leq i\leq n}$ of $\mathscr{O}$.
2. A morphism $\{X_i\}_{1\leq i\leq m} \to \{Y_j\}_{1\leq j\leq n}$ over a morphism $\langle m\rangle \xrightarrow{f} \langle n\rangle$ is a collection

$$\left\{\phi_j \in \mathrm{Mul}_{\mathscr{O}}(\{X_i\}_{i\in f^{-1}\{j\}}, Y_j)\right\}_{1\leq j\leq n}.$$

3. Composition is defined using composition in $\mathscr{O}$.

Remark 3.1.4.1. Let $\mathscr{O}$ be a colored operad, and consider the functor $\mathscr{O}^\otimes \xrightarrow{p} \mathrm{Fin}_*$. It satisfies the following conditions.

1. Let $\langle m\rangle \xrightarrow{f} \langle n\rangle$ be an inert morphism, and $\{X_i\}_{1\leq i\leq m}$ an object of $\mathscr{O}^\otimes_{\langle m\rangle}$. One can check that the following is a p-coCartesian arrow over f. Since f is inert, by letting $Y_{f(i)} = X_i$, we get a well defined object $\{Y_j\} \in \mathscr{O}^\otimes_{\langle n\rangle}$. Then the arrow

$$\{X_i\}_{1\leq i\leq n} \xrightarrow{\left\{\mathrm{id}_{X_i}\right\}_{1\leq j\leq n}} \{Y_j\}_{1\leq j\leq n}$$

is p-coCartesian.

2. For any $n \geq 0$, any product of induced functors

$$\mathscr{O}^\otimes_{\langle n\rangle} \xrightarrow{(\rho^i)_{1\leq i\leq n)}} \prod_{1\leq i\leq n} \mathscr{O}^\otimes_{\langle 1\rangle}$$

is an equivalence of categories.

On the other hand, we can reconstruct the colored operad $\mathscr{O}$ from the functor $\mathscr{O}^{\otimes} \to \mathrm{Fin}_*$ using the above two conditions.

Variation 3.1.5. If $\mathscr{O}$ is a simplicial operad, then performing Construction 3.1.4 in the simplicially enriched setting gives a simplicial category $\mathscr{O}^{\otimes}$. We denote the simplicial nerve of $\mathscr{O}^{\otimes}$ by $N^{\otimes}(\mathscr{O})$, and call it the *operadic nerve of* $\mathscr{O}$.

Definition 3.1.6. Let $\mathscr{O}^{\otimes} \xrightarrow{p} N(\mathrm{Fin}_*)$ be a functor of ∞-categories. We say that it is an ∞*-operad*, if it satisfies the following two conditions.

(1) If $\langle m \rangle \xrightarrow{f} \langle n \rangle$ is an inert morphism, and X is an object of $\mathscr{O}^{\otimes}_{\langle m \rangle}$, then there exists a p-coCartesian edge in $\mathscr{O}^{\otimes}$ over f.

(2) Let $C \in \mathscr{O}^{\otimes}_{\langle m \rangle}$, $C' \in \mathscr{O}^{\otimes}_{\langle n \rangle}$ and $\langle m \rangle \xrightarrow{f} \langle n \rangle$ a morphism in Fin_*. Choosing p-coCartesian morphisms over the inert morphisms $\langle n \rangle \xrightarrow{\rho^i} \langle 1 \rangle$, $1 \leq i \leq n$, the induced map

$$\mathrm{Map}_{\mathscr{O}^{\otimes}/f}(C, C') \to \prod_{1 \leq i \leq n} \mathrm{Map}_{\mathscr{O}^{\otimes}/\rho^i \circ f}(C, C'_i)$$

is an equivalence of Kan complexes.

(3) For any $n \geq 0$, the induced arrow

$$\mathscr{O}^{\otimes}_{\langle n \rangle} \xrightarrow{(\rho^i_!)} \prod_{1 \leq i \leq n} \mathscr{O}^{\otimes}_{\langle 1 \rangle}$$

is an equivalence of ∞-categories.

Notation 3.1.7. We denote $\mathscr{O}^{\otimes}_{\langle 1 \rangle}$ by $\mathscr{O}$ and call it the *underlying category of* $\mathscr{O}$. Let $n \geq 0$. Via the equivalence $\mathscr{O}^{\otimes}_{\langle n \rangle} \simeq \mathscr{O}^{\times n}$, an object $X \in \mathscr{O}^{\otimes}_{\langle n \rangle}$ corresponds to some n-tuple $(X_1, \ldots, X_n)$ of objects of $\mathscr{O}$. In this case, we might write $X_1 \oplus \cdots \oplus X_n$ for the object in $\mathscr{O}^{\otimes}_{\langle n \rangle}$. Let $Y \in \mathscr{O}$ be another object. We will denote by $\mathrm{Mul}_{\mathscr{O}}(\{X_i\}_{1 \leq i \leq n}, Y)$ the union of those components of $\mathrm{Map}_{\mathscr{O}^{\otimes}}(X_1 \oplus \cdots \oplus X_n, Y)$ which lie over the unique active morphism $\langle n \rangle \to \langle 1 \rangle$.

Definition 3.1.8. Let f be a morphism in an ∞-operad $\mathscr{O}^{\otimes} \xrightarrow{p} N(\mathrm{Fin}_*)$. We say that f is *inert*, if it is a p-coCartesian arrow over an inert arrow in $N(\mathrm{Fin}_*)$. We say that f is *active* if it is over an active arrow in $N(\mathrm{Fin}_*)$.

Definition 3.1.9. A map $\mathscr{O}^{\otimes} \xrightarrow{f} \mathscr{O}'^{\times}$ in $(\mathrm{Set}_{\Delta})_{/N(\mathrm{Fin}_*)}$ between ∞-operads is an ∞-*operad map*, if it takes inert morphisms into inert morphisms. If moreover f is a categorical fibration, we call it a *fibration of* ∞-*operads*.

Notation 3.1.10. Let $\mathscr{C}^{\otimes} \xrightarrow{p} \mathscr{O}^{\otimes}$ be a fibration of ∞-operads, and $\mathscr{O}'^{\otimes} \xrightarrow{\alpha} \mathscr{O}^{\otimes}$ a map of ∞-operads. We let $\mathrm{Alg}_{\mathscr{O}'/\,/\mathscr{O}}(\mathscr{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathscr{O}^{\otimes}}(\mathscr{O}'^{\otimes}, \mathscr{O}^{\otimes})$ spanned by the ∞-operad maps. If $\alpha = \mathrm{id}_{\mathscr{O}}$, we let $\mathrm{Alg}_{/\mathscr{O}}(\mathscr{C})$ denote $\mathrm{Alg}_{\mathscr{O}'/\,/\mathscr{O}}(\mathscr{C})$.

Definition 3.1.11. Let $\mathscr{O}^{\otimes}$ be an ∞-operad, and $\mathscr{C}^{\otimes} \xrightarrow{p} \mathscr{O}^{\otimes}$ a coCartesian fibration. We say that p is a *coCartesian fibration of* ∞-*operads*, if the following equivalent conditions hold [**Lur14**, Proposition 2.1.2.12]

(1) The composite $\mathscr{C}^{\otimes} \xrightarrow{p} \mathscr{O}^{\otimes} \to N(\mathrm{Fin}_*)$ exhibits $\mathscr{C}^{\otimes}$ as an ∞-operad.

(2) For any object $T \simeq T_1 \oplus \cdots \oplus T_n \in \mathscr{O}^{\otimes}_{\langle n \rangle}$, the inert morphisms $T \to T_i$ induce an equivalence of ∞-categories

$$\mathscr{C}^{\otimes}_T \to \prod_{1 \leq i \leq n} T_i.$$

In this case, we also say that *p exhibits $\mathscr{C}$ as an $\mathscr{O}$-monoidal* ∞-*category*.

Definition 3.1.12. Let $\mathscr{C}^{\otimes} \xrightarrow{p} \mathscr{O}^{\otimes}$ and $\mathscr{D}^{\otimes} \xrightarrow{q} \mathscr{O}^{\otimes}$ be coCartesian fibrations of ∞-operads. A functor of $\mathscr{O}$-monoidal ∞-categories $\mathscr{C}^{\otimes} \to \mathscr{D}^{\otimes}$ is an *$\mathscr{O}$-monoidal functor*, if it takes p-coCartesian edges to q-coCartesian edges.

Definition 3.1.13. Let us define a simplicial category $\mathrm{Op}^{\Delta}_{\infty}$ as follows.

(1) Its objects are the small ∞-operads.

(2) The mapping space $\mathrm{Map}_{\mathrm{Op}^{\Delta}_{\infty}}(\mathscr{O}\otimes, \mathscr{O}'^{\otimes})$ is the Kan complex of ∞-operad maps $\mathrm{Alg}_{\mathscr{O}}(\mathscr{O}')$.

The ∞-*category of* ∞-*operads* Op_{∞} is the simplicial nerve $N(\mathrm{Op}^{\Delta}_{\infty})$.

3.2. Cartesian symmetric monoidal ∞-categories

Definition 3.2.1. The *commutative operad* **Comm** is the colored operad with one object and all whose morphism sets are singletons. The corresponding ∞-operad is given by the identity map $N(\mathrm{Fin}_*) \to N(\mathrm{Fin}_*)$, which we denote by $\mathrm{Comm}^{\otimes}$ and call it the *commutative* ∞-*operad*.

Definition 3.2.2. Let $\mathscr{C}^{\otimes}$ is an ∞-operad such that its structure map is a coCartesian fibration of ∞-operads $\mathscr{C}^{\otimes} \xrightarrow{p} \mathrm{Comm}^{\otimes} = N(\mathrm{Fin}_*)$. In this case, we say that $\mathscr{C}$ is a *symmetric monoidal* ∞-*category*.

If $\langle 2\rangle \xrightarrow{\alpha} \langle 1\rangle$ is the unique active map, then we have a chain

$$\mathscr{C}^{\times 2} \xleftarrow{\simeq} \mathscr{C}^{\otimes}_{\langle 2\rangle} \xrightarrow{\alpha_!} \mathscr{C}.$$

We will denote the image of $X \oplus Y \in \mathscr{C}^{\otimes}_{\langle 2\rangle}$ by $X \otimes Y$. Since $\mathscr{C}_{\langle 0\rangle}$ is equivalent to a final object in Set_{Δ}, it is contractible. The unique map $\langle 0\rangle \to \langle 1\rangle$ induces a p-coCartesian map $\mathscr{C}^{\otimes}_{\langle 0\rangle} \to \mathscr{C}$. Therefore there is a *unit object of* $\mathscr{C}$, which is well defined up to equivalence. We will denote it by $1_{\mathscr{C}}$.

Remark 3.2.2.1. Let $\mathscr{C} \xrightarrow{\iota} \mathscr{C}^{\otimes}_{\langle 2\rangle}$ be a p-coCartesian arrow above the map $\langle 1\rangle \xrightarrow{1\mapsto 1} \langle 2\rangle$. Its composite with the canonical equivalence $\mathscr{C}^{\otimes}_{\langle 2\rangle} \to \mathscr{C}^{\times 2}$ has the following components. Similarly, starting from the map $\langle 1\rangle \xrightarrow{1\mapsto 2} \langle 2\rangle$, we can get an equivalence $\mathscr{C} \xrightarrow{C\mapsto 1\otimes C} \mathscr{C}$.

1. The one on the left is a p-coCartesian map over $\mathrm{id}_{\langle 1\rangle}$.
2. The one on the right is equivalent to a map factoring through a p-coCartesian map $\mathscr{C}^{\otimes}_{\langle 0\rangle} \to \mathscr{C}$.

Therefore, up to equivalence the action on objects of the composite with the product map $\alpha_! \circ \iota$ is $C \mapsto C \otimes 1$. Since this map is p-coCartesian over $\mathrm{id}_{\langle 1\rangle}$, it is an equivalence.

Definition 3.2.3. A Comm-monoidal functor $\mathscr{C}^{\otimes} \to \mathscr{D}^{\otimes}$ is called a *symmetric monoidal functor*. We define a subcategory $\mathrm{Cat}^{\otimes}_{\infty} \subseteq \mathrm{Op}_{\infty}$ as follows.

1. Its objects are the symmetric monoidal ∞-categories.
2. Its morphisms are the symmetric monoidal functors.

We call this category the *∞-category of symmetric monoidal ∞-categories*.

Definition 3.2.4. Let $\mathscr{C}$ be an ∞-category. A symmetric monoidal ∞-category and an isomorphism between its underlying ∞-category and $\mathscr{C}$ is called a *symmetric monoidal structure on $\mathscr{C}$*. A symmetric monoidal structure on $\mathscr{C}$ is *Cartesian*, if the following statements hold.

1. The unit object $1_{\mathscr{C}} \in \mathscr{C}$ is final.
2. For any $C, D \in \mathscr{C}$, the canonical maps

$$C \xrightarrow{\simeq} C \otimes 1 \leftarrow C \otimes D \to 1 \otimes D \xleftarrow{\simeq} D$$

   exhibit $C \otimes D$ as a product of C and D in the ∞-category $\mathscr{C}$.

Proposition 3.2.5. [**Lur14**, Corollary 2.4.1.9]

Let $\mathrm{Cat}_\infty^{\otimes,\times} \subseteq \mathrm{Cat}_\infty^{\otimes}$ *denote the full subcategory spanned by the Cartesian symmetric monoidal* ∞*-categories, and* $\mathrm{Cat}_\infty^{\mathrm{Cart}}$ *denote the full subcategory spanned by the* ∞*-categories which admit finite products. Then the forgetful functor*

$$\mathrm{Cat}_\infty^{\otimes,\times} \to \mathrm{Cat}_\infty^{\mathrm{Cart}}$$

is an equivalence of ∞*-categories.*

Notation 3.2.6. Let $\mathscr{C}$ be an ∞-category which admits finite products. We will denote by $\mathscr{C}^\times$ a choice of a Cartesian symmetric monoidal ∞-category with underlying set isomorphic to $\mathscr{C}$.

Definition 3.2.7. Let $\mathscr{C}^\otimes \xrightarrow{p} N(\mathrm{Fin}_*)$ be an ∞-operad. A *lax Cartesian structure on* $\mathscr{C}^\otimes$ is a functor $\mathscr{C}^\otimes \xrightarrow{\pi} \mathscr{D}$ between ∞-categories, such that it satisfies the following condition.

(1) For any $n \geq 0$ and $C \simeq C_1 \oplus \cdots \oplus C_n \in \mathscr{C}^\otimes_{\langle n \rangle}$, the images of p-coCartesian edges $\pi(C) \to \pi(C_i)$ exhibit $\pi(C)$ as a product of the $\pi(C_i)$ in $\mathscr{D}$.

We also call π an $\mathscr{C}$*-monoid*. We say that π is a *weak Cartesian structure*, if moreover the following conditions hold.

(2) The ∞-operad $\mathscr{C}^\otimes$ is a symmetric monoidal ∞-category.

(3) If $C \xrightarrow{f} C'$ is a p-coCartesian edge over an active morphism of the form $\langle n \rangle \to \langle 1 \rangle$, then $\pi(f)$ is an equivalence.

We say that π is a *Cartesian structure*, if in addition to all this, the following condition is satisfied.

(4) The functor π restricts to an equivalence $\mathscr{C} \to \mathscr{D}$.

Depending on the context, we denote the full subcategory of $\mathrm{Fun}(\mathscr{C}^\otimes, \mathscr{D})$ spanned by lax Cartesian structures or $\mathscr{C}$-monoids by $\mathrm{Fun}^{\mathrm{lax}}(\mathscr{C}^\otimes, \mathscr{D})$ or $\mathrm{Mon}_{\mathscr{C}}(\mathscr{D})$.

Proposition 3.2.8. [**Lur14**, Proposition 2.4.2.5]

Let $\mathscr{C}^\otimes \xrightarrow{\pi} \mathscr{D}$ *be a Cartesian structure, and* $\mathscr{O}^\otimes$ *an* ∞*-operad. Then postcomposition with* π *induces an equivalence of* ∞*-categories*

$$\mathrm{Alg}_{\mathscr{O}}(\mathscr{C}) \to \mathrm{Mon}_{\mathscr{O}}(\mathscr{D}).$$

3.3. Monoidal ∞-categories and tensored ∞-categories

Definition 3.3.1. The *associative operad* **Ass** is defined as follows.

(1) Its object set is the singleton $\{*\}$.

(2) For each finite set I, the morphism set $\mathrm{Mul}_{\mathbf{Comm}}(\{*\}_{i\in I}, *)$ is the set of linear orderings on $\langle n\rangle^{\circ}$.

(3) Let $I \xrightarrow{f} J$ be a morphism of finite sets. The composition map

$$\prod_{j\in J} \mathrm{Mul}_{\mathbf{Comm}}(\{X_i\}_{i\in f^{-1}\{j\}}, Y_j) \times \mathrm{Mul}_{\mathbf{Comm}}(\{Y_j\}_{j\in J}, Z)$$

is defined as follows. Suppose given linear orderings $\leq_j$ on $f^{-1}\{j\}$ for each $j \in J$, and another one $\leq'$ on J. Composition takes this to the linear ordering $\leq$ on I, where $i \leq i'$ if $f(i) < f(i')$, or $f(i) = f(i') = j$, and $i \leq_j i'$.

We denote by $\mathbf{Ass}^{\otimes} \to \mathrm{Fin}_*$ the corresponding cofibered category, and by $\mathrm{Ass}^{\otimes}$ the ∞-operad $N(\mathbf{Ass}^{\otimes})$. We call it the *associative* ∞*-operad*.

Since the hom sets $\phi \in \mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{i\in I}, Y)$ are defined by linear orderings on I, we have a canonical map $\mathbf{LM} \to \mathbf{Ass}$, inducing a canonical map $\mathrm{LM}^{\otimes} \to \mathrm{Ass}^{\otimes}$.

Definition 3.3.2. Let $\mathscr{C}^{\otimes} \xrightarrow{p} \mathrm{Ass}^{\otimes}$ be a fibration of ∞-operads. Then we call it a *planar* ∞*-operad*. If moreover p is a coCartesian fibration of ∞-operads, then we call it a *monoidal* ∞*-category*.

Remark 3.3.2.1. One can describe the cofibered category $\mathbf{Ass}^{\otimes} \to \mathrm{Fin}_*$ as follows.

(1) Its objects are those of Fin_*.

(2) A morphism over $\langle m\rangle \overset{\langle}{\to} n\rangle$ is given by linear orderings $\leq_i$ on each preimage $f^{-1}\{i\}$, $1 \leq i \leq n$.

Notation 3.3.3. Let $\mathscr{C}^{\otimes} \xrightarrow{p} \mathrm{Ass}^{\otimes}$ be a monoidal ∞-category. Consider a p-coCartesian map $\mathscr{C}^{\otimes}_{\langle 2\rangle} \to \mathscr{C}$ over the morphism $\langle 2\rangle \to \langle 1\rangle$ in $\mathrm{Ass}^{\otimes}$, which is the unique active morphism, and $\langle 2\rangle^{\circ}$ is given its canonical ordering. We will denote the action of this map on objects by $C \oplus D \mapsto C \otimes D$.

Definition 3.3.4. Let $\mathscr{C}^{\otimes} \xrightarrow{p} \mathrm{Ass}^{\otimes}$ be a monoidal ∞-category. We say that $\mathscr{C}$ *is left closed*, if for any $C \in \mathscr{C}$, any composite of up to equivalence canonical arrows

$$\mathscr{C} \xrightarrow{D\mapsto(C,D)} \mathscr{C}^{\times 2} \overset{\simeq}{\leftarrow} \mathscr{C}_{\langle 2\rangle} \xrightarrow{C\oplus D\mapsto C\otimes D} \mathscr{C},$$

which we denote by $D \mapsto C\otimes D$ admits a right adjoints. We define right closed and closed monoidal ∞-categories similarly.

Definition 3.3.5. Let $\mathscr{C}$ be an ∞-category. A *monoid object* of $\mathscr{C}$ is a functor $N(\Delta)^{\mathrm{op}} \xrightarrow{X} \mathscr{C}$ such that for each $1 \leq i \leq n$, the maps

$$X(\{0, \dots, i\}) \leftarrow X([n]) \rightarrow X(\{i, \dots, n\})$$

exhibit $(X[n])$ as a product $X(\{0, \dots, i\}) \times X(\{i, \dots, n\})$. We denote by $\mathrm{Mon}(\mathscr{C})$ the full subcategory of $\mathrm{Fun}(N(\Delta)^{\mathrm{op}}, \mathscr{C})$ spanned by monoid objects.

Remark 3.3.5.1. Let X be a group object of $\mathscr{C}$. Then in addition to being a monoid object, it satisfies conditions of the following type. The square

$$\begin{tikzcd} X_2 \arrow[r, "d_1"] \arrow[d, "d_2"'] & X_1 \arrow[d] \\ X_1 \arrow[r] & X_0 \end{tikzcd}$$

is Cartesian, and X_0 is a final object of $\mathscr{C}$, which can be interpreted as follows. Let $T \xrightarrow{f} X_1$ be a morphism of simplicial sets, and let $T \xrightarrow{1} X_1$ be one factoring through the degeneracy map $X_0 \to X_1$. Via the equivalence $X_2 \simeq X_1^{\times 2}$, the map $X_2 \xrightarrow{d_2} X_1$ can be interpreted as a projection map, the map $X_2 \xrightarrow{d_1} X_1$ as a multiplication map, for which images of the map $X_0 \xrightarrow{r_0} X_1$ are left and right units. In this setting, the pair of maps $T \underset{1}{\overset{f}{\rightrightarrows}} X_1$ induce a map $T \to X_2$ which can be interpreted as describing a pair (f, g) such that $fg = 1$.

Proposition 3.3.6. [**Lur14**, Proposition 4.1.2.6]

There exists a functor $N(\Delta)^{\mathrm{op}} \xrightarrow{\mathrm{Cut}} \mathrm{Ass}^{\otimes}$*, which satisfies the following property. For any* ∞*-category* $\mathscr{C}$ *which admits finite products, precomposition with* Cut *induces an equivalence*

$$\mathrm{Mon}_{\mathrm{Ass}}(\mathscr{C}) \rightarrow \mathrm{Mon}(\mathscr{C}).$$

Definition 3.3.7. Let us define a colored operad **LM** as follows.

(1) It has two objects $\mathfrak{a}$ and $\mathfrak{m}$.

(2) Let $\{X_i\})_{i \in I}$ be a finite collection of objects, and Y an another one. The morphism set $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{1 \leq i \leq n}, Y)$ is defined as follows.

  - (a) Suppose $Y = \mathfrak{a}$. Then $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{1\le i\le n}, Y)$ is the set of all linear orderings on the set I if every $X_i = \mathfrak{a}$, and empty otherwise.
  - (b) Suppose $Y = \mathfrak{m}$. Then $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{1\le i\le n}, Y)$ is the set of all linear orderings $\{i_1 < \ldots < i_n\}$ on the set I such that $X_{i_j} = \mathfrak{a}$ for all $1 \le j < n$, and $X_{i_n} = \mathfrak{m}$.

(3) The composition law is determined by the composition of linear orderings.

We denote by $\mathbf{LM}^{\otimes} \to \mathrm{Fin}_*$ the corresponding cofibered category, and by $\mathrm{LM}^{\otimes}$ the corresponding ∞-operad. The inclusion $\{\mathfrak{a}\} \to \mathrm{LM}^{\otimes}$ determines a morphism of ∞-operads $\mathrm{Ass}^{\otimes} \to \mathrm{LM}^{\otimes}$, which is the inclusion of the full subcategory spanned by the objects of the form $\mathfrak{a}^{\oplus n}$.

Notation 3.3.8. Let $\mathscr{C}^{\otimes} \to \mathrm{LM}^{\otimes}$ be a fibration of ∞-operads. We denote by $\mathscr{C}^{\otimes}_{\mathfrak{a}}$ the fiber product $\mathscr{C}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathrm{Ass}^{\otimes}$, by $\mathscr{C}_{\mathfrak{a}}$ its underlying category $\mathscr{C}^{\otimes}_{\mathrm{LM}^{\otimes}}\{\mathfrak{a}\}$, and by $\mathscr{C}^{\otimes}_{\mathfrak{m}}$ the fiber product $\mathscr{C}^{\otimes}_{\mathrm{LM}^{\otimes}}\{\mathfrak{m}\}$.

Definition 3.3.9. Let $\mathscr{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$ be a planar ∞-operad, and $\mathscr{M}$ an ∞-category. A *weak enrichment of $\mathscr{M}$ over $\mathscr{C}^{\otimes}$* is a fibration of ∞-operads $\mathscr{O}^{\otimes} \xrightarrow{q} \mathrm{LM}^{\otimes}$ together with isomorphisms $\mathscr{O}^{\otimes}_{\mathfrak{a}} \cong \mathscr{C}^{\otimes}$, $\mathscr{O}_{\mathfrak{m}} \cong \mathscr{M}$. In this case, we also say that *q exhibits $\mathscr{M}$ as weakly enriched over $\mathscr{C}$*.

Definition 3.3.10. Let $\mathscr{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$ be a planar ∞-operad, and $\mathscr{M}$ an ∞-category. Suppose that a fibration of ∞-operads $\mathscr{O}^{\otimes} \xrightarrow{q} \mathrm{LM}^{\otimes}$ exhibits $\mathscr{M}$ as weakly enriched over $\mathscr{C}^{\otimes}$. We let $\mathrm{LMod}(\mathscr{M})$ denote the ∞-category $\mathrm{Alg}_{/\,\mathrm{LM}}(\mathscr{O})$, and call it the *$\infty$-category of left module objects of $\mathscr{M}$*. Precomposition with the inclusion map $\mathrm{Ass}^{\otimes} \to \mathrm{LM}^{\otimes}$ induces a categorical fibration

$$\mathrm{LMod}(\mathscr{M}) \to \mathrm{Alg}(\mathscr{C}).$$

For $A \in \mathrm{Alg}(\mathscr{C})$, we let $\mathrm{LMod}_A(\mathscr{M})$ denote the fiber product $\mathrm{LMod}(\mathscr{M}) \times_{\mathrm{Alg}(\mathscr{C})} \{A\}$, and call it the *$\infty$-category of left A-module objects of $\mathscr{M}$*.

Definition 3.3.11. Let $\mathscr{C}^{\otimes} \xrightarrow{q} \mathrm{LM}^{\otimes}$ be a fibration of ∞-operads. We say that *q exhibits $\mathscr{C}_{\mathfrak{m}}$ as left-tensored over $\mathscr{C}_{\mathfrak{a}}$*, if q is a coCartesian fibration of ∞-operads.

Definition 3.3.12. Let $\mathscr{C}$ be an ∞-category and M a monoid object of $\mathscr{C}$. A morphism of simplicial objects $M \xrightarrow{\alpha} M'$ is a *left M-action object of $\mathscr{C}$*, if for each $n \ge 0$, the maps $M'([n]) \to M'(\{n\})$ and $M'([n]) \to M([n])$ exhibit $M'([n])$ as a product $M'(\{n\}) \times M([n])$. We let $\mathrm{LMon}(\mathscr{C}) \subseteq \mathrm{Fun}(N(\Delta)^{\mathrm{op}} \times \Delta^1, \mathscr{C})$ the full subcategory spanned by left action objects.

Proposition 3.3.13. [**Lur14**, Proposition 4.2.2.9]

There is a functor $N(\Delta)^{\mathrm{op}} \times \Delta^1 \xrightarrow{\gamma} \mathrm{LM}^{\otimes}$ *such that precomposition with* γ *induces an equivalence of* ∞*-categories*

$$\mathrm{Mon}_{\mathrm{LM}}(\mathscr{C}) \to \mathrm{LMon}(\mathscr{C}).$$

Variation 3.3.14. We define the colored operad of right modules **RM**, the corresponding ∞-operad $\mathrm{RM}^{\otimes}$, weak enrichments of ∞-categories by planar ∞-operards from the right, ∞-categories of right module objects $\mathrm{RMod}(\mathscr{M})$, and ∞-categories right-tensored over monoidal ∞-categories accordingly.

Definition 3.3.15. One can define a colored operad **BM** as follows.

(1) It has 3 objects $\mathfrak{a}_-$, $\mathfrak{m}$ and $\mathfrak{a}_+$.

(2) Let $\{X_i\}_{i \in I}$ be a finite collection of objects, and Y another one. If $Y = \mathfrak{a}_-$, then $\mathrm{Mul}_{\mathbf{BM}}(\{X_i\}_{i\in I}, Y)$ is the set of all linear orderings on I is $X_i = \mathfrak{a}_-$ for all $i \in I$ and empty otherwise. If $Y = \mathfrak{a}_+$, then $\mathrm{Mul}_{\mathbf{BM}}(\{X_i\}_{i\in I}, Y)$ is the set of all linear orderings on I is $X_i = \mathfrak{a}_+$ for all $i \in I$ and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{BM}}(\{X_i\}_{i\in I}, Y)$ is the set of all linear orderings $\{i_1 < \cdots < i_n\}$ on I such that there is a unique $1 \leq j \leq n$ such that $X_{i_k} = \mathfrak{m}$, we have $X_{i_j} = \mathfrak{a}_-$ for all $1 \leq j < k$, and we have $X_{i_j} = \mathfrak{a}_+$ for all $k < j \leq n$.

(3) Composition is defined as composition of linear orderings.

The colored operad **BM** is called the *bimodule operad*. The corresponding ∞-operad $\mathrm{BM}^{\otimes}$ is called the *bimodule* ∞*-operad*.

Here also we have a canonical map $\mathrm{BM}^{\otimes} \to \mathrm{Ass}^{\otimes}$.

Notation 3.3.16. We have two inclusions of full subcategories $\mathrm{LM}^{\otimes}, \mathrm{RM}^{\otimes} \to \mathrm{BM}^{\otimes}$ spanned by finite sequences of objects of **BM** all of which are in $\{\mathfrak{a}_-, \mathfrak{m}\}$ and $\{\mathfrak{m}, \mathfrak{a}_+\}$ respectively. These two embedding restrict to two different embeddings of $\mathrm{Ass}^{\otimes}$, which we denote by $\mathrm{Ass}^{\otimes}_-$ and $\mathrm{Ass}^{\otimes}_+$ respectively.

Notation 3.3.17. Let $\mathscr{C}^{\otimes} \to \mathrm{BM}^{\otimes}$ be a fibration of ∞-operads. We let $\mathscr{C}^{\otimes}_-$ denote the fiber product $\mathscr{C}^{\otimes} \times_{\mathrm{BM}^{\otimes}} \mathrm{Ass}^{\otimes}_-$ with underlying category $\mathscr{C}_- = \mathscr{C}^{\otimes} \times_{\mathrm{BM}} \{\mathfrak{a}_-\}$, and similarly for $\mathscr{C}^{\otimes}_+$ and $\mathscr{C}_+$. We let $\mathscr{C}_{\mathfrak{m}}$ denote $\mathscr{C}^{\otimes} \times_{\mathrm{BM}} \{\mathfrak{m}\}$.

Definition 3.3.18. Let $\mathscr{C}^{\otimes} \to \mathrm{BM}^{\otimes}$ be a fibration of ∞-operads with $\mathscr{M} = \mathscr{C}_{\mathfrak{m}}$. We denote by $\mathrm{BMod}(\mathscr{M})$ the ∞-category $\mathrm{Alg}_{/\,\mathrm{BM}}(\mathscr{C})$, and call it the *$\infty$-category of bimodule objects on $\mathscr{M}$.*

Composition with the inclusions $\mathrm{Ass}^{\otimes}_{-}, \mathrm{Ass}^{\otimes}_{+} \to \mathrm{BM}^{\otimes}$ determines a categorical fibration

$$\mathrm{BMod}(\mathscr{M}) \to \mathrm{Alg}(\mathscr{C}_{-}) \times \mathrm{Alg}(\mathscr{C}_{+}).$$

Its fiber over $(A, B) \in \mathrm{Alg}(\mathscr{C}_{-}) \times \mathrm{Alg}(\mathscr{C}_{+})$ is denoted by ${}_{A}\mathrm{BMod}_{B}(\mathscr{M})$ and called the *∞-category of A-B-bimodule objects of $\mathscr{M}$.*

Definition 3.3.19. Let $\mathscr{C}^{\otimes} \to \mathrm{BM}^{\otimes}$ be a fibration of ∞-operads. We say that *q exhibits $\mathscr{C}_{\mathfrak{m}}$ as bitensored over $\mathscr{C}_{-}$ and $\mathscr{C}_{+}$*, if the map q is a coCartesian fibration.

Construction 3.3.20. Let $\mathscr{C}^{\otimes} \xrightarrow{q} \mathrm{BM}^{\otimes}$ be a fibration of ∞-operads. Let $\mathrm{LM}^{\otimes} \times \mathrm{RM}^{\otimes} \xrightarrow{\mathrm{pr}_{\mathrm{RM}^{\otimes}}} \mathrm{RM}^{\otimes}$ denote the projection map, and let $\mathrm{LM}^{\otimes} \times \mathrm{RM}^{\otimes} \xrightarrow{\mathbf{Pr}} \mathrm{BM}^{\otimes}$ denote the functor defined in [**Lur14**, Construction 4.3.2.1]. Let $\overline{\mathrm{LMod}}(\mathscr{C}_{\mathfrak{m}})^{\otimes} \to \mathrm{RM}^{\otimes}$ denote the ∞-category $(\mathrm{pr}_{\mathrm{RM}^{\otimes}})_{*}\,\mathbf{Pr}^{*}\,\mathscr{C}$ over $\mathrm{RM}^{\otimes}$. Then we have a natural isomorphism

$$\mathrm{Hom}_{(\mathrm{Set}_{\Delta})_{/\,\mathrm{RM}^{\otimes}}}(K, \overline{\mathrm{LMod}}(\mathscr{C}_{\mathfrak{m}})) \cong \mathrm{Hom}_{(\mathrm{Set}_{\Delta})_{/\,\mathrm{BM}^{\otimes}}}(\mathrm{LM}^{\otimes} \times K, \mathscr{C}^{\otimes}).$$

Let $\mathrm{LMod}(\mathscr{C}_{\mathfrak{m}})^{\otimes}$ denote the full simplicial subset spanned by vertices corresponding to ∞-operad morphisms $\mathrm{LM}^{\otimes} \to \mathscr{C}^{\otimes}$ over $\mathrm{BM}^{\otimes}$.

Proposition 3.3.21. [**Lur14**, Proposition 4.3.2.5]
Let $\mathscr{C}^{\otimes} \xrightarrow{q} \mathrm{BM}^{\otimes}$ be a fibration of ∞-operads. Then the following assertions hold.

1. *The induced map $\mathrm{LMod}(\mathscr{C}_{\mathfrak{m}})^{\otimes} \xrightarrow{p} \mathrm{RM}^{\otimes}$ is also a fibration of ∞-operads.*
2. *If q is a coCartesian fibration of operads, then so is p.*

Theorem 3.3.22. [**Lur14**, Theorem 4.3.2.7]
Let $\mathscr{C}^{\otimes} \xrightarrow{q} \mathrm{BM}^{\otimes}$ be a fibration of ∞-operads. Then the morphism induced by precomposition by $\mathbf{Pr}$*:*

$$\mathrm{BMod}(\mathscr{C}_{\mathfrak{m}}) \to \mathrm{RMod}(\mathrm{LMod}(\mathscr{C}_{\mathfrak{m}}))$$

is an equivalence of ∞-categories.

3.4. The functor $\Theta\colon A \mapsto \mathrm{RMod}_A$

Definition 3.4.1. Let S be a simplicial set and $\mathscr{O}$ an ∞-operad. Let $\mathscr{C}^{\otimes} \xrightarrow{q} \mathscr{O}^{\otimes} \times S$ be a coCartesian fibration. We will say that q *is a coCartesian S-family of $\mathscr{O}$-monoidal ∞-categories*, if for each vertex $s \in S$, the fiber $\mathscr{C}^{\otimes} \times_S \{s\} \xrightarrow{q_s} \mathscr{O}^{\otimes}$ is a coCartesian fibration of ∞-operads. If $\mathscr{O}^{\otimes} = \mathrm{Ass}^{\otimes}$, we call q a *coCartesian S-family of monoidal ∞-categories*.

Construction 3.4.2. Consider the evaluation map $\mathrm{Ass}^{\otimes} \times \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \xrightarrow{\mathrm{ev}} \mathrm{Cat}_{\infty}$, which corresponds to the inclusion $\mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \to \mathrm{Fun}(\mathrm{Ass}^{\otimes}, \mathrm{Cat}_{\infty})$. It classifies a coCartesian fibration

$$\widetilde{\mathrm{Mon}}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \xrightarrow{q} \mathrm{Ass}^{\otimes} \times \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}).$$

Remark 3.4.2.1. We will now show how q is a universal coCartesian family of monoidal ∞-categories. Suppose that $\mathscr{C}^{\otimes} \xrightarrow{q'} \mathrm{Ass}^{\otimes} \times S$ is a coCartesian S-family of monoidal ∞-categories. It is classified by a map $\mathrm{Ass}^{\otimes} \times S \xrightarrow{f'} \mathrm{Cat}_{\infty}$. Since the fibers q'_s are monoidal objects, the naturality of the covariant unstraightening functor (Proposition 2.4.11$^{\mathrm{op}}$) implies that the map f' factors as a composite

$$\mathrm{Ass}^{\otimes} \times S \xrightarrow{g} \mathrm{Ass}^{\otimes} \times \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \xrightarrow{\mathrm{ev}} \mathrm{Cat}_{\infty},$$

so that evoking naturality of unstraightening again implies that we have a Cartesian square of the form

$$\begin{tikzcd}
\mathscr{C}^{\otimes} \arrow[r, "q'"] \arrow[d] & \mathrm{Ass}^{\otimes} \times S \arrow[d, "g"] \\
\widetilde{\mathrm{Mon}}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \arrow[r, "q"] & \mathrm{Ass}^{\otimes} \times \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}).
\end{tikzcd}$$

Construction 3.4.3. Let $\mathscr{C}^{\otimes} \xrightarrow{q} \mathrm{Ass}^{\otimes} \times S$ be a coCartesian S-family of monoidal ∞-categories. By Lemma 3.4.3.1, the simplicial set over S defined by

$$\widetilde{\mathrm{Alg}}(\mathscr{C}) =_{\mathrm{df}} \mathrm{Fun}(\mathrm{Ass}^{\otimes}, \mathscr{C}^{\otimes}) \times_{\mathrm{Fun}(\mathrm{Ass}^{\otimes}, \mathrm{Ass}^{\otimes} \times S)} S$$

can be described by the following universal property. We have an isomorphism natural in maps of simplicial sets $K \to S$:

$$\mathrm{Hom}_{(\mathrm{Set}_{\Delta})_{/S}}(K, \widetilde{\mathrm{Alg}}(\mathscr{C})) \cong \mathrm{Hom}_{(\mathrm{Set}_{\Delta})_{/(\mathrm{Ass}^{\otimes} \times S)}}(\mathrm{Ass}^{\otimes} \times S, \mathscr{C}^{\otimes}).$$

In particular, its vertices over $s \in S$ are sections of the monoidal ∞-category $\mathscr{C}_s^{\otimes} \xrightarrow{q_s} \mathrm{Ass}^{\otimes}$. We let

$$\mathrm{Alg}(\mathscr{C}) \subseteq \widetilde{\mathrm{Alg}}(\mathscr{C})$$

denote the full subcategory spanned by the vertices of the form $A \in \mathrm{Alg}_{\mathrm{Ass}}(\mathscr{C}_s^{\otimes})$.

We let $\mathrm{Cat}_{\infty}^{\mathrm{Alg}}$ denote $\mathrm{Alg}(\widetilde{\mathrm{Mon}}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}))$. Note that for $\mathscr{C}^{\otimes} \in \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})$, we have a Cartesian square

$$\begin{tikzcd}
\mathscr{C}^{\otimes} \arrow[r] \arrow[d] & \mathrm{Ass}^{\otimes} \times \{\mathscr{C}^{\otimes}\} \arrow[d] \\
\widetilde{\mathrm{Mon}}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \arrow[r] & \mathrm{Ass}^{\otimes} \times \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}).
\end{tikzcd}$$

Therefore, the fiber of the map $\mathrm{Cat}_{\infty}^{\mathrm{Alg}}(\mathrm{Cat}_{\infty}) \to \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})$ over $\mathscr{C}^{\otimes} \in \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})$ has vertex set $\mathrm{Alg}_{\mathrm{Ass}}(\mathscr{C}^{\otimes})$.

Lemma 3.4.3.1. *Let $Q \xrightarrow{q} X \times S$ be a morphism in a closed Cartesian category $\mathscr{C}$. Consider the object $\tilde{Q} =_{\mathrm{df}} Q^X \times_{(X\times S)^X} S$, where $Q^X \xrightarrow{q^X} (X \times S)^X$ is the postcomposition map, and $S \xrightarrow{\eta_S} (X \times S)^X$ is the unit map. The object $\tilde{Q}$ with its projection map satisfies the following universal propery. We have a natural isomorphism*

$$\mathrm{Hom}_{\mathscr{C}_{/S}}(K, \tilde{Q}) \cong \mathrm{Hom}_{\mathscr{C}_{/X\times S}}(X \times K, Q),$$

where $X \times K \xrightarrow{\mathrm{id}\times f} X \times S$ is the base change of f.

Proof. We have a natural isomorphism

$$\mathrm{Hom}_S(K, Q^X \times_{(X\times S)^X} S) \cong \mathrm{Hom}_{(X\times S)^X}(K, Q^X)$$

by the adjunction $(\eta_S)_! \dashv (\eta_S)^*$. Then we have a natural isomorphism

$$\mathrm{Hom}_{(X\times S)^X}(K, Q^X) \cong \mathrm{Hom}_{X\times S}(X \times K, Q),$$

since naturality of the adjunction $(\bullet \times X) \dashv (\bullet^X)$ implies commutativity of the diagram of Hom sets in $\mathscr{C}$:

$$\begin{tikzcd}
\mathrm{Hom}(K, Q^X) \arrow[r, "(q^X)\circ"] \arrow[d, "\cong"] & \mathrm{Hom}(K, (X \times S)^X) \arrow[d, "\cong"] \\
\mathrm{Hom}(X \times K, Q) \arrow[r, "q\circ"] & \mathrm{Hom}(X \times K, X \times S),
\end{tikzcd}$$

and the right vertical arrow takes η_S to $(\mathrm{id} \times f)$ by the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}(S, (X \times S)^X) \arrow[r, "\circ f"] \arrow[d, "\cong"] & \mathrm{Hom}(K, (X \times S)^X) \arrow[d, "\cong"] \\
\mathrm{Hom}(X \times S, X \times S) \arrow[r, "\circ(\mathrm{id} \times f)"] & \mathrm{Hom}(X \times K, X \times S).
\end{tikzcd}$$

□

Variation 3.4.4. Let $\mathscr{M}^\otimes \to \mathrm{LM}^\otimes \times S$ be a coCartesian S-family of ∞-categories left-tensored over $\mathscr{C}^\otimes =_{\mathrm{df}} \mathscr{M}^\otimes_{\mathfrak{a}}$. We let

$$\mathrm{LMod}(\mathscr{M}) \subseteq \mathrm{Fun}(\mathrm{LM}^\otimes, \mathscr{M}^\otimes) \times_{\mathrm{Fun}(\mathrm{LM}^\otimes, \mathrm{LM}^\otimes \times S)} S$$

denote the full subcategory spanned by vertices of the form $M \in \mathrm{Alg}_{\mathrm{LM}}(\mathscr{M}^\otimes_s)$ for some $s \in S$.

Note that the collection of restriction maps

$$\mathrm{Hom}_{\mathrm{LM}^\otimes \times S}(\mathrm{LM}^\otimes \times K, \mathscr{M}^\otimes) \to \mathrm{Hom}_{\mathrm{Ass}^\otimes \times S}(\mathrm{Ass}^\otimes \times K, \mathscr{C}^\otimes)$$

induces a functor $\mathrm{LMod}(\mathscr{M}) \to \mathrm{Alg}(\mathscr{C})$.

Definition 3.4.5. Let $\mathscr{C}^\otimes \xrightarrow{p} \mathrm{Ass}^\otimes \times S$ be a coCartesian S-family of monoidal ∞-categories. An *S-family of algebra objects of $\mathscr{C}^\otimes$* is a section $S \xrightarrow{A} \mathrm{Alg}(\mathscr{C})$. In this situation, if $\mathscr{M}^\otimes \xrightarrow{q} \mathrm{LM}^\otimes \times S$ is a coCartesian S-family of ∞-categories left-tensored over $\mathscr{C}^\otimes$, then we let $\mathrm{LMod}_A(\mathscr{M})$ denote the fiber product $\mathrm{LMod}(\mathscr{M}) \times_{\mathrm{Alg}(\mathscr{C})} S$.

Construction 3.4.6. Let $\mathbf{A}$ be the Cartesian model category of marked simplicial sets. Then A° can be endowed with a simplicial colored operad structure via the mapping sets

$$\mathrm{Mul}_{\mathbf{A}^\circ}(\{S_i\}_{i \in I}, T) = \mathrm{Map}_{\mathbf{A}}\left(\prod_{i \in I} S_i, T\right),$$

and these mapping sets are fibrant [**Lur14**, Proposition 4.1.3.10, Remark 4.1.3.11]. We let $\mathrm{Cat}_\infty^\otimes$ denote the operadic nerve $N^\otimes(\mathbf{A}^\circ)$, which is a symmetric monoidal ∞-category. Let P denote the poset of subsets of the set of small simplicial sets. We define a subcategory $\mathscr{M} \subseteq \mathrm{Cat}_\infty^\otimes \times N(P)$ as follows.

(1) Its objects are the pairs $(X_1 \oplus \cdots \oplus X_n, \mathscr{K})$ such that each X_i admits K-indexed colimits.

(2) A morphism $(X_1 \oplus \cdots \oplus X_n, \mathscr{K}) \to (Y_1 \oplus \cdots \oplus Y_m, \mathscr{K}')$ over $\langle n\rangle \xrightarrow{\alpha} \langle m\rangle$ is in $\mathscr{M}$ if for all $1 \leq j \leq m$, the induced functor

$$\prod_{i \in \alpha^{-1}\{j\}} X_i \to Y_j$$

commutes with $\mathscr{K}$-indexed colimits separately in each variable (note that $\mathscr{K} \subseteq \mathscr{K}'$ by construction).

For $\mathscr{K} \in P$, we denote by $\mathrm{Cat}_\infty(\mathscr{K})^\otimes$ the fiber $\mathscr{M} \times_{N(P)} \{\mathscr{K}\}$. It is a symmetric monoidal ∞-category, and the inclusion $\mathrm{Cat}_\infty(\mathscr{K})^\otimes \to \mathrm{Cat}_\infty^\otimes$ is a lax symmetric monoidal functor [**Lur14**, Corollary 4.8.1.4].

Definition 3.4.7. Let $\mathscr{K}$ be a set of simplicial sets. A coCartesian S-family of monoidal ∞-categories $\mathscr{C}^\otimes \xrightarrow{q} \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_\infty) \times S$ is *compatible with $\mathscr{K}$-indexed colimits*, if the following assertions hold.

(1) For each $s \in S$, the fiber $\mathscr{C}^s$ admits $\mathscr{K}$-indexed colimits.

(2) For each $s \in S$, the tensor product functor $\mathscr{C}_s \times \mathscr{C}_s \xrightarrow{\otimes} \mathscr{C}_s$ commutes with $\mathscr{K}$-indexed colimits separately in each variable.

(3) For each edge $t \to s$ in S, the induced functor $\mathscr{C}_t \to \mathscr{C}_s$ commutes with $\mathscr{K}$-indexed colimits.

Variation 3.4.8. Let $\mathscr{K}$ be a set of simplicial sets. We denote by $\mathrm{Mon}_{\mathrm{Ass}}^{\mathscr{K}}(\mathrm{Cat}_\infty) \subseteq \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)$ the subcategory with objects the monoidal ∞-categories $\mathscr{C}^\otimes \to \mathrm{Ass}^\otimes$ which are compatible with $\mathscr{K}$-indexed colimits, and morphisms the monoidal functors $\mathscr{C}^\otimes \to \mathscr{D}^\otimes$ such that the underlying functor $\mathscr{C} \to \mathscr{D}$ commutes with $\mathscr{K}$-indexed colimits. Let $\widetilde{\mathrm{Mon}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)$ denote the fiber product $\widetilde{\mathrm{Mon}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty) \times_{\mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)} \mathrm{Mon}_{\mathrm{Ass}}^{\mathscr{K}}(\mathrm{Cat}_\infty)$. The induced map $\widetilde{\mathrm{Mon}}_{\mathrm{Ass}}^{\mathscr{K}} \to \mathrm{Ass}^\otimes \times \mathrm{Mon}_{\mathrm{Ass}}^{\mathscr{K}}(\mathrm{Cat}_\infty)$ is a universal family of monoidal ∞-categories compatible with $\mathscr{K}$-indexed colimits. We let $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K})$ denote the fiber product $\mathrm{Cat}_\infty^{\mathrm{Alg}} \times_{\mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)} \mathrm{Mon}_{\mathrm{Ass}}^{\mathscr{K}}(\mathrm{Cat}_\infty)$.

Construction 3.4.9. Let $\mathscr{K}$ be a set of simplicial sets containing $N(\Delta)^{\mathrm{op}}$, $\mathscr{C}^{\otimes} \to \mathrm{Ass}^{\otimes} \times S$ a coCartesian S-family of monoidal ∞-categories compatible with $\mathscr{K}$-indexed colimits, and $S \xrightarrow{A} \mathrm{Alg}(\mathscr{C})$ an S-family of algebra objects of $\mathscr{C}^{\otimes}$. Letting $\mathbf{Pr}_0$ denote the composite

$$\mathrm{LM}^{\otimes} \times \mathrm{RM}^{\otimes} \xrightarrow{\mathbf{Pr}} \mathrm{BM}^{\otimes} \to \mathrm{Ass}^{\otimes}$$

we get get a diagram of two Cartesian squares of solid arrows

$$\begin{tikzcd}
\bar{\mathscr{C}}_{\mathfrak{a}}^{\otimes} \arrow[r] \arrow[d] & \bar{\mathscr{C}}^{\otimes} \arrow[r] \arrow[d] & \mathscr{C}^{\otimes} \arrow[d, "q"] \\
\mathrm{Ass}^{\otimes} \times \mathrm{LM}^{\otimes} \times S \arrow[r, "\mathrm{incl} \times \mathrm{id}_{\mathrm{LM}^{\otimes} \times S}"] \arrow[u, dashed, bend left, "\bar{A}"] & \mathrm{RM}^{\otimes} \times \mathrm{LM}^{\otimes} \times S \arrow[r, "\mathbf{Pr}_0 \times \mathrm{id}_S"] \arrow[u, dashed, bend left] & \mathrm{Ass}^{\otimes} \times S. \arrow[u, dashed, bend left, "A"]
\end{tikzcd}$$

That is, $\bar{\mathscr{C}}^{\otimes}$ is an $(\mathrm{LM}^{\otimes} \times S)$-family of ∞-categories right-tensored over $\bar{\mathscr{C}}_{\mathfrak{a}}^{\otimes}$, which commutes with $\mathscr{K}$-indexed colimits. By construction, the S-family of algebra objects A corresponds to a section of q depicted in the diagram as the rightmost dashed arrow. Since the solid squares are Cartesian, this section can be lifted to give an $(\mathrm{LM}^{\otimes} \times S)$-family of algebra objects $\mathrm{LM}^{\otimes} \times S \xrightarrow{\bar{A}} \mathrm{Alg}(\bar{\mathscr{C}}_{\mathfrak{a}}^{\otimes})$. Now we can form the following diagram of Cartesian squares.

$$\begin{tikzcd}
\mathscr{C}'^{\otimes} \arrow[r] \arrow[d] & \mathrm{RMod}_A(\mathscr{C})^{\otimes} \arrow[r] \arrow[d] & \mathrm{RMod}(\bar{\mathscr{C}}) \arrow[d] \\
\mathrm{Ass}^{\otimes} \times S \arrow[r, "\mathrm{incl} \times \mathrm{id}_S"] & \mathrm{LM}^{\otimes} \times S \arrow[r, "\bar{A}"] & \mathrm{Alg}(\bar{\mathscr{C}}_{\mathfrak{a}}).
\end{tikzcd}$$

The rightmost vertical arrow is a coCartesian fibration [**Lur14**, Lemma 4.8.3.15], thus so are the other two vertical arrows. In other words, $\mathrm{RMod}_A(\mathscr{C})^{\otimes}$ is a coCartesian $(LM \otimes \times S)$-family of ∞-categories left-tensored over $\mathscr{C}'\otimes$. By construction, the identity map $\mathrm{RMod}(\bar{\mathscr{C}}) \to \mathrm{RMod}(\bar{\mathscr{C}})$ corresponds to a map $\mathrm{RM}^{\otimes} \times \mathrm{RMod}(\bar{\mathscr{C}}) \to \bar{\mathscr{C}}$, which fits into the commutative diagram

$$\begin{tikzcd}
\mathscr{C}'^{\otimes} \arrow[r] \arrow[d] & \{\mathfrak{m}\} \times \mathrm{RMod}(\bar{\mathscr{C}}) \arrow[r] \arrow[d] & \mathrm{RM}^{\otimes} \times \mathrm{RMod}(\bar{\mathscr{C}}) \arrow[r] \arrow[d] & \bar{\mathscr{C}} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\
\mathrm{Ass}^{\otimes} \times S \arrow[r] & \{\mathfrak{m}\} \times \mathrm{LM}^{\otimes} \times S \arrow[r] & \mathrm{RM}^{\otimes} \times \mathrm{LM}^{\otimes} \times S \arrow[r, "\mathrm{id}"] & \mathrm{RM}^{\otimes} \times \mathrm{LM}^{\otimes} \times S \arrow[r, "\mathbf{Pr}_0 \times \mathrm{id}_S"] & \mathrm{Ass}^{\otimes} \times S.
\end{tikzcd}$$

The composite of top horizontal arrows $\mathscr{C}'^{\otimes} \to \mathscr{C}^{\otimes}$ is a categorical equivalence [**Lur14**, Construction 4.8.3.21].

Notation 3.4.10. We let $\mathrm{Cat}_\infty^{\mathrm{Mod}}$ denote the ∞-category of $\mathrm{LM}^{\otimes}$-monoid objects $\mathrm{Mon}_{\mathrm{LM}}(\mathrm{Cat}_\infty)$. Let $\mathscr{K}$ be a set of simplicial sets. We denote by $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathscr{K}) \subseteq \mathrm{Cat}_\infty^{\mathrm{Mod}}$ the subcategory with objects those pairs $(\mathscr{C}, \mathscr{M})$ of ∞-categories $\mathscr{M}$ left-tensored over monoidal ∞-categories $\mathscr{C}^{\otimes}$ which satisfy the following conditions.

(1) The underlying ∞-categories $\mathscr{C}$ and $\mathscr{M}$ admit $\mathscr{K}$-indexed colimits.
(2) The product functors

$$\mathscr{C} \times \mathscr{C} \to \mathscr{C} \text{ and } \mathscr{C} \times \mathscr{M} \to \mathscr{M}$$

commute with $\mathscr{K}$-indexed colimits separately in each variable.

An edge $(\mathscr{C}^{\otimes}, \mathscr{M}) \to (\mathscr{C}'^{\otimes}, \mathscr{M}')$ in $\mathrm{Cat}_\infty^{\mathrm{Mod}}$ is contained in $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathscr{K})$, if the underlying functors $\mathscr{C} \to \mathscr{C}'$ and $\mathscr{M} \to \mathscr{M}'$ commute with colimits over $\mathscr{K}$.

Remark 3.4.10.1. The inclusion $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathscr{K}) \to \mathrm{Fun}(\mathrm{LM}^{\otimes}, \mathrm{Cat}_\infty)$ classifies a universal coCartesian family of $\mathrm{LM}^{\otimes}$-monoidal ∞-categories which commutes with $\mathscr{K}$-indexed colimits

$$\widetilde{\mathrm{Mon}}^{\mathscr{K}}_{\mathrm{LM}}(\mathrm{Cat}_\infty) \to \mathrm{LM}^{\otimes} \times \mathrm{Mon}^{\mathscr{K}}_{\mathrm{LM}}(\mathrm{Cat}_\infty).$$

Construction 3.4.11. Suppose that $\mathscr{K}$ is a set of simplicial sets, which contains $N(\Delta)^{\mathrm{op}}$. Consider the Cartesian square of solid arrows

$$\begin{tikzcd}
\mathscr{C}^{\otimes} \arrow[r] \arrow[d] & \widetilde{\mathrm{Mon}}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty) \arrow[d] \\
\mathrm{Ass}^{\otimes} \times \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K}) \arrow[u, dashed, bend left, "A"] \arrow[r] \arrow[ur, dashed] & \mathrm{Ass}^{\otimes} \times \mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty).
\end{tikzcd}$$

By construction, the identity map $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K}) \to \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K})$ corresponds to a map which is depicted as the dashed diagonal arrow in the diagram. It induces the $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K})$-family of algebra objects $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K}) \xrightarrow{A} \mathrm{Alg}(\mathscr{C})$, which is depicted as the vertical dashed arrow in the diagram. Using this, we can construct the coCartesian $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K})$-family of $\mathrm{LM}^{\otimes}$-monoidal ∞-categories $\mathrm{RMod}_A(\mathscr{C})^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K})$, which in turn is classified by a map

$$\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K}) \xrightarrow{\Theta} \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathscr{K}).$$

Remark 3.4.11.1. Note that up to equivalence, the action of Θ on vertices can be described as follows. Over a monoidal ∞-category $\mathscr{C}^{\otimes} \in \mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})$, it takes an algebra $A \in \mathrm{Alg}(\mathscr{C}^{\otimes})$ to the category left-tensored over $\mathscr{C}$ of right A-modules $\mathrm{RMod}_A(\mathscr{C}) \in \mathrm{Mon}_{\mathrm{LM}}(\mathscr{C})$.

Remark 3.4.11.2. Let us leave the set of simplicial sets $\mathscr{K}$ out of the notation. Composition with the inclusion $\mathrm{Ass}^{\otimes} \to \mathrm{LM}^{\otimes}$ induces a map $\mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \to \mathrm{Mon}_{\mathrm{LM}}(\mathrm{Cat}_{\infty})$, which corresponds to viewing monoidal ∞-categories as left-tensored over themselves. This map classifies a coCartesian $\mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})$-family of $\mathrm{LM}^{\otimes}$-monoidal ∞-categories $\mathscr{M} \to \mathrm{LM}^{\otimes} \times \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})$. Consider the diagram of Cartesian squares

$$\begin{tikzcd}
\mathscr{C}'^{\otimes} \arrow[rrrr] \arrow[dr] \arrow[dd] & & & & \widetilde{\mathrm{Mon}}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \arrow[dr] \arrow[dd] & \\
& \mathrm{RMod}_A(\mathscr{C})^{\otimes} \arrow[rr] \arrow[dd] & & \widetilde{\mathrm{Mon}}_{\mathrm{LM}}(\mathrm{Cat}_{\infty}) \arrow[rr] \arrow[dd] & & \mathscr{M} \arrow[dd] \\
\mathrm{Ass}^{\otimes} \times \mathrm{Cat}_{\infty}^{\mathrm{Alg}} \arrow[rr, dashed] \arrow[dr] & & \mathrm{Ass}^{\otimes} \times \mathrm{Mon}_{\mathrm{LM}}(\mathrm{Cat}_{\infty}) \arrow[rr] \arrow[dr] & & \mathrm{Ass}^{\otimes} \times \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) \arrow[dr] & \\
& \mathrm{LM}^{\otimes} \times \mathrm{Cat}_{\infty}^{\mathrm{Alg}} \arrow[rr, "\mathrm{id}_{\mathrm{LM}^{\otimes}} \times \Theta"] & & \mathrm{LM}^{\otimes} \times \mathrm{Mon}_{\mathrm{LM}}(\mathrm{Cat}_{\infty}) \arrow[rr, "\mathrm{id}_{\mathrm{LM}^{\otimes}} \times \mathrm{res}"] & & \mathrm{LM}^{\otimes} \times \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})
\end{tikzcd}$$

Since the $\mathrm{Cat}_{\infty}^{\mathrm{Alg}}$-family of monoidal ∞-categories $\mathscr{C}'^{\otimes}$ is equivalent to $\mathscr{C}^{\otimes}$ [**Lur14**, Construction 4.8.3.21], the back face of the diagram shows that the composite $\mathrm{res} \circ \Theta$ classifies $\mathscr{C}^{\otimes}$. This implies that the diagram

$$\begin{tikzcd}
\mathrm{Cat}_{\infty}^{\mathrm{Alg}} \arrow[rr, "\Theta"] \arrow[dr] & & \mathrm{Cat}_{\infty}^{\mathrm{Mod}} \arrow[dl, "\mathrm{res}"] \\
& \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty}) &
\end{tikzcd}$$

is commutative up to equivalence.

To get a tensor structure on the stack of perfect complexes, we need to promote Θ to a functor of symmetric monoidal ∞-categories.

Notation 3.4.12. Let $\mathscr{K}$ be a set of small simplicial sets. Let $\mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})^{\otimes} \subseteq \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_{\infty})^{\times}$ denote subcategory defined as follows.

(1) Its objects are those $(\mathscr{C}_1^{\otimes}, \ldots, \mathscr{C}_m^{\otimes}) \in \mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)^{\times}$ such that the following conditions are satisfied.

   (a) The underlying ∞-categories $\mathscr{C}_i$ admit $\mathscr{K}$-colimits.

   (b) The tensor product morphisms

$$\mathscr{C}_i \times \mathscr{C}_i \xleftarrow{\simeq} (\mathscr{C}_i^{\otimes})_{\langle 2 \rangle} \to \mathscr{C}_i$$

   are compatible with $\mathscr{K}$-indexed colimits separately in each variable.

(2) Its morphisms over $\langle m \rangle \xrightarrow{\alpha} \langle n \rangle$ are those $(\mathscr{C}_1^{\otimes}, \ldots, \mathscr{C}_m^{\otimes}) \to (\mathscr{D}_1^{\otimes}, \ldots, \mathscr{D}_n^{\otimes})$ such that for each $1 \leq j \leq n$, the map

$$\prod_{i \in \alpha^{-1}\{j\}} \mathscr{C}_i \to \mathscr{D}_i$$

commutes with $\mathscr{K}$-indexed colimits separately in each variable.

We define $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathscr{K})^{\otimes} \subseteq (\mathrm{Cat}_\infty^{\mathrm{Mod}})^{\times}$ similarly. We let $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K})$ denote the fiber product $(\mathrm{Cat}_\infty^{\mathrm{Alg}})^{\times} \times_{\mathrm{Mon}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)^{\times}} \mathrm{Mon}_{\mathrm{Ass}}^{\mathscr{K}}(\mathrm{Cat}_\infty)^{\otimes}$.

Construction 3.4.13. Let $\mathscr{K}$ be a set of small simplicial sets containing $N(\Delta)^{\mathrm{op}}$. Then the functor

$$\mathrm{Cat}_\infty^{\mathrm{Alg}}(\{N(\Delta)^{\mathrm{op}}\}) \xrightarrow{\Theta} \mathrm{Cat}_\infty^{\mathrm{Mod}}(\{N(\Delta)^{\mathrm{op}}\})$$

lifts to a symmetric monoidal functor

$$\mathrm{Cat}_\infty^{\mathrm{Alg}}(\{N(\Delta)^{\mathrm{op}}\})^{\times} \xrightarrow{\Theta^{\times}} \mathrm{Cat}_\infty^{\mathrm{Mod}}(\{N(\Delta)^{\mathrm{op}}\})^{\times},$$

which in turn restricts to a symmetric monoidal functor

$$\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K})^{\otimes} \xrightarrow{\Theta^{\otimes}} \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathscr{K})^{\otimes}$$

[**Lur14**, Theorem 4.8.5.16].

Construction 3.4.14. Let $\mathscr{K}$ be a set of small simplicial sets containing $N(\Delta)^{\mathrm{op}}$, and let $\mathscr{C}^{\otimes} \in \mathrm{Mon}_{\mathrm{Comm}}^{\mathscr{K}}(\mathrm{Cat}_\infty)$. By Lemma 3.4.14.1, we can lift $\mathscr{C}^{\otimes}$ to a commutative monoid object $N(\mathrm{Fin}_*) \to$

$\mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)^\otimes$. We can then form the following diagram with commutative squares

$$\begin{tikzcd}
\mathrm{Alg}(\mathscr{C})^\otimes \arrow[rr, "\Theta^\otimes_{\mathscr{C}}"] \arrow[dr] \arrow[ddr] & & \mathrm{LMod}_{\mathscr{C}}(\mathrm{Cat}_\infty(\mathscr{K}))^\otimes \arrow[dr] \arrow[ddl] & \\
& \mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathscr{K})^\otimes \arrow[rr, "\Theta^\otimes"] \arrow[ddr] & & \mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathscr{K})^\otimes \arrow[ddl] \\
& N(\mathrm{Fin}_*) \arrow[dr, "\mathscr{C}^\otimes"'] & & \\
& & \mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)^\otimes &
\end{tikzcd}$$

Applying Lemma 3.4.14.1 again, the precomposition by the bifunctor $\mathrm{Comm}^\otimes \times \mathrm{Ass}^\otimes \xrightarrow{f} \mathrm{Comm}^\otimes$ gives a trivial fibration $\mathrm{CAlg}(\mathrm{Alg}(\mathscr{C})) \to \mathrm{CAlg}(\mathscr{C})$, thus choosing a quasi-inverse, we get a functor defined as the composite

$$\mathrm{CAlg}(\mathscr{C}) \to \mathrm{CAlg}(\mathrm{Alg}(\mathscr{C})) \xrightarrow{\mathrm{CAlg}(\Theta^\otimes_{\mathscr{C}})} \mathrm{CAlg}(\mathrm{LMod}_{\mathscr{C}}(\mathrm{Cat}_\infty(\mathscr{K}))).$$

Lemma 3.4.14.1. *The unique ∞-operad bifunctor* $\mathrm{Comm}^\otimes \times \mathrm{Ass}^\otimes \to \mathrm{Comm}^\otimes$ *exhibits* $\mathrm{Comm}^\times$ *as a tensor product of* $\mathrm{Comm}^\otimes$ *and* $\mathrm{Ass}^\otimes$*.*

Proof. Let $\mathrm{Comm}^\otimes \times \mathrm{Ass}^\otimes \xrightarrow{f} \mathrm{Comm}^\otimes$ be the unique ∞-operad bifunctor, and let $\mathrm{Comm}^\otimes \times \mathrm{Ass}^\otimes \xrightarrow{\mathrm{pr}} \mathrm{Comm}^\otimes$ be the projection map. Let $\mathscr{C}^\otimes$ be an ∞-operad. Let $\mathrm{CAlg}(\mathscr{C})^\otimes \subseteq \mathrm{pr}_* f^* \mathscr{C}^\otimes$ denote the full subcategory over $\mathrm{Comm}^\otimes$ the vertices of which over $\langle m \rangle \in \mathrm{Comm}^\otimes$ correspond to commutative algebra objects via the chain of natural isomorphisms coming from adjunctions

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Comm}^\otimes}(\Delta^0, \mathrm{pr}_* f^* \mathscr{C}^\otimes) &\cong \mathrm{Hom}_{\mathrm{Comm}^\otimes \times \mathrm{Ass}^\otimes}(\mathrm{pr}^* \Delta^0, f^* \mathscr{C}^\otimes) \\
&\cong \mathrm{Hom}_{\mathrm{Comm}^\otimes}(f_! \mathrm{pr}^* \Delta^0, \mathscr{C}^\otimes) \\
&= \mathrm{Hom}_{\mathrm{Comm}^\otimes}(\mathrm{Comm}^\otimes \times \{\langle m \rangle\}, \mathscr{C}^\otimes).
\end{aligned}$$

By [**Lur14**, Proposition 3.2.4.10], the ∞-operad $\mathrm{CAlg}(\mathscr{C})$ is a coCartesian symmetric monoidal ∞-category. Therefore, by [**Lur14**, Proposition 2.4.3.9], the restriction map $\mathrm{Alg}(\mathrm{CAlg}(\mathscr{C})) \xrightarrow{F \mapsto F|\{[1]\}} \mathrm{CAlg}(\mathscr{C})$ is an equivalence.

We have embeddings

$$\mathrm{Alg}(\mathrm{CAlg}(\mathscr{C})) \to \mathrm{Fun}_{\mathrm{Comm}^{\otimes}}(\mathrm{Ass}^{\otimes}, \mathrm{CAlg}(\mathscr{C})^{\otimes}) \to \mathrm{Fun}_{\mathrm{Comm}^{\otimes}}(\mathrm{Comm}^{\otimes} \times \mathrm{Ass}^{\otimes}, \mathscr{C}^{\otimes})$$

by construction, such that moreover the image of $\mathrm{Alg}(\mathrm{CAlg}(\mathscr{C}))$ is precisely $\mathrm{BiFun}(\mathrm{Comm}, \mathrm{Ass}; \mathscr{C})$. Let $\mathrm{Comm}^{\otimes} \times \{[1]\} \xrightarrow{i} \mathrm{Comm}^{\otimes} \times \mathrm{Ass}^{\otimes}$ denote the inclusion map. Then precomposition gives an equivalence

$$\mathrm{BiFun}(\mathrm{Comm}, \mathrm{Ass}; \mathscr{C}) \xrightarrow{\circ i} \mathrm{Alg}_{\mathrm{Comm}}(\mathscr{C}).$$

Since we have $f \circ i = \mathrm{id}_{\mathrm{Comm}^{\otimes}}$ by construction, this implies that the precomposition map

$$\mathrm{Alg}_{\mathrm{Comm}}(\mathscr{C}) \xrightarrow{\circ f} \mathrm{BiFun}(\mathrm{Comm}, \mathrm{Ass}; \mathscr{C})$$

is also an equivalence, as claimed.

□

3.5. The stack of perfect complexes

From now on, we will fix a commutative ring k.

Construction 3.5.1. Let $\mathrm{Ch}(k)$ denote the category of cochain complexes over k. Tensor products of complexes endow $\mathrm{Ch}(k)$ with a symmetric monoidal structure, which can be promoted to a symmetric monoidal model structure, which is determined as follows [**Lur14**, Proposition 7.1.2.11].

(1) A morphism of complexes $C^{\bullet} \to D^{\bullet}$ is a weak equivalence, if it is a quasi-isomorphism.
(2) A morphism of complexes $C^{\bullet} \to D^{\bullet}$ is a fibration, if it is a componentwise surjection.

This model structure is referred to as the *projective model structure*. As in Construction 3.4.6, it induces a symmetric monoidal ∞-category, which we will denote by $\mathscr{D}(k)$. We will also denote it by Mod_k. Note that its homotopy category is the unbounded derived category $\mathbf{D}(k)$.

Construction 3.5.2. Let $\mathscr{K}$ denote the set of all small simplicial sets. Applying Construction 3.4.14 to $\mathscr{C} = \mathrm{Mod}_k$, we get a functor

$$\mathrm{CAlg}(\mathrm{Mod}_k) \xrightarrow{\mathfrak{M}_1^{\otimes}} \mathrm{CAlg}(\mathrm{LMod}_{\mathrm{Mod}_k}(\mathrm{Cat}_{\infty}(\mathscr{K}))).$$

We let CAlg_k denote $\mathrm{CAlg}(\mathrm{Mod}_k)$. Let $\mathrm{Pr}^{\mathrm{L}}_k$ denote the module category $\mathrm{LMod}_{\mathrm{Mod}_k}(\mathrm{Pr}^{\mathrm{L}})$, and let $\mathrm{Pr}^{\mathrm{L}}_{\omega,k} \subseteq \mathrm{Pr}^{\mathrm{L}}_k$ denote the full subcategory spanned by compactly generated k-linear categories. By Lemma 3.5.2.1, we can write

$$\mathrm{CAlg}_k \xrightarrow{\mathfrak{M}_1^{\otimes}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega,k}).$$

Lemma 3.5.2.1. *Let $A \in \mathrm{Mod}_k$. Then the presentable ∞-category $\mathrm{RMod}_A \in \mathrm{Pr}^{\mathrm{L}}$ is compactly generated.*

Proof. Let $\mathscr{C}$ be a stable ∞-category. The proof of [**Lur14**, Corollary 1.4.4.2] implies that if there exists a κ-compact object $X \in \mathscr{C}$ which generates $\mathscr{C}$ in the sense that for any $Y \in \mathscr{C}$, if $\pi_0 \operatorname{Map}_{\mathscr{C}}(X, Y) \cong *$, then $Y \simeq 0$. On the other hand, by [**Lur14**, Theorem 7.1.2.1], the ∞-category RMod_A contains a compact object X, which generates it in the sense that for any $Y \in \mathrm{RMod}_A$, if $\mathrm{Ext}^i_{\mathrm{RMod}_A}(X, Y) = \pi_0 \operatorname{Map}_{\mathrm{RMod}_A}(X, Y[i]) = 0$ for all integers i, then $Y \simeq 0$. Taking $\sum_{i \in \mathbf{Z}}(X[i])$, we get a compact generator in the stronger sense.

□

Construction 3.5.3. Let $\mathrm{Aff}_k = \mathrm{CAlg}_k^{\mathrm{op}}$. Since precomposition by the opposite of the Yoneda embedding restricts to an equivalence $\mathrm{Fun}^{\mathrm{R}}(\mathscr{P}(\mathrm{Aff}_k)^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega,k})) \to \mathrm{Fun}(\mathrm{Aff}_k^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega,k}))$ [**Lur09**, Theorem 5.1.5.6], the functor $\mathrm{Aff}_k^{\mathrm{op}} \xrightarrow{\mathfrak{M}_1^{\otimes}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega,k})$ has an essential preimage $\mathscr{P}(\mathrm{Aff}_k)^{\mathrm{op}} \xrightarrow{\widetilde{\mathrm{QC}}^{\otimes}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega,k})$. Let $\mathrm{St}_k \xrightarrow{i} \mathscr{P}(\mathrm{Aff}_k)$ denote the canonical inclusion of the ∞-topos of derived stacks, and let $\mathrm{QC}^{\otimes} = \widetilde{\mathrm{QC}}^{\otimes} \circ i^{\mathrm{op}}$. This presheaf satisfies flat hyperdescent, because its restriction to underlying categories $\mathrm{St}_k^{\mathrm{op}} \xrightarrow{\mathrm{QC}} \mathrm{Pr}^{\mathrm{L}}_{\omega,k}$ does [**Pan11**, Proposition 2.2.17], and that is enough [**Lur14**, Corollary 3.2.2.4].

Definition 3.5.4. Let $\mathrm{Perf}_k^{\otimes}$ denote the postcomposite of the functor $\mathrm{St}_k^{\mathrm{op}} \xrightarrow{\mathrm{QC}^{\otimes}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega,k})$ with the restriction to full subcategories of compact objects functor $\mathrm{Pr}^{\mathrm{L}}_{\omega,k} \xrightarrow{(-)^{\omega}} \mathrm{LMod}_k(\mathrm{Cat}_{\infty})$. It still satisfies flat hyperdescent [**Pan11**, Lemmas 2.3.1 and 2.3.5]. For $X \in \mathrm{St}_k$, the *stack of perfect complexes on X* is a representing object for the sheaf defined as the composite

$$((\mathrm{St}_k)_{/X})^{\mathrm{op}} \to (\mathrm{St}_k)^{\mathrm{op}} \xrightarrow{\mathrm{Perf}^{\otimes}} \mathrm{CAlg}(\mathrm{LMod}_k(\mathrm{Cat}_{\infty})).$$

CHAPTER 4

The stack of generalized $\mathfrak{sl}_n$-forms

We will construct the stack of generalized $\mathfrak{sl}_n$-forms on a smooth, proper S-scheme X as the essential image of the functor $\mathscr{T}^{\mathrm{perf}}_{X/S} \xrightarrow{F \mapsto \mathfrak{sl}(F)} \mathrm{LAlg}\,(\mathrm{QC}_k(X))$, mapping a totally supported sheaf to the traceless part of its endomorphism complex. In §3.1, we define the additive ∞-operad $\mathrm{Lie}^{\otimes}$, and prove that the Lie-algebra objects it induces give weak Lie algebras in the homotopy category. In §3.2 and §3.3, we define the two functors

$$\mathrm{QC}_k(X)^{\simeq} \xrightarrow{E \mapsto \mathrm{End}(E)} \mathrm{Alg}(\mathrm{QC}_k(X)) \xrightarrow{A \mapsto \mathfrak{sl}(A)} \mathrm{LAlg}(\mathrm{QC}_k(X)).$$

4.1. Lie algebras in an additive symmetric monoidal ∞-category

In this section, we define the Lie ∞-operad, and prove that algebras on it in any additive symmetric monoidal ∞-category are weak Lie algebras. The classical Lie algebra operad is defined in any additive symmetric monoidal 1-category, and in [**Lur14**], the category of operators of a classical operad, the nerve of which is an ∞-operad, is only defined for operads in the Cartesian symmetric monoidal 1-categories $(\mathrm{Set}, \times)$ and $(\mathrm{Set}_{\Delta}, \times)$. However, additive ∞-categories are defined in [*ibid.*, §2.4.5], and we can build up on the following machinery for 1-categories.

Definition 4.1.1. Let **C** be a 1-category. We say that **C** is *pre-additive*, if it satisfies the following conditions.

(1) **C** admits finite products and coproducts.
(2) **C** is pointed.
(3) For all $X, Y \in \mathbf{C}$, the canonical map $X \sqcup Y \to X \times Y$ fitting into the commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\cong"'] & X \sqcup Y \arrow[d] & Y \arrow[l] \arrow[d, "\cong"] \\
X \times \{\emptyset\} \arrow[r] & X \times Y & \emptyset \times Y \arrow[l]
\end{tikzcd}$$

is an isomorphism.

In this case, finite products and coproducts are referred to as *biproducts*, and the biproduct operation is denoted by the symbol $\oplus$. We say that $\mathbf{C}$ is *additive*, if moreover, for any $X \in \mathbf{C}$, the shear morphism

$$X \oplus X \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)} X \oplus X$$

is an isomorphism.

Remark 4.1.1.1. If $\mathscr{O}^{\otimes}$ is an ∞-operad, and $X \in \mathscr{O}^{\otimes}$ is an object over $\langle n \rangle$, the image of which by the canonical map

$$\mathscr{O}^{\otimes}_{\langle n \rangle} \to \mathscr{O}^{\times n}$$

is $(X_1, \ldots, X_n)$, then in the notation of **[Lur14]**, we write $X = X_1 \oplus \cdots \oplus X_n$. It will be clear from the context, in which meaning are we using $\oplus$.

Proposition 4.1.2. **[Wei94]**

The following assertions hold.

(1) *Let* $\mathbf{C}$ *be an additive 1-category. Then there is a unique way to promote* $\mathbf{C}$ *to an* Ab*-enriched category: the sum of the morphisms* $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ *can be gotten as the composite*

$$X \xrightarrow{(f,g)} X \oplus X \xrightarrow{\nabla} X.$$

(2) *Let* $\mathbf{C}$ *be an* Ab*-enriched 1-category. Then it is an additive 1-category precisely when it admits finite coproducts.*

(3) *Let* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ *be a functor between additive categories. Then it is* Ab*-enriched precisely when it commutes with finite coproducts.*

Definition 4.1.3. Let $\mathscr{C}$ be an ∞-category. We say that $\mathscr{C}$ is *additive*, if the following assertions hold.

(1) $\mathscr{C}$ admits finite products.

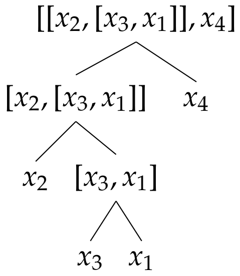
(2) $\mathscr{C}$ admits finite coproducts.

(3) The homotopy category $\mathrm{h}\mathscr{C}$ is additive.

Let $\mathscr{C} \xrightarrow{F} \mathscr{D}$ be a functor between additive ∞-categories. We say that F is *additive*, if it is compatible with finite coproducts. We let $\mathrm{Fun}^{\pi}(\mathscr{C}, \mathscr{D}) \subseteq \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ denote the full subcategory spanned by additive functors.

Construction 4.1.4. Let the classical symmetric operad **Lie** in the additive symmetric monoidal 1-category of abelian groups be defined as follows.

(1) For $n \geq 0$, let $L(n)$ denote the free $\mathbf{Z}$-Lie algebra on $\{x_1, \ldots, x_n\}$. Let $\mathbf{Lie}(n) \subseteq L(n)$ denote the $\mathbf{Z}$-submodule generated by the monomials in which all of $x_1, \ldots, x_n$ appear. These generators can be represented by the binary trees having $\{x_1, \ldots, x_n\}$ as the set of leaves. Consider the following example.



Note this set of generators is not linearly independent if $n > 1$. Moreover, $\mathbf{Lie}(0)$ is the 0 module. If I is a finite set of cardinality n, then we let $\mathbf{Lie}(I)$ denote $\mathbf{Lie}_n$.

(2) The S_n-action on $\mathbf{Lie}(n)$ is given by permuting the x_i.

(3) For a morphism of finite sets $I \xrightarrow{\alpha} J$, let the composition morphism

$$\prod_{j \in J} \mathbf{Lie}(\alpha^{-1}\{j\}) \times \mathbf{Lie}(J) \xrightarrow{(f, g_j : j \in J) \mapsto f(g_j : j \in J)} \mathbf{Lie}(I)$$

be defined by substitution in the appropriate free $\mathbf{Z}$-Lie algebra. In the binary tree representation, this can be pictured as grafting. Note that including a 0 makes the entire Lie product 0, therefore composites can be nonzero only if α is surjective.

Let $\mathbf{Lie}^{\otimes} \to \mathrm{Fin}_*$ denote the corresponding category of operators, that is, it is defined as follows.

(1) It has a single object over each $\langle n \rangle \in \mathrm{Fin}_*$.

(2) Its morphism sets are of the form

$$\coprod_{I \xrightarrow{\alpha} J} \mathbf{Lie}(\alpha^{-1}\{j\}).$$

(3) Composition is induced from that on **Lie**.

Variation 4.1.5. We introduce an enlargment $\mathbf{Lie}^{\otimes,\mathrm{Ab}} \to \mathrm{Fin}_*$ of $\mathbf{Lie}^{\otimes} \to \mathrm{Fin}_*$, the fibers of which are additive categories. It is defined as follows.

(1) Over each $\langle n\rangle$, it has a countable set of objects $\left\{\langle n\rangle^{\oplus k}\right\}_{k\in\mathbf{Z}_{>0}}$.
(2) The morphism sets are defined as follows.

$$\mathrm{Hom}_{\mathbf{Lie}^{\otimes,\mathrm{Ab}}}\left(\langle n\rangle^{\oplus k}, \langle m\rangle^{\oplus \ell}\right) = \coprod_{\langle m\rangle \xrightarrow{\alpha} \langle n\rangle} \mathrm{Mat}_{\mathbf{Z}}^{\ell\times k}\left(\bigoplus_{1\leq j\leq n} \mathbf{Lie}(\alpha^{-1}\{j\})\right).$$

(3) Composition is defined by that on $\mathbf{Lie}^{\otimes}$ and matrix multiplication. It can be described as follows. Suppose given a pair of composable morphisms

$$\langle m_1\rangle^{\oplus \ell_1} \xrightarrow{\alpha} \langle m_2\rangle^{\oplus \ell_2} \xrightarrow{\beta} \langle m_3\rangle^{\oplus \ell_3},$$

and two morphisms over these:

$$A \in \mathrm{Mat}^{\ell_2\times\ell_1}\left(\oplus_{1\leq k\leq m_2} \mathbf{Lie}(\alpha^{-1}\{k\})\right), \quad B \in \mathrm{Mat}^{\ell_3\times\ell_2}\left(\oplus_{1\leq k\leq m_3} \mathbf{Lie}(\beta^{-1}\{k\})\right).$$

We define $A \circ B \in \mathrm{Mat}^{\ell_3\times\ell_1}\left(\oplus_{1\leq k\leq m_3} \mathbf{Lie}((\beta\circ\alpha)^{-1}\{k\})\right)$ using the formula

$$\left((A\circ B)_{ij}\right)_k = \sum_{\ell=1}^{\ell_2} (B_{i\ell})\left((A_{\ell j})_{k'} : k' \in \alpha^{-1}\{k\}\right),$$

where $1 \leq i \leq \ell_3$, $1 \leq j \leq \ell_1$, and $1 \leq k \leq m_3$.

We will denote its nerve by $\mathrm{Lie}^{\otimes} \to N(\mathrm{Fin}_*)$, and call it the *∞-Lie operad*.

Remark 4.1.5.1. Let f, g be maps in $\mathbf{Lie}^{\otimes}$ over $\langle m\rangle \xrightarrow{\alpha} \langle n\rangle$. The sum $f+g$ is the composite of the horizontal arrows in the diagram

$$\begin{tikzcd}
 & \langle n\rangle & \\
\langle m\rangle \arrow[ur, bend left, "f"] \arrow[r, "\begin{pmatrix} f \\ g \end{pmatrix}"] \arrow[dr, "g"'] & \langle n\rangle^{\oplus 2} \arrow[u, "(1\ 0)"'] \arrow[r, "(1\ 1)"] \arrow[d, "(0\ 1)"] & \langle n\rangle. \\
 & \langle n\rangle &
\end{tikzcd}$$

Proposition 4.1.6. *Let $\langle m\rangle \xrightarrow{\alpha} \langle n\rangle$ be an inert morphism, and $k > 0$. Then there exists a p-coCartesian edge with source $\langle m\rangle^{\oplus k} \in \mathbf{Lie}^{\otimes,\mathrm{Ab}}$ over α.*

Proof. Since α is inert, for each $1 \leq j \leq n$, there exists a unique $1 \leq i_j \leq m$ such that $\alpha(i_j) = j$. Therefore, a p-coCartesian edge $\langle m\rangle^{\oplus k} \to \langle n\rangle^{\oplus k}$ in $\mathbf{Lie}^{\otimes,\mathrm{Ab}}$ can be given as

$$I \in \mathrm{Mat}^{k\times k}\left(\bigoplus_{1\leq j\leq n} \mathbf{Lie}(\alpha^{-1}\{j\})\right) \cong \mathrm{Mat}^{k\times k}\,\mathbf{Z}^{\oplus n}.$$

□

Definition 4.1.7. Let $\mathscr{C}^{\otimes}$ be a symmetric monoidal ∞-category. We say that it is *additive*, if it is compatible with small coproducts.

Remark 4.1.7.1. Let $\mathscr{K}$ denote the set of all small sets. Then a symmetric monoidal category $\mathscr{C}^{\otimes}$ is additive precisely when a coclassifying map $\mathrm{Comm}^{\otimes} \to \mathrm{Cat}_{\infty}$ factors through $\mathrm{Cat}_{\infty}(\mathscr{K})$. Note that this implies that if $\mathscr{C}^{\otimes}$ is an additive symmetric monoidal ∞-category, then its underlying ∞-category $\mathscr{C}$ is additive, and for any edge $\langle m\rangle \xrightarrow{\alpha} \langle n\rangle$ in $N(\mathrm{Fin}_*)$, the pushforward functor $\mathscr{C}^{\otimes}_{\langle m\rangle} \xrightarrow{\alpha_!} \mathscr{C}^{\otimes}_{\langle n\rangle}$ is additive.

Definition 4.1.8. Let $\mathscr{C}^{\otimes}$ be an additive symmetric monoidal ∞-category, let $\langle m\rangle \xrightarrow{\alpha} \langle 1\rangle$ be an edge in $N(\mathrm{Fin}_*)$, and let f, g, h be three edges $C \to D$ over α. Let

$$K = \Delta^2 \sqcup_{\Delta^{\{0,1\}}} \Delta^3 \sqcup_{\Delta^{\{0,1\}}} \Delta^2,$$

and let $K \xrightarrow{\sigma} \mathscr{C}^{\otimes}$ be a functor of the form

$$\begin{tikzcd}
& D & & \\
C \arrow[ur, "f"] \arrow[r, "{(f,g)}"] \arrow[dr, "g"'] & D\times D \arrow[u] \arrow[d] \arrow[r, "\beta"] & D\sqcup D \arrow[r, "\nabla"] & D, \\
& D & &
\end{tikzcd}$$

such that the longest horizontal edge in the diagram is h, and β is a quasi-inverse to the canonical map $D \sqcup D \to D\times D$. In this case, we say that σ *exhibits h as a sum of f and g*, and we write $f + g = h$.

Note that this endows the set of homotopy classes of maps $\mathrm{Hom}_{\mathrm{h}\mathscr{C}^{\otimes}}(C, D)_{\alpha}$ with an abelian group structure.

Definition 4.1.9. Let $\mathscr{C}$ be an additive symmetric monoidal ∞-category. A *Lie algebra object of* $\mathscr{C}$ is a functor $\mathrm{Lie}^{\otimes} \xrightarrow{L} \mathscr{C}^{\otimes}$, which satisfies the following criteria.

(1) The functor L is over $N(\mathrm{Fin}_*)$.
(2) It takes inert morphisms to inert morphisms.
(3) The images of the canonical maps $\langle n\rangle \to \langle n\rangle^{\oplus k}$ exhibit $L(\langle n\rangle^{\oplus n})$ as a coproduct of n copies of $L(\langle n\rangle)$.

We let the *∞-category of Lie algebras on $\mathscr{C}$* be the full subcategory $\mathrm{LAlg}(\mathscr{C}) \subseteq \mathrm{Fun}^{\pi}_{N(\mathrm{Fin}_*)}(\mathrm{Lie}^{\otimes}, \mathscr{C})$ spanned by the Lie algebra objects.

Proposition 4.1.10. *Let $\mathscr{C} \xrightarrow{p} N(\mathrm{Fin}_*)$ be an additive symmetric monoidal ∞-category, and $\mathrm{Lie}^{\otimes} \xrightarrow{L} \mathscr{C}$ a Lie algebra on $\mathscr{C}$. Then L defines a weak Lie algebra on $\mathscr{C}$.*

Proof. Let E denote the image $L(\langle 1\rangle) \in \mathscr{C}$. By construction, there exists a p-coCartesian arrow $L(\langle 2\rangle) \xrightarrow{\otimes_E} E \otimes E$ over the active edge $\langle 2\rangle \xrightarrow{\mu} \langle 1\rangle$. Therefore, by the coCartesian property, there exists a factorization unique up to equivalence

$$\begin{tikzcd}
 & E \otimes E \arrow[dr, "{[\,,]}"] & \\
L(\langle 2\rangle) \arrow[ur, "\otimes_E"] \arrow[rr, "{L([x_1,x_2])}"] & & E.
\end{tikzcd}$$

Let $\langle 2\rangle \xrightarrow{\sigma} \langle 2\rangle$ be the edge which swaps 1 and 2. We get a diagram

$$\begin{tikzcd}
L(2) \arrow[r, "\otimes_E"] \arrow[d, "\sigma_E"'] \arrow[drr, bend left, "{L([x_2,x_1])}"] & E \otimes E \arrow[d, "\mathrm{sh}"] \arrow[dr, "{[\,,]'}"] & \\
L(2) \arrow[r, "\otimes_E"] \arrow[rr, bend right, "{L([x_1,x_2])}"'] & E \otimes E \arrow[r, "{[\,,]}"] & E.
\end{tikzcd}$$

By construction, we have $[x_1, x_2] + [x_2, x_1] = 0$ in $\mathrm{Hom}_{\mathbf{Lie}^{\otimes,\mathrm{Ab}}}(\langle 2\rangle, \langle 1\rangle)_\mu$. This implies $L([x_1, x_2]) + L([x_2, x_1]) = 0$ by Lemma 4.1.10.1, which in turn implies $[\ ,\ ] + [\ ,\ ]' = [\ ,\ ] + ([\ ,\ ] \circ \mathrm{sh}) = 0$ by Lemma 4.1.10.2. The Jacobian property can be proven a similar way.

□

Lemma 4.1.10.1. *Let $\langle m\rangle \xrightarrow{\alpha} \langle 1\rangle$ be an edge in $N(\mathrm{Fin}_*)$. Then L induces a morphism of abelian groups*

$$\mathrm{Hom}_{\mathbf{Lie}^{\otimes,\mathrm{Ab}}}(\langle m\rangle, \langle 1\rangle)_\alpha \to \mathrm{Hom}_{\mathrm{h}\mathscr{C}^{\otimes}}(L(m), L(1))_\alpha.$$

Proof. Let the sum of two edges $\langle m\rangle \underset{g}{\overset{f}{\rightrightarrows}} \langle 1\rangle$ in $\mathrm{Lie}^{\otimes}$ be exhibited by a diagram σ of the form

$$\begin{tikzcd}
 & \langle 1\rangle & & \\
\langle m\rangle \arrow[ur, "f"] \arrow[r] \arrow[dr, "g"'] & \langle 1\rangle \times \langle 1\rangle \arrow[u] \arrow[d] \arrow[r, "\beta"] & \langle 1\rangle \sqcup \langle 1\rangle \arrow[r, "\nabla"] & \langle 1\rangle, \\
 & \langle 1\rangle & &
\end{tikzcd}$$

where β is a quasi-inverse to the canonical map. Since L is additive, we get a diagram of the form

$$\begin{tikzcd}
 & E & & & \\
L(m) \arrow[ur, "L(f)"] \arrow[r] \arrow[dr, "L(g)"'] & L(\langle 1\rangle \times \langle 1\rangle) \arrow[u] \arrow[d] \arrow[r, "c"] & E \times E \arrow[r, "\hat{\beta}"] & E \sqcup E \arrow[r, "\nabla"] & E, \\
 & E & & &
\end{tikzcd}$$

where $\hat{\beta}$ is a quasi-inverse to the canonical map. Why restricting to the subdiagram without the $L(\langle 1\rangle \times \langle 1\rangle)$ does not exhibit $L(f) + L(g) = L(f + g)$ on the nose is that its long vertical edge is not $L(f + g)$. But as $\hat{\beta} \circ c$ is homotopic to $L(\beta)$ by construction, we can replace the diagram by an appropriate one if we graft a homotopy diagram

$$\begin{tikzcd}
 & E \sqcup E \arrow[d, "\mathrm{id}"] \\
L(\langle 1\rangle \times \langle 1\rangle) \arrow[ur, "L(\beta)"] \arrow[r, "\hat{\beta}\circ c"] & E \sqcup E
\end{tikzcd}$$

on it.

□

Lemma 4.1.10.2. *Let $\mathscr{C}^{\otimes} \xrightarrow{p} N(\mathrm{Fin}_*)$ be an additive symmetric monoidal ∞-category, let $\langle m\rangle \xrightarrow{\alpha} \langle 1\rangle$ be an edge in $N(\mathrm{Fin}_*)$, and let $C \xrightarrow{s} \alpha_! C$ be a p-coCartesian map over α. Then the bijection induced by precomposition by f:*

$$\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(\alpha_! C, D) \xrightarrow{h_{[f]}} \mathrm{Hom}_{\mathrm{h}\mathscr{C}^{\otimes}}(C, D)_{\alpha}$$

is an isomorphism of abelian groups.

Proof. Let $C \overset{f}{\underset{g}{\rightrightarrows}} D$ be two edges over α. Suppose that $K \xrightarrow{\sigma} \mathscr{C}^{\otimes}$ exhibits $f + g = h$. Let $\mathscr{D}$ denote $N(\mathrm{Fin}_*)$. Since s is an α-coCartesian edge, the following diagram of solid arrows can be completed with a dashed arrow.

$$\begin{tikzcd}
 & \mathscr{C}^{\otimes}_{s/} \arrow[d] \\
K' \arrow[ur, dashed, "\bar{\sigma}"] \arrow[r, "{(\sigma,\mathrm{const}_1)}"] & \mathscr{C}^{\otimes}_{C/} \times_{\mathscr{D}_{\langle m\rangle/}} \mathscr{D}_{\alpha/}
\end{tikzcd}$$

The diagram $\bar{\sigma}$ has the form

$$\begin{tikzcd}
 & & D & & \\
C \arrow[r, "s"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & \alpha_! C \arrow[r, "{(\bar{f},\bar{g})}"] \arrow[ur, "\bar{f}"] \arrow[dr, "\bar{g}"'] & D \times D \arrow[r, "\beta"] \arrow[u] \arrow[d] & D \sqcup D \arrow[r, "\nabla"] & D, \\
 & & D & &
\end{tikzcd}$$

Let $\bar{h}$ denote its edge $\alpha_! C \to D$. Then restriction to the appropriate triangles proves that $h_{[f]}$ takes $\bar{f}$, $\bar{g}$ and $\bar{h}$ to f, g and h respectively, and restriction to the sub-simplicial set without the C exhibits $\bar{f} + \bar{g} = \bar{h}$.

□

4.2. The functor $E \mapsto \underline{\mathrm{End}}(E)$

Construction 4.2.1. Let $\mathscr{K}$ be a set of small simplicial sets. Let $\mathscr{S}(\mathscr{K}) \subseteq \mathscr{S}$ be the smallest subcategory containing Δ^0, and closed under colimits over $\mathscr{K}$. Then for every functor $\Delta^0 \xrightarrow{f_0} \mathscr{D}$ to an ∞-category which admits colimits over $\mathscr{K}$, the following hold [**Lur09**, Remark 5.3.5.9]

(1) There exists a left Kan extension $\mathscr{S}(K) \xrightarrow{f} \mathscr{D}$ of f_0.
(2) A functor $\mathscr{S}(K) \xrightarrow{f} \mathscr{D}$ is a left Kan extension of f_0 precisely when it commutes with colimits over $\mathscr{K}$.

Since the product functor $\mathscr{S} \times \mathscr{S} \to \mathscr{S}$ commutes with $\mathscr{K}$-colimits separately in each variable, the category $\mathscr{S}(\mathscr{K})$ admits Cartesian monoidal ∞-category structure, and we can choose Δ^0 as a unit object $\mathbf{1}$. Then $\mathscr{S}(\mathscr{K}) \in \mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)$ and $(\mathscr{S}(\mathscr{K}), \mathbf{1}) \in \mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathscr{K})$ are initial objects [**Lur14**, Lemma 4.8.5.3].

Let $\mathfrak{M} \in \mathrm{Mon}^{\mathscr{K}}_{\mathrm{LM}}(\mathrm{Cat}_\infty)$ be defined by the action of $\mathscr{S}(\mathscr{K})^\times$ on its underlying category. Then the forgetful functor $\mathrm{RMod}_{\mathbf{1}}(\mathscr{S}(\mathscr{K})) \to \mathscr{S}(\mathscr{K})$ gives an equivalence $\Theta(\mathscr{S}(\mathscr{K})^\times, \mathbf{1}) \to \mathfrak{M}$ in $\mathrm{LMod}_{\mathscr{S}(\mathscr{K})}(\mathrm{Cat}_\infty)$ [**Lur14**, Corollary 4.2.3.3 and Proposition 4.2.4.9]. Let us denote by Θ_* the composite

$$\mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathscr{K}) \to \mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathscr{K})_{/(\mathscr{S}(\mathscr{K}),\mathbf{1})} \to \mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathscr{K})_{/\mathfrak{M}},$$

where the arrow on the left is a quasi-inverse of the projection map, and the arrow on the right is induced by Θ.

Proposition 4.2.2. *Let $\mathscr{K}$ be a set of small simplicial sets. Consider the map given in Construction 4.2.1*

$$\mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathscr{K}) \xrightarrow{\Theta_*} \mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathscr{K})_{/\mathfrak{M}}.$$

Let $(\mathscr{C}^\otimes, \mathscr{M}^\otimes) \in \mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathscr{K})$. Then the fiber $\left(\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathscr{K})_{/\mathfrak{M}}\right)_{\mathscr{M}}$ is equivalent to the classifying space $\mathscr{M}^\simeq$.

Proof. Let $\mathscr{S}' \subseteq \mathscr{S}$ denote the smallest subcategory which contains Δ^0, and closed under $\mathscr{K}$-colimits. We have

$$\left(\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathscr{K})_{/\mathfrak{M}}\right)_{\mathscr{M}} = \mathrm{Hom}^{\mathrm{R}}_{\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathscr{K})}(\mathfrak{M}, \mathscr{M}) \simeq \mathrm{Map}_{\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathscr{K})}(\mathfrak{M}, \mathscr{M})$$

by [**Lur09**, §2.2]. Since the restriction map $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathscr{K}) \to \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathscr{K})$ is a Cartesian fibration by Lemma 4.2.2.1, we get a fiber sequence [**Lur09**, Proposition 2.4.4.2]

$$\mathrm{Map}_{\mathrm{Mon}_{\mathrm{LM}}^{\mathscr{K}}(\mathrm{Cat}_\infty)_{\mathscr{S}'}}(\mathfrak{M}, f^*\mathscr{M}) \to \mathrm{Map}_{\mathrm{Mon}_{\mathrm{LM}}^{\mathscr{K}}(\mathrm{Cat}_\infty)}(\mathfrak{M}, \mathscr{M}) \to \mathrm{Map}_{\mathrm{Mon}_{\mathrm{Ass}}^{\mathscr{K}}(\mathrm{Cat}_\infty)}(\mathscr{S}', \mathscr{C}).$$

Since $\mathscr{S}' \in \mathrm{Mon}_{\mathrm{Ass}}^{\mathscr{K}}(\mathrm{Cat}_\infty)$ is an initial object [**Lur14**, Proposition 4.8.5.3], the rightmost Kan complex is contractible, so the left arrow is a weak equivalence.

By construction,

$$\mathrm{Map}_{\mathrm{Mon}_{\mathrm{LM}}^{\mathscr{K}}(\mathrm{Cat}_\infty)}(\mathfrak{M}, f^*\mathscr{M}) \subseteq \mathrm{Map}_{\mathrm{Fun}(\mathrm{LM}^\otimes, \mathrm{Cat}_\infty)}(\mathfrak{M}, f^*\mathscr{M})$$

is the connected component with vertices corresponding to natural morphisms $\mathfrak{M} \to f^*\mathscr{M}$ which commute with $\mathscr{K}$-colimits on the underlying ∞-categories. We have a natural equivalence [**GHN15**, Proposition 5.1]

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{LM}^\otimes, \mathrm{Cat}_\infty)}(\mathfrak{M}, f^*\mathscr{M}) \simeq \int_{X \in \mathrm{LM}^\otimes} \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathfrak{M}(X), (f^*\mathscr{M})(X)).$$

Since we have isomorphisms by construction [**Lur09**, Definition 3.0.0.1 and §3.1.3]

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathfrak{M}(X), (f^*\mathscr{M})(X)) \cong \mathrm{Fun}^{\simeq}(\mathfrak{M}(X), (f^*\mathscr{M})(X)) \cong \mathrm{Map}^{\sharp}_{\mathrm{Set}_\Delta^+}(\mathfrak{M}(X)^{\natural}, (f^*\mathscr{M})(X)^{\natural}),$$

the mapping complex $\mathrm{Map}_{\mathrm{Fun}(\mathrm{LM}^\otimes, \mathrm{Cat}_\infty)}(\mathfrak{M}, f^*\mathscr{M})$ is equivalent to the interior Hom object $[\mathfrak{M}, f^*\mathscr{M}]$ of the simplicial functor category $(\mathrm{Set}_\Delta^+)^{\mathrm{LM}}$ [**Kel05**, §2.2]. This in turn is naturally isomorphic to the mapping complex $\mathrm{Map}_{((\mathrm{Set}_\Delta^+)^{\mathrm{LM}})^\circ}(\mathfrak{M}, f^*\mathscr{M})$ given in [**Lur09**, §3.2.4] by Lemma 4.2.2.2. Via the co-unstraightening functor we get an equivalence

$$\mathrm{Map}_{((\mathrm{Set}_\Delta^+)^{\mathrm{LM}})^\circ}(\mathfrak{M}, f^*\mathscr{M}) \simeq \mathrm{Map}_{((\mathrm{Set}_\Delta^+)_{\mathrm{LM}^\otimes})^\circ}(\mathrm{coUn}(\mathfrak{M}), \mathrm{coUn}(f^*\mathscr{M})),$$

and by construction the latter mapping complex is naturally isomorphic to the interior of the category of monoidal functors $\mathrm{Fun}_{\mathrm{LM}^\otimes}^{\otimes,\simeq}(\mathrm{coUn}(\mathfrak{M}), \mathrm{coUn}(f^*\mathscr{M}))$.

The same argument works for the associative operad also. Let Ass $\xrightarrow{f}$ LM denote the canonical map. Restricting to the functors which commute with $\mathscr{K}$-colimits, the naturality of the co-unstraightening functor [**GHN15**, Corollary A.31] gives a homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Mon}^{\mathscr{K}}_{\mathrm{LM}}(\mathrm{Cat}_\infty)}(\mathscr{M}, f^*\mathscr{M}) \arrow[r, "\mathrm{coUn}", "\simeq"'] \arrow[d, "f^*"] & \mathrm{Fun}^{\otimes,\mathscr{K},\simeq}_{\mathrm{LM}^\otimes}(\mathrm{coUn}(\mathfrak{M}), \mathrm{coUn}(f^*\mathscr{M})) \arrow[d, "f^*"] \\
\mathrm{Map}_{\mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)}(\mathscr{S}', \mathscr{S}') \arrow[r, "\mathrm{coUn}", "\simeq"'] & \mathrm{Fun}^{\otimes,\mathscr{K},\simeq}_{\mathrm{Ass}^\otimes}(\mathrm{coUn}(\mathscr{S}'), \mathrm{coUn}(\mathscr{S}')).
\end{tikzcd}$$

We get an equivalence of homotopy fibers

$$\mathrm{Map}_{\mathrm{Mon}^{\mathscr{K}}_{\mathrm{LM}}(\mathrm{Cat}_\infty)_{\mathscr{S}'}}(\mathfrak{M}, f^*\mathscr{M}) \simeq \mathrm{LinFun}^{\mathscr{K},\simeq}_{\mathscr{S}'}(\mathfrak{M}, f^*\mathscr{M}).$$

Since $\mathfrak{M} \simeq \Theta(\mathscr{S}', \mathbf{1})$, we have

$$\mathrm{LinFun}^{\mathscr{K},\simeq}_{\mathscr{S}'}(\mathfrak{M}, f^*\mathscr{M}) \simeq \mathrm{LMod}^{\simeq}_{\mathbf{1}}(f^*\mathscr{M}),$$

which in turn is equivalent to $(f^*\mathscr{M})^{\simeq}$ [**Lur14**, Proposition 4.2.4.9]. We can finish by evoking that the Cartesian edge $f^*\mathscr{M} \to \mathscr{M}$ is an equivalence on the underlying ∞-categories.

□

Lemma 4.2.2.1. *Let $\mathscr{K}$ be a set of small simplicial sets. Then the restriction map* $\mathrm{Mon}^{\mathscr{K}}_{\mathrm{LM}}(\mathrm{Cat}_\infty) \to \mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)$ *is a Cartesian fibration.*

Proof. For any operad $\mathscr{O}^{\otimes}$, postcomposition with the composite of canonical maps

$$\mathrm{Cat}_\infty(\mathscr{K}) \to \mathrm{Cat}^{\times}_\infty \to \mathrm{Cat}_\infty$$

gives an equivalence $\mathrm{Alg}_{\mathscr{O}}(\mathrm{Cat}_\infty(\mathscr{K})) \to \mathrm{Mon}^{\mathscr{K}}_{\mathscr{O}}(\mathrm{Cat}_\infty)$ [**Lur14**, Proposition 2.4.2.5 and Remark 4.8.1.9]. Letting the horizontal arrows in the diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{LM}}(\mathrm{Cat}_\infty(\mathscr{K})) \arrow[r] \arrow[d, "p"] & \mathrm{Mon}^{\mathscr{K}}_{\mathrm{LM}}(\mathrm{Cat}_\infty) \arrow[d, "q"] \\
\mathrm{Alg}_{\mathrm{Ass}}(\mathrm{Cat}_\infty(\mathscr{K})) \arrow[r] & \mathrm{Mon}^{\mathscr{K}}_{\mathrm{Ass}}(\mathrm{Cat}_\infty)
\end{tikzcd}$$

be like this, and let the vertical arrows be given by precomposition with the canonical inclusion $\mathrm{Ass}^{\otimes} \to \mathrm{LM}^{\otimes}$. Since p is a Cartesian fibration [**Lur14**, Corollary 4.2.3.2], so is q.

□

Lemma 4.2.2.2. *Let* $\mathbf{C}$ *be a simplicial category, let* $\mathbf{C} \xrightarrow{F} \mathrm{Set}_{\Delta}^{+}$ *be a functor, and let* K *be a simplicial set. Let* $F \boxtimes K^{\sharp}$ *denote the functor*

$$\mathbf{C} \xrightarrow{C \mapsto FC \times K^{\sharp}} \mathrm{Set}_{\Delta}^{\sharp}.$$

Let $\mathbf{C} \xrightarrow{G} \mathrm{Set}_{\Delta}^{+}$ *be another simplicial functor. Let* $\mathrm{Map}'(F, G)$ *denote the simplicial set defined by*

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(K, \mathrm{Map}'(F, G)) = \mathrm{Hom}_{(\mathrm{Set}_{\Delta}^{+})^{\mathbf{C}}}(F \boxtimes K^{\sharp}, G).$$

Then there exists a natural isomorphism

$$\mathrm{Map}'(F, G) \to \int_{C \in \mathbf{C}} \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GC).$$

Proof. By definition, an end is an equalizer of the form

$$\int_{C \in \mathbf{C}} \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GC) \to \prod_{C \in \mathbf{C}} \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GC) \rightrightarrows \prod_{C, D \in \mathbf{C}} \mathrm{Map}_{\mathrm{Set}_{\Delta}}(\mathbf{C}(C, D), \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GD)),$$

where the two arrows on the right are given by precomposition and postcomposition respectively. To be precise, the precomposition map

$$\mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GC) \to \mathrm{Map}_{\mathrm{Set}_{\Delta}}(\mathbf{C}(D, C), \mathrm{Map}^{+}_{\mathrm{Set}_{\Delta}}(FD, GC))$$

corresponds to the simplicial map

$$\mathbf{C}(C, D) \times \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GC) \to \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}}(FC, GD),$$

which takes a map $K \xrightarrow{f} \mathbf{C}(C, D) \times \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GC)$ to the composite

$$K \xrightarrow{f} \mathbf{C}(C, D) \times \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GC) \xrightarrow{(G_{CD}, \mathrm{id})} \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(GC, GD) \times \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GC) \xrightarrow{\circ} \mathrm{Map}^{\sharp}_{\mathrm{Set}_{\Delta}^{+}}(FC, GD).$$

Let $\mathrm{Map}^+(FC, GC)$ denote the marked mapping complex. Let us define a map $\mathrm{Map}'(F, G) \to \prod_{C\in\mathbf{C}} \mathrm{Map}^\sharp_{\mathrm{Set}^+_\Delta}(FC, GC)$ as follows. Its component over $C \in \mathbf{C}$ is given by the composite

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}'(F, G)) \cong \mathrm{Hom}_{(\mathrm{Set}^+_\Delta)^{\mathbf{C}}}(K^\sharp \boxtimes F, G) &\xrightarrow{\mathrm{ev}_C} \mathrm{Hom}_{\mathrm{Set}^+_\Delta}(K^\sharp \times FC, GC)\\ &\cong \mathrm{Hom}_{\mathrm{Set}^+_\Delta}(K^\sharp, \mathrm{Map}^+(FC, GC)) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}^\sharp_{\mathrm{Set}^+_\Delta}(FC, GC)).\end{aligned}$$

By definition, a collection of maps $\{K^\sharp \times FC \to GC\}_{C\in\mathbf{C}}$ determines a natural map $K^\sharp \times F \to G$ precisely when its images by the precomposition and postcomposition maps agree.

□

4.3. Trivial forms: $\mathfrak{sl}_n(F)$ for totally supported sheaves

Construction 4.3.1. Let us construct an extension $\mathbf{Ass}^{\otimes,\mathrm{Ab}}$ of the category of operations $\mathbf{Ass}^\otimes \to \mathrm{Fin}_*$ similar to the Lie case.

(1) The objects over $\langle n\rangle \in \mathrm{Fin}_*$ are denoted by $\langle n\rangle^{\oplus k}$ for $k \geq 0$.
(2) The hom sets are of the following form.

$$\mathrm{Hom}_{\mathbf{Ass}^{\otimes,\mathrm{Ab}}}(\langle m\rangle^{\oplus k}, \langle n\rangle^{\oplus \ell}) = \coprod_{\langle m\rangle \xrightarrow{\alpha} \langle n\rangle} \mathrm{Mat}^{\ell\times k}_{\mathbf{Z}}\left(\oplus_{1\leq j\leq n} \mathbf{Z}\,\mathbf{Ass}(n)\right).$$

(3) Composition is defined by matrix multiplication and the composition in $\mathbf{Ass}$.

Note that as $\mathbf{Ass}(n)$ is the set of linear orderings on the set $\{x_1, \dots, x_n\}$, the free abelian group $\mathbf{Z}\,\mathbf{Ass}(n)$ can be regarded as the subgroup $T^n_{\mathbf{Z}}(x_1, \dots, x_n) \subset T_{\mathbf{Z}}(x_1, \dots, x_n)$ of the tensor algebra on the free abelian group $x_1\mathbf{Z} \oplus \cdots \oplus x_n\mathbf{Z}$. Let $\mathrm{Ass}^{\otimes,\mathrm{Ab}} \to N(\mathrm{Fin}_*)$ denote the simplicial nerve of $\mathbf{Ass}^{\otimes,\mathrm{Ab}} \to \mathrm{Fin}_*$.

Definition 4.3.2. Let $\mathscr{C}$ be an additive symmetric monoidal ∞-category. An *additive associative algebra object* is a functor $\mathrm{Ass}^\otimes \xrightarrow{A} \mathscr{C}^\otimes$, which satisfies the following criteria.

(1) The functor A is over $N(\mathrm{Fin}_*)$.
(2) It takes inert morphisms to inert morphisms.
(3) The images of the canonical maps $\langle n\rangle \to \langle n\rangle^{\oplus k}$ exhibit $L(\langle n\rangle^{\oplus n})$ as a coproduct of n copies of $L(\langle n\rangle)$.

We let the *∞-category of additive associative algebras on $\mathscr{C}$* be the full subcategory $\mathrm{Alg}^{\mathrm{Ab}}(\mathscr{C}) \subseteq \mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathrm{Ass}^{\otimes,\mathrm{Ab}}, \mathscr{C})$ spanned by the additive associative algebra objects.

Remark 4.3.2.1. This is a special case of the generalization to adjoining relative colimits of [**Lur09**, Proposition 5.3.6.2].

Construction 4.3.3 (From algebras to Lie algebras). Let $\mathscr{C}^{\otimes} \xrightarrow{q} N(\mathrm{Fin}_*)$ be an additive symmetric monoidal ∞-category which admits all small colimits. Consider the restriction map

$$\mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathrm{Ass}^{\otimes,\mathrm{Ab}}, \mathscr{C}^{\otimes}) \to \mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathrm{Ass}^{\otimes}, \mathscr{C}^{\otimes}).$$

It has a quasi-inverse given by taking left Kan extensions over $N(\mathrm{Fin}_*)$ [**Lur09**, Proposition 4.3.2.15], and Lemma 4.3.3.1 implies that by restricting a quasi-inverse we get a map

$$\mathrm{Alg}(\mathscr{C}) \to \mathrm{Alg}^{\mathrm{Ab}}(\mathscr{C}).$$

Composing this by the precomposition by the canonical morphism of operads $\mathrm{Lie}^{\otimes} \to \mathrm{Ass}^{\otimes,\mathrm{Ab}}$ map, we get a functor

$$\mathrm{Alg}(\mathscr{C}) \to \mathrm{LAlg}(\mathscr{C}).$$

Lemma 4.3.3.1. *Let $A \in \mathrm{Alg}(\mathscr{C})$ be an algebra object in $\mathscr{C}$, and let $\mathrm{Ass}^{\otimes,\mathrm{Ab}} \xrightarrow{\bar{A}} \mathscr{C}^{\otimes}$ be a left Kan extension of it over $N(\mathrm{Fin}_*)$. Then for any $n, k \geq 0$, the functor $\bar{A}$ takes $\langle n\rangle^{\oplus k}$ to an $N(\mathrm{Fun}_*)$-coproduct of k copies of $A(\langle n\rangle)$.*

Proof. Let $\mathrm{Ass}^{\otimes,\mathrm{Ab}} \xrightarrow{q'} N(\mathrm{Fin}_*)$ denote the structure map. The diagram $(\mathrm{Ass}^{\otimes,\mathrm{Ab}}_{/\langle n\rangle^{\oplus k}})^{\triangleright} \xrightarrow{\bar{A}_{\langle n\rangle^{\oplus k}}} \mathscr{C}^{\otimes}$ induced by $\bar{A}$ is an $N(\mathrm{Fin}_*)$-colimit of the diagram $\mathrm{Ass}^{\otimes,\mathrm{Ab}}_{/\langle n\rangle^{\oplus k}} \xrightarrow{A_{\langle n\rangle^{\oplus k}}} \mathscr{C}^{\otimes}$ induced by A by construction. This gives half of the diagram of Cartesian squares

$$\begin{tikzcd}
\mathscr{C}^{\otimes}_{\bar{A}_{\langle n\rangle^{\oplus k}}|\{e_i\to\infty\}/} \arrow[r] \arrow[d] & \mathscr{C}^{\otimes}_{\bar{A}_{\langle n\rangle^{\oplus k}}/} \arrow[r] \arrow[d] & N(\mathrm{Fin}_*)_{q'\bar{A}_{\langle n\rangle^{\oplus k}}/} \arrow[d] \\
\mathscr{C}^{\otimes}_{A_{\langle n\rangle^{\oplus k}}|\{e_i\}/} \arrow[r] & \mathscr{C}^{\otimes}_{A_{\langle n\rangle^{\oplus k}}/} \arrow[r] & N(\mathrm{Fin}_*)_{q'A_{\langle n\rangle^{\oplus k}}/}.
\end{tikzcd}$$

The other square is a preimage diagram, where $\langle n\rangle \xrightarrow{e_i} \langle n\rangle^{\oplus k}$ is a standard basis element in $\mathrm{Hom}_{\mathrm{Ass}^{\otimes,\mathrm{Ab}},\mathrm{id}_{\langle n\rangle}}(\langle n\rangle, \langle n\rangle^{\oplus k}) = \mathrm{Mat}^{k\times 1}_{\mathbf{Z}} \mathrm{Hom}_{\mathrm{Ass}^{\otimes},\mathrm{id}_{\langle n\rangle}}(\langle n\rangle, \langle n\rangle)$. Inspecting the images, the outer Cartesian square factors through the Cartesian square

$$\begin{tikzcd}
\mathscr{C}^{\otimes}_{\bar{A}_{\langle n\rangle^{\oplus k}}|\{e_i\to\infty\}/} \arrow[r] \arrow[d] & N(\mathrm{Fin}_*)_{q'\bar{A}_{\langle n\rangle^{\oplus k}}|\{e_i\to\infty\}/} \arrow[d] \\
\mathscr{C}^{\otimes}_{A_{\langle n\rangle^{\oplus k}}|\{e_i\}/} \arrow[r] & N(\mathrm{Fin}_*)_{q'A_{\langle n\rangle^{\oplus k}}|\{e_i\}/},
\end{tikzcd}$$

which exhibits $\bar{A}(\langle n\rangle^{\oplus k})$ as an $N(\mathrm{Fun}_*)$-coproduct of k copies of $A(\langle n\rangle)$.

□

Construction 4.3.4 (The trace of an algebra). Let $\mathscr{C}$ be a symmetric monoidal ∞-category, which admits all colimits. Let $\mathrm{Ass}^{\otimes} \xrightarrow{p'} N(\mathrm{Fin}_*)$ and $\mathrm{LM}^{\otimes} \xrightarrow{p} N(\mathrm{Fin}_*)$ denote the structure maps. Via the morphism $\mathrm{LM}^{\otimes} \xrightarrow{f} \mathrm{Ass}^{\otimes}$ and the natural transformation $p' \circ f \xrightarrow{\iota} p$ defined in [**Lur14**, Construction 4.6.3.7], we get a composite functor

$$\mathrm{Alg}(\mathscr{C}) \xrightarrow{\circ f} \mathrm{Fun}_{p'\circ f}(\mathrm{LM}^{\otimes}, \mathscr{C}^{\otimes}) \xrightarrow{\iota_*} \mathrm{LMod}(\mathscr{C}),$$

which maps an algebra object $A \in \mathrm{Alg}(\mathscr{C})$ to the *evaluation module* $A^e \in \mathrm{LMod}_{A\otimes A^{\mathrm{rev}}}(\mathscr{C})$. Postcomposing this functor with the composite

$$\mathrm{LMod}(\mathscr{C}) \xleftarrow{\simeq} \mathrm{RMod}_{\mathbf{1}}(\mathrm{LMod}\,\mathscr{C}) \subseteq \mathrm{RMod}(\mathrm{LMod}\,\mathscr{C}) \xleftarrow{\simeq} \mathrm{BMod}(\mathscr{C}),$$

where the two left arrows are equivalences of ∞-categories by [**Lur14**, Propositions 4.2.4.9 and 4.3.2.7] respectively, gives us a functor $A \mapsto {}_{A\otimes A^{\mathrm{rev}}}A_{\mathbf{1}}$. Postcomposition with the reversal functor $\mathrm{BMod}(\mathscr{C}) \xrightarrow{\mathrm{rev}} \mathrm{BMod}(\mathscr{C})$ [**Lur14**, Construction 4.6.3.1] gives a functor $A \mapsto {}_{\mathbf{1}}A_{A\otimes A^{\mathrm{rev}}}$, the images of which are called *coevaluation modules*, and they are denoted by A^c.

Let us define a functor

$$\mathrm{Alg}(\mathscr{C}) \times \Delta^1 \to \mathrm{BMod}(\mathscr{C}) \times_{\mathrm{Alg}(\mathscr{C})} \mathrm{BMod}(\mathscr{C})$$

with the following components.

Consider the functor

$$\mathrm{Alg}(\mathscr{C}) \to \mathrm{RMod}(\mathscr{C}) \times_{\mathrm{Alg}(\mathscr{C})} \mathrm{Fun}(\Delta^1, \mathrm{Alg}\,\mathscr{C})$$

with left component the map $A \mapsto A^c$, and right component given as the composite

$$\mathrm{Alg}(\mathscr{C}) \overset{\simeq}{\leftarrow} \mathrm{Alg}(\mathscr{C})^{\mathbf{1}/} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Alg}\,\mathscr{C}).$$

Postcomposing with a section of the functor

$$\mathrm{Fun}(\Delta^1, \mathrm{RMod}\,\mathscr{C})^{\mathrm{cart}} \to \mathrm{RMod}(\mathscr{C}) \times_{\mathrm{Alg}\,\mathscr{C}} \mathrm{Fun}(\Delta^1, \mathrm{Alg}\,\mathscr{C})$$

which exists because of Lemma 4.3.4.1, we get a map

$$\mathrm{Alg}(\mathscr{C}) \xrightarrow{A \mapsto ({}_{\mathbf{1}}A_{\mathbf{1}} \to {}_{\mathbf{1}}A_{A \otimes A^{\mathrm{rev}}})} \mathrm{Fun}(\Delta^1, \mathrm{BMod}(\mathscr{C})).$$

The right component is defined as the composite

$$\mathrm{Alg}(\mathscr{C}) \xrightarrow{A \mapsto A^e} \mathrm{BMod}(\mathscr{C}) \overset{\simeq}{\leftarrow} \mathrm{BMod}(\mathscr{C})^{{}_{\mathbf{1}}\mathbf{1}_{\mathbf{1}}/} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{BMod}\,\mathscr{C}).$$

Postcomposing with the relative tensor product functor [**Lur14**, Example 4.4.2.11]

$$\mathrm{BMod}(\mathscr{C}) \times_{\mathrm{Alg}(\mathscr{C})} \mathrm{BMod}(\mathscr{C}) \to \mathrm{BMod}(\mathscr{C}),$$

we get a functor

$$\mathrm{Alg}(\mathscr{C}) \xrightarrow{A \mapsto \left({}_{\mathbf{1}}A_{\mathbf{1}} \xrightarrow{a \mapsto a \otimes 1} A \otimes_{A \otimes A^{\mathrm{rev}}} A\right)} \mathrm{Fun}(\Delta^1, \mathrm{BMod}\,\mathscr{C})$$

Lemma 4.3.4.1. *Let* $\mathscr{C} \xrightarrow{p} \mathscr{D}$ *be a Cartesian fibration of* ∞*-categories, and let* $\mathrm{Fun}(\Delta^1, \mathscr{C})^{\mathrm{cart}} \subseteq \mathrm{Fun}(\Delta^1, \mathscr{C})$ *denote the full subcategory spanned by p-Cartesian edges. Then the functor*

$$\mathrm{Fun}(\Delta^1, \mathscr{C})^{\mathrm{cart}} \to \mathscr{C} \times_{\mathscr{D}} \mathrm{Fun}(\Delta^1, \mathscr{D})$$

is an equivalence of ∞*-categories.*

Proof. We need to prove that all commutative diagrams of solid arrows

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathscr{C})^{\mathrm{cart}} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathscr{C} \times_{\mathscr{D}} \mathrm{Fun}(\Delta^1, \mathscr{D}),
\end{tikzcd}$$

can be augmented with a dashed arrow in a commutative manner. This is true for $n = 0$ because $\mathscr{C} \to \mathscr{D}$ is a Cartesian fibration. If $n > 0$, the diagram can be regarded as diagram of marked simplicial sets

$$\begin{tikzcd}
\big((\partial\Delta^n)^\flat \times (\Delta^1)^\sharp\big) \coprod_{(\partial\Delta^n)^\flat \times \{1\})} \big((\Delta^n)^\flat \times \{1\}\big) \arrow[r] \arrow[d] & \mathscr{C}^\natural \arrow[d] \\
(\Delta^n)^\flat \times (\Delta^1)^\sharp \arrow[r] \arrow[ur, dashed] & \mathscr{D}^\sharp.
\end{tikzcd}$$

Since the left vertical arrow is marked anodyne [**Lur09**, Proposition 3.1.1.5], a dashed arrow exists [**Lur09**, Proposition 3.1.1.6].

□

Definition 4.3.5. Let $\mathscr{C}$ be an additive symmetric monoidal ∞-category, which admits all small colimits, and let $L \in \mathrm{LAlg}(\mathscr{C})$ be a Lie algebra object. We say that L *is an abelian Lie algebra*, if the image of every non-inert morphism in $\mathrm{Lie}^{\otimes}$ is a zero morphism.

Proposition 4.3.6. *Let $\mathscr{K} \subset \mathrm{Fun}(\Delta^1, \mathrm{LAlg}\,\mathscr{C})$ be the full subcategory spanned by functors with targets abelian Lie algebras, and let $\mathscr{K}_0 \subseteq \mathrm{LAlg}\,\mathscr{C} \times_{\mathscr{C}} \mathrm{Fun}(\Delta^1, \mathscr{C})$ be the full subcategory on pairs $(L, L(1) \xrightarrow{f} M(1))$ such that precomposing f with the image by L of any non-inert morphism, we get a zero morphism. Then the morphism*

$$\mathscr{K} \to \mathscr{K}_0$$

with components the restriction maps to the source and to the underlying categories respectively is a trivial Kan fibration.

Proof. Let $\mathrm{LAlg}_0(\mathscr{C}) \subseteq \mathrm{LAlg}(\mathscr{C})$ denote the full subcategory spanned by abelian Lie algebras. We claim that the restriction map $\mathrm{LAlg}_0(\mathscr{C}) \to \mathscr{C}$ is a trivial Kan fibration. Let $\mathrm{Lie}^{\otimes}_{\mathrm{triv}} \subseteq \mathrm{Lie}^{\otimes}$ be the subcategory spanned by inert morphisms, and let $\mathrm{LAlg}_{\mathrm{triv}}(\mathscr{C})$ denote the category of additive $\mathrm{Lie}_{\mathrm{triv}}$-algebras. Then the restriction map $\mathrm{LAlg}_{\mathrm{triv}}(\mathscr{C}) \to \mathscr{C}$ is a trivial Kan fibration [**Lur14**, Example

2.1.3.5], therefore it is enough to show that the restriction map $\mathrm{LAlg}_{\mathrm{Ab}}(\mathscr{C}) \to \mathrm{LAlg}_{\mathrm{triv}}(\mathscr{C})$ is a trivial Kan fibration. That is, we need to find a dashed arrow in any commutative diagram of solid arrows of the form

$$\begin{tikzcd}
(\partial\Delta^n \times \mathrm{Lie}^{\otimes}) \coprod_{\partial\Delta^n \times \mathrm{Lie}^{\otimes}_{\mathrm{triv}}} (\Delta^n \times \mathrm{Lie}^{\otimes}_{\mathrm{triv}}) \arrow[r] \arrow[d] & \mathscr{C}^{\otimes} \arrow[d] \\
\Delta^n \times \mathrm{Lie}^{\otimes} \arrow[r] \arrow[ur, dashed] & N(\mathrm{Fin}_*)
\end{tikzcd}$$

such that the horizontal morphisms take inert morphisms to inert morphisms, and not inert morphisms to zero morphisms. We will fill in the simplices by induction on their dimension $m \geq 0$. Since we want to extend from a $\mathrm{Lie}_{\mathrm{triv}}$-algebra, every vertex already has an image, and similarly for inert edges, so for $m = 1$ all that remains is to map each non-inert edge to a zero morphism.

Let us now prove the induction statement for $m > 1$, assuming that it holds for simplices of all dimensions less than m. That is, let us be given an m-simplex $\Delta^m \xrightarrow{\sigma} \mathrm{Lie}^{\otimes}$ not all of which edges are inert, and such that we know the image of its boundary $\partial\Delta^m \xrightarrow{\tau} \mathscr{C}^{\otimes}$. We claim that $\tau(\Delta^{\{0,m\}})$ is a zero morphism. This is immediate, if $\sigma(\Delta^{\{0,m\}}) \in \mathrm{Lie}^{\otimes}$ is not inert. If it is, then by Lemma 4.3.6.1, the edge $\sigma(\Delta^{\{0,m\}})$ is a zero morphism, thus so need to be $\tau(\Delta^{\{0,m\}})$ by additivity.

In order to apply Lemma 4.3.6.2, we need to extend τ to a diagram $\partial\Delta^k \sqcup_{\mathrm{sk}_2 \Delta^k} \mathrm{sk}_2 \Delta^{k+1} \xrightarrow{\tilde{\tau}} \mathscr{C}^{\otimes}$, where $\Delta^m \xrightarrow{d^i} \Delta^{m+1}$ is the inclusion of an inner face: $0 < i < m + 1$, and $\tilde{\tau}(\{i\})$ is a zero object. Since not all edges of σ are inert, there exists a $0 \leq j < m$ such that $\sigma(\Delta^{\{j,j+1\}})$ is not inert. Therefore, $\tau(\Delta^{\{j,j+1\}})$ is a zero morphism. This implies that we can define $\tilde{\tau}$ on $\mathrm{sk}_2 \Delta^{m+1}$ as follows. Since $\tau(\Delta^{\{j,j+1\}})$ is a zero morphism, we can extend it to a 2-simplex of the form

$$\begin{tikzcd}
& 0 \arrow[dr] & \\
\tau(\{j\}) \arrow[ur] \arrow[rr, "\tau(\Delta^{\{j,j+1\}})"] & & \tau(\{j+1\}).
\end{tikzcd}$$

Starting from $k = j - 1$ and decreasing it one by one to make the new edges agree (or just replacing them with equivalent ones), we can use that 0 is a final object to get 2-simplices of the form

$$\begin{tikzcd}
 & 0 & \\
\tau(\{k\}) \arrow[ur] \arrow[rr, "\tau(\Delta^{\{k,k+1\}})"] & & \tau(\{k+1\}). \arrow[ul]
\end{tikzcd}$$

Similarly, we can start from $k = j + 1$, and increase it one by one so that using that 0 is an initial object, we get 2-simplices of the form

$$\begin{tikzcd}
 & 0 \arrow[dl] \arrow[dr] & \\
\tau(\{k\}) \arrow[rr, "\tau(\Delta^{\{k,k+1\}})"] & & \tau(\{k+1\}).
\end{tikzcd}$$

Now we can apply Lemma 4.3.6.2 and restrict along d^i to get an extension of τ as desired.

Let us now prove that the morphism $\mathscr{K} \to \mathscr{K}_0$ is a trivial Kan fibration. That is, we need to show that we can lift any diagram of the form

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathscr{K} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathscr{K}_0.
\end{tikzcd}$$

That is, we need to lift a diagram

$$(\partial\Delta^n \times \Delta^1 \times \mathrm{Lie}^{\otimes}) \cup (\Delta^n \times \Delta^1 \times \mathrm{Lie}^{\otimes}) \cup (\Delta^n \times \{0\} \times \mathrm{Lie}^{\otimes}) \xrightarrow{\sigma} \mathscr{C}^{\otimes}$$

to $\Delta^n \times \Delta^1 \times \mathrm{Lie}^{\otimes}$. By what we have done above, we can extend σ over the simplices over $\Delta^n \times \{1\} \times \mathrm{Lie}^{\otimes}$ to form a simplex of $\mathrm{LAlg}_{\mathrm{Ab}}\,\mathscr{C}$, and moreover, via the canonical equivalences $\mathscr{C}^{\otimes}_{\langle m \rangle} \to \mathscr{C}^{\times m}$, we can extend σ over the entire 1-skeleton $\mathrm{sk}_1(\Delta^n \times \Delta^1 \times \mathrm{Lie}^{\otimes})$. We now want to finish the way we did in the previous claim, and for that, we need to show that for $m > 1$ and any subsimplex

$$\Delta^m \subseteq (\Delta^n \times \Delta^1 \times \mathrm{Lie}^{\otimes}) \setminus \Big((\Delta^n \times \Delta^1 \times \mathrm{Lie}^{\otimes}_{\mathrm{triv}}) \cup (\Delta^n \times \mathrm{sk}_0(\Delta^1) \times \mathrm{Lie}^{\otimes})\Big),$$

we have that $\sigma(\Delta^{\{0,m\}})$ is a zero morphism. The edge $\sigma(\Delta^{\{0,m\}})$ is a composite of the edges $\sigma(\Delta^{\{m-1,m\}}), \sigma(\Delta^{\{0,1\}})$. Since σ is already defined over the fibers over $\mathrm{sk}_0\,\Delta^1$, we can enlarge this composition chain to contain the image of an edge $\Delta^{\{i,i+1\}}$ over $\mathrm{sk}_0\,\Delta^n \times \Delta^1 \times \mathrm{sk}_0\,N(\mathrm{Fin}_*)$. If not all edges $\Delta^{\{j,j+1\}}$ for $0 \leq j < i$ are inert, then the composite of their images is either inert and thus zero by Lemma 4.3.6.1, or it is not inert and thus the composite of that with $\Delta^{\{i,i+1\}}$ is zero by construction. Similarly, if not all edges $\Delta^{\{j,j+1\}}$ for $i < j \leq m$ are inert, then the composite of their images is either inert and thus zero by Lemma 4.3.6.1, or it is not inert and thus zero, since the restriction of σ to the fiber over $\{1\} \subset \Delta^1$ is a diagram in $\mathrm{LAlg}_{\mathrm{Ab}}\,\mathscr{C}$. We can finish by applying Lemma 4.3.6.2 as in the previous claim.

□

Lemma 4.3.6.1. *Let* $\Delta^m \xrightarrow{\sigma} \mathrm{Lie}^{\otimes}$ *be an m-simplex such that not all of its edges are inert, but* $\sigma(\Delta^{\{0,m\}})$ *is inert. Then* $\sigma(\Delta^{\{0,m\}})$ *is a zero arrow.*

Proof. Since not all edges of σ are inert, in its image in $N(\mathrm{Fin}_*)$:

$$\langle n_0 \rangle \to \cdots \to \langle n_i \rangle \xrightarrow{\alpha_i} \langle n_{i+1} \rangle \to \cdots \to \langle n_j \rangle \xrightarrow{\alpha_j} \langle n_{j+1} \rangle \to \cdots \to \langle n_m \rangle$$

there has to be an α_i such that $k \in \langle n_{i+1} \rangle^\circ$ hasn't got a preimage, and it's taken to $\ell \in \langle n_j \rangle^\circ$, which is not the unique element in its α_j-fiber. In the hom set over α_i, the k-component is going to be zero, and when we compose arrows, that is going to be substituted to an input of the Lie bracketing given by α_j, thus the composite is the zero map.

□

Lemma 4.3.6.2. *Let* $k \geq 2$ *and let* $\partial\Delta^k \sqcup_{\mathrm{sk}_2\,\Delta^k} \mathrm{sk}_2\,\Delta^{k+1} \subseteq K \subseteq \Delta^{k+1}$ *be an intermediate simplex, where* $\Delta^k \xrightarrow{d^i} \Delta^{k+1}$ *is an intermediate face map, that is* $0 < i < m+1$. *Then every diagram* $K \xrightarrow{\tau_0} \mathscr{C}^{\otimes}$ *such that* $\tau_0(\{i\})$ *is a zero object of* $\mathscr{C}^{\otimes}$ *can be extended to an* $(k+1)$*-simplex, up to possibly changing some of the faces of dimension* > 1 *containing* $\{i\}$.

Proof. We prove the statement by induction on k. If $k = 2$, then we can leave out one of the 2-dimensional faces containing $\{i\}$ to get an inner horn, and use that $\mathscr{C}^{\otimes}$ is an ∞-category. Supposing that the statement holds for k, we can prove it for $k+1$ by filling in the faces containing $\{i\}$ using the induction assumption, then leaving out a k-dimensional face containing $\{i\}$ to get an inner horn, from which we can extend again via the ∞-category property.

□

Construction 4.3.7 (Taking kernels). Let $\mathscr{D}$ be an ∞-category, which admits pullbacks and a zero object $0 \in \mathscr{D}$. We have a canonical isomorphism

$$\mathrm{Fun}(\Lambda^2_2, \mathrm{LAlg}\,\mathscr{D}) \cong \mathrm{Fun}(\Delta^1, \mathscr{D}) \times_{\mathscr{D}} \mathrm{Fun}(\Delta^1, \mathscr{D}),$$

along which the full subcategory $\mathscr{K} \subseteq \mathrm{Fun}(\Lambda^2_2, \mathscr{D})$ spanned by functors which take 0 to 0 corresponds to the subcategory $\mathrm{Fun}(\Delta^1, \mathscr{D}) \times_{\mathscr{D}} \mathscr{D}^{0/}$. Therefore, the restriction map

$$\mathrm{Fun}(\Lambda^2_2, \mathscr{D}) \xrightarrow{\circ d_0} \mathrm{Fun}(\Delta^1, \mathscr{D})$$

has a section, which maps into $\mathscr{K}$. Since an edge in $\mathrm{Fun}(\Delta^1, \mathscr{D})$ is Cartesian with respect to $\mathrm{Fun}(\Delta^1, \mathscr{D}) \xrightarrow{\mathrm{ev}_1} \mathscr{D}$ precisely when it is a Cartesian square, Lemma 4.3.4.1 implies that the restriction map

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{D}) \to \mathrm{Fun}(\Lambda^2_2, \mathscr{D})$$

has a section which maps into the full subcategory spanned by Cartesian squares. Composing the two sections, and restricting to the top-left vertex, we get a kernel map

$$\mathrm{Fun}(\Delta^1, \mathscr{D}) \to \mathscr{D}.$$

Definition 4.3.8. Let $S = \mathrm{Spec}\, k$ be an affine scheme, and $X \to S$ a smooth and proper morphism of schemes. Let $\mathscr{T}^{\mathrm{perf}}_{X/S}(n)$ denote the 1-stack of perfect totally supported sheaves on X of rank n at each maximal point of each geometric fiber of $X \to S$, as defined in [**Lie09**, §3.2]. The *stack of generalized* $\mathfrak{sl}_n$*-forms* is the essential image of the composite of morphisms in the ∞-topos St_k:

$$\mathscr{T}^{\mathrm{perf}}_{X/S}(n) \to \mathrm{QC}_k(X)^{\simeq} \xrightarrow{E \mapsto \underline{\mathrm{End}}(E)} \mathrm{Alg}(\mathrm{QC}_k(X)) \xrightarrow{A \mapsto \ker(A \to A \otimes_{A \otimes A^{\mathrm{rev}}} A)} \mathrm{LAlg}(\mathrm{QC}_k(X)).$$

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