Problems in Algebraic Vision

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Abstract

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This thesis studies several fundamental mathematical problems that arise from computer vision using techniques in algebraic geometry and optimization.

Chapters 2 and 3 consider the fundamental question of the existence of a fundamental (resp. essential) matrix given m point correspondences in two views. We present a complete answer for the existence of fundamental matrices for any value of m. We disprove the widely held beliefs that fundamental matrices always exist whenever $m \leq 7$. At the same time, we prove that they exist unconditionally when $m \leq 5$. Under a mild genericity condition, we show that an essential matrix always exists when $m \leq 4$. We also characterize the six and seven point configurations in two views for which all matrices satisfying the epipolar constraint have rank at most one. Chapter 3 discusses the existence of an essential matrix when $m = 6, 7$ via Chow forms.

We shift gears in Chapters 4 and 5 to study the problem of minimizing the Euclidean distance to a set which arises frequently in various applications, and in particular in computer vision. When the set is algebraic, a measure of complexity of this optimization problem is its number of critical points. We provide a general framework to compute and count the real smooth critical points of a data matrix on an orthogonally invariant set of matrices, for

instance, the set of essential matrices. The technique relies on “transfer principles” that allow calculations to be done in the space of singular values of the matrices in the orthogonally invariant set. As a result, the calculations often simplify greatly and yield transparent formulas. We illustrate the method on several examples, and compare our results to the recently introduced notion of Euclidean distance degree of an algebraic variety.

In contrast to Chapter 4, Chapter 5 considers the complex regular critical points of the Euclidean distance problem. We show that the Euclidean distance degree of an orthogonally invariant matrix variety equals the Euclidean distance degree of its restriction to diagonal matrices. We illustrate how this result can greatly simplify calculations in concrete circumstances.

Chapter 6 studies critical points for two-view triangulation. Two-view triangulation is a problem of minimizing a quadratic polynomial under an equality constraint. We derive a polynomial that encodes the local minimizers of this problem using the theory of Lagrange multipliers. This offers a simpler derivation of the critical points that are given in Hartley-Sturm [36].

Finally, Chapter 7 studies the connection between the existence of a projective reconstruction and the existence of a fundamental matrix satisfying the epipolar constraints.

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DEDICATION

to my family

Chapter 1

INTRODUCTION

This dissertation was motivated by the fundamental problem of three-dimensional (3D) reconstruction in computer vision. The problem of 3D reconstruction concerns recovering a 3D structure from a set of two-dimensional (2D) views. Its mathematical underpinning, called *multi-view geometry* [35], is based on linear algebra and projective geometry. Recently computational algebraic methods like Gröbner basis were brought into this area; see e.g. [47,68]. In this thesis we apply algebraic methods to study problems in computer vision. This general methodology has come to be called *algebraic vision*.

This thesis investigates two themes in algebraic vision, one concerns 3D reconstruction given two 2D views, the other one is about Euclidean distance minimization.

1.1 Existence of Epipolar matrices

Chapters 2 and 3 come from a paper coauthored with Sameer Agarwal, Bernd Sturmfels and Rekha Thomas, which is to appear in the *International Journal of Computer Vision* [2], and a previous version which can be found on the arXiv [1].

1.1.1 Background

We begin with the mathematical definition of a *(pinhole) camera*. Roughly speaking, a camera projects a point in the 3D world into a point in a 2D viewing plane. Denote by $\mathbb{P}^n_{\mathbb{R}}$ the n-dimensional projective space over the field $\mathbb{R}$ of real numbers. A camera is a real 3×4 matrix P such that its left 3×3 block is invertible. Its kernel is a point $c \in \mathbb{P}^3_{\mathbb{R}}$ up to scale, called the *camera center*. This matrix P maps a *world point* $w \in \mathbb{P}^3_{\mathbb{R}} \setminus \{c\}$ to its *image* $Pw \in \mathbb{P}^2_{\mathbb{R}}$; see Figure 1.1.

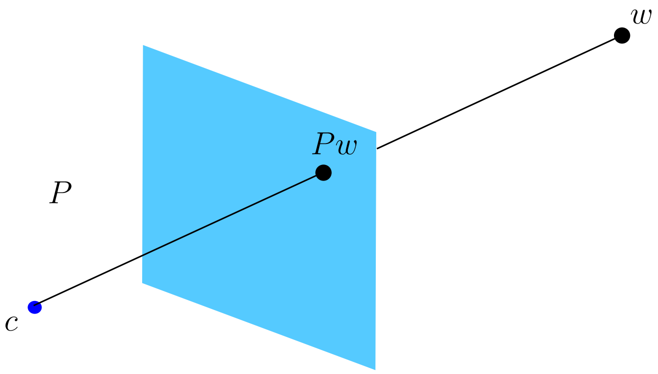


Figure 1.1: The image Pw is the intersection of the line joining the world point w and the camera center c, and the viewing plane.

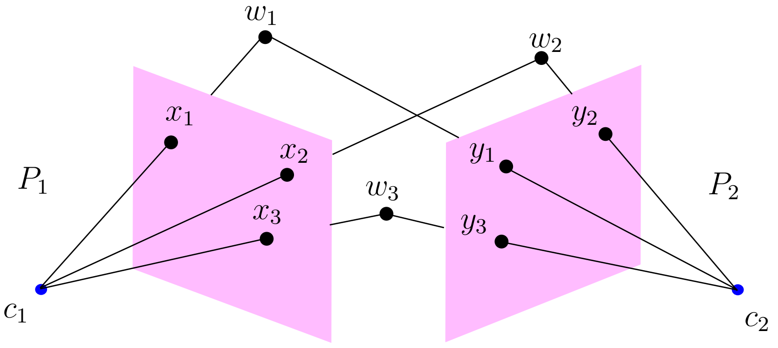


Figure 1.2: A two-view reconstruction for the given three point correspondences (x_i, y_i), $i = 1, 2, 3$.

A camera matrix $P = \begin{pmatrix} A & b \end{pmatrix}$ can be decomposed as $P = K \begin{pmatrix} R & t \end{pmatrix}$ where R is a 3×3 rotation matrix, $t \in \mathbb{R}^3$, and K is an upper triangular matrix whose diagonal entries are all positive. The matrix K stores the intrinsics of the camera, and it is called the *calibration matrix* of the camera. Any camera projection is indeed a rigid motion followed by a calibration. If K is known then P is said to be *calibrated*, otherwise P is *uncalibrated*.

The basic building block of 3D reconstruction is two-view reconstruction: recovering a 3D scene from two images. Two-view reconstruction can be stated as follows: Given m finite point correspondences $\{(x_i, y_i) \in \mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^2_{\mathbb{R}} \, : \, i = 1, \ldots, m\}$, construct two cameras P_1, P_2 and world points $w_1, \ldots, w_m$ such that $P_1 w_i = x_i$, $P_2 w_i = y_i$ for all $i = 1, \ldots, m$; see Figure 1.2. According to the projective reconstruction theorem [35, Theorem 10.1], if the cameras to be constructed are uncalibrated, then under mild assumptions, a two-view reconstruction can be acquired by finding a rank two 3×3 real matrix F, called the *fundamental matrix*, that solves a system of m linear equations, called the *epipolar constraints*:

$$y_i^\top F x_i = 0, \; i = 1, \ldots, m. \tag{1.1.1}$$

If the cameras P_1, P_2 are required to be calibrated by calibration matrices K_1, K_2 respectively, then instead of looking for a fundamental matrix F, one finds a 3×3 real matrix E of rank two whose first two singular values are equal, called an *essential matrix*, such that the following variant of the above epipolar constraints are satisfied:

$$y_i^\top K_2^{-\top} E K_1^{-1} x_i = 0, \; i = 1, \ldots, m. \tag{1.1.2}$$

Any fundamental matrix can recover a pair of uncalibrated cameras, while any essential matrix can recover a pair of rigid motions, up to change of coordinates. By an *epipolar matrix* we mean either a fundamental matrix or an essential matrix.

1.1.2 The Problem

Chapters 2 and 3 study the following problem: Given a set of m finite point correspondences $(x_i, y_i) \in \mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^2_{\mathbb{R}}$ for $i = 1, \ldots, m$, when does there exist a fundamental (resp. essential) matrix satisfying the epipolar constraints (1.1.1) (resp. (1.1.2))? This problem dates back to the 1800's, and is referred to as a Chasle's problem. Recall that the existence of an epipolar matrix for m point correspondences is almost equivalent to the existence of a two-view reconstruction.

1.1.3 Results

We give a complete answer to the existence question for fundamental matrices for all values of m and for essential matrices for $m \leq 4$. The problem of checking the existence of an epipolar matrix for an arbitrary value of m reduces to one where $m \leq 9$. The situations of $m = 8, 9$ are easy and thus the work needed is for $m \leq 7$.

- We have the following results for fundamental matrices:

    1. For $m \leq 5$ there always exists a fundamental matrix.
    2. For $m = 6, 7$ there may not exist a fundamental matrix, and we will provide an exact test for its existence. For $m \leq 7$, the popular statement of the so called *seven point algorithm* will have you believe that there always exists a fundamental matrix. We will show that this is not true. The problem is, that the matrix returned by the seven point algorithm is only guaranteed to be rank deficient, it is not guaranteed to have rank two.

  A fundamental matrix can fail to exist in several ways. An important such case is when all matrices that run for competition have rank at most one. We fully characterize this phenomenon directly in terms of the geometry of the input point correspondences. The key technical task in all this is to establish conditions for the existence of a real point in the intersection of a subspace and a fixed set of 3×3 matrices.

- For esseential matrices we prove the following:

    1. For $m \leq 3$ there always exists an essential matrix.

2. For $m = 4$ there exists an essential matrix, under a mild genericity assumption.
3. For $m = 7$ there is an exact test for checking for the existence of an essential matrix.

Notice that the cases $m = 5, 6$ are still open. When $m = 6$ we complexify the problem and understand it using Chow forms.

1.2 Euclidean distance to orthogonally invariant matrix sets

Chapters 4 and 5 originate from two papers. The first one is coauthored with Dmitriy Drusvyatskiy and Rekha Thomas, which appeared in the *SIAM Journal of Matrix Analysis and Applications* [26]. The second one is coauthored with the authors just mentioned, together with Giorgio Ottaviani, and is to appear in *Israel Journal of Mathematics* [25].

1.2.1 Background

We consider the fundamental problem of finding a point in a set $\mathcal{V}$ closest to a given point, with respect to Euclidean distance. The points in $\mathcal{V}$ that satisfy the first order optimality condition are called *critical points*. Chapters 4 and 5 provide transparent formulae for the critical points and the number of critical points, provided that the given set sits inside a matrix space and has a property called orthogonal invariance.

Let $\mathbb{R}^{n\times t}$ be the space of real $n \times t$ matrices, and assume that $n \leq t$. A subset $\mathcal{M}$ of $\mathbb{R}^{n\times t}$ is *orthogonally invariant* if $UXV^\top \in \mathcal{M}$ for any $X \in \mathcal{M}$ and orthogonal matrices U, V of correct sizes. For example, the set of fundamental matrices and the set of essential matrices are orthogonally invariant. Because of singular value decomposition (SVD), an orthogonally invariant set is uniquely determined by its diagonal restriction $S := \{x \in \mathbb{R}^n \,:\, \mathrm{Diag}(x) \in \mathcal{M}\}$, where $\mathrm{Diag}\,(x)$ denotes the $n \times t$ diagonal matrix whose diagonal is x. In many occasions working on the set S is easier than $\mathcal{M}$.

Assume $\mathcal{M}$ is orthogonally invariant. Consider the following Euclidean distance (ED) minimization problem given a *data point* $Y \in \mathcal{M}$:

$$\min_{X \in \mathcal{M}} \|Y - X\|^2. \tag{1.2.1}$$

We call $X \in \mathcal{M}$ a *smooth critical point* of the problem (1.2.1) if X satisfies the first order optimality condition of (1.2.1), that is, if $\mathcal{M}$ is smooth around X and $Y - X$ is orthogonal to the tangent space of $\mathcal{M}$ at X. If $\mathcal{M}$ is a variety (that is, the zero set of finitely many polynomials) in $\mathbb{R}^{n\times t}$, then we let $\mathcal{M}_{\mathbb{C}}$ be the smallest variety in $\mathbb{C}^{n\times t}$ that contains $\mathcal{M}$, and call $X \in \mathcal{M}_{\mathbb{C}}$ a *regular critical point* of (1.2.1) if $\mathcal{M}_{\mathbb{C}}$ is regular at X, and $Y - X$ is orthogonal to the tangent space of $\mathcal{M}_{\mathbb{C}}$ at X. If Y is general then the number of regular critical points of (1.2.1) is a constant, called the *ED degree* of $\mathcal{M}$ [22].

1.2.2 Results

We suggest efficient methods to compute and count the smooth critical points and regular critical points of the problem 1.2.1. A *transfer principle* describes how a property about an orthogonally invariant matrix set $\mathcal{M}$ is inherited from that of its diagonal restriction S. Our main results are the following four new transfer principles.

1. For a general data point $Y \in \mathbb{R}^{n\times t}$, let $\sigma(Y)$ be the vector of singular values of Y, in nonincreasing order. Then the smooth critical points of (1.2.1) have the form $U\mathrm{Diag}\,(x)V^\top$, where x is a smooth critical point of

$$\min_{x \in S} \|\sigma(Y) - x\|^2, \tag{1.2.2}$$

   assuming Y has SVD, $Y = U\mathrm{Diag}\,(\sigma(Y))V^\top$.

2. $\mathcal{M}$ is a variety if and only if S is a variety.

3. Assume $\mathcal{M}$ is a variety. For a general data point $Y \in \mathbb{R}^{n\times t}$, the regular critical points of (1.2.1) have the form $U\mathrm{Diag}\,(x)V^\top$, where x is a regular critical point of (1.2.2) assuming Y has SVD, $Y = U\mathrm{Diag}\,(\sigma(Y))V^\top$.

4. If $\mathcal{M}$ is a variety, then the ED degree of $\mathcal{M}$ equals the ED degree of S.

1.2.3 Impact

First, our results generalize several classical results in linear algebra and computer vision, including the Eckart-Young theorem, closest orthogonal matrix, closest essential matrix etc. Second, computing the ED degree of a complicated orthogonally invariant matrix variety using Gröbner basis [22] can be expensive. Our results can reduce this task to computing the ED degree of a simpler variety. This can significantly save on computations.

1.3 Critical points for two-view triangulation

Chapter 6 studies the two-view triangulation problem: given two pinhole cameras P_1, P_2 and $u_1, u_2 \in \mathbb{R}^2$, solve the minimization problem

$$\min_{W \in \mathbb{R}^3} \sum_{j=1}^{2} \left\| \frac{(P_j\widehat{W})_1}{(P_j\widehat{W})_3} - u_{j1} \right\|^2 + \left\| \frac{(P_j\widehat{W})_2}{(P_j\widehat{W})_3} - u_{j2} \right\|^2 .$$

(Note: for any $x \in \mathbb{R}^n$, the point $\widehat{x}$ denotes $(x^\top \ 1)^\top \in \mathbb{P}^n_{\mathbb{R}}$). It can be reformulated as a constrained least squares problem:

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}^2} \quad & \|x_1 - u_1\|^2 + \|x_2 - u_2\|^2 \\ \text{subject to} \quad & \widehat{x}_2^\top F \widehat{x}_1 = 0 \end{aligned}$$

where F is the fundamental matrix associated to the two given cameras. The local minimizers can be computed via solving the KKT equations. It is known that these local minimizers can be found by solving a degree six univariate polynomial [36]. In this chapter we provide a simple algebraic derivation of this univariate polynomial.

1.4 Existence of a Projective Reconstruction

Chapter 7 gives a proof of the following fundamental result using linear algebra: There exists a two-view reconstruction for $\{(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ i = 1, \ldots, m\}$ if and only if there exists a fundamental matrix F satisfying the epipolar constraints and (x_i, y_i) satisfy a regularity condition.

Chapter 2

ON THE EXISTENCE OF EPIPOLAR MATRICES

2.1 Introduction

A set of point correspondences $\{(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2,\ i = 1, \ldots, m\}$ are the images of m points in $\mathbb{R}^3$ in two uncalibrated (resp. calibrated) cameras only if there exists a fundamental matrix F (resp. essential matrix E) such that the (x_i, y_i) satisfy the *epipolar constraints* [35, Chapter 9]. Under mild genericity conditions on the point correspondences, the existence of these matrices is also sufficient for the correspondences (x_i, y_i) to be the images of a 3D scene [27, 38, 39, 54]; see also Chapter 7. This brings us to the following basic question in multi-view geometry:

Question 2.1.1. *Given a set of m point correspondences $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$, when does there exist a fundamental (essential) matrix relating them via the epipolar constraints?*

The answer to this question is known in several special cases [10, 37], but even in the minimally constrained and under-constrained cases ($m \leq 7$ for fundamental matrices and $m \leq 5$ for essential matrices) our knowledge is incomplete.

For instance, in the uncalibrated case, for $m \leq 7$, the popular statement of the so called *seven point algorithm* will have you believe that there always exists a fundamental matrix [40, 70]. We will show that this is not true. The problem is, that the matrix returned by the seven point algorithm is only guaranteed to be rank deficient, it is not guaranteed to have rank two.

In the calibrated case, when $m = 5$, there exist up to 10 distinct complex essential matrices [19], but it is not known when we can be sure that one of them is real. Similarly, it is unknown whether there always exists a real essential matrix for $m \leq 4$ point correspondences.

The common mistake in many of the usual existence arguments is the reliance on dimension counting. This unfortunately works only on algebraically closed fields, which the field of real numbers is not.

In this chapter we give a complete answer to the existence question for fundamental matrices for all m and for essential matrices for $m \leq 4$. The problem of checking the existence of a fundamental (resp. essential) matrix for an arbitrary value of m reduces to one where $m \leq 9$. The situations of $m = 8, 9$ are easy and thus the work needed is for $m \leq 7$. We prove the following results:

(1) For $m \leq 5$ there always exists a fundamental matrix.

(2) For $m = 6, 7$ there may not exist a fundamental matrix, and we will provide an exact test for checking for its existence.

(3) For $m \leq 4$ there always exists an essential matrix.

It is relatively easy to prove the existence of a fundamental matrix when $m \leq 4$. We give a much more sophisticated proof to extend this result to $m \leq 5$ in (1). Similarly, it is elementary to see that there is always an essential matrix when $m \leq 3$. The proof of (3) is much more complicated.

A fundamental matrix can fail to exist in several ways. An important such case is when all matrices that run for competition have rank at most one. We fully characterize this phenomenon directly in terms of the geometry of the input point correspondences.

The key technical task in all this is to establish conditions for the existence of a real point in the intersection of a subspace and a fixed set of 3×3 matrices.

In the remainder of this section we establish our notation and some basic facts about cameras, epipolar matrices, projective varieties and linear algebra. Section 2.2 considers the existence problem for the fundamental matrix and Section 2.3 does so for the essential matrix. We conclude in Section 2.4 with a discussion of the results and directions for future work.

2.1.1 Notation

Capital roman letters (say E, F, X, Y, Z) denote matrices. For a matrix F, the corresponding lower case letter f denotes the vector obtained by concatenating the rows of F. Upper case calligraphic letters denote sets of matrices (say $\mathcal{E}, \mathcal{F}$).

For a field $\mathbb{F}$ such as $\mathbb{R}$ or $\mathbb{C}$, the projective space $\mathbb{P}^n_{\mathbb{F}}$ is $\mathbb{F}^{n+1} \setminus \{0\}$ in which we identify u and v if $u = \lambda v$ for some $\lambda \in \mathbb{F} \setminus \{0\}$. For example $(1, 2, 3)$ and $(4, 8, 12)$ are the same point in $\mathbb{P}^2_{\mathbb{R}}$, denoted as $(1, 2, 3) \sim (4, 8, 12)$. The set of $m \times n$ matrices with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{m\times n}$, and by $\mathbb{P}^{m\times n}_{\mathbb{F}}$ if the matrices are only up to scale. For $v \in \mathbb{R}^3$,

$$[v]_\times := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

is a skew-symmetric matrix whose rank is two unless $v = 0$. Also, $[v]_\times w = v \times w$, where $\times$ denotes the vector cross product. For $A \in \mathbb{F}^{m\times n}$, we have $\ker_{\mathbb{F}}(A) = \{u \in \mathbb{F}^n : Au = 0\}$, and $\operatorname{rank}(A) = n - \dim(\ker_{\mathbb{F}}(A))$. We use $\det(A)$ to denote the determinant of A. Points x_i and y_i in $\mathbb{F}^2$ will be identified with their homogenizations $(x_{i1}, x_{i2}, 1)^\top$ and $(y_{i1}, y_{i2}, 1)^\top$ in $\mathbb{P}^2_{\mathbb{F}}$. Also, $y_i^\top \otimes x_i^\top := \begin{pmatrix} y_{i1}x_{i1} & y_{i1}x_{i2} & y_{i1} & y_{i2}x_{i1} & y_{i2}x_{i2} & y_{i2} & x_{i1} & x_{i2} & 1 \end{pmatrix} \in \mathbb{F}^{1\times 9}$.

If P and Q are finite dimensional subspaces, then $P \otimes Q$ is the span of the pairwise Kronecker products of the basis elements of P and Q.

2.1.2 Linear algebra

Below we list five facts from linear algebra that will be helpful in this chapter.

Lemma 2.1.2. [65, pp. 399] *If $x_0, \ldots, x_{n+1}$ and $y_0, \ldots, y_{n+1}$ are two sets of $n + 2$ points in $\mathbb{R}^{n+1}$ such that no $n+1$ points in either set are linearly dependent. Then there is an invertible matrix $H \in \mathbb{R}^{(n+1)\times(n+1)}$ such that*

$$Hx_i \sim y_i \quad \textit{for any } i = 0, 1, \ldots, n + 1.$$

Lemma 2.1.3. [59, Theorem 3] *Suppose V is a linear subspace of $\mathbb{R}^{n\times n}$ of dimension rn, such that for any $A \in V$, $\operatorname{rank}(A) \leq r$. Then either $V = W \otimes \mathbb{R}^n$ or $V = \mathbb{R}^n \otimes W$, for some r-dimensional subspace $W \subseteq \mathbb{R}^n$.*

Lemma 2.1.4. [29, Theorem 1] *Suppose V is a linear subspace of $\mathbb{R}^{m\times n}$ and r is the maximum rank of an element of V. Then $\dim(V) \leq r \cdot \max\{m, n\}$.*

Lemma 2.1.5 (Matrix Determinant Lemma)**.** [43, Theorem 18.1.1] *If $A \in \mathbb{R}^{n\times n}$ is invertible and $u, v \in \mathbb{R}^n$, then $\det(A + uv^\top) = (1 + v^\top A^{-1} u) \det(A)$.*

In the following lemma we identify points in $\mathbb{R}^2$ with their homogenizations in $\mathbb{P}^2_{\mathbb{R}}$ as mentioned earlier. The proof of the lemma is in Section 2.5.1.

Lemma 2.1.6. *Given two lines l, m in $\mathbb{R}^2$, and $x_0 \in l$, $y_0 \in m$, there is an invertible matrix $H \in \mathbb{R}^{3\times 3}$ such that*

(1) $Hx_0 = y_0$; and

(2) for any $x \in l$, $Hx \in m$.

2.1.3 Projective varieties

We recall some basic notions from algebraic geometry [13, 33, 66]. Let $\mathbb{F}[u] = \mathbb{F}[u_1, \ldots, u_n]$ denote the ring of all polynomials with coefficients in the field $\mathbb{F}$.

Definition 2.1.7 (Homogeneous Polynomial)**.** *A polynomial in $\mathbb{F}[u]$ is homogeneous (or called a* form*) if all its monomials have the same total degree.*

For example, $u_1^2 u_2 + u_1 u_2^2$ is a form of degree three but $u_1^3 + u_2$ is not a form.

Definition 2.1.8 (Projective Variety and Subvariety)**.** *A subset $\mathcal{V} \subseteq \mathbb{P}^n_{\mathbb{F}}$ is a (projective) variety if there are homogeneous polynomials $h_1, \ldots, h_t \in \mathbb{F}[u]$ such that $\mathcal{V} = \{u \in \mathbb{P}^n_{\mathbb{F}} : h_1(u) = \ldots = h_t(u) = 0\}$. A variety $\mathcal{V}_1$ is a subvariety of $\mathcal{V}$ if $\mathcal{V}_1 \subseteq \mathcal{V}$.*

Given homogeneous polynomials $h_1, \ldots, h_t \in \mathbb{R}[u]$, let $\mathcal{V}_\mathbb{C} := \{u \in \mathbb{P}^n_\mathbb{C} : h_i(u) = 0 \text{ for } i = 1, \ldots, t\}$ be their projective variety over the complex numbers, and $\mathcal{V}_\mathbb{R} := \mathcal{V}_\mathbb{C} \cap \mathbb{P}^n_\mathbb{R}$ be the set of real points in $\mathcal{V}_\mathbb{C}$.

Definition 2.1.9 (Irreducibility)**.** *A projective variety $\mathcal{V} \subseteq \mathbb{P}^n_\mathbb{F}$ is irreducible if it is not the union of two nonempty proper subvarieties of $\mathbb{P}^n_\mathbb{F}$.*

We define the dimension of a projective variety over $\mathbb{C}$ in a form that is particularly suitable to this chapter.

Definition 2.1.10 (Dimension)**.** [66, Corollary 1.6] *The dimension* $\dim(\mathcal{V})$ *of a projective variety $\mathcal{V} \subseteq \mathbb{P}^n_\mathbb{C}$ is d where $n - d - 1$ is the maximum dimension of a linear subspace of $\mathbb{P}^n_\mathbb{C}$ disjoint from $\mathcal{V}$.*

As a special case, if $\mathcal{L}$ is a l-dimensional linear subspace in $\mathbb{C}^{n+1}$, it can be viewed as an irreducible projective variety in $\mathbb{P}^n_\mathbb{C}$ of dimension $l - 1$.

The following result shows how dimension counting can be used to infer facts about the intersection of a variety and a linear subspace in $\mathbb{P}^n_\mathbb{C}$. It is a consequence of the more general statement in [66, Theorem 1.24]. This result does not extend to varieties over $\mathbb{R}$.

Theorem 2.1.11. *Consider an irreducible projective variety $\mathcal{V}_\mathbb{C} \subseteq \mathbb{P}^n_\mathbb{C}$ of dimension d and a linear subspace $\mathcal{L} \subseteq \mathbb{P}^n_\mathbb{C}$ of dimension l. If $d + l = n$ then $\mathcal{V}$ must intersect $\mathcal{L}$. If $d + l > n$ then $\mathcal{V}$ intersects $\mathcal{L}$ at infinitely many points.*

Observe that the above theorem only applies over the complex numbers. As a simple illustration the curve $x^2 - y^2 + z^2 = 0$ in $\mathbb{P}^2_\mathbb{C}$ is guaranteed to intersect the subspace $y = 0$ in two complex points since they have complementary dimensions in $\mathbb{P}^2_\mathbb{C}$. However, neither of these intersection points is real.

If $\mathcal{V} \subseteq \mathbb{P}^n_\mathbb{C}$ is a projective variety, then it intersects any linear subspace of dimension $n - \dim(\mathcal{V})$ in $\mathbb{P}^n_\mathbb{C}$. If the subspace is general, then the cardinality of this intersection is a constant which is an important invariant of the variety.

Definition 2.1.12 (Degree)**.** [33, Definition 18.1] *The degree of a projective variety* $\mathcal{V} \subseteq \mathbb{P}^n_{\mathbb{C}}$*, denoted by degree*$(\mathcal{V})$*, is the number of intersection points with a general linear subspace of dimension* $n - \dim(\mathcal{V})$ *in* $\mathbb{P}^n_{\mathbb{C}}$ *.*

2.1.4 Camera Matrices

A general projective camera can be modeled by a matrix $P \in \mathbb{P}^{3\times 4}_{\mathbb{R}}$ with $\text{rank}(P) = 3$. Partitioning a camera as $P = \begin{pmatrix} A & b \end{pmatrix}$ where $A \in \mathbb{R}^{3\times 3}$, we say that P is a *finite camera* if A is nonsingular. In this chapter we restrict ourselves to finite cameras.

A finite camera P can be written as $P = K \begin{pmatrix} R & t \end{pmatrix}$, where $t \in \mathbb{R}^3$, K is an upper triangular matrix with positive diagonal entries known as the *calibration matrix*, and $R \in \text{SO}(3)$ is a rotation matrix that represents the orientation of the camera coordinate frame. If the calibration matrix K is known, then the camera is said to be *calibrated*, and otherwise the camera is *uncalibrated*. The *normalization* of a calibrated camera $P = K \begin{pmatrix} R & t \end{pmatrix}$ is the camera $K^{-1}P = \begin{pmatrix} R & t \end{pmatrix}$.

By dehomogenizing (i.e. scaling the last coordinate to be 1), we can view the image $x = Pw$ as a point in $\mathbb{R}^2$. If x is the image of w in the calibrated camera P, then $K^{-1}x$ is called the *normalized image* of w, or equivalently, it is the image of w in the normalized camera $K^{-1}P$. This allows us to remove the effect of the calibration K by passing to the normalized camera $K^{-1}P$ and normalized images $\tilde{x} := K^{-1}x$.

2.1.5 Epipolar Matrices

In this chapter we use the name *epipolar matrix* to refer to either a fundamental matrix or essential matrix derived from the *epipolar geometry* of a pair of cameras. These matrices are explained and studied in [35, Chapter 9].

An *essential matrix* is any matrix in $\mathbb{P}^{3\times 3}_{\mathbb{R}}$ of the form $E = SR$ where S is a skew-symmetric matrix and $R \in \text{SO}(3)$. Essential matrices are characterized by the property that they have rank two (and hence one zero singular value) and two equal non-zero singular values. An essential matrix depends on six parameters, three each from S and R, but since it is only defined up to scale, it has five degrees of freedom.

The essential matrix of the two normalized cameras $\begin{pmatrix} I & 0 \end{pmatrix}$ and $\begin{pmatrix} R & t \end{pmatrix}$ is $E = [t]_\times R$. For every pair of normalized images $\tilde{x}$ and $\tilde{y}$ in these cameras of a point $w \in \mathbb{P}^3_\mathbb{R}$, the triple $(\tilde{x}, \tilde{y}, E)$ satisfies the *epipolar constraint*

$$\tilde{y}^\top E \tilde{x} \;=\; 0. \tag{2.1.1}$$

Further, any $E = SR$ is the essential matrix of a pair of cameras as shown in [35, Section 9.6.2].

If the calibrations K_1 and K_2 of the two cameras were unknown, then for a pair of corresponding images (x, y) in the two cameras, the epipolar constraint becomes

$$0 \;=\; \tilde{y}^\top E \tilde{x} \;=\; y^\top K_2^{-\top} E K_1^{-1} x. \tag{2.1.2}$$

The matrix $F := K_2^{-\top} E K_1^{-1}$ is the *fundamental matrix* of the two uncalibrated cameras. This is a rank two matrix but its two non-zero singular values are no longer equal. Conversely, any real 3×3 matrix of rank two is the fundamental matrix of a pair of cameras [35, Section 9.2]. A fundamental matrix has seven degrees of freedom since it satisfies the rank two condition and is only defined up to scale. The set of fundamental matrices can be parametrized as $F = [b]_\times H$, where b is a non-zero vector and H is an invertible matrix 3×3 matrix [35, Section 9.6.2].

2.1.6 X, Y and Z

Suppose we are given m point correspondences (normalized or not) $\{(x_i, y_i), i = 1, \ldots, m\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$. We homogenize this data and represent it by three matrices with m rows:

$$X = \begin{pmatrix} x_1^\top \\ \vdots \\ x_m^\top \end{pmatrix} \in \mathbb{R}^{m\times 3}, \tag{2.1.3}$$

$$Y = \begin{pmatrix} y_1^\top \\ \vdots \\ y_m^\top \end{pmatrix} \in \mathbb{R}^{m\times 3}, \text{ and} \tag{2.1.4}$$

$$Z = \begin{pmatrix} y_1^\top \otimes x_1^\top \\ \vdots \\ y_m^\top \otimes x_m^\top \end{pmatrix} \in \mathbb{R}^{m\times 9}. \tag{2.1.5}$$

The ranks of X and Y are related to the geometry of the point sets $\{x_i\}$ and $\{y_i\}$. This is made precise by the following lemma which is stated in terms of X but obviously also applies to Y.

Lemma 2.1.13.

$$\operatorname{rank}(X) = \begin{cases} 1 & \textit{If } x_i\textit{'s, as points in } \mathbb{R}^2, \textit{ are all equal.} \\ 2 & \textit{If all the } x_i\textit{'s are collinear in } \mathbb{R}^2 \\ & \textit{but not all equal.} \\ 3 & \textit{If the } x_i\textit{'s are noncollinear in } \mathbb{R}^2. \end{cases}$$

Notice that every row of X (resp. Y) can be written as a linear combination of $\operatorname{rank}(X)$ (resp. $\operatorname{rank}(Y)$) rows of it. From this and the bilinearity of Kronecker product, it is evident that:

Lemma 2.1.14. *For any m,*

$$\operatorname{rank}(Z) \le \operatorname{rank}(X)\operatorname{rank}(Y) \le 9.$$

In particular, if all points x_i are collinear in $\mathbb{R}^2$ then $\operatorname{rank}(Z) \leq 6$. If all points x_i are equal in $\mathbb{R}^2$ then $\operatorname{rank}(Z) \leq 3$.

We study Question 2.1.1 via the the subspace $\ker_{\mathbb{R}}(Z)$. Observe that for all m, $\ker_{\mathbb{R}}(Z) = \ker_{\mathbb{R}}(Z')$ for a submatrix Z' of Z consisting of $\operatorname{rank}(Z)$ linearly independent rows. Therefore, we can replace Z with Z' in order to study $\ker_{\mathbb{R}}(Z)$ which allows us to restrict our investigations to the values of m such that

$$\boxed{1 \leq m = \operatorname{rank}(Z) \leq 9. \qquad (2.1.6)}$$

In light of the above discussion, it is useful to keep in mind that even though all our results are stated in terms of $m \leq 9$, we are in fact covering all values of m.

2.2 Fundamental matrices

Following Section 2.1.5, a fundamental matrix is any matrix in $\mathbb{P}_{\mathbb{R}}^{3\times 3}$ of rank two [35, Section 9.2.4]. In our notation, we denote the set of fundamental matrices as

$$\mathcal{F} \;:=\; \{f \in \mathbb{P}_{\mathbb{R}}^{8} \,:\, \operatorname{rank}(F) = 2\}, \tag{2.2.1}$$

where the vector f is the concatenation of the rows of the matrix F. This notation allows us to write the epipolar constraints (2.1.2) as

$$Zf = 0. \tag{2.2.2}$$

Hence a fundamental matrix F exists for the m given point correspondences if and only if the linear subspace $\ker_{\mathbb{R}}(Z)$ intersects the set $\mathcal{F}$, i.e.,

$$\ker_{\mathbb{R}}(Z) \cap \mathcal{F} \neq \emptyset. \tag{2.2.3}$$

This geometric reformulation of the existence question for F is well-known in multi-view geometry [35, 58].

We now introduce two complex varieties that are closely related to $\mathcal{F}$.

Let $\mathcal{R}_1 := \{a \in \mathbb{P}^8_{\mathbb{C}} : \mathrm{rank}(A) \leq 1\}$ be the set of matrices in $\mathbb{P}^{3\times 3}_{\mathbb{C}}$ of rank one. It is an irreducible variety with $\dim(\mathcal{R}_1) = 4$ and $\mathrm{degree}(\mathcal{R}_1) = 6$.

Let $\mathcal{R}_2 := \{a \in \mathbb{P}^8_{\mathbb{C}} : \mathrm{rank}(A) \leq 2\}$ be the set of matrices in $\mathbb{P}^{3\times 3}_{\mathbb{C}}$ of rank at most two. It is an irreducible variety with $\dim(\mathcal{R}_2) = 7$ and $\mathrm{degree}(\mathcal{R}_2) = 3$. Observe that

$$\mathcal{R}_2 = \{a \in \mathbb{P}^8_{\mathbb{C}} : \det(A) = 0\}. \tag{2.2.4}$$

The set of fundamental matrices can now be written as $\mathcal{F} = (\mathcal{R}_2 \backslash \mathcal{R}_1) \cap \mathbb{P}^8_{\mathbb{R}}$ which is not a variety over $\mathbb{R}$.

In this section we will give a complete answer to the question of existence of fundamental matrices for any number m of point correspondences. Recall from Section 2.1.6 (2.1.6) that assuming $m = \mathrm{rank}(Z)$, we only need to consider the cases $1 \leq m \leq 9$.

2.2.1 Case: $m = 9$

If $m = 9$, then $\ker_{\mathbb{R}}(Z) \subseteq \mathbb{P}^8_{\mathbb{R}}$ is empty, and Z has no fundamental matrix.

2.2.2 Case: $m = 8$

If $m = 8$, then $\ker_{\mathbb{R}}(Z) \subseteq \mathbb{P}^8_{\mathbb{R}}$ is a point $a \in \mathbb{P}^8_{\mathbb{R}}$ corresponding to the matrix $A \in \mathbb{P}^{3\times 3}_{\mathbb{R}}$. It is possible for A to have rank one, two or three. Clearly, Z has a fundamental matrix if and only if A has rank two.

2.2.3 Case: $m = 7$

The majority of the literature in computer vision deals with the case of $m = 7$ which falls under the category of “minimal problems”; see for example [68, Chapter 3]. The name refers to the fact that $m = 7$ is the smallest value of m for which $\ker_{\mathbb{C}}(Z) \cap \mathcal{R}_2$ is finite, making the problem of estimating F well-posed (at least over $\mathbb{C}$).

Indeed, when $m = 7$, $\ker_{\mathbb{C}}(Z)$ is a one-dimensional subspace of $\mathbb{P}^8_{\mathbb{C}}$ and hence by Theorem 2.1.11, generically it will intersect $\mathcal{R}_2$ in three points, of which at least one is real since

$\det(A)$ is a degree three polynomial. Therefore, there is always a matrix of rank at most two in $\ker_{\mathbb{R}}(Z)$. This leads to the common belief that when $m = 7$, there is always a fundamental matrix for Z.

We first show an example for which $\ker_{\mathbb{R}}(Z)$ contains only matrices of ranks either one or three.

Example 2.2.1. Consider

$$X = \begin{pmatrix} \frac{1}{5} & -1 & 1 \\ -1 & -7 & 1 \\ \frac{-1}{2} & 0 & 1 \\ -2 & -12 & 1 \\ \frac{-57}{4} & 8 & 1 \\ 2 & 8 & 1 \\ 0 & \frac{-1}{9} & 1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 5 & 1 \\ 3 & \frac{-5}{12} & 1 \\ 4 & 7 & 1 \\ 5 & \frac{-11}{8} & 1 \\ 6 & 9 & 1 \end{pmatrix}.$$

Here, $\operatorname{rank}(Z) = 7$ and $\ker_{\mathbb{R}}(Z)$ is spanned by the rank three matrices

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 5 & 4 & -2 \\ -15 & 3 & 11 \end{pmatrix}.$$

For any $u_1, u_2 \in \mathbb{R}$, one obtains

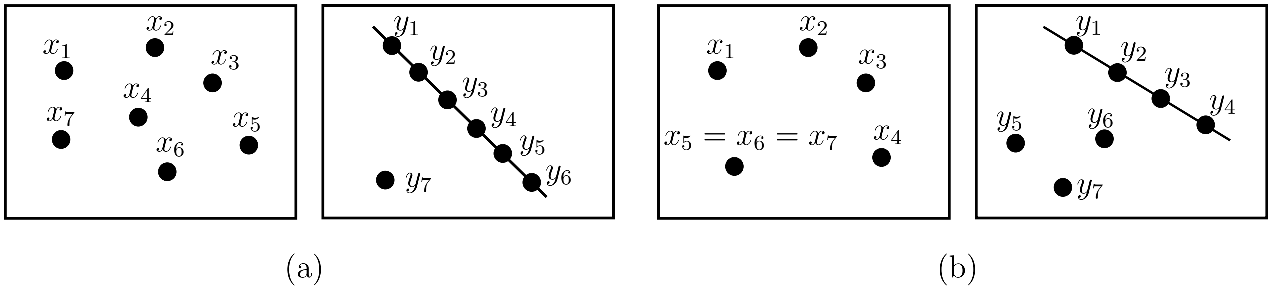
$$\det(Iu_1 + A_2u_2) = (u_1 + 5u_2)^3.$$

If $\det(Iu_1 + A_2u_2) = 0$, then $u_1 = -5u_2$ and

$$Iu_1 + A_2u_2 = u_2(A_2 - 5I) = u_2 \begin{pmatrix} -5 & 1 & 2 \\ 5 & -1 & -2 \\ -15 & 3 & 6 \end{pmatrix}$$

which has rank at most one. Thus for $(u_1, u_2) \neq (0, 0)$,

$$\operatorname{rank}(Iu_1 + A_2u_2) = \begin{cases} 1 & \text{if } u_1 + 5u_2 = 0 \\ 3 & \text{if } u_1 + 5u_2 \neq 0. \end{cases}$$



(a) (b)

Figure 2.1: Two examples where the conditions for Theorem 2.2.2 are satisfied and there does not exist a fundamental matrix for $m = 7$ because $\ker_{\mathbb{R}}(Z) \subseteq \mathcal{R}_1$.

Hence $\ker_{\mathbb{R}}(Z)$ consists of matrices of rank either one or three, and Z does not have a fundamental matrix.

Another way for Z to not have a fundamental matrix is if $\ker_{\mathbb{R}}(Z)$ is entirely in $\mathcal{R}_1$. The following theorem whose proof can be found in Section 2.5.3, characterizes this situation. See Figure 2.1 for illustrations.

Theorem 2.2.2. *If* $m = 7$, $\ker_{\mathbb{R}}(Z) \subseteq \mathcal{R}_1$ *if and only if one of the following holds:*

(1) There is a nonempty proper subset $\tau \subset \{1, \ldots, 7\}$ *such that as points in* $\mathbb{R}^2$, $\{y_i : i \in \tau\}$ *are collinear and* $x_i = x_j$ *for all* $i, j \notin \tau$.

(2) There is a nonempty proper subset $\tau \subset \{1, \ldots, 7\}$ *such that as points in* $\mathbb{R}^2$, $\{x_i : i \in \tau\}$ *are collinear and* $y_i = y_j$ *for all* $i, j \notin \tau$.

2.2.4 Case: $m = 6$

When $m \leq 6$, by Theorem 2.1.11, $\ker_{\mathbb{C}}(Z) \cap \mathcal{R}_2$ is infinite, and here the conventional wisdom is that there are infinitely many fundamental matrices for Z and thus these cases deserve no study.

Indeed, it is true that for six points in two views in general position, there exists a fundamental matrix relating them. To prove this, we first note the following fact which is a generalization of a result of Chum et al. [10]. Its proof can be found in Section 2.5.2.

Lemma 2.2.3. *Assume there is a real* 3×3 *invertible matrix* H *such that for at least* $m-2$ *of the indices* $i \in \{1, \ldots, m\}$, $y_i \sim Hx_i$. *Then* Z *has a fundamental matrix.*

An immediate consequence of Lemma 2.2.3 with $m = 6$ and Lemma 2.1.2 with $n = 2$ is the following.

Theorem 2.2.4. *If* $m = 6$ *and* τ *is a subset of* $1, \ldots, 6$ *with four elements such that no set of three points in either* $\{x_i \ : \ i \in \tau\}$ *or* $\{y_i \ : \ i \in \tau\}$ *is collinear in* $\mathbb{R}^2$, *then* Z *has a fundamental matrix.*

Note that in [37, Theorem 1.2], Hartley shows that a fundamental matrix associated with six point correspondences is uniquely determined under certain geometric assumptions on the point correspondences and world points. One of Hartley's assumptions is the as that in Theorem 2.2.4.

Theorem 2.2.4 hints at the possibility that collinearity of points in any one of the two views may prevent a fundamental matrix from existing. The following theorem, whose proof can be found in Section 2.5.4, characterizes the conditions under which $\ker_{\mathbb{R}}(Z) \subseteq \mathcal{R}_1$ when $m = 6$. No fundamental matrix can exist in this case.

Theorem 2.2.5. *If* $m = 6$, $\ker_{\mathbb{R}}(Z) \subseteq \mathcal{R}_1$ *if and only if either all points* x_i *are collinear in* $\mathbb{R}^2$ *or all points* y_i *are collinear in* $\mathbb{R}^2$.

We remark that when $m = 6$, it is impossible that as points in $\mathbb{R}^2$, all x_i are collinear and all y_i are collinear. If this were the case, then by Lemmas 2.1.13 and 2.1.14, $\operatorname{rank}(Z) \leq 4 < 6 = m$ which violates our assumption (2.1.6).

2.2.5 Existence of fundamental matrices in general

In the previous two sections, we have demonstrated that dimension counting is not enough to argue for the existence of a fundamental matrix for $m = 6$ and $m = 7$. We have also described particular configurations in two views which guarantee the existence and non-existence of a fundamental matrix. We are now ready to tackle the general existence question for fundamental matrices for $m \leq 8$. To do this, we first need the following key structural lemma. It provides a sufficient condition for $\ker_{\mathbb{R}}(Z)$ to have a matrix of rank two.

Lemma 2.2.6. *Let $\mathcal{L}$ be a positive dimensional subspace in $\mathbb{P}_{\mathbb{R}}^{3\times 3}$ that contains a matrix of rank three. If the determinant restricted to $\mathcal{L}$ is not a power of a linear form, then $\mathcal{L}$ contains a real matrix of rank two.*

The proof of this lemma can be found in Section 2.5.5, but we elaborate on its statement. If $\{A_1, \ldots, A_t\}$ is a basis of a subspace $\mathcal{L}$ in $\mathbb{P}_{\mathbb{R}}^{3\times 3}$, then any matrix in $\mathcal{L}$ is of the form $A = u_1A_1 + \cdots + u_tA_t$ for scalars $u_1, \ldots, u_t \in \mathbb{R}$, and $\det(A)$ is a polynomial in $u_1, \ldots, u_t$ of degree at most three. Lemma 2.2.6 says that if $\det(A)$ is not a power of a linear form $a_1u_1 + \cdots + a_tu_t$ where $a_1, \ldots, a_t \in \mathbb{R}$, then $\mathcal{L}$ contains a matrix of rank two.

It is worth noting that Lemma 2.2.6 is only a sufficient condition and not necessary for a subspace $\mathcal{L} \subseteq \mathbb{P}_{\mathbb{R}}^{3\times 3}$ to have a rank two matrix. This is illustrated by the following example:

Example 2.2.7. *For*

$$X = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ 6 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & 1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 1 & 0 & 1 \\ \frac{1}{3} & 0 & 1 \\ \frac{1}{3} & -1 & 1 \\ 1 & -1 & 1 \\ \frac{1}{2} & -1 & 1 \\ 4 & -2 & 1 \\ 2 & -2 & 1 \end{pmatrix},$$

$\ker_{\mathbb{R}}(Z)$ is spanned by

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\det(A_1u_1 + A_2u_2) = u_1^3$. *Since* $\operatorname{rank}(A_2) = 2$, *$A_2$ is a fundamental matrix of Z.*

We now present a general theorem that characterizes the existence of a fundamental matrix for $m \leq 8$.

Theorem 2.2.8. *For a basis* $\{A_1, \ldots, A_t\}$ *of* $\ker_{\mathbb{R}}(Z)$*, define* $M(u) := \sum_{i=1}^{t} A_i u_i$*, and set* $d(u) := \det(M(u))$*.*

1. *If* $d(u)$ *is the zero polynomial then* Z *has a fundamental matrix if and only if some* 2×2 *minor of* $M(u)$ *is non-zero.*

2. *If* $d(u)$ *is a non-zero polynomial that is not a power of a linear form in* u *then* Z *has a fundamental matrix.*

3. *If* $d(u) = (b^\top u)^k$ *for some* $k \geq 1$ *and non-zero vector* $b \in \mathbb{R}^t$*, then* Z *has a fundamental matrix if and only if some* 2×2 *minor of* $M\big(u - \frac{b^\top u}{b^\top b} b\big)$ *is non-zero.*

Proof: Note that $M(u)$ is a parametrization of $\ker_{\mathbb{R}}(Z)$ and $d(u)$ is a polynomial in u of degree at most three.

1. If $d(u)$ is the zero polynomial, then all matrices in $\ker_{\mathbb{R}}(Z)$ have rank at most two. In this case, Z has a fundamental matrix if and only if some 2×2 minor of $M(u)$ is a non-zero polynomial in u.

2. If $d(u)$ is a non-zero polynomial in u, then we factor $d(u)$ and see if it is the cube of a linear form. If it is not, then by Lemma 2.2.6, Z has a fundamental matrix.

3. Suppose $d(u) = (b^\top u)^k$ for some $k \geq 1$ and non-zero vector b. Then the set of rank deficient matrices in $\ker_{\mathbb{R}}(Z)$ is $\mathcal{M} := \big\{M(u) : u \in b^\perp\big\}$ where, $b^\perp := \{u \in \mathbb{R}^t \ : b^\top u = 0\}$. The hyperplane $b^\perp$ consists of all vectors $u - \frac{b^\top u}{b^\top b} b$ where $u \in \mathbb{R}^t$. Therefore, $\mathcal{M} = \Big\{M\big(u - \frac{b^\top u}{b^\top b} b\big) : u \in \mathbb{R}^t\Big\}$. As a result, Z has a fundamental matrix if and only if some 2×2 minor of $M\big(u - \frac{b^\top u}{b^\top b} b\big)$ is non-zero.

$\square$

2.2.6 Cases: $m \leq 5$

While Theorem 2.2.8 provides a general existence condition for fundamental matrices for $m \leq 8$, we now show that for $m \leq 5$ there always exists a fundamental matrix.

Theorem 2.2.9. *Every three-dimensional subspace of $\mathbb{P}_{\mathbb{R}}^{3\times 3}$ contains a rank two matrix. In particular, if $m \leq 5$, then Z has a fundamental matrix.*

Proof: Suppose $\mathcal{L}$ is a three-dimensional subspace in $\mathbb{P}_{\mathbb{R}}^{3\times 3}$ generated by the basis $\{A_1, \dots, A_4\}$, and suppose $\mathcal{L}$ does not contain a rank two matrix. Since the dimension of $\mathcal{L}$ as a linear subspace is four, by applying Lemma 2.1.4 with $m = n = 3$ and $r = 1$, we see that $\mathcal{L}$ cannot be contained in the variety of rank one matrices. Therefore, we may assume that A_4 has rank three. Since $\mathcal{L}$ is assumed to have no matrices of rank two, by Lemma 2.2.6 we also have that

$$\begin{aligned} &\det(\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4) \\ &= (a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4)^3 \end{aligned} \tag{2.2.5}$$

where $\lambda_1, \cdots, \lambda_4$ are variables. Note that $a_4 \neq 0$ since otherwise, choosing $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = 1$ we get $\det(A_4) = 0$ which is impossible.

By a change of coordinates, we may assume that

$$\det(\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4) = \lambda_4^3, \tag{2.2.6}$$

and in particular, $\det(A_4) = 1$. Indeed, consider

$$\begin{aligned} \widetilde{A}_1 &:= A_1 - \frac{a_1}{a_4} A_4, & \widetilde{A}_2 &:= A_2 - \frac{a_2}{a_4} A_4 \\ \widetilde{A}_3 &:= A_3 - \frac{a_3}{a_4} A_4, & \widetilde{A}_4 &:= \frac{1}{a_4} A_4 \end{aligned}$$

which also form a basis of $\mathcal{L}$. Then using (2.2.5) with the variables $\eta_1, \eta_2, \eta_3, \eta_4$, we obtain

$$\begin{aligned} &\det(\eta_1 \widetilde{A}_1 + \eta_2 \widetilde{A}_2 + \eta_3 \widetilde{A}_3 + \eta_4 \widetilde{A}_4) = \\ &\det\left(\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3 + \frac{(\eta_4 - \eta_1 a_1 - \eta_2 a_2 - \eta_3 a_3)}{a_4} A_4\right) \\ &= (a_1\eta_1 + a_2\eta_2 + a_3\eta_3 + (\eta_4 - \eta_1 a_1 - \eta_2 a_2 - \eta_3 a_3))^3 = \eta_4^3, \end{aligned}$$

which is the desired conclusion.

Setting $\lambda_4 = 0$ in (2.2.6) we get $\det(\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3) = 0$. Hence, $\mathrm{span}\{A_1, A_2, A_3\}$ consists only of rank one matrices since there are no rank two matrices in $\mathcal{L}$. Therefore, by Lemma 2.1.3 with $n = 3$ and $r = 1$, up to taking transposes of all A_i, there are column vectors $u, v_1, v_2, v_3 \in \mathbb{P}^2_{\mathbb{R}}$ such that $A_j = uv_j^\top$ for all $j = 1, 2, 3$.

Now setting $\lambda_4 = 1$, by the Matrix Determinant Lemma, we have

$$\begin{aligned} 1 &= \det(\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + A_4) \\ &= \det(A_4 + u(\lambda_1 v_1^\top + \lambda_2 v_2^\top + \lambda_3 v_3^\top)) \\ &= 1 + (\lambda_1 v_1^\top + \lambda_2 v_2^\top + \lambda_3 v_3^\top) A_4^{-1} u. \end{aligned}$$

Hence $(\lambda_1 v_1^\top + \lambda_2 v_2^\top + \lambda_3 v_3^\top) A_4^{-1} u$ is the zero polynomial, and so $A_4^{-1} u$ is a non-zero vector orthogonal to $\mathrm{span}\{v_1, v_2, v_3\}$. This means that v_1, v_2, v_3 are linearly dependent, and so are A_1, A_2, A_3, which is impossible. This completes the proof of the first statement.

If $m \leq 5$, then $\mathrm{rank}(Z) \leq 5$ (cf. (2.1.6)) and so $\ker_{\mathbb{R}}(Z)$ is a subspace in $\mathbb{P}^8_{\mathbb{R}}$ of dimension at least three. By the first statement of the theorem, Z has a fundamental matrix. ☐

Note that when $m \leq 4$ there is a simpler proof (Section 2.5.6) for the existence of a fundamental matrix associated to the point correspondences, but it does not extend to the case of $m = 5$.

2.2.7 Comments

As far as we know, the seven and eight point algorithms are the only general methods for checking the existence of a fundamental matrix. They work by first computing the matrices in $\ker_{\mathbb{C}}(Z) \cap \mathcal{R}_2$ and then checking if there is a real matrix of rank two in this collection.

While such an approach might decide the existence of a fundamental matrix for a given input X and Y, it does not shed light on the structural requirements of X and Y to have a fundamental matrix. The goal of this chapter is to understand the existence of epipolar matrices in terms of the input data.

When the input points x_i and y_i are rational, the results in this section also certify the existence of a fundamental matrix by exact rational arithmetic in polynomial time. The only calculation that scales with m is the computation of a basis for $\ker_{\mathbb{R}}(Z)$, which can be done in polynomial time using Gaussian elimination.

2.3 Essential Matrices

We now turn our attention to calibrated cameras and essential matrices. The set of essential matrices is the set of real 3×3 matrices of rank two with two equal (non-zero) singular values [28]. In particular, all essential matrices are fundamental matrices and hence, contained in $\mathcal{R}_2 \setminus \mathcal{R}_1$. We denote the set of essential matrices by

$$\mathcal{E}_{\mathbb{R}} = \{e \in \mathbb{P}^8_{\mathbb{R}} : \sigma_1(E) = \sigma_2(E) \text{ and } \sigma_3(E) = 0\}, \tag{2.3.1}$$

where $\sigma_i(E)$ denotes the i^{th} singular value of the matrix E. Demazure [19] showed that

$$\mathcal{E}_{\mathbb{R}} = \left\{e \in \mathbb{P}^8_{\mathbb{R}} : p_j(e) = 0 \text{ for } j = 1, \ldots, 10\right\}, \tag{2.3.2}$$

where the p_j's are homogeneous polynomials of degree three defined as

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{pmatrix} := 2EE^\top E - \operatorname{Tr}(EE^\top)E, \text{ and} \tag{2.3.3}$$

$$p_{10} := \det(E). \tag{2.3.4}$$

Therefore, $\mathcal{E}_{\mathbb{R}}$ is a real projective variety in $\mathbb{P}^8_{\mathbb{R}}$.

Passing to the common complex roots of the cubics $p_1, \ldots, p_{10}$, we get

$$\mathcal{E}_{\mathbb{C}} := \{e \in \mathbb{P}^8_{\mathbb{C}} : p_j(e) = 0, \forall j = 1, \ldots, 10\}. \tag{2.3.5}$$

This is an irreducible projective variety with $\dim(\mathcal{E}_{\mathbb{C}}) = 5$ and $\operatorname{degree}(\mathcal{E}_{\mathbb{C}}) = 10$ (see [19]), and $\mathcal{E}_{\mathbb{R}} = \mathcal{E}_{\mathbb{C}} \cap \mathbb{P}^8_{\mathbb{R}}$. See [58] for many interesting facts about $\mathcal{E}_{\mathbb{C}}$ and $\mathcal{E}_{\mathbb{R}}$ and their role in reconstruction problems in multi-view geometry.

As before, our data consists of m point correspondences, which are now normalized image coordinates. For simplicity we will denote them as $\{(x_i, y_i),\ i = 1, \ldots, m\}$ instead of $\{(\tilde{x}_i, \tilde{y}_i),\ i = 1, \ldots, m\}$.

As in the uncalibrated case, we can write the epipolar constraints (cf. (2.1.1)) as $Ze = 0$ where $e \in \mathcal{E}_{\mathbb{R}}$, and Z has an essential matrix if and only if

$$\ker_{\mathbb{R}}(Z) \cap \mathcal{E}_{\mathbb{R}} \neq \emptyset. \tag{2.3.6}$$

Hence the existence of an essential matrix for a given Z is equivalent to the intersection of a subspace with a fixed real projective variety being non-empty. This formulation can also be found in [58, Section 5.2].

2.3.1 Cases: $m = 8, 9$

As in the previous section it is easy to settle the existence of E for $m = 8, 9$. If $m = 8$, then the subspace $\ker_{\mathbb{R}}(Z) \subseteq \mathbb{P}^8_{\mathbb{R}}$ is a point a in $\mathbb{P}^8_{\mathbb{R}}$, and Z has an essential matrix if and only if A satisfies the conditions of (2.3.1) or (2.3.2). If $m = 9$, then $\ker_{\mathbb{R}}(Z) \subseteq \mathbb{P}^8_{\mathbb{R}}$ is empty, and Z has no essential matrix.

2.3.2 Cases: $5 \leq m \leq 7$

The "minimal problem" for essential matrices is the case of $m = 5$ where, by Definition 2.1.12, $\mathcal{E}_{\mathbb{C}} \cap \ker_{\mathbb{C}}(Z)$ is a finite set of points. Since degree$(\mathcal{E}_{\mathbb{C}}) = 10$, generically we expect ten distinct complex points in this intersection. An essential matrix exists for Z if and only if one of these points is real. There can be ten distinct real points in $\mathcal{E}_{\mathbb{C}} \cap \ker_{\mathbb{C}}(Z)$ as shown in [58, Theorem 5.14]. On the other extreme, it can also be that no point in $\mathcal{E}_{\mathbb{C}} \cap \ker_{\mathbb{C}}(Z)$ is real as we show below.

Example 2.3.1. *We verified using* `Maple` *that the following set of five point correspondences* *has no essential matrix.*

$$X = \begin{pmatrix} 3 & 0 & 1 \\ 9 & 1 & 1 \\ 1 & 2 & 1 \\ 8 & 8 & 1 \\ 4 & 8 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 & 0 & 1 \\ 5 & 4 & 1 \\ 9 & 6 & 1 \\ 2 & 5 & 1 \\ 1 & 4 & 1 \end{pmatrix}.$$

None of the ten points in $\ker_{\mathbb{C}}(Z) \cap \mathcal{E}_{\mathbb{C}}$ *are real.*

As we have mentioned earlier, the existence of an essential matrix is equivalent to existence of a real point in the intersection $\ker_{\mathbb{C}}(Z) \cap \mathcal{E}_{\mathbb{C}}$. In general, this is a hard question which falls under the umbrella of *real algebraic geometry*.

The reason we were able to give a general existence result for fundamental matrices is because we were able to exploit the structure of the set of rank 2 matrices (Lemma 2.2.6). We believe that a general existence result for essential matrices would require a similar result about the variety of essential matrices. One that still eludes us, and therefore, we are unable to say more about the existence of essential matrices for the case $5 \leq m \leq 7$.

In theory, the non-existence of a real solution to a system of polynomials can be characterized by the *real Nullstellensatz* [57] and checked degree by degree via *semidefinite programming* [74]. Or given a Z we could solve the Demazure cubics together with the linear equations cutting out $\ker_{\mathbb{R}}(Z)$ and check if there is a real solution among the finitely many complex solutions [61, 68]. In both of these approaches, it is a case by case computation for each instance of Z and will not yield a characterization of those Z's for which there is an essential matrix.

2.3.3 Cases: $m \leq 4$

We now consider the cases of $m \leq 4$ where $\mathcal{E}_{\mathbb{C}} \cap \ker_{\mathbb{C}}(Z)$ is infinite and the conventional wisdom is that an essential matrix always exists. It turns out that an essential matrix does indeed exist when $m \leq 4$. Again, such a result does not follow from dimension counting for complex varieties since we have to exhibit the existence of a real matrix in $\mathcal{E}_{\mathbb{C}} \cap \ker_{\mathbb{C}}(Z)$.

When $m \leq 3$, there is a short proof that Z always has an essential matrix using the fact that an essential matrix can be written in the form $E = [t]_\times R$ where t is a non-zero vector in $\mathbb{R}^3$ and $R \in \mathrm{SO}(3)$.

Theorem 2.3.2. *If $m \leq 3$ then Z has an essential matrix.*

Proof: Without loss of generality we assume $m = 3$. Choose a rotation matrix R so that $y_1 \sim Rx_1$. Then consider $t \in \mathbb{R}^3 \setminus \{0\}$ which is orthogonal to both $y_2 \times Rx_2$ and $y_3 \times Rx_3$. Now check that for each $i = 1, 2, 3$, $y_i^\top [t]_\times R x_i = 0$ and hence $[t]_\times R$ is an essential matrix for Z. It helps to recall that $y_i^\top [t]_\times R x_i = t^\top (y_i \times R x_i)$. □

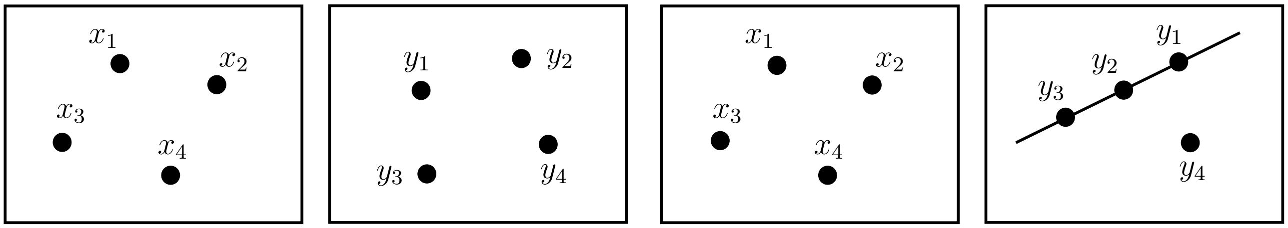
The above argument does not extend to $m = 4$. Our main result in this section is Theorem 2.3.4 which proves the existence of E when $m = 4$ under the mild assumption that all the x_i's (respectively, y_i's) are distinct. This result will need the following key lemma which is a consequence of Theorems 5.19 and 5.21 in [55].

Lemma 2.3.3. *If there is a matrix $H \in \mathbb{R}^{3\times 3}$ of rank at least two such that for each i, either $y_i \sim Hx_i$ or $Hx_i = 0$, then Z has an essential matrix.*

Theorem 2.3.4. *If $m = 4$ and all the x_i's are distinct points and all the y_i's are distinct points for $i = 1, \ldots, 4$, then Z has an essential matrix.*

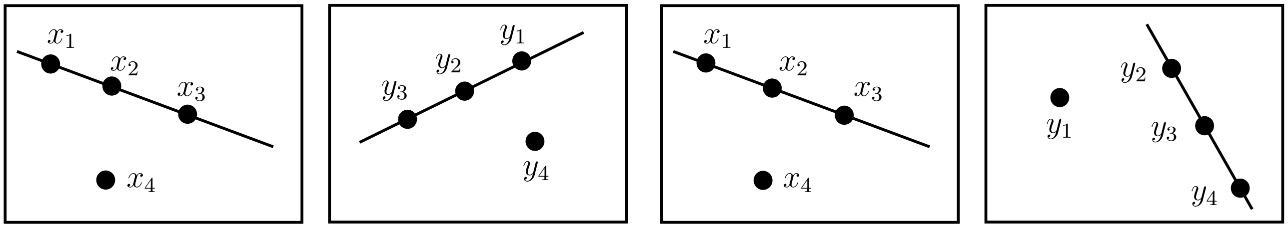
Proof: We divide the proof into several cases; see Figure 2.2. The first is the generic situation in which the x_i's and y_i's are in general position. In the remaining cases the input data satisfy special non-generic conditions. Together, these cases exhaust all possibilities, up to rearranging indices and swapping x with y.

In the first three cases, the proof proceeds by exhibiting an explicit matrix H that satisfies the assumption of Lemma 2.3.3. Cases 1 and 2 are easy to check. The H in case 3 is quite a bit more involved, although it only suffices to verify that it satisfies Lemma 2.3.3, which



Case 1: No three points in the two images are collinear.

Case 2: No three points in x_1, x_2, x_3, x_4 are collinear, and y_1, y_2, y_3 are collinear.



Case 3: x_1, x_2, x_3 are collinear, while x_4 is not on the line, and y_1, y_2, y_3 are collinear while y_4 is not on the line.

Case 4: x_1, x_2, x_3 are collinear, and y_2, y_3, y_4 are collinear.

Figure 2.2: The four point configurations (up to rearranging indices and swapping x with y) for $m = 4$. Theorem 2.3.4 proves the existence of an E matrix for $m = 4$ by treating each of these cases separately.

is mechanical. The last case uses a different argument to construct an essential matrix associated to Z.

1. *No three of the x_i's are collinear in $\mathbb{R}^2$, and no three of the y_i's are collinear in $\mathbb{R}^2$; see Figure 2.2.*

   In this case, there is an invertible matrix $H \in \mathbb{R}^{3\times 3}$ such that $y_i \sim Hx_i$ by Lemma 2.1.2 with $n = 2$. The conclusion now follows from Lemma 2.3.3.

2. *No three points in x_1, x_2, x_3, x_4 are collinear in $\mathbb{R}^2$, and the points y_1, y_2, y_3 are collinear*

in $\mathbb{R}^2$*; see Figure 2.2.*

By Lemma 2.1.2, we can choose an invertible matrix $H_1 \in \mathbb{R}^{3\times 3}$ such that

$$H_1x_1 \sim (1,1,1)^\top,\ H_1x_2 \sim (0,0,1)^\top,$$
$$H_1x_3 \sim (0,1,0)^\top \text{ and } H_1x_4 \sim (1,0,0)^\top.$$

On the other hand, by Lemma 2.1.6, there is an invertible matrix $H_2 \in \mathbb{R}^{3\times 3}$ such that

$$H_2y_1 = (0,0,1)^\top,\ H_2y_2 = (0,\alpha,1)^\top,\ H_2y_3 = (0,\beta,1)^\top$$

for some non-zero distinct real numbers α and β. Consider the rank two matrix

$$H_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha\beta & \alpha\beta \\ 0 & -\alpha & \beta \end{pmatrix}.$$

Then we obtain

$$H_3(1,1,1)^\top \sim H_2y_1,\ H_3(0,0,1)^\top \sim H_2y_2,$$
$$H_3(0,1,0)^\top \sim H_2y_3 \text{ and } H_3(1,0,0)^\top = (0,0,0)^\top.$$

Consequently, if we consider the rank two matrix $H := H_2^{-1}H_3H_1$, then $y_i \sim Hx_i$ for $i = 1,2,3$ and $Hx_4 = 0$. Thus, the result follows from Lemma 2.3.3.

3. *The points* x_1, x_2, x_3 *are collinear in* $\mathbb{R}^2$ *while* x_4 *is not on the line, and the points* y_1, y_2, y_3 *are collinear in* $\mathbb{R}^2$ *while* y_4 *is not on the line; see Figure 2.2.*

   Using Lemma 2.1.6, by multiplying two invertible matrices from the left to x_i's and y_i's if necessary, we may assume $x_1 = (0,0)$, $x_2 = (0,\alpha)$, $x_3 = (0,\beta)$, $x_4 = (x_{41}, x_{42})$, $y_1 = (0,0)$, $y_2 = (0,\gamma)$, $y_3 = (0,\delta)$ and $y_4 = (y_{41}, y_{42})$, where $x_{41}, \alpha, \beta, \gamma, \delta$ are non-zero real numbers, $\alpha \neq \beta$ and $\gamma \neq \delta$. Then there is a matrix $H \in \mathbb{R}^{3\times 3}$ such that

$$\begin{aligned} Hx_1 &= \alpha\beta x_{41}(\gamma - \delta)y_1 \\ Hx_2 &= \alpha\delta x_{41}(\alpha - x_{41})y_2 \\ Hx_3 &= \beta\gamma x_{41}(\alpha - x_{41})y_3 \\ Hx_4 &= x_{41}[\beta\gamma(\alpha - x_{42}) - \alpha\delta(\beta - x_{42})]y_4, \end{aligned}$$

given by

$$H := \begin{pmatrix} H_{11} & 0 & 0 \\ H_{21} & H_{22} & 0 \\ 0 & H_{32} & H_{33} \end{pmatrix}$$

where

$$\begin{aligned} H_{11} &= (\alpha - x_{42})\beta\gamma y_{41} - (\beta - x_{42})\alpha\delta y_{41} \\ H_{21} &= -\alpha x_{42}\gamma\delta + \beta x_{42}\gamma\delta + \alpha\beta\gamma y_{42} - \beta x_{42}\gamma y_{42} \\ &\quad - \alpha\beta\delta y_{42} + \alpha x_{42}\delta y_{42} \\ H_{22} &= (\alpha - \beta)x_{41}\gamma\delta \\ H_{32} &= (\alpha\delta - \beta\gamma)x_{41} \\ H_{33} &= (\gamma - \delta)x_{41}\alpha\beta. \end{aligned}$$

Notice that $H_{22}H_{33} \neq 0$, which implies $\operatorname{rank}(H) \geq 2$. Then, the result follows using Lemma 2.3.3.

4. *The points x_1, x_2, x_3 are collinear in $\mathbb{R}^2$, and the points y_2, y_3, y_4 are collinear in $\mathbb{R}^2$; see Figure 2.2.*

Let P_X be the plane in $\mathbb{R}^3$ containing $(0, 0, 0)$ and the common line l_X joining x_1, x_2, x_3. Let P_Y be the plane in $\mathbb{R}^3$ containing $(0, 0, 0)$ and the common line joining y_2, y_3, y_4. Take a unit vector $u \in P_X$ so that $u^\top x_1 = 0$. Let U be the orthogonal matrix given by

$$U := \left(\frac{x_1}{\|x_1\|},\ u,\ \frac{x_1}{\|x_1\|} \times u \right)$$

Let $w \in P_Y$ be a unit vector so that $w^\top y_4 = 0$. We consider the orthogonal matrix

$$W := \left(\frac{y_4}{\|y_4\|},\ w,\ \frac{y_4}{\|y_4\|} \times w \right).$$

Let R be an orthogonal matrix so that $RU = W$, namely, $R := WU^\top$. Then, $R\frac{x_1}{\|x_1\|} = \frac{y_4}{\|y_4\|}$ and $Ru = w$. If $x \in l_X$, then $x = \alpha\frac{x_1}{\|x_1\|} + \beta u$ for some real numbers α, β. Thus we have

$$Rx = \alpha R\frac{x_1}{\|x_1\|} + \beta Ru = \alpha\frac{y_4}{\|y_4\|} + \beta w \in P_Y.$$

Consider the essential matrix $E = [y_4]_\times R$. One has

$$y_4^\top E x_4 = y_4^\top [y_4]_\times R x_4 = 0^\top R x_4 = 0 \text{ and}$$
$$y_1^\top E x_1 = y_1^\top [y_4]_\times R x_1 \sim y_1^\top [y_4]_\times y_4 = 0.$$

For $i = 2, 3$, since $Rx_i \in P_Y = \operatorname{span}\{y_i, y_4\}$, one obtains

$$y_i^\top E x_i \sim [y_i \times y_4]^\top R x_i = 0.$$

Hence E is an essential matrix of Z.

□

Corollary 2.3.5. *An essential matrix always exists when $m \leq 4$ provided all the x_i's are distinct and all the y_i's are distinct.*

2.4 Discussion

In this chapter, we have settled the existence problem for fundamental matrices for all values of m and essential matrices for $m \leq 4$ (equivalently, for all m for which $\operatorname{rank}(Z) \leq 4$). In doing so, we have shown that pure dimension counting arguments are not enough to reason about the existence of real valued epipolar matrices.

As we mentioned in the previous section, the conditions for the existence of an essential matrix for $5 \leq m \leq 7$ appear to be difficult, and are unknown for $m = 5, 6$. For $m = 7$, we did find a test for the existence of an essential matrix. This uses the classical theory of *Chow forms* [14, 31]. Chow forms also provide a test for whether $\ker_{\mathbb{C}}(Z) \cap \mathcal{E}_{\mathbb{C}} \neq \emptyset$ when $m = 6$. For more details see Chapter 3.

Even though our results are phrased in terms of the matrix Z, we have shown that they can be reinterpreted in terms of the input X and Y in most cases. We are curious about the set of six and seven point correspondences in two views for which there is no fundamental matrix. Theorems 2.2.2 and 2.2.5 characterized the point configurations for which there is no fundamental matrix because $\ker_{\mathbb{R}}(Z) \subseteq \mathcal{R}_1$. It would also be interesting to understand the configurations for which $\ker_{\mathbb{R}}(Z)$ contains only matrices of ranks one and three as in Example 2.2.1.

The results in this chapter show that reasoning over real numbers is both a source of surprises and complications. We believe that similar surprises and complications lurk in other existence problems in multi-view geometry and are worthy of study.

2.5 Proof of results

2.5.1 Proof of Lemma 2.1.6

Since $l - x_0$ and $m - y_0$ are lines in $\mathbb{R}^2$ passing through the origin, one can choose an orthogonal matrix $W \in \mathbb{R}^{2\times 2}$ such that $m - y_0 = W(l - x_0)$. It follows that

$$m = W(l - x_0) + y_0 = Wl - Wx_0 + y_0 = Wl + z$$

where $z := y_0 - Wx_0$ is a point in $\mathbb{R}^2$. Then, for the 3×3 matrix $H := \left(\begin{smallmatrix} W & z \\ 0 & 1 \end{smallmatrix}\right)$, one has $\left(\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right) = H \left(\begin{smallmatrix} l \\ 1 \end{smallmatrix}\right)$ which verifies the statement (2). In addition, $H \left(\begin{smallmatrix} x_0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} y_0 \\ 1 \end{smallmatrix}\right)$, and thus the assertion (1) also holds.

2.5.2 Proof of Lemma 2.2.3

Recall that a fundamental matrix can be written in the form $F = [b]_\times H$ where b is a non-zero vector in $\mathbb{R}^3$ and $H \in \mathbb{R}^{3\times 3}$ is an invertible matrix. Then the epipolar constraints can be rewritten as

$$
\begin{aligned}
& y_i^\top F x_i = 0, \forall i = 1, \dots, m. \\
\iff & y_i^\top [b]_\times H x_i = 0, \forall i = 1, \dots, m. \\
\iff & y_i^\top (b \times H x_i) = 0, \forall i = 1, \dots, m. && (2.5.1) \\
\iff & b^\top (y_i \times H x_i) = 0, \forall i = 1, \dots, m. && (2.5.2) \\
\iff & b^\top \begin{pmatrix} \cdots & y_i \times H x_i & \cdots \end{pmatrix} = 0.
\end{aligned}
$$

A non-zero b exists in the expression for F if and only if

$$\operatorname{rank} \begin{pmatrix} \cdots & y_i \times H x_i & \cdots \end{pmatrix} < 3. \tag{2.5.3}$$

The equivalence of (2.5.1) and (2.5.2) follows from the fact that $p^\top(q \times r) = -q^\top(p \times r)$. The matrix in (2.5.3) is of size $3 \times m$. A sufficient condition for it to have rank less than 3 is for $m - 2$ or more columns to be equal to zero. This is the case if we take $H = A$ given in the assumption.

The observation about the scalar triple product and the resulting rank constraint has also been used by Kneip *et al.* [46] but only in the calibrated case.

2.5.3 Proof of Theorem 2.2.2

If part: Suppose (1) holds and let τ be the set given in (1). Then there is a $u \in \mathbb{P}^2_{\mathbb{R}}$ such that $u^\top y_i = 0$ for any $i \in \tau$. Let x_k be the single element in the set $\{x_i\}_{i \notin \tau}$. Consider a basis $\{v_1, v_2\} \subseteq \mathbb{P}^2_{\mathbb{R}}$ of the orthogonal complement of x_k. For $j = 1, 2$, define $A_j = u v_j^\top \in \mathbb{P}^{3\times 3}_{\mathbb{R}}$ and let $a_j \in \mathbb{P}^8_{\mathbb{R}}$ be its vectorization. Then $\{a_1, a_2\}$ is a linearly independent set spanning a subset of $\mathcal{R}_1$. Moreover for any $i = 1, \dots, 7$ and $j = 1, 2$, $y_i^\top A_j x_i = (y_i^\top u)(v_j^\top x_i) = 0$. Hence $a_j \in \ker_{\mathbb{R}}(Z)$ for $j = 1, 2$. As $\operatorname{rank}(Z) = 7$ (cf. (2.1.6)), $\ker_{\mathbb{R}}(Z) = \operatorname{span}\{a_1, a_2\} \subseteq \mathcal{R}_1$. The same idea of proof works if (2) holds.

Only if part: Consider a basis $\{a_1, a_2\} \subseteq \mathbb{P}^8_{\mathbb{R}}$ of $\ker_{\mathbb{R}}(Z)$, which is inside $\mathcal{R}_1$, and assume a_j is the vectorization of $A_j \in \mathbb{P}^{3\times 3}_{\mathbb{R}}$ for $j = 1, 2$. For any j, $\operatorname{rank}(A_j) = 1$, so $A_j = u_j v_j^\top$

for some $u_j, v_j \in \mathbb{P}^2_{\mathbb{R}}$. Since $\operatorname{rank}(A_1 + A_2) = 1$, a simple check shows that either $\{u_1, u_2\}$ or $\{v_1, v_2\}$ is linearly dependent. Thus, up to scaling, we may assume either $u_1 = u_2$ or $v_1 = v_2$. If $u_1 = u_2$, then $\{v_1, v_2\}$ is linearly independent. In addition, $0 = y_i^\top A_j x_i = (y_i^\top u)(v_j^\top x_i)$ for each $i = 1, \dots, 6$, $j = 1, 2$. Thus, either $y_i^\top u = 0$ or $x_i \in \operatorname{span}\{v_1, v_2\}^\perp$. Notice that $\operatorname{span}\{v_1, v_2\}^\perp$ is a singleton in $\mathbb{P}^2_{\mathbb{R}}$. As $\operatorname{rank}(Z) = 7$, by the paragraph after Lemma 2.1.14, neither " $y_i^\top u = 0$ for all i" nor "$x_i \in \operatorname{span}\{v_1, v_2\}^\perp$ for all i" can happen. Hence (1) holds with the nonempty proper subset $\tau := \{i \ : \ y_i^\top u = 0\}$ of $\{1, \dots, 7\}$. If $v_1 = v_2$, by the same idea one sees that (2) holds.

2.5.4 Proof of Theorem 2.2.5

Recall that we are assuming that Z has full row rank, i.e., $m = \operatorname{rank}(Z) = 6$. By Lemma 2.1.14, this can only be true for $m = 6$ if x_i and y_i are not simultaneously collinear, i.e. one of X or Y has to have full row rank.

If part: If all points y_i are collinear in $\mathbb{R}^2$, then there is $u \in \mathbb{P}^2_{\mathbb{R}}$ such that $u^\top y_i = 0$ for any $i = 1, \dots, 6$. Let $e_1 = (1, 0, 0)^\top$, $e_2 = (0, 1, 0)^\top$, $e_3 = (0, 0, 1)^\top$. Consider the 3×3 matrices

$$A_j = u e_j^\top \text{ for } j = 1, 2, 3$$

and their vectorizations $a_j \in \mathbb{P}^8_{\mathbb{R}}$. Then, $\{a_1, a_2, a_3\}$ is a linearly independent set spanning a subset of $\mathcal{R}_1$. Moreover, for any $i = 1, \dots, 6$ and $j = 1, 2, 3$, $y_i^\top A_j x_i = (y_i^\top u)(x_i^\top e_j) = 0$. Hence $a_j \in \ker_{\mathbb{R}}(Z)$. As $\operatorname{rank}(Z) = 6$ (cf. (2.1.6)), $\ker_{\mathbb{R}}(Z) = \operatorname{span}\{a_1, a_2, a_3\} \subseteq \mathcal{R}_1$. The same idea of proof works if all points x_i are collinear in $\mathbb{R}^2$.

Only if part: Consider a basis $\{a_1, a_2, a_3\} \subseteq \mathbb{P}^8_{\mathbb{R}}$ of $\ker_{\mathbb{R}}(Z)$, which is inside $\mathcal{R}_1$, and assume a_j is the vectorization of $A_j \in \mathbb{P}^{3\times 3}_{\mathbb{R}}$ for $j = 1, 2, 3$. Then, by Lemma 2.1.3 with $n = 3$ and $r = 1$, up to taking transpose of all A_j, there are non-zero vectors $u, v_1, v_2, v_3 \in \mathbb{P}^2_{\mathbb{R}}$ such that $A_j = u v_j^\top$ for $j = 1, 2, 3$. The vectors v_j are linearly independent as A_j are. Moreover $0 = y_i^\top A_j x_i = (y_i^\top u)(x_i^\top v_j)$ for any $i = 1, \dots, 6$, $j = 1, 2, 3$. We fix $i \in \{1, \dots, 6\}$ and claim that $y_i^\top u = 0$. Indeed, if $y_i^\top u \neq 0$, then $x_i^\top v_j = 0$ for any $j = 1, 2, 3$. As vectors v_j are linearly independent we have $x_i = 0$. This is impossible because x_i as a point in $\mathbb{P}^2_{\mathbb{R}}$ has non-zero third coordinate. Hence our claim is true and thus all points y_i are collinear in $\mathbb{R}^2$. If it is necessary to replace A_j by $A_j^\top$, it follows that all points x_i are collinear in $\mathbb{R}^2$.

2.5.5 Proof of Lemma 2.2.6

We first consider the case when L is a projective line, i.e.,

$$L = \{A\mu + B\eta \,:\, \mu, \eta \in \mathbb{R}\}$$

for some $A, B \in \mathbb{R}^{3\times 3}$, with B invertible. Then $B^{-1}L = \{B^{-1}A\mu + I\eta \,:\, \mu, \eta \in \mathbb{R}\}$ is an isomorphic image of L and contains a matrix of rank two if and only if L does. Hence we can assume $L = \{M\mu - I\eta \,:\, \mu, \eta \in \mathbb{R}\}$ for some $M \in \mathbb{R}^{3\times 3}$. The homogeneous cubic polynomial $\det(M\mu - I\eta)$ is not identically zero on L. When dehomogenized by setting $\mu = 1$, it is the characteristic polynomial of M. Hence the three roots of $\det(M\mu - I\eta) = 0$ in $\mathbb{P}^1$ are $(\mu_1, \eta_1) \sim (1, \lambda_1), (\mu_2, \eta_2) \sim (1, \lambda_2)$ and $(\mu_3, \eta_3) \sim (1, \lambda_3)$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of M. At least one of these roots is real since $\det(M\mu - I\eta)$ is a cubic. Suppose (μ_1, η_1) is real. If $\text{rank}(M\mu_1 - I\eta_1) = \text{rank}(M - I\lambda_1) = 2$, then L contains a rank two matrix. Otherwise, $\text{rank}(M - I\lambda_1) = 1$. Then λ_1 is a double eigenvalue of M and hence equals one of λ_2 or λ_3. Suppose $\lambda_1 = \lambda_2$. This implies that (μ_3, η_3) is a real root as well. If it is different from (μ_1, η_1), then it is a simple real root. Hence, $\text{rank}(M\mu_3 - I\eta_3) = 2$, and L has a rank two matrix. So suppose $(\mu_1, \eta_1) \sim (\mu_2, \eta_2) \sim (\mu_3, \eta_3) \sim (1, \lambda)$ where λ is the unique eigenvalue of M. In that case, $\det(M\mu - I\eta) = \alpha \cdot (\eta - \lambda\mu)^3$ for some constant α. This finishes the case $\dim(L) = 1$.

Now suppose $\dim(L) \geq 2$. If det restricted to L is not a power of a homogeneous linear polynomial, then there exists a projective line L' in L such that det restricted to L' is also not the power of a homogeneous linear polynomial. The projective line L' contains a matrix of rank two by the above argument.

2.5.6 A proof for the existence of a fundamental matrix when $m \leq 4$

Theorem 2.5.1. *If $m \leq 4$, then Z has a fundamental matrix.*

Proof: If $m \leq 3$, by adding point pairs if necessary we can assume $m = 3$. One can always construct an invertible matrix H such that $y_1 \sim Hx_1$ which implies that $y_1 \times Hx_1 = 0$ and equation (2.5.3) is satisfied.

Let us now consider the case $m = 4$. Since $\operatorname{rank}(Z) = 4$, by Lemma 2.1.14, $\operatorname{rank}(X) \geq 2$ and $\operatorname{rank}(Y) \geq 2$. If we can find two indices i and j such that the matrices $\begin{pmatrix} x_i & x_j \end{pmatrix}$ and $\begin{pmatrix} y_i & y_j \end{pmatrix}$ both have rank 2 then we can construct an invertible matrix H such that $y_i \sim Hx_i$ and $y_j \sim Hx_j$ and that would be enough for (2.5.3). Without loss of generality let us assume that the matrix $\begin{pmatrix} x_1 & x_2 \end{pmatrix}$ is of rank 2, i.e., $x_1 \not\sim x_2$. If $\begin{pmatrix} y_1 & y_2 \end{pmatrix}$ has rank 2 we are done. So let us assume that this is not the case and $y_2 \sim y_1$. Since $\operatorname{rank}(Y) \geq 2$, we can without loss of generality assume that $y_3 \not\sim y_1$. Since $x_1 \not\sim x_2$, either, $x_3 \not\sim x_1$ or $x_3 \not\sim x_2$. In the former case, $i = 1, j = 3$ is the pair we want, otherwise $i = 2, j = 3$ is the pair we want. □

2.5.7 Proof of Lemma 2.3.3

Denote by $O(3)$ the group of real orthogonal 3×3 matrices. We need the following lemma.

Lemma 2.5.2. *If H is a 3×3 real matrix of rank at least 2, then $H \sim R + tn^\top$ for some $R \in O(3)$, $t \in \mathbb{R}^3 \setminus \{0\}$ and $n \in \mathbb{R}^3$.*

Proof: If a scalar multiple of the matrix H is orthogonal, one can take $R = H$, $t \neq 0$ and $n = 0$ to make the claim hold. From now on, we assume that no scalar mulitple of the matrix H is orthogonal. Suppose that the singular value decomposition of the matrix H is $H = X\operatorname{Diag}(a, b, c)V^\top$ with $a \geq b \geq c \geq 0$, and $X, V \in O(3)$. Since $\operatorname{rank}(H) \geq 2$, one has $b \neq 0$. Thus, by dividing H by b we can assume $b = 1$ and work with the resulting scaled version of H since our claim is only up to scaling. Renaming this scaled version of H again as H, we have $H = X\operatorname{Diag}(a, 1, c)V^\top$. Let v_1, v_2, v_3 be the column vectors of V. We now have $H^\top H = V\operatorname{Diag}(a^2, 1, c^2)V^\top$ and indeed,

$$H^\top H v_1 = a^2 v_1,\ H^\top H v_2 = v_2,\ H^\top H v_3 = c^2 v_3.$$

Recall that $\{v_1, v_2, v_3\}$ is an orthonormal set. Moreover, $\|Hv_2\| = 1$ because $\|Hv_2\|^2 = (Hv_2)^\top(Hv_2) = v_2^\top H^\top Hv_2 = v_2^\top v_2 = 1$. Now, we define

$$u := \frac{\sqrt{1-c^2}v_1 + \sqrt{a^2-1}v_3}{\sqrt{a^2-c^2}}.$$

First $\|u\| = 1$ for

$$u^\top u = \frac{1-c^2+a^2-1}{a^2-c^2} = 1.$$

We claim $\|Hu\| = 1$. Indeed, one has

$$H^\top Hu = \frac{a^2\sqrt{1-c^2}v_1 + c^2\sqrt{a^2-1}v_3}{\sqrt{a^2-c^2}}$$

and hence

$$\|Hu\|^2 = u^\top H^\top Hu = \frac{a^2(1-c^2)+(a^2-1)c^2}{a^2-c^2} = 1.$$

Also, $Hu \perp Hv_2$ because

$$(Hu)^\top Hv_2 = u^\top H^\top Hv_2 = u^\top v_2 = 0$$

as v_2 is orthogonal to $\langle v_1, v_3\rangle$. Let $U, W \in O(3)$ be given by

$$U := [v_2, u, v_2 \times u] \text{ and } W := [Hv_2, Hu, (Hv_2)\times(Hu)].$$

Choose a matrix $R \in O(3)$ satisfying $RU = W$; namely, $R := WU^\top$. Then we know $Rv_2 = Hv_2$ and $Ru = Hu$. In addition, set $n := v_2 \times u$ and $t := (H-R)n$. We have $H = R + tn^\top$ by showing $(R+tn^\top)U = HU$:

$$\begin{aligned}
(R+tn^\top)v_2 &= Rv_2 + t(n^\top v_2) = Hv_2 + t(0) = Hv_2,\\
(R+tn^\top)u &= Ru + t(n^\top u) = Hu + t(0) = Hu,\\
(R+tn^\top)n &= Rn + (H-R)n(n^\top n) = Rn + (H-R)n(1) = Hn.
\end{aligned}$$

Since no scalar multiple of H is orthogonal, one has $t \neq 0$. The lemma is then seen to be true. $\square$

Here is the proof of Lemma 2.3.3:

Proof: By the previous lemma we know $H \sim R + tn^\top$ for some $R \in O(3)$, $t \in \mathbb{R}^3 \setminus \{0\}$ and $n \in \mathbb{R}^3$. Next we consider the essential matrix $E := [t]_\times R$. Notice that

$$H^\top E \sim (R^\top + nt^\top)[t]_\times R = R^\top [t]_\times R + 0 = R^\top [t]_\times R.$$

Let i be fixed. If $y_i \sim Hx_i$, then we have

$$y_i^\top E x_i \sim x_i^\top H^\top E x_i \sim x_i^\top R^\top [t]_\times R x_i = (Rx_i)^\top (t \times Rx_i) = 0.$$

If $Hx_i = 0$, then $(R + tn^\top)x_i = 0$ and hence $Rx_i = -tn^\top x_i$. One has

$$y_i^\top E x_i = y_i^\top [t]_\times R x_i = -y_i^\top [t]_\times t n^\top x_i = -y_i^\top (t \times t)(n^\top x_i) = 0.$$

As a result, E is an essential matrix corresponding to the given point correspondences. $\square$

Chapter 3

EXISTENCE OF AN ESSENTIAL MATRIX AND CHOW FORMS

3.1 Introduction

The notations we use in this chapter follow that in the previous chapter. This chapter we discuss the existence of an essential matrix for the cases $m = 6$ and $m = 7$, which is not tackled by the previous chapter. We will establish conditions on Z such that $\ker_{\mathbb{R}}(Z) \cap \mathcal{E}_{\mathbb{R}} \neq \emptyset$, when $m = 7$. For $m = 6$, we characterize the data Z that satisfy $\ker_{\mathbb{C}}(Z) \cap \mathcal{E}_{\mathbb{C}} \neq \emptyset$. This is certainly necessary for a (real) essential matrix of Z to exist, and, as we shall see, already involves non-trivial mathematics. These ranks can be dealt with via a tool from algebraic geometry that we explain next.

3.2 The Chow form of a projective variety

Let $\mathcal{V} \subseteq \mathbb{P}^n_{\mathbb{C}}$ be an irreducible projective variety of dimension d and degree δ. Recall that a generic subspace $L \subseteq \mathbb{P}^n_{\mathbb{C}}$ of dimension $n-d$ will intersect $\mathcal{V}$ in δ points up to multiplicity, but if $\dim(L) < n-d$ then L will usually not intersect $\mathcal{V}$. We consider all subspaces of dimension exactly $n-d-1$ that do intersect $\mathcal{V}$. To be concrete, let us express an $(n-d-1)$-dimensional subspace L as $\ker_{\mathbb{C}}(A)$ for some $A \in \mathbb{C}^{(d+1)\times(n+1)}$. Then there is a single irreducible polynomial in the entries of A called the *Chow form* of $\mathcal{V}$, denoted as $\mathrm{Ch}_{\mathcal{V}}(A)$, that defines the set of all $(n-d-1)$-dimensional subspaces L that intersect $\mathcal{V}$. More explicity, for a scalar matrix $A \in \mathbb{C}^{(d+1)\times(n+1)}$, $\mathrm{Ch}_{\mathcal{V}}(A) = 0$ if and only if $\ker_{\mathbb{C}}(A) \cap \mathcal{V} \neq \emptyset$. Hence $\mathrm{Ch}_{\mathcal{V}}$ certifies exactly when an $(n-d-1)$-dimensional subspace intersects $\mathcal{V}$. Further, $\mathrm{Ch}_{\mathcal{V}}(A)$ is a homogeneous polynomial of degree $(d+1)\delta$ in the entries of A. The computation of a Chow form is a problem in *elimination theory* [13, Chapter 3] and we illustrate the method in Example 3.2.1.

An accessible introduction to Chow forms can be found in [14], with algorithms for computing them in Section 3.1. For an in-depth treatment see [31, Section 3.2 B].

Example 3.2.1. *[14, 1.2] Consider the cubic curve in* 3*-space given parametrically as*

$$\mathcal{C} \;=\; \{(\lambda^3\,,\,\lambda^2\mu\,,\,\lambda\mu^2\,,\,\mu^3) \in \mathbb{P}^3_{\mathbb{C}} \,:\, \lambda,\mu \in \mathbb{C}\}.$$

Here $n = 3, d = 1$ *and* $\delta = 3$*. The Chow form* $\mathrm{Ch}_{\mathcal{C}}$ *characterizes the set of all lines* $L \subseteq \mathbb{P}^3_{\mathbb{C}}$ *that intersect* $\mathcal{C}$*. Writing* $L = \ker_{\mathbb{C}} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$*,*

$$\mathrm{Ch}_{\mathcal{C}}(A) = \det \begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} & a_{11}a_{24} - a_{14}a_{21} \\ a_{11}a_{23} - a_{13}a_{21} & \substack{a_{11}a_{24}-a_{14}a_{21} \\ +a_{12}a_{23}-a_{13}a_{22}} & a_{12}a_{24} - a_{14}a_{22} \\ a_{11}a_{24} - a_{14}a_{21} & a_{12}a_{24} - a_{14}a_{22} & a_{13}a_{24} - a_{14}a_{23} \end{pmatrix}. \tag{3.2.1}$$

This has degree $6 = 2\cdot 3 = (d+1)\cdot\delta$ *as a polynomial in the entries of* A*. We have* $\mathrm{Ch}_{\mathcal{C}}(A) = 0$ *if and only if* $L \cap \mathcal{C} \neq \emptyset$ *if and only if there exists* $(\lambda,\mu) \neq (0,0)$ *with*

$$a_{11}\lambda^3 + a_{12}\lambda^2\mu + a_{13}\lambda\mu^2 + a_{14}\mu^3 = 0, \text{ and} \tag{3.2.2}$$

$$a_{21}\lambda^3 + a_{22}\lambda^2\mu + a_{23}\lambda\mu^2 + a_{24}\mu^3 = 0. \tag{3.2.3}$$

These two binary cubics have a common root if and only if their resultant *vanishes. Thus in this example,* $\mathrm{Ch}_{\mathcal{C}}$ *shown in* (3.2.1) *is given by the Bézout formula for the resultant of two binary cubics [71, §4.1]*

The resultant of two binary cubics mentioned above carries further information that will be useful for us later. The two cubics share a unique common root if and only if the *Bézout matrix* shown in (3.2.1) (whose determinant is $\mathrm{Ch}_{\mathcal{C}}(A)$) has rank exactly two [44, pp. 42]. In this case, this unique common root is real. Otherwise, the two cubics (assuming they are not scalar multiples of each other), share two roots and hence their greatest common divisor (gcd) is a quadratic polynomial. The roots of this quadratic are the two common roots and they are real or complex depending on sign of the discriminant of the quadratic gcd.

Note that we can also determine whether $\mathcal{E}_{\mathbb{C}} \cap \ker_{\mathbb{C}}(Z) \neq \emptyset$ by computing a Gröbner basis of the set of Demazure polynomials and the epipolar constraints. By Hilbert's Nullstellensatz, the intersection is empty if and only if the Gröbner basis contains 1. However, this is a blackbox that needs to be invoked for each Z. The Chow form on the other hand is a fixed polynomial, depending on $\mathcal{E}_{\mathbb{C}}$, that just needs to be computed once. Both approaches are based on elimination theory.

3.3 Existence of a complex essential matrix for $m = 6$

Recall that $\mathcal{E}_{\mathbb{C}} \subseteq \mathbb{P}^8_{\mathbb{C}}$ is an irreducible variety of dimension 5 and degree 10. Hence, its Chow form $\mathrm{Ch}_{\mathcal{E}_{\mathbb{C}}}$ is a polynomial of degree $60 = 6 \times 10$ in the entries of a 6×9 matrix A. This polynomial tells precisely when $\ker_{\mathbb{C}}(Z) \cap \mathcal{E}_{\mathbb{C}} \neq \emptyset$.

Theorem 3.3.1. *Suppose $m = 6$. Then $\ker_{\mathbb{C}}(Z) \cap \mathcal{E}_{\mathbb{C}} \neq \emptyset$ if and only if $\mathrm{Ch}_{\mathcal{E}_{\mathbb{C}}}(Z) = 0$.*

The Chow form $\mathrm{Ch}_{\mathcal{E}_{\mathbb{C}}}$ is a polynomial of degree 60 in the 54 entries of the matrix Z. It has total degree 120 when expressed in the coordinates of the points x_i and y_i. So, this is a huge polynomial. It would be desirable to find a compact determinantal representation as that in Example 3.2.1, and it is done by Fløystad et al. [30].

Nevertheless, we can use Gröbner basis methods to compute specializations of $\mathrm{Ch}_{\mathcal{E}_{\mathbb{C}}}$ to few unknowns. For instance, when the given data Z depends on one or two parameters, we may wish to identify all parameter values for which an essential matrix exists. The following example illustrates that application of the Chow form.

Despite the large size of $\mathrm{Ch}_{\mathcal{E}_{\mathbb{C}}}$, a formula for it can be computed offline, once and for all, in a single preprocessing step. Thus, from the point of view of computational complexity, we can compute $\mathrm{Ch}_{\mathcal{E}_{\mathbb{C}}}$ in constant time. For any given data set with $m = 6$, we can decide whether $\mathrm{Ch}_{\mathcal{E}_{\mathbb{C}}}(Z) = 0$, and hence whether $\ker_{\mathbb{C}}(Z) \cap \mathcal{E}_{\mathbb{C}} \neq \emptyset$, by Theorem 3.3.1. This furnishes an algorithm that runs in polynomial time in m for deciding whether a given Z has an essential matrix over $\mathbb{C}$.

3.4 Existence for an essential matrix for $m = 7$

In the case $m = 7$, we present the following method to decide when Z has an essential matrix.

Suppose that $u, v \in \mathbb{R}^9$ is a basis of $\ker_{\mathbb{R}}(Z)$. Since any element of $\ker_{\mathbb{R}}(Z)$ is of the form $\lambda u + \mu v$ for $\lambda, \mu \in \mathbb{R}$, we know Z has an essential matrix if and only if there exist $\lambda, \mu \in \mathbb{R}$ such that for every $j = 1, \ldots, 10$,

$$p_j(\lambda u + \mu v) = r_{j1}\lambda^3 + r_{j2}\lambda^2\mu + r_{j3}\lambda\mu^2 + r_{j4}\mu^3 = 0$$

where p_j is a cubic in (2.3.2). We may express all 10 equations together as

$$R\omega = 0 \quad \text{where } R = (r_{ij}) \in \mathbb{R}^{10\times 4} \tag{3.4.1}$$

and $\omega := \begin{pmatrix} \lambda^3 & \lambda^2\mu & \lambda\mu^2 & \mu^3 \end{pmatrix}^\top$ parametrizes the real twisted cubic, denoted by $\mathcal{C}$. Therefore, Z has an essential matrix if and only if there is a real point in $\ker_{\mathbb{R}}(R) \cap \mathcal{C}$.

If $R = 0$ then Z has an essential matrix. If $\operatorname{rank}(R) = 1$, then one takes a nonzero row of R, say the i^{th} row. Since the binary cubic $r_{i1}\lambda^3 + r_{i2}\lambda^2\mu + r_{i3}\lambda\mu^2 + r_{i4}\mu^3$ must have a real root, Z has an essential matrix.

If $\operatorname{rank}(R) = 2$, then we take two independent rows (say the i-th row and the j-th row) and set up the Bézout matrix

$$B := \begin{pmatrix} r_{i1}r_{j2} - r_{12}r_{j1} & r_{i1}r_{j3} - r_{i3}r_{j1} & r_{i1}r_{j4} - r_{i4}r_{j1} \\ r_{i1}r_{j3} - r_{i3}r_{j1} & \substack{r_{i1}r_{j4} - r_{i4}r_{j1} \\ + r_{i2}r_{j3} - r_{i3}r_{j2}} & r_{i2}r_{j4} - r_{i4}r_{j2} \\ r_{i1}r_{j4} - r_{i4}r_{j1} & r_{i2}r_{j4} - r_{i4}r_{j2} & r_{i3}r_{j4} - r_{i4}r_{j3} \end{pmatrix}.$$

By Example 3.2.1, $\det(B)$ is the resultant of the two binary cubics

$$\begin{gathered} r_{i1}\lambda^3 + r_{i2}\lambda^2\mu + r_{i3}\lambda\mu^2 + r_{i4}\mu^3 \text{ and} \\ r_{j1}\lambda^3 + r_{j2}\lambda^2\mu + r_{j3}\lambda\mu^2 + r_{j4}\mu^3. \end{gathered} \tag{3.4.2}$$

Hence Z does not have an essential matrix when $\operatorname{rank}(B) = 3$. When $\operatorname{rank}(B) = 2$, the system (3.4.2) has a unique root and thus Z has a unique essential matrix. When $\operatorname{rank}(B) = 1$, then the sign of the discriminant of the quadratic, given by the gcd of the two cubics in (3.4.2) while $\mu = 1$, tells if Z has an essential matrix.

If $\operatorname{rank}(R) = 3$, then we know if Z has an essential matrix by checking if the unique ω in $\ker_{\mathbb{R}}(R)$ lies in $\mathcal{C}$.

If $\operatorname{rank}(R) = 4$, then Z does not have an essential matrix.

Chapter 4

EUCLIDEAN DISTANCE TO ORTHOGONALLY INVARIANT MATRIX SETS

4.1 Introduction

Finding an element of a subset $\mathcal{V}$ in $\mathbb{R}^n$ closest to a specified point y is a common task in computational mathematics, often called the *Euclidean distance (ED) minimization problem*:

$$\text{minimize} \sum_{i=1}^{n}(x_i - y_i)^2 \quad \text{subject to} \quad x \in \mathcal{V}. \tag{4.1.1}$$

Our current work is motivated by the systematic study of the "critical points" of the problem (4.1.1) in an algebraic setting, initiated in [22] and continued in [21, 48, 63]. There, the basic assumption is that $\mathcal{V}$ is a *real variety* — a zero set of finitely many polynomials with real coefficients. Consider now the set $\mathcal{V}_{\mathbb{C}}$ of complex points satisfying the defining equations of $\mathcal{V}$. Then a *critical point* of $y \in \mathbb{C}^n$ with respect to (4.1.1) is any smooth (possibly complex) point x of $\mathcal{V}_{\mathbb{C}}$ such that $y - x$ lies in the normal space of $\mathcal{V}_{\mathbb{C}}$ at x, meaning that $y - x$ lies in the span of the gradients of the defining equations at x. It was shown in [22] that for a general data point $y \in \mathbb{C}^n$, the number of (complex) critical points of (4.1.1) is a constant. This constant is the *Euclidean distance degree* of $\mathcal{V}$, denoted $\mathrm{EDdegree}(\mathcal{V})$, and is a measure of the algebraic complexity of expressing a minimizer of (4.1.1) as a function of y.

The work in this chapter is geared toward understanding the (real) critical points of *orthogonally invariant* matrix sets. Our theme is best illustrated with an example. Fix positive integers $r \leq n \leq t$ and consider the matrix set

$$\mathbb{R}_r^{n\times t} := \{X \in \mathbb{R}^{n\times t} \ : \ \mathrm{rank}(X) \leq r\}.$$

Finding the closest matrix of rank at most r to a given matrix Y arises in many applications. The set $\mathbb{R}_r^{n\times t}$ is a real variety, and the authors of [22] established that $\mathrm{EDdegree}(\mathbb{R}_r^{n\times t}) = \binom{n}{r}$.

They also provide a recipe for all the critical points of Y on $\mathbb{R}_r^{n\times t}$, which may be viewed as a generalization of the *Eckart-Young theorem.*

In the context of this chapter, what is important about $\mathbb{R}_r^{n\times t}$ is that it is *orthogonally invariant*, meaning that if $X \in \mathbb{R}_r^{n\times t}$ then for all orthogonal matrices U and V of appropriate sizes, $UXV^\top$ is also in $\mathbb{R}_r^{n\times t}$. Alternately, membership of X in the set $\mathbb{R}_r^{n\times t}$ is determined solely by the vector of singular values $\sigma(X) = (\sigma_1(X), \ldots, \sigma_n(X))$. Indeed, a matrix X lies in $\mathbb{R}_r^{n\times t}$ if and only if its vector of singular values $\sigma(X)$ lies in the set

$$\mathbb{R}_r^n := \{x \in \mathbb{R}^n : \operatorname{rank}(x) \leq r\},$$

where $\operatorname{rank}(x)$ denotes the number of nonzero coordinates of x. Observe that the geometry of the piecewise linear set $\mathbb{R}_r^n$ is much simpler than that of the highly nonlinear set $\mathbb{R}_r^{n\times t}$. In particular, $\mathbb{R}_r^n$ is also a variety and $\text{EDdegree}(\mathbb{R}_r^n) = \binom{n}{r}$. The equality $\text{EDdegree}(\mathbb{R}_r^n) = \text{EDdegree}(\mathbb{R}_r^{n\times t})$ is not accidental; in this chapter, we elucidate this phenomenon.

As alluded to above, our focus in this chapter is on a (naturally defined) real analog of critical points for (4.1.1). Namely, we say that $x \in \mathbb{R}^n$ is a (real) *ED critical point* of $y \in \mathbb{R}^n$ relative to a set $\mathcal{V} \subseteq \mathbb{R}^n$ (not necessarily a variety) if x is a smooth point of $\mathcal{V}$ and $y - x$ lies in the normal space to $\mathcal{V}$ at x; see Definition 4.2.1 for details. Our main result shows that one can always obtain the ED critical points of an orthogonally invariant matrix set by restricting to diagonal matrices, or equivalently, to an *absolutely symmetric* set obtained from the singular values of the matrices. In the last section, we explore the connection between ED critical points and the critical points in the sense of [22] that happen to be real, a surprisingly subtle topic. In particular, ED critical point calculations often help to understand $\text{EDdegree}(\mathcal{V})$, when $\mathcal{V}$ is a variety.

Sets constrained via their singular values are numerous in applications. Define $\mathcal{E} \subseteq \mathbb{R}^{3\times 3}$ to be the set of rank deficient matrices with two equal singular values:

$$\mathcal{E} := \{X \in \mathbb{R}^{3\times 3} : \sigma_1(X) = \sigma_2(X),\, \sigma_3(X) = 0\}. \tag{4.1.2}$$

A matrix $X \in \mathcal{E}$ is called an *essential matrix* in 3D computer vision and represents a pair of calibrated pinhole cameras [35, Chapter 9]. The set $\mathcal{E}$ is also a real variety cut out by the following ten cubic equations in the entries of X [28, Proposition 4]:

$$\det(X) = 0, \;\; 2XX^\top X - \operatorname{tr}(XX^\top)X = 0. \tag{4.1.3}$$

Observe that $\mathcal{E}$ is orthogonally invariant as membership of a matrix X in $\mathcal{E}$ depends only on its singular values. We will show that one can obtain the ED critical points of $\mathcal{E}$ from the simpler set $E_{3,2}$ of vectors in $\mathbb{R}^3$ with one coordinate zero and the other two equal in absolute value.

The idea of studying orthogonally invariant matrix sets $\mathcal{M} \subseteq \mathbb{R}^{n\times t}$ via their diagonal restrictions $S = \{x : \operatorname{Diag} x \in \mathcal{M}\}$ is not new, and goes back at least to von Neumann's theorem on *unitarily invariant matrix norms* [75]. In recent years, the general theme has become clear: various analytic properties of $\mathcal{M}$ and S are in one-to-one correspondence. This philosophy is sometimes called the "transfer principle"; see for instance, [17]. For example, $\mathcal{M}$ is C^p-smooth around a matrix X if and only if S is C^p-smooth around $\sigma(X)$ [15,50,67,73]. Other properties, such as *convexity* [18], *positive reach* [16], *partial smoothness* [15], and *Whitney conditions* [24] follow the same paradigm. We note in passing that the setting of eigenvalue constrained sets of nonsymmetric matrices is more complicated; see e.g. [8,52]. In the current work, we derive a transfer theorem for the ED critical points of an orthogonally invariant matrix set $\mathcal{M}$ and illustrate it on several examples. The manuscripts [16,51,53] play a central role in our work. The transfer paradigm has many useful features. The first is that in many instances, the calculation over the set $S \subseteq \mathbb{R}^n$ is simpler than the original one over $\mathcal{M}$. This can make the formulas for the number of ED critical points of $\mathcal{M}$ much more transparent as compared to the direct calculation in matrix space. The case of $\mathbb{R}^{n\times t}_r$ is an example of this. Second, our method focuses on (real) ED critical points as opposed to all complex critical points in the setting of algebraic varieties. This is useful in applications and allows us to gauge the difference between real and complex situations (under appropriate conditions).

This chapter is structured as follows. In Section 4.2 we establish basic notation and define the key notion of ED critical points of a set in $\mathbb{R}^n$ with respect to a data point y. Then in

Section 4.3 we derive our main theorem that transfers the study of ED critical points of an orthogonally invariant set of matrices to that of its diagonal restriction. Section 4.4 illustrates the technique on some concrete examples deriving formulas for the number of ED critical points in each case. Finally in Section 4.5 we establish the relationships between our work and the results in [22], by restricting to algebraic varieties. In particular, we compare our formulas to those for EDdegree in several instances. We note in passing, that many of the results in the chapter also hold for complex matrices with respect to the standard Hermitian inner product.

4.2 ED critical points of subsets of $\mathbb{R}^n$

Throughout this chapter we consider the n-dimensional Euclidean space $\mathbb{R}^n$, with a fixed orthonormal basis. Let $\langle\cdot,\cdot\rangle$ denote the inner product in this setting, and $\|\cdot\|$ denote the induced norm. The *distance* and the *projection* of a (data) point $y \in \mathbb{R}^n$ onto a subset $S \subseteq \mathbb{R}^n$, are defined by

$$\begin{aligned} \mathrm{dist}_S(y) &:= \inf_{x\in S} \|y-x\|, \qquad \text{and} \\ \mathrm{proj}_S(y) &:= \{x \in S \,:\, \mathrm{dist}_S(y) = \|y-x\|\}. \end{aligned}$$

Computing the distance of y to S amounts to solving the optimization problem:

$$\inf_{x\in S} \frac{1}{2}\|y-x\|^2 = \inf_{x\in S} \frac{1}{2}\sum_{i=1}^{n}(y_i - x_i)^2. \tag{4.2.1}$$

A classical first-order necessary condition for a putative point $x \in S$ to be optimal for (4.2.1) is that the gradient of the object function, namely, $x - y$, makes an acute angle with every vector v in the *tangent cone*[1]

$$\mathcal{T}_S(x) := \mathbb{R}_+\Big\{ \lim_{z_i\to x} \frac{z_i - x}{\|z_i - x\|} : z_i \in S\Big\}.$$

One can regard such points as generalized critical points of the distance minimization problem (4.2.1). On the other hand, in order to compare and unify our work with that in [22], we

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[1] If x is an isolated point of S, then $\mathcal{T}_S(x)$ is declared to consist only of the origin.

will impose an extra smoothness condition on a point $x \in S$ in order for it to be considered critical for (4.2.1). To this end, throughout the manuscript, we fix $p \in \{2, 3, \ldots, \infty, \omega\}$, and say that a point $x \in S$ is *C^p-smooth* if there is a neighborhood Ω of x such that $S \cap \Omega$ is an embedded C^p-smooth manifold (recall that C^ω means *real analytic*). In this case, the tangent cone $\mathcal{T}_S(x)$ is the usual tangent space in the sense of differentiable manifolds, and the criticality condition above amounts to the inclusion $y - x \in \mathcal{N}_S(x)$, where $\mathcal{N}_S(x)$ denotes the *normal space* to S at x — the orthogonal complement of $\mathcal{T}_S(x)$.

From now on we abbreviate "C^p-smooth" to "smooth". We will use S^* to denote the set of all smooth points of S. Here then is our main definition, attuned to the one considered in [22].

Definition 4.2.1 (ED critical points)**.** Consider a set $S \subseteq \mathbb{R}^n$ and a point y in $\mathbb{R}^n$. A point $x \in S$ is an *ED critical point* of y on S if the following conditions hold:

(i) $x \in S^*$, and

(ii) $y - x \in \mathcal{N}_S(x)$.

The symbol $\mathrm{C}_S(y)$ will denote the set of all ED critical points of y on S, while the cardinality of $\mathrm{C}_S(y)$ will be denoted by $C_S^{\#}(y)$.

By definition, all ED critical points of y on S are real and smooth. The number of ED critical points $C_S^{\#}(y)$ varies with y. For example, consider the parabola S shown in Figure 4.1, with an additional curve called its *ED discriminant* or *evolute*. All points y above the evolute have three ED critical points while below the evolute they have one ED critical point. Since S is an algebraic variety, we can compute its EDdegree which is three, i.e., $S_{\mathbb{C}}$ has three distinct regular complex critical points almost everywhere. We will comment more on the ED discriminant in the appendix.

Letting Θ be the collection of all Lebesgue null subsets of $\mathbb{R}^n$, the following worst-case measure of criticality arises naturally:

$$C^{\#}(S) := \inf_{\Gamma \in \Theta} \sup_{y \in \Gamma^c} C_S^{\#}(y),$$

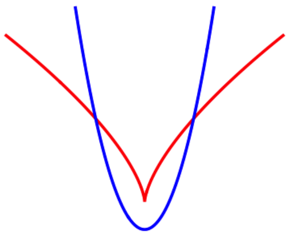

Figure 4.1: A parabola in the plane with its evolute or ED discriminant.

where Γ^c is the set complement of Γ in the ambient space. Indeed, intuitively if one believes that any single zero measure set Γ can be discarded, then $C^{\#}(S)$ measures the maximal value of $C^{\#}_S(y)$ that has a non-negligible chance of being encountered. For our purposes, one could think of $C^{\#}(S)$ as a real analog of the EDdegree considered in [22]. We discuss this further in Section 4.5.

We note in passing that this criticality measure can be infinite in pathological situations (e.g. union of countably many co-centric circles in $\mathbb{R}^2$). On the other hand, for structured sets, such as those that are semi-algebraic, this number is finite.

Proposition 4.2.2 (ED critical points of semi-algebraic sets)**.** *For any semi-algebraic set S in $\mathbb{R}^n$, the number $C^{\#}(S)$ is finite.*

Proof: Standard quantifier elimination shows that the set S^* is semi-algebraic. Define the manifolds $M_i := \{x \in S^* : \dim \mathcal{T}_S(x) = i\}$ for $i = 1, \ldots, n$. Again, quantifier elimination shows that the normal bundle

$$\Omega := \bigsqcup_i \{(x, v) : x \in M_i,\ v \in \mathcal{N}_{M_i}(x)\}$$

is a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^n$ having dimension n. Consider now the mapping $\phi\colon \Omega \to \mathbb{R}^n$ defined by $\phi(x, v) = x + v$. Notice that $C_S(y)$ coincides with the projection of the preimage $\phi^{-1}(y)$ onto x. For dimensional reasons, there is a full-measure set $D \subseteq \mathbb{R}^n$ so that for every $y \in D$ the preimage $\phi^{-1}(y)$ has finite cardinality. Moreover, since preimages of a semi-algebraic map have a uniformly bounded number of connected components (see e.g. [11, Theorem 3.12]), the quantity $C_S^{\#}(y)$ is uniformly bounded on D. The result follows. $\square$

Next we consider ED critical points on a union of finitely many sets. Consider a finite collection of sets $\{S_i\}_{i\in\mathcal{I}}$ in $\mathbb{R}^n$ and define the union $U := \bigcup_{i\in\mathcal{I}} S_i$. In general, the two sets $C_U(y)$ and $\bigcup_{i\in\mathcal{I}} C_{S_i}(y)$ can be vastly different because of the way the sets S_i intersect. For instance, think of two half spaces whose union is $\mathbb{R}^n$. A simple situation in which more can be said is when locally around each critical point $x \in C_U(y)$, the set U coincides with S_i for some $i \in \mathcal{I}$. In that situation, if $x \in C_U(y)$ then x is also in $C_{S_i}(y)$. An important situation in this chapter is the case of all S_i being affine subspaces. We say that a finite collection of sets $\{S_i\}_{i\in\mathcal{I}}$ in $\mathbb{R}^n$ is *minimally defined* if no S_i is contained in any S_j for distinct indices i and j.

Proposition 4.2.3 (ED critical points of affine complexes)**.** *Consider a finite collection of affine subspaces $\{S_i\}_{i\in\mathcal{I}}$ in $\mathbb{R}^n$, that is minimally defined, and let $U := \bigcup_{i\in\mathcal{I}} S_i$. Then we have*

$$U^* = \{x \in \mathbb{R}^n : \text{ there exists unique } i \in \mathcal{I} \text{ with } x \in S_i\}. \tag{4.2.2}$$

Consequently for any $y \in \mathbb{R}^n$, we have

$$\mathrm{C}_U(y) = \bigsqcup_{i\in\mathcal{I}} \left(\mathrm{proj}_{S_i}(y) \cap U^*\right), \tag{4.2.3}$$

and the equality

$$C^{\#}(U) = |\mathcal{I}|.$$

Proof: The inclusion $\supseteq$ in (4.2.2) follows since the sets S_i are affine. To see the reverse inclusion, observe that the tangent cone to U at any point x coincides with the union $\bigcup_{i\in\mathcal{I}:\, x\in S_i}(S_i - x)$. For $x \in U^*$, the cone $\mathcal{T}_U(x)$ is itself a linear subspace, and hence, by the minimality of the collection $\{S_i\}_{i\in\mathcal{I}}$, there exists a unique index $i \in \mathcal{I}$ satisfying $x \in S_i$. This establishes (4.2.2). Equation (4.2.3) then follows immediately.

To see the last claim, consider the set

$$Z := \{y \in \mathbb{R}^n : \exists i \in \mathcal{I} \text{ with } \mathrm{proj}_{S_i}(y) \cap U^* = \emptyset\}.$$

We will show that Z is a finite union of proper affine subspaces of $\mathbb{R}^n$. To see this, consider a point $y \in Z$ along with an index $i \in \mathcal{I}$ satisfying $\mathrm{proj}_{S_i}(y) \cap U^* = \emptyset$. From (4.2.2), we conclude that there exists $j \in \mathcal{I}$, distinct from i, satisfying $\mathrm{proj}_{S_i}(y) \cap S_j \neq \emptyset$. Thus y lies in the set $S_i^\perp + (S_i \cap S_j)$. Since the collection $\{S_i\}$ is minimally defined, the intersection $S_i \cap S_j$ has dimension strictly smaller than that of S_i. Consequently the affine space $S_i^\perp + (S_i \cap S_j)$ has dimension strictly smaller than n. Taking the union over all pairs of distinct indices $i, j \in \mathcal{I}$, we deduce that Z is a finite union of proper affine subspaces of $\mathbb{R}^n$. Therefore, by (4.2.3), $C_U^\#(y) = |\mathcal{I}|$ for all $y \notin Z$ which proves that $C^\#(U) = |\mathcal{I}|$. □

The following two elementary examples illustrate Proposition 4.2.3; these are essentially the piecewise linear examples alluded to in the introduction. In the next section, we will use them to obtain the ED critical points of the (nonlinear) matrix sets $\mathbb{R}_r^{n\times t}$ and $\mathcal{E}$. In what follows we set $[n] := \{1, \ldots, n\}$.

Example 4.2.4 (Union of r-dimensional coordinate subspaces)**.** For $x \in \mathbb{R}^n$, define the rank of x, denoted by $\mathrm{rank}(x)$, to be the number of nonzero coordinates of x. Fix an integer $r \in [n]$ and recall the set

$$\mathbb{R}_r^n = \{x \in \mathbb{R}^n \ : \ \mathrm{rank}(x) \leq r\}. \tag{4.2.4}$$

Define $\mathcal{I}$ to be the collection of distinct cardinality r subsets of $[n]$. Then $|\mathcal{I}| = \binom{n}{r}$, and

$$\mathbb{R}_r^n = \bigcup_{S \in \mathcal{I}} \mathrm{span}\,\{e_i\}_{i \in S},$$

where e_i denotes the i'th coordinate vector in $\mathbb{R}^n$. This representation of $\mathbb{R}_r^n$ is minimal and therefore Proposition 4.2.3 implies the equality $C^\#(\mathbb{R}_r^n) = |\mathcal{I}| = \binom{n}{r}$.

Example 4.2.5 (k nonzero entries equal up to sign)**.** Fix an integer $k \in [n]$, and let $E_{n,k}$ be the set of points in $\mathbb{R}^n$ with the property that k of their coordinates are equal in absolute value

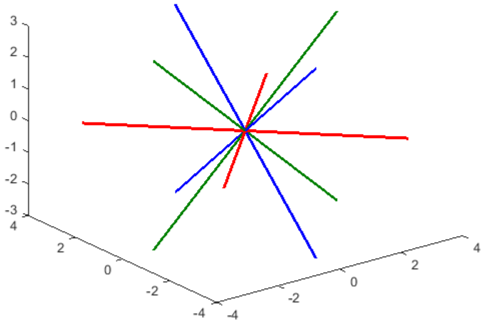

Figure 4.2: The set $E_{3,2}$

and the other $n-k$ coordinates are zero. Note that $E_{n,k}$ can be written as the union of $2^{k-1}\binom{n}{k}$ linear subspaces (minimally defined). Proposition 4.2.3 then implies that $C^{\#}(E_{n,k}) = 2^{k-1}\binom{n}{k}$. As a special case, the set $E_{3,2}$, which is a union of six lines in $\mathbb{R}^3$ (see Figure 4.2), satisfies $C^{\#}(E_{3,2}) = 6$.

4.3 ED critical points of orthogonally invariant matrix sets

In this section we describe our main result which yields an elegant technique for counting the ED critical points of orthogonally invariant matrix sets, based on the tools established in [16, 51, 53]. Setting the notation, let $\mathbb{R}^{n\times t}$ denote the set of real $n \times t$ matrices, where we assume without loss of generality that $n \leq t$. The singular value map $\sigma\colon \mathbb{R}^{n\times t} \to \mathbb{R}^n$ assigns to each matrix $X \in \mathbb{R}^{n\times t}$ the vector of its singular values $\sigma(X) := (\sigma_1(X), \ldots, \sigma_n(X))$ arranged in non-increasing order. The corresponding inverse map is defined as

$$\sigma^{-1}(S) := \{X \in \mathbb{R}^{n\times t} : \sigma(X) \in S\} \qquad \text{for any subset } S \text{ of } \mathbb{R}^n.$$

We will be interested in subsets of $\mathbb{R}^{n\times t}$ that are invariant under multiplication on the left and right by orthogonal matrices. In what follows, we let $\mathcal{O}^s$ denote the group of real $s \times s$ orthogonal matrices.

Definition 4.3.1 (Orthogonal invariance)**.**
A set $\mathcal{M} \subseteq \mathbb{R}^{n\times t}$ is *orthogonally invariant* if

$$\mathcal{M} = U\mathcal{M}V^\top \qquad \text{for all } U \in \mathcal{O}^n \text{ and } V \in \mathcal{O}^t.$$

An example of an orthogonally invariant matrix set is the essential variety

$$\mathcal{E} = \{X \in \mathbb{R}^{3\times 3} \ : \ \sigma_1(X) = \sigma_2(X),\, \sigma_3(X) = 0\}$$

considered in the introduction. At first sight, $\mathcal{E}$ is a complicated set; it is a highly nonlinear real variety cut out by the polynomials in (4.1.3). Since orthogonally invariant matrix sets are precisely those matrix sets for which membership is determined solely by the singular values of its elements, the restriction of such sets to the subspace of diagonal matrices plays an important role. Our strategy will be to exploit this observation to calculate the ED critical points of orthogonally invariant matrix sets.

A mapping $\pi : [n] \to \{\pm 1, \ldots, \pm n\}$ is a *signed permutation* if the assignment $i \mapsto |\pi(i)|$ is a permutation on $[n]$ in the usual sense. We let $\Pi_n^\pm$ denote the set of signed permutations on $[n]$. Note that any signed permutation $\pi \in \Pi_n^\pm$ induces a linear map $\mathbb{R}^n \to \mathbb{R}^n$ which we also denote by π.

Definition 4.3.2 (Absolute symmetry)**.** A set $S \subseteq \mathbb{R}^n$ is said to be *absolutely symmetric* if

$$S = \pi S \qquad \text{for all } \pi \in \Pi_n^\pm.$$

For any set $S \subseteq \mathbb{R}^n$, we call $\Pi_n^\pm S := \{\pi x \ : \ \pi \in \Pi_n^\pm \text{ and } x \in S\}$ the *absolute symmetrization* of S.

For a vector $x \in \mathbb{R}^n$, let $\operatorname{Diag} x \in \mathbb{R}^{n\times t}$ denote the matrix with x in its principal diagonal and zeros elsewhere. If $\mathcal{M} \subseteq \mathbb{R}^{n\times t}$ is orthogonally invariant, then the set $\{x \in \mathbb{R}^n : \operatorname{Diag} x \in \mathcal{M}\}$ is absolutely symmetric, and we have the following basic observation (see e.g. [53, Proposition 5.1]).

Theorem 4.3.3 (Diagonal Correspondence)**.** *A set $\mathcal{M} \subseteq \mathbb{R}^{n\times t}$ is orthogonally invariant if and only if there exists an absolutely symmetric set $S \subseteq \mathbb{R}^n$ such that $\mathcal{M} = \sigma^{-1}(S)$.*

Thus the assignment σ^{-1} from the family of absolutely symmetric sets in $\mathbb{R}^n$ to the family of orthogonally invariant sets in $\mathbb{R}^{n\times t}$ is a bijection. As discussed in the introduction, it so happens that various analytic properties of absolutely symmetric sets S and orthogonally invariant sets $\sigma^{-1}(S)$ are in one-to-one correspondence; see e.g. [15–18, 24, 50, 67, 73, 75]. This may seem surprising since σ is a highly nonsmooth mapping; it is the absolute symmetry of the underlying set S that makes up for the fact. As an illustration, we show in Theorem 4.3.4 how the transfer principle for algebraicity can be used to see that the set of essential matrices $\mathcal{E}$ is a variety, even without explicitly knowing its defining equations (4.1.3). The proof is entirely analogous to the symmetric case in [17, Proposition 1.1]; we provide an argument for completeness.

Recall that a set $\mathcal{V} \subseteq \mathbb{R}^n$ is called a *real variety* if there exist polynomials $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ such that $\mathcal{V} = \{x \in \mathbb{R}^n \,:\, f_1(x) = \cdots = f_s(x) = 0\}$. Note that if $x \in \mathcal{V}$, then $\sum_{i=1}^{s} f_i^2(x) = 0$, and conversely if a point $x \in \mathbb{R}^n$ satisfies $\sum_{i=1}^{s} f_i^2(x) = 0$, then $f_i(x) = 0$ for all $i = 1, \ldots, s$, and hence, $x \in \mathcal{V}$. Therefore, $\mathcal{V}$ can be described as the set of real zeros of a single sum of squares polynomial with real coefficients, which we call a defining polynomial of $\mathcal{V}$.

Theorem 4.3.4 (Transfer of algebraicity)**.** *Suppose $S \subseteq \mathbb{R}^n$ is an absolutely symmetric set. Then $\sigma^{-1}(S) \subseteq \mathbb{R}^{n\times t}$ is a real variety if and only if S is a real variety.*

Proof: If $\sigma^{-1}(S)$ is a real variety, then $S = \{x \in \mathbb{R}^n : \operatorname{Diag} x \in \sigma^{-1}(S)\}$ is the real variety cut out by the equations defining $\sigma^{-1}(S)$ when x_{ij} are set to zero for all $i \neq j$. Suppose conversely that S is a real variety with defining polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$. Consider the polynomial $\hat{f}(x) := \sum_{\pi \in \Pi_n^{\pm}} f^2(\pi x)$. Note that $\hat{f}(x)$ is also a defining polynomial of S, and hence, $\sigma^{-1}(S)$ is the zero level set of $\hat{f} \circ \sigma$. To finish the proof, we just need to show that $\hat{f} \circ \sigma(X)$ is a polynomial in the entries of X. To this end, since $\hat{f}$ is invariant under sign changes, it is easy to see that $\hat{f}$ is a symmetric polynomial in the squares $x_1^2, \ldots, x_n^2$, that is we may write $\hat{f}(x) = g(x_1^2, \ldots, x_n^2)$ for some symmetric polynomial g. By the fundamental theorem of symmetric polynomials (see e.g. [56]) we may write g as a polynomial of elementary symmetric polynomials $\epsilon_1, \ldots, \epsilon_n$. On the other hand, the expressions $\epsilon_i(\sigma_1^2(X), \ldots, \sigma_n^2(X))$ coincide with the coefficients of the characteristic polynomial of $X^\top X$ and are hence polynomial expressions in the entries of X. $\square$

Another useful illustration of transfer principles concerns the distance to orthogonally invariant matrix sets. Recall that for a set $S \subseteq \mathbb{R}^n$, the *distance* and *projection* of a point $y \in \mathbb{R}^n$ to (respectively, onto) S are defined by $\operatorname{dist}_S(y) := \inf_{x \in S} \|y - x\|$, and $\operatorname{proj}_S(y) := \{x \in S \,:\, \operatorname{dist}_S(y) = \|y - x\|\}$. (The distance and the projection in the matrix space $\mathbb{R}^{n \times t}$ are defined analogously with respect to the *Frobenius norm* $\|Y\| := \sqrt{\sum_{i,j} Y_{ij}^2}$.) Then for an absolutely symmetric set $S \subseteq \mathbb{R}^n$, the following holds [16, Proposition 8]:

$$\operatorname{dist}_{\sigma^{-1}(S)}(Y) = \operatorname{dist}_S(\sigma(Y)). \tag{4.3.1}$$

This in turn implies the following result, which was essentially proved in [16, Proposition 8], though not formally recorded. We provide a proof sketch for completeness.

Proposition 4.3.5 (Projections onto orthogonally invariant matrix sets)**.** *If $S \subseteq \mathbb{R}^n$ is an absolutely symmetric set, then for any matrix $Y \in \mathbb{R}^{n \times t}$, the projection $\operatorname{proj}_{\sigma^{-1}(S)}(Y)$ is precisely the set*

$$\left\{ U(\operatorname{Diag} x)V^\top \,:\, U \in \mathcal{O}^n, V \in \mathcal{O}^t \text{ where } \begin{array}{l} Y = U(\operatorname{Diag} \sigma(Y))V^\top, \\ x \in \operatorname{proj}_S(\sigma(Y)) \end{array} \right\}. \tag{4.3.2}$$

Proof: Consider first matrices $U \in \mathcal{O}^n, V \in \mathcal{O}^t$ with $Y = U(\operatorname{Diag} \sigma(Y))V^\top$ and a vector $x \in \operatorname{proj}_S(\sigma(Y))$. Define $X := U(\operatorname{Diag} x)V^\top$ and observe the equalities:

$$\|X - Y\| = \|x - \sigma(Y)\| = \operatorname{dist}_S(\sigma(Y)) = \operatorname{dist}_{\sigma^{-1}(S)}(Y),$$

where the last equality follows from (4.3.1). Hence the inclusion $X \in \operatorname{proj}_{\sigma^{-1}(S)}(Y)$ is valid, as claimed. Conversely, for any matrix $X \in \operatorname{proj}_{\sigma^{-1}(S)}(Y)$ observe

$$\operatorname{dist}_{\sigma^{-1}(S)}(Y) = \|X - Y\| \geq \|\sigma(X) - \sigma(Y)\| \geq \operatorname{dist}_S(\sigma(Y)),$$

where the first inequality follows from von Neumann's trace inequality [45, p. 182]; see also [53, Theorem 4.6]. Equation (4.3.1) implies equality throughout. In particular we get $\sigma(X) \in \operatorname{proj}_S(\sigma(Y))$. Moreover, applying the equality characterization in the trace inequality [53, Theorem 4.6], we conclude that X and Y admit a simultaneous ordered singular value decomposition, that is, there exist matrices $U \in \mathcal{O}^n, V \in \mathcal{O}^t$ with $Y = U(\mathrm{Diag}\ \sigma(Y))V^\top$ and $X = U(\mathrm{Diag}\ \sigma(X))V^\top$. This completes the proof. $\square$

A consequence of Proposition 4.3.5 is a convenient representation of the normal space at a point in an orthogonally invariant manifold. First, the set $\sigma^{-1}(S)$ is smooth around X if and only if S is smooth around $\sigma(X)$ (see e.g. [15, Theorem 2.4]). Next recall that at any C^2-smooth point x of a set M, the following equivalence holds:

$$z \in \mathcal{N}_M(x) \quad \Longleftrightarrow \quad \{x\} = \operatorname{proj}_M(x + \lambda z) \quad \text{for some} \quad \lambda > 0.$$

Proposition 4.3.5 and a short computation implies that at any smooth point X of $\sigma^{-1}(S)$ the following formula holds:

$$\mathcal{N}_{\sigma^{-1}(S)}(X) = \left\{ U(\mathrm{Diag}\, z)V^\top \,:\, U \in \mathcal{O}^n, V \in \mathcal{O}^t \text{ with } \begin{array}{c} X = U(\mathrm{Diag}\, \sigma(X))V^\top, \\ z \in \mathcal{N}_S(\sigma(X)) \end{array} \right\}.$$

A more general expression without smoothness assumptions can be found in [53, Theorem 7.1], with a much simplified argument in [23]. Using the above facts, we now derive a transfer principle for ED critical points which is the main result of this section. The proof uses the following lemma.

Lemma 4.3.6 (Transfer of Lebesgue null sets)**.** *Consider an absolutely symmetric set $S \subseteq \mathbb{R}^n$. Then S is Lebesgue null if and only if $\sigma^{-1}(S)$ is Lebesgue null.*

We have placed the proof of the lemma above at the end of the section so as to not stray from the narrative.

Theorem 4.3.7. (ED critical points of orthogonally invariant matrix sets)

Consider an absolutely symmetric set $S \subseteq \mathbb{R}^n$ and a matrix $Y \in \mathbb{R}^{n \times t}$ along with a singular *value decomposition* $Y = \overline{U}(\operatorname{Diag}\sigma(Y))\overline{V}^\top$. *Suppose moreover, that* Y *has all distinct singular values. Then for the orthogonally invariant set* $\mathcal{M} := \sigma^{-1}(S)$,

$$\mathrm{C}_{\mathcal{M}}(Y) = \{\overline{U}\big(\operatorname{Diag}\omega\big)\overline{V}^\top \ : \ \omega \in \mathrm{C}_S(\sigma(Y))\}. \tag{4.3.3}$$

Consequently, we also obtain the equality $C^{\#}(\mathcal{M}) = C^{\#}(\mathcal{S})$.

Proof: Consider a critical point $X \in \mathrm{C}_{\mathcal{M}}(Y)$. Then there exist orthogonal matrices U and V, and a vector $z \in \mathcal{N}_S(\sigma(X))$ satisfying

$$X = U(\operatorname{Diag}\sigma(X))V^\top \quad \text{and} \quad Y = X + U(\operatorname{Diag} z)V^\top.$$

Hence we deduce

$$Y = U(\operatorname{Diag}(\sigma(X) + z))V^\top.$$

By uniqueness of singular values and singular vectors, there exists a signed permutation matrix $\pi\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\sigma(Y) = \pi(\sigma(X)) + \pi(z).$$

On the other hand, it is easy to verify the inclusion $\pi(z) \in \mathcal{N}_S(\pi\sigma(X))$. Therefore, we obtain $\pi(\sigma(X)) \in C_S(\sigma(Y))$.

Define now the extended signed permutation matrix $\hat{\pi} := \left(\begin{smallmatrix} \pi & 0 \\ 0 & I \end{smallmatrix}\right) \in \mathbb{R}^{t \times t}$. Observe then the equality

$$Y = U(\operatorname{Diag}\pi^{-1}(\sigma(Y)))V^\top = (U\pi)(\operatorname{Diag}\sigma(Y))(V\hat{\pi})^\top.$$

Changing the left and the right singular vector pairs in U and in V by the same sign change in the first place, if needed, we can ensure $U\pi = \overline{U}$ and $V\hat{\pi} = \overline{V}$. Hence,

$$X = U(\operatorname{Diag}\sigma(X))V^\top = \overline{U}(\operatorname{Diag}\pi(\sigma(X)))\overline{V}^\top,$$

as claimed.

Now to prove the reverse inclusion, consider a point $\omega \in \mathrm{C}_S(\sigma(Y))$. Thus there exists a vector $z \in \mathcal{N}_S(\omega)$ satisfying

$$\sigma(Y) - \omega = z.$$

Choose a signed permutation π so that $\pi(\omega)$ is nonnegative and nonincreasing. Observe then that

$$\pi(\sigma(Y)) - \pi(\omega) = \pi(z) \in \mathcal{N}_S(\pi(\omega)).$$

Defining

$$X := \overline{U}(\operatorname{Diag}\omega)\overline{V}^{\top} = (\overline{U}\pi)(\operatorname{Diag}\pi(\omega))(\overline{V}\hat{\pi})^{\top},$$

we deduce

$$X + (\overline{U}\pi)(\operatorname{Diag}\pi(z))(\overline{V}\hat{\pi})^{\top} = (\overline{U}\pi)(\operatorname{Diag}\pi(\sigma(Y)))(\overline{V}\hat{\pi})^{\top} = Y.$$

The inclusion $X \in \mathrm{C}_S(Y)$ follows. This establishes (4.3.3).

Define now the set

$$\Delta := \{x \in \mathbb{R}^n : |x_i| = |x_j| \text{ for some indices } i \neq j\}.$$

Consider a Lebesgue null set $\Gamma \subseteq \mathbb{R}^n$. Then clearly the absolutely symmetric set $\widehat{\Gamma} := \Delta \cup \Pi^{\pm}_n \Gamma$ is Lebesgue null as well. Lemma 4.3.6 shows that $\sigma^{-1}(\widehat{\Gamma})$ is also Lebesgue null. Note for all $\pi \in \Pi^{\pm}_n$ and $y \in \mathbb{R}^n$, one has $C_S(\pi y) = \pi C_S(y)$ and hence $C^{\#}_S(\pi y) = C^{\#}_S(y)$. Combining these observations, we obtain

$$C^{\#}(\mathcal{M}) \leq \sup_{Y \in \sigma^{-1}(\widehat{\Gamma}^c)} C^{\#}_{\mathcal{M}}(Y) = \sup_{y \in \widehat{\Gamma}^c} C^{\#}_S(y) \leq \sup_{y \in \Gamma^c} C^{\#}_S(y),$$

where the middle equality follows from (4.3.3). Since Γ was an arbitrary null set, we deduce $C^{\#}(\mathcal{M}) \leq C^{\#}(S)$.

To see the reverse inequality, consider a Lebesgue null set $\Gamma \subseteq \mathbb{R}^{n \times t}$. Consider the following subset of Γ:

$$\widehat{\Gamma} := \{Y \in \Gamma : UYV^{\top} \in \Gamma \text{ for all } U \in \mathcal{O}^n, V \in \mathcal{O}^t\}.$$

Clearly $\widehat{\Gamma}$ is an orthogonally invariant Lebesgue null set. Now trivially we have

$$\sup_{Y \in \widehat{\Gamma}^c} C^{\#}_{\mathcal{M}}(Y) \geq \sup_{Y \in \Gamma^c} C^{\#}_{\mathcal{M}}(Y).$$

On the other hand, it is easy to verify the equation $C_{\mathcal{M}}(UYV^\top) = U(C_{\mathcal{M}}(Y))V^\top$ and thus $C^\#_{\mathcal{M}}(UYV^\top) = C^\#_{\mathcal{M}}(Y)$ for all $Y \in \mathbb{R}^{n\times t}$ and $U \in \mathcal{O}^n, V \in \mathcal{O}^t$. Hence

$$\sup_{Y\in\widehat{\Gamma}^c} C^\#_{\mathcal{M}}(Y) = \sup_{Y\in\Gamma^c} C^\#_{\mathcal{M}}(Y).$$

Notice $\widetilde{\Gamma} := \widehat{\Gamma} \cup \sigma^{-1}(\Delta)$ is also a Lebesgue null orthogonally invariant set. Hence we deduce

$$\sup_{Y\in\Gamma^c} C^\#_{\mathcal{M}}(Y) \geq \sup_{Y\in\widetilde{\Gamma}^c} C^\#_{\mathcal{M}}(Y) = \sup_{y\in\sigma(\widetilde{\Gamma})^c} C^\#_S(y) \geq C^\#(S),$$

where the middle equality follows from equation (4.3.3) and last inequality follows from Lemma 4.3.6. Since Γ was an arbitrary Lebesgue null set, we deduce the reverse inequality $C^\#(\mathcal{M}) \geq C^\#(S)$, and hence equality, as claimed. □

The assumption that singular values of the data point Y are distinct is essential in Theorem 4.3.7, as the following example shows.

Example 4.3.8 (Nondistinct singular values)**.** Consider the orthogonally invariant set $\mathcal{M} = \{X \in \mathbb{R}^{2\times 2} \ : \ \det(X) = 0\}$. We may write $\mathcal{M} = \sigma^{-1}(S)$ where $S := \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1x_2 = 0\}$ is absolutely symmetric. Then $\mathrm{C}_S((1, 1)) = \{(1, 0), (0, 1)\}$. On the other hand, one can verify that $uu^\top \in \mathrm{C}_{\mathcal{M}}(I_2)$ for each $u \in \mathbb{R}^2$ of norm one, where I_2 is the 2×2 identity matrix.

We now apply Theorem 4.3.7 to our two running examples.

Example 4.3.9 (Matrices of rank at most r)**.** Fix $r \leq n$ and recall the orthogonally invariant set

$$\mathbb{R}^{n\times t}_r = \{X \in \mathbb{R}^{n\times t} \ : \ \mathrm{rank}(X) \leq r\}.$$

This set is a determinantal variety cut out by the $(r + 1) \times (r + 1)$ minors of a symbolic $n \times t$ matrix. One sees that $\sigma^{-1}(\mathbb{R}^n_r) = \mathbb{R}^{n\times t}_r$ where $\mathbb{R}^n_r$ is the absolutely symmetric set defined in (4.2.4). Theorem 4.3.7 and Example 4.2.4 together imply that

$$C^\#(\mathbb{R}^{n\times t}_r) = C^\#(\mathbb{R}^n_r) = \binom{n}{r}.$$

This number is also the ED degree of $\mathbb{R}^{n\times t}_r$ (cf. [22, Example 2.3]); we will revisit this point in Section 4.5. In particular, the ED critical points of a general data matrix Y, with singular value decomposition $Y = U(\text{Diag}\ \sigma(Y))V^\top$, are the $\binom{n}{r}$ matrices obtained by setting to zero all possible choices of $n-r$ singular values in this singular value decomposition. The matrix

$$U\ (\text{Diag}\,(\sigma_1(Y), \ldots, \sigma_r(Y), \underbrace{0, \ldots, 0}_{n-r}))\ V^\top$$

is a nearest element of $\mathbb{R}^{n\times t}_r$ to Y. This is precisely the statement of the classical *Eckart-Young theorem.*

Example 4.3.10 (Essential variety)**.** The essential variety $\mathcal{E}$ defined in (4.1.2) is orthogonally invariant with $\mathcal{E} = \sigma^{-1}(E_{3,2})$ where $E_{3,2}$ is the absolutely symmetric set defined in Example 4.2.5. Then by Theorem 4.3.7 and Example 4.2.5,

$$C^{\#}(\mathcal{E}) = C^{\#}(E_{3,2}) = 6.$$

Furthermore, for a matrix $Y \in \mathbb{R}^{3\times 3}$ along with a singular value decomposition $Y = U(\text{Diag}\,\sigma(Y))V^\top$, by (4.3.2), the matrix

$$U\ \left(\text{Diag}\ \left(\frac{\sigma_1(Y)+\sigma_2(Y)}{2}, \frac{\sigma_1(Y)+\sigma_2(Y)}{2}, 0\right)\right)\ V^\top$$

is a nearest element of $\mathcal{E}$ to Y. This is precisely Hartley's result [34, Theorem 5] which is well known in the computer vision community. The six ED critical points of a general $Y = U(\text{Diag}\ \sigma(Y))V^\top$ are the matrices $U(\text{Diag}\ x)V^\top$ where x varies over the following vectors:

$$\begin{array}{ll}
\left(\frac{\sigma_1(Y)+\sigma_2(Y)}{2}, \frac{\sigma_1(Y)+\sigma_2(Y)}{2}, 0\right), & \left(\frac{\sigma_1(Y)-\sigma_2(Y)}{2}, \frac{-\sigma_1(Y)+\sigma_2(Y)}{2}, 0\right), \\
\left(\frac{\sigma_1(Y)+\sigma_3(Y)}{2}, 0, \frac{\sigma_1(Y)+\sigma_3(Y)}{2}\right), & \left(\frac{\sigma_1(Y)-\sigma_3(Y)}{2}, 0, \frac{-\sigma_1(Y)+\sigma_3(Y)}{2}\right), \\
\left(0, \frac{\sigma_2(Y)+\sigma_3(Y)}{2}, \frac{\sigma_2(Y)+\sigma_3(Y)}{2}\right), & \left(0, \frac{\sigma_2(Y)-\sigma_3(Y)}{2}, \frac{-\sigma_2(Y)+\sigma_3(Y)}{2}\right).
\end{array}$$

We now give the proof of the lemma evoked in the proof of Theorem 4.3.7.

Proof of Lemma 4.3.6. Before we get to the main part of the proof, we need some notation. To this end, define $\mathcal{O}^t_n$ to be the set of $t \times n$ matrices with orthonormal columns. Note that

$\mathcal{O}^t_n$ is a smooth manifold of dimension $nt - \frac{n(n+1)}{2}$, and is called a *Stiefel manifold*; see for example [7, Proposition A.4]. In particular $\mathcal{O}^n$ is a manifold of dimension $\frac{n(n-1)}{2}$. We deduce that the product manifold $\mathcal{O}^n \times \mathcal{O}^t_n \times \mathbb{R}^n$ has dimension exactly nt, same as $\mathbb{R}^{n\times t}$. For any vector $x \in \mathbb{R}^n$ let sDiag x denote the $n \times n$ diagonal matrix with x on the diagonal. Define now the mapping $\Gamma\colon \mathcal{O}^n \times \mathcal{O}^t_n \times \mathbb{R}^n \to \mathbb{R}^{n\times t}$ by setting $\Gamma(U, V, x) = U(\text{sDiag } x)V^\top$. Notice that for the absolutely symmetric set S, we have equality $\Gamma(\mathcal{O}^n \times \mathcal{O}^t_n \times S) = \sigma^{-1}(S)$.

Suppose now that S is a Lebesgue null set. Then clearly the set $\mathcal{O}^n \times \mathcal{O}^t_n \times S$ is Lebesgue null in the manifold $\mathcal{O}^n \times \mathcal{O}^t_n \times \mathbb{R}^n$. Since Γ is smooth, the image $\Gamma(\mathcal{O}^n \times \mathcal{O}^t_n \times S)$, which coincides with $\sigma^{-1}(S)$, is Lebesgue null $\mathbb{R}^{n\times t}$, as claimed. Conversely, suppose that $\sigma^{-1}(S)$ is Lebesgue null. It is well-known that Γ is a local diffeomorphism on an open full-measure subset of the domain manifold $\mathcal{O}^n \times \mathcal{O}^t_n \times \mathbb{R}^n$. To see this quickly, define the set

$$\Delta = \{x \in \mathbb{R}^n : |x_i| = |x_j| \text{ for some indices } i \neq j\}.$$

Observe that since Γ is a semi-algebraic map, we may stratify $\mathcal{O}^n \times \mathcal{O}^t_n \times \Delta^c$ into finitely many smooth manifolds $\mathcal{M}_i$ so that the restriction of Γ to each $\mathcal{M}_i$ has constant rank. Suppose now for the sake of contradiction that the derivative of Γ has deficient rank on some maximal dimensional manifold $\mathcal{M}_i$. Then by the constant rank theorem [49, Theorem 5.13], there exists a nontrivial smooth path $(U(t), V(t), x(t))$ in $\mathcal{M}_i$ so that Γ is constant on the path, meaning that the matrices $U(t)(\text{sDiag } x(t))V(t)^\top$ are equal for all t. On the other hand, taking into account that the coordinates of $x(t)$ are distinct and appealing to uniqueness of singular values and singular vectors, we obtain a contradiction. Hence Γ is a local diffeomorphism on an open full measure subset of $\mathcal{O}^n \times \mathcal{O}^t_n \times \mathbb{R}^n$. It follows immediately that S is Lebesgue null, since otherwise the image $\Gamma(\mathcal{O}^n \times \mathcal{O}^t_n \times S)$ would fail to be Lebesgue null. This completes the proof. $\square$

4.4 Applications

We now apply the techniques from the last section to calculate and count the ED critical points of several orthogonally invariant matrix sets.

Example 4.4.1 (Matrices orthogonally equivalent to a given matrix)**.** Fix a matrix $A \in \mathbb{R}^{n\times t}$. We say $X \in \mathbb{R}^{n\times t}$ is *orthogonally equivalent* to A if $X = UAV^\top$ for some $U \in \mathcal{O}^n$ and $V \in \mathcal{O}^t$. Let $\mathcal{M}_A \subseteq \mathbb{R}^{n\times t}$ be the set of matrices that are orthogonally equivalent to A. Then $\mathcal{M}_A$ is orthogonally invariant and $\mathcal{M}_A = \sigma^{-1}(\sigma(A))$. By Theorem 4.3.7,

$$C^{\#}(\mathcal{M}_A) = |\Pi_n^{\pm}\{\sigma(A)\}| = 2^n \frac{n!}{n_1! \dots n_k!}$$

where $\sigma(A) = (\underbrace{\sigma_1, \dots, \sigma_1}_{n_1}, \dots, \underbrace{\sigma_k, \dots, \sigma_k}_{n_k})$.

Example 4.4.2 (Real orthogonal group)**.** The real orthogonal group $\mathcal{O}^n$ is a special case of the previous example because $\mathcal{O}^n$ is the set of $n \times n$ real matrices that are orthogonally equivalent to the $n \times n$ identity matrix, namely,

$$\mathcal{O}^n = \sigma^{-1}((\underbrace{1, \dots, 1}_{n})).$$

Hence $C^{\#}(\mathcal{O}^n) = 2^n$ which is also the ED degree of $\mathcal{O}^n$ (cf. [21, Theorem 3.2]).

By Proposition 4.3.5, for any $Y \in \mathbb{R}^{n\times n}$ with singular value decomposition $Y = U(\operatorname{Diag}\sigma(Y))V^\top$ where $U, V \in \mathcal{O}^n$, one has $UV^\top \in \operatorname{proj}_{\mathcal{O}^n}(Y)$. Moreover, if the singular values of Y are distinct, then by Theorem 4.3.7, the set of critical points of Y on $\mathcal{O}^n$ is $\{U(\operatorname{Diag} x)V^\top : x_i = \pm 1 \text{ for all } i\}$. These results were also obtained by Draisma and Baaijens [21] using algebraic methods.

Example 4.4.3 (Unit sphere of Schatten d-norm)**.** The Schatten d-norm of a matrix $X \in \mathbb{R}^{n\times t}$ is defined as

$$\|X\|_d := \left[\sum_{i=1}^{n} (\sigma_i(X))^d\right]^{1/d}.$$

Note the following fact: $\|UXV^\top\|_d = \|X\|_d$ for any $X \in \mathbb{R}^{n\times t}$, $U \in \mathcal{O}^n$, $V \in \mathcal{O}^t$. Hence the unit sphere of the Schatten d-norm

$$\mathcal{F}_{n,t,d} := \{X \in \mathbb{R}^{n\times t} \;:\; \|X\|_d = 1\}$$

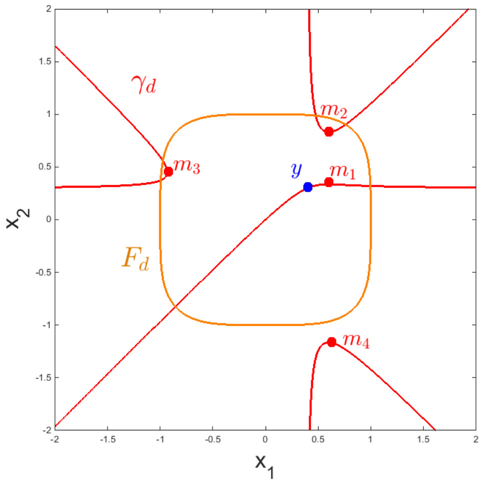

Figure 4.3: The curves F_d and γ_d in Example 4.4.3 for $d = 4$

is orthogonally invariant. Assume $d \geq 2$ is an even integer. One has $\mathcal{F}_{n,t,d} = \sigma^{-1}(F_{n,d})$ where $F_{n,d}$ is the *affine Fermat hypersurface*

$$F_{n,d} := \left\{ x \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} x_i^d = 1 \right\}$$

which is absolutely symmetric. Therefore, $C^{\#}(\mathcal{F}_{n,t,d}) = C^{\#}(F_{n,d})$ for all t.

Consider the special case $n = 2$. For simplicity, we set $F_d := F_{2,d}$. Then F_2 is the unit circle and $C^{\#}(F_2) = 2$. Suppose $d \geq 4$. For any $y \in \mathbb{R}^2$, a point x is an ED critical point of y on F_d if and only if x lies on both F_d and the curve

$$\gamma_d := \{x \in \mathbb{R}^2 \ : \ x_1^{d-1}(x_2 - y_2) = x_2^{d-1}(x_1 - y_1)\}.$$

The graph of γ_4 is shown in Figure 4.3. By symmetry, we may assume that $y_1 > y_2 > 0$. When y_1 is small, the "optimal" points, m_1, m_2, m_3, m_4, on the pieces of γ_d in the coordinate directions lie inside the curve F_d, namely,

$$m_i \in \{x \in \mathbb{R}^2 \ : \ x_1^d + x_2^d < 1\}$$

for $i = 1, \ldots, 4$. Hence, $C^{\#}_{F_d}(y) \leq 8$ for any point y except on a null set, and there is an open set V in $\mathbb{R}^2$ such that $C^{\#}_{F_d}(y) = 8$ for any $y \in V$. To sum up, by Theorem 4.3.7, we have

$$C^{\#}(\mathcal{F}_{2,t,d}) = C^{\#}(F_d) = \begin{cases} 2 & \text{if } d = 2 \\ 8 & \text{if } d \geq 4. \end{cases}$$

Example 4.4.4 ($SL_n^{\pm}$)**.** This example was considered in [21, Section 4]. Define the orthogonally invariant set

$$SL_n^{\pm} := \{X \in \mathbb{R}^{n\times n} \ : \ \det(X) = \pm 1\} = \sigma^{-1}(H_n)$$

where H_n is the absolutely symmetric set defined as

$$H_n := \{x \in \mathbb{R}^n \ : \ x_1 \cdots x_n = \pm 1\}.$$

Consider the special case $n = 2$. We evaluate $C^{\#}(SL_2^{\pm})$ by computing $C^{\#}(H_2)$. The set H_2 is a disjoint union of two hyperbolas:

$$H_2^{\pm} := \{x \in \mathbb{R}^2 \ : \ x_1 x_2 = \pm 1\}.$$

Since these hyperbolas are disjoint, for any y, one has

$$C^{\#}_{H_2}(y) = C^{\#}_{H_2^+}(y) + C^{\#}_{H_2^-}(y).$$

Notice that each of the sets $\mathrm{C}_{H_2^{\pm}}(y)$ is defined by the roots of a univariate quartic. Indeed,

$$C_{H_2^+}(y) = \left\{\left(x, \frac{1}{x}\right) \in \mathbb{R}^2 \ : \ q_y^+(x) := x^4 - x^3 y_1 + x y_2 - 1 = 0\right\}$$

and

$$C_{H_2^-}(y) = \left\{\left(x, \frac{-1}{x}\right) \in \mathbb{R}^2 \ : \ q_y^-(x) := x^4 - x^3 y_1 - x y_2 - 1 = 0\right\}.$$

By computing the trace forms of these quartics and applying Sylvester's criterion (see e.g. [71, Corollary 2.9]) we acquire these results:

$$C^{\#}_{H_2}(y) = \begin{cases} 6 & \text{if } D^+(y) > 0 \text{ or } D^-(y) > 0 \\ 4 & \text{if } D^+(y) < 0 \text{ and } D^-(y) < 0 \end{cases}$$

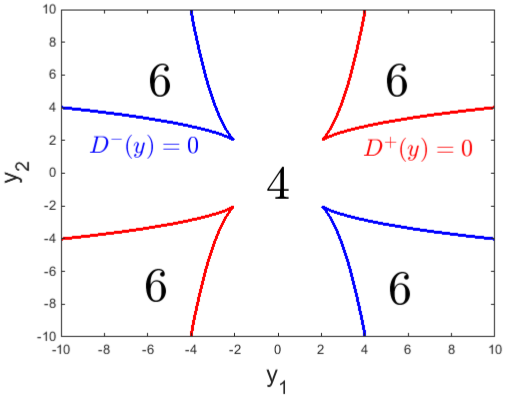

Figure 4.4: Values of $C^{\#}_{H_2}(y)$ for general y, in Example 4.4.4

where the bivariate polynomials

$$D^+(y) := -256 + 192y_1y_2 + 6y_1^2y_2^2 + 4y_1^3y_2^3 - 27y_1^4 - 27y_2^4$$

and

$$D^-(y) := -256 - 192y_1y_2 + 6y_1^2y_2^2 - 4y_1^3y_2^3 - 27y_1^4 - 27y_2^4$$

are the discriminants of q_y^+ and q_y^-, respectively; see Figure 4.4. Then by Theorem 4.3.7, we know

$$C^{\#}(SL_2^{\pm}) = 6.$$

4.5 Connections to the Euclidean distance degree

As mentioned in the introduction, this work was inspired by the general framework in [22] for counting the critical points of the squared Euclidean distance function on an algebraic variety. In this section we comment on the relationships between our results and those in [22], beginning with some basic facts about real varieties [5, 76].

Consider a real variety $\mathcal{V} \subseteq \mathbb{R}^n$ whose vanishing ideal is generated by the polynomials $f_1, \dots, f_s \in \mathbb{R}[x_1, \dots, x_n]$. Then the *Zariski closure* of $\mathcal{V}$ is $\mathcal{V}_{\mathbb{C}} := \{x \in \mathbb{C}^n \; : \; f_1(x) = \cdots =$

$f_s(x) = 0\}$, the smallest complex variety containing $\mathcal{V}$. The real part of $\mathcal{V}_{\mathbb{C}}$, i.e., $\mathcal{V}_{\mathbb{C}} \cap \mathbb{R}^n$, is precisely $\mathcal{V}$ and hence the vanishing ideal of $\mathcal{V}_{\mathbb{C}}$ in $\mathbb{C}[x_1, \ldots, x_n]$ is also generated by $f_1, \ldots, f_s$. Also, both $\mathcal{V}$ and $\mathcal{V}_{\mathbb{C}}$ have the same dimension over $\mathbb{R}$ and $\mathbb{C}$ respectively. Recall that if $\mathcal{V}_{\mathbb{C}} = \bigcup_{j=1}^{t} \mathcal{W}^j$ is a minimal irreducible decomposition of $\mathcal{V}_{\mathbb{C}}$, then for any j, the real part of $\mathcal{W}^j$, denoted by $\mathcal{U}^j$, is an irreducible real variety whose Zariski closure is $\mathcal{W}^j$. Moreover, $\mathcal{V} = \bigcup_{j=1}^{t} \mathcal{U}^j$ is a minimal irreducible decomposition of $\mathcal{V}$. For simplicity, in the rest of this section we will assume that $\mathcal{V}_{\mathbb{C}}$ is irreducible.

Denote by $J(f)$ the $s \times n$ *Jacobian* matrix whose (i,j)-entry is $\frac{\partial f_i}{\partial x_j}$. Then a point $x \in \mathcal{V}_{\mathbb{C}}$ is said to be *regular* if the rank of the Jacobian matrix evaluated at x, $\operatorname{rank}(J(f)(x))$, equals the codimension of $\mathcal{V}_{\mathbb{C}}$. Let $\mathcal{V}_{\mathbb{C}}^{\text{reg}}$ denote the regular points of $\mathcal{V}_{\mathbb{C}}$. It is known that $\mathcal{V}_{\mathbb{C}}^{\text{reg}}$ is a Zariski open subset of $\mathcal{V}_{\mathbb{C}}$, and its complement is a proper subvariety in $\mathcal{V}_{\mathbb{C}}$, denoted as $\operatorname{Sing}(\mathcal{V}_{\mathbb{C}})$, and called the *singular locus* of $\mathcal{V}_{\mathbb{C}}$. The set $\mathcal{V}^{\text{reg}}$ of regular points in $\mathcal{V}$ is precisely the set of real points in $\mathcal{V}_{\mathbb{C}}^{\text{reg}}$. However, while $\mathcal{V}_{\mathbb{C}}^{\text{reg}}$ is dense in $\mathcal{V}_{\mathbb{C}}$ in the Euclidean topology, $\mathcal{V}^{\text{reg}}$ may not be dense in $\mathcal{V}$ or even $\mathcal{V}^*$, the set of smooth (i.e., C^p-smooth) points in $\mathcal{V}$. The following example illustrates this behavior.

Example 4.5.1 (Cartan umbrella)**.** The *Cartan umbrella* is the real variety

$$\mathcal{V} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_3(x_1^2 + x_2^2) - x_1^3 = 0\}.$$

Here $\mathcal{V}_{\mathbb{C}}$ is irreducible, as is $\mathcal{V}$. On the other hand, $\mathcal{V}$ is also the union of a surface (which is not a real variety) and the x_3-axis (Figure 4.5). No point on the x_3-axis is regular while all points on the x_3-axis, except the origin, are smooth.

This example shows an important distinction between smooth and regular points on a real variety. Our goal in this section is to elaborate on this difference in the context of critical points as defined in [22]. Recall that the normal space at a regular point $x \in \mathcal{V}_{\mathbb{C}}$, denoted as $\mathcal{N}_{\mathcal{V}_{\mathbb{C}}}^{\text{alg}}(x)$, is the row space of $J(f)(x)$. Given a data point $y \in \mathbb{C}^n$, and a real variety $\mathcal{V}$, the authors of [22] study the set of critical points

$$C_{\mathcal{V}_{\mathbb{C}}}^{\text{reg}}(y) := \left\{ x \in \mathcal{V}_{\mathbb{C}}^{\text{reg}} \,:\, y - x \in \mathcal{N}_{\mathcal{V}_{\mathbb{C}}}^{\text{alg}}(x) \right\}.$$

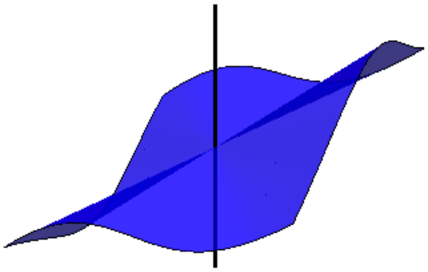

Figure 4.5: The Cartan umbrella

Note that in the above algebraic setting, even though our input is a real variety $\mathcal{V} \subseteq \mathbb{R}^n$, there is a natural passage into the complex numbers; the real variety $\mathcal{V}$ is replaced by the complex variety $\mathcal{V}_{\mathbb{C}}$, the data point can be any $y \in \mathbb{C}^n$, and the critical points of y are regular points in $\mathcal{V}_{\mathbb{C}}$. This complexification yields the first key result in [22], namely that for a fixed $\mathcal{V}$, the number of critical points of a general data point $y \in \mathbb{C}^n$ is a constant. In other words, the cardinality of $C^{\text{reg}}_{\mathcal{V}_{\mathbb{C}}}(y)$, which we will denote as $C^{\#\,\text{reg}}_{\mathcal{V}_{\mathbb{C}}}(y)$, is a constant. This constant is called the *Euclidean distance degree (EDdegree)* of $\mathcal{V}$ in [22]. The phrase "general data point" refers to the fact that $C^{\#\,\text{reg}}_{\mathcal{V}_{\mathbb{C}}}(y)$ is different from EDdegree($\mathcal{V}$) only if y lies on certain proper subvarieties in $\mathbb{C}^n$ which we will describe in the appendix. We remark that if $\mathcal{V}_{\mathbb{C}}$ is reducible, then EDdegree($\mathcal{V}$) is the sum of the ED degrees of its irreducible components.

Our results in this chapter provide a method to compute the (smooth) ED critical points of a real orthogonally invariant set of matrices. We did not require this set to be a real variety, and the notion of smoothness (of critical points) was geometric and not algebraic. Therefore, in order to connect our work to that in [22], we need to restrict to real varieties $\mathcal{V} \subseteq \mathbb{R}^n$ and data points $y \in \mathbb{R}^n$, and to understand the relationships between the following two sets of critical points of y:

- $C_{\mathcal{V}}(y)$, the set of ED critical points of y on $\mathcal{V}$ as in Section 4.2, and

- $C_{\mathcal{V}}^{\mathrm{reg}}(y)$, the set of regular critical points of y on $\mathcal{V}$.

As usual we denote the cardinalities of these sets by $C_{\mathcal{V}}^{\#}(y)$ and $C_{\mathcal{V}}^{\#\,\mathrm{reg}}(y)$ respectively. Also, note that $C_{\mathcal{V}}^{\mathrm{reg}}(y) = C_{\mathcal{V}_{\mathbb{C}}}^{\mathrm{reg}}(y) \cap \mathbb{R}^n = C_{\mathcal{V}_{\mathbb{C}}}^{\mathrm{reg}}(y) \cap \mathcal{V}$. Since a regular point of $\mathcal{V}$ is C^p-smooth for $p \in \{\infty, \omega\}$, for any y we have

$$C_{\mathcal{V}}^{\mathrm{reg}}(y) \subseteq C_{\mathcal{V}}(y) \quad \text{and so} \quad C_{\mathcal{V}}^{\#\,\mathrm{reg}}(y) \leq C_{\mathcal{V}}^{\#}(y). \tag{4.5.1}$$

However, for a given y, this inequality can be strict since there can be smooth points on $\mathcal{V}$ that are not regular as we saw in the Cartan umbrella. In fact, in the Cartan umbrella there is a dense set of y for which the inequality in (4.5.1) is strict since for any y with $y_3 \neq 0$, there will be an ED critical point on the x_3-axis.

Moving on to invariants of the entire set $\mathcal{V}$, on the one hand, we have the quantity $C^{\#}(\mathcal{V}) = \inf_{\Gamma \in \Theta} \sup_{y \in \Gamma^c} C_{\mathcal{V}}^{\#}(y)$ from Section 4.2, and on the other hand, we have the constant EDdegree($\mathcal{V}$) which is an upper bound to $C_{\mathcal{V}_{\mathbb{C}}}^{\#\,\mathrm{reg}}(y)$ whenever $C_{\mathcal{V}_{\mathbb{C}}}^{\mathrm{reg}}(y)$ is finite. Since the inequality in (4.5.1) can be strict, it is not clear that the number $C^{\#}(\mathcal{V})$ is always a lower bound to EDdegree($\mathcal{V}$). However, it is true under certain conditions.

Theorem 4.5.2. *If $\mathcal{V}$ is a real variety such that $\mathcal{V}^{\mathrm{reg}}$ is Euclidean dense in $\mathcal{V}^*$, then we have the inequality $C^{\#}(\mathcal{V}) \leq \mathrm{EDdegree}(\mathcal{V})$.*

This is an immediate consequence of the following lemma. Note that the Cartan umbrella does not satisfy the assumptions of Theorem 4.5.2.

Lemma 4.5.3. *Consider a real variety $\mathcal{V} \subseteq \mathbb{R}^n$ (possibly reducible) such that $\mathcal{V}^{\mathrm{reg}}$ is Euclidean dense in $\mathcal{V}^*$. Then there is an open set of points $y \in \mathbb{R}^n$ such that*

(i) for any irreducible component $\mathcal{W}$ of $\mathcal{V}$, *we have* $C_{\mathcal{W}_{\mathbb{C}}}^{\#\,\mathrm{reg}}(y) = \mathrm{EDdegree}(\mathcal{W})$;

(ii) $C_{\mathcal{V}}^{\#}(y) = C^{\#}(\mathcal{V})$;

(iii) $C_{\mathcal{V}}(y) \subseteq C_{\mathcal{V}}^{\mathrm{reg}}(y)$.

Proof: Fix an irreducible component $\mathcal{W}$ of $\mathcal{V}$. Then from [22, Theorem 4.1] the set of all $y \in \mathbb{C}^n$ for which $C^{\#\,\mathrm{reg}}_{\mathcal{W}_{\mathbb{C}}}(y) \neq \mathrm{EDdegree}(\mathcal{W})$ lies in a proper subvariety D in $\mathbb{C}^n$. Consider the real variety $D_{\mathbb{R}} := D \cap \mathbb{R}^n$ which contains all the real points y with $C^{\#\,\mathrm{reg}}_{\mathcal{W}_{\mathbb{C}}}(y) \neq \mathrm{EDdegree}(\mathcal{W})$. The Zariski closure of $D_{\mathbb{R}}$ is a subvariety of D with complex dimension equal to the real dimension of $D_{\mathbb{R}}$. Therefore, the real dimension of $D_{\mathbb{R}}$ is smaller than n, and hence $D_{\mathbb{R}}$ is a proper subvariety in $\mathbb{R}^n$. In particular, the complement of $D_{\mathbb{R}}$ in $\mathbb{R}^n$ is an open full measure set. Taking the intersection of all such open full measure sets as we vary over the irreducible components of $\mathcal{V}$, we still have an open full measure set, and any point y in this set satisfies (i).

Since $\mathcal{V}$ is a variety, one has $C^{\#}(\mathcal{V}) < \infty$ by Proposition 4.2.2. Then by quantifier elimination, the set $\{y \in \mathbb{R}^n \ : \ C^{\#}_{\mathcal{V}}(y) = C^{\#}(\mathcal{V})\}$ is semialgebraic. Moreover, this set cannot be Lebesgue null by the definition of $C^{\#}(\mathcal{V})$. Therefore, it must contain an open subset. Hence (ii) holds for all y in some open set.

We now show that (iii) holds for any point y in an open full measure set, which will prove the lemma. For any $i = 0, 1, \ldots, n$ consider the submanifold

$$M_i := \{x \in \mathcal{V}^* \ : \ \dim(\mathcal{T}_{\mathcal{V}}(x)) = i\}.$$

These submanifolds M_i clearly partition $\mathcal{V}^*$. We first claim $M_i \cap \mathrm{cl}(M_j) = \emptyset$ for all distinct pairs $i, j \in [n]$. Note the symbol "cl" here denotes the Euclidean closure. Indeed, if there existed some point $x \in M_i \cap \mathrm{cl}(M_j)$, then we would deduce for all points $y \in M_j$ sufficiently near x, the equality $j = \dim \mathcal{T}_{\mathcal{V}}(y) = \dim \mathcal{T}_{\mathcal{V}}(x) = i$, a contradiction. The middle equality follows from the fact that $\dim \mathcal{T}_{\mathcal{V}}(\cdot)$ is constant in a neighborhood of x in $\mathcal{V}$.

Now since $\mathcal{V}^{\mathrm{reg}}$ is dense in $\mathcal{V}^*$, the inclusion $\mathcal{V}^* \subseteq \mathrm{cl}(\mathcal{V}^{\mathrm{reg}}) = \bigcup_i \mathrm{cl}(\mathcal{V}^{\mathrm{reg}} \cap M_i)$ holds. We claim that for any i, the set $\mathcal{V}^{\mathrm{reg}} \cap M_i$ is Euclidean dense in M_i, namely, $M_i \subseteq \mathrm{cl}(\mathcal{V}^{\mathrm{reg}} \cap M_i)$. To see this, it suffices to establish $M_i \cap \mathrm{cl}(\mathcal{V}^{\mathrm{reg}} \cap M_j) = \emptyset$ for any $j \neq i$. This follows immediately by observing $M_i \cap \mathrm{cl}(\mathcal{V}^{\mathrm{reg}} \cap M_j) \subseteq M_i \cap \mathrm{cl}(M_j) = \emptyset$. It follows that for any i, the strict inequality

$$\dim(M_i \setminus (\mathcal{V}^{\mathrm{reg}} \cap M_i)) < \dim M_i$$

holds (see [12, Proposition 3.16]). By dimension arguments, the dimension of the set

$$\bigcup_{x \in M_i \setminus (M_i \cap \mathcal{V}^{\mathrm{reg}})} (x + \mathcal{N}_{\mathcal{V}}(x))$$

is strictly less than n. In particular, its interior is empty. Taking the union over i,

$$\bigcup_{x \in \mathcal{V}^* \setminus \mathcal{V}^{\mathrm{reg}}} (x + \mathcal{N}_{\mathcal{V}}(x))$$

also has empty interior. Any y which is not in this set satisfies (iii). □

We conclude the chapter by comparing $C^{\#}(\mathcal{V})$ and EDdegree($\mathcal{V}$) in some of the examples we saw in Sections 4.3 and 4.4.

Example 4.5.4 ($SL_n^{\pm}$)**.** Every point in the Zariski closure of $SL_n^{\pm}$ (c.f. Example 4.4.4) is regular. Therefore by Theorem 4.5.2, we obtain

$$C^{\#}(SL_n^{\pm}) \leq \mathrm{EDdegree}(SL_n^{\pm}) = n2^n.$$

The formula for EDdegree was derived in [21]. We saw in Example 4.4.4 that strict inequality holds when $n = 2$.

Example 4.5.5 (Unit sphere of Schatten d-norm)**.** Consider the set $\mathcal{F}_{2,2,d} \subseteq \mathbb{R}^{2\times 2}$ given in Example 4.4.3. For any d the Zariski closure of $\mathcal{F}_{2,2,d}$ is regular everywhere. By Theorem 4.5.2,

$$\mathrm{EDdegree}(\mathcal{F}_{2,2,d}) \geq C^{\#}(\mathcal{F}_{2,2,d}) = 8,$$

for $d \geq 4$. From `Macaulay2` [32] computations, EDdegree($\mathcal{F}_{2,2,d}$) equals $16, 34, 64, 98$ when $d = 4, 6, 8, 10$ respectively. This suggests the gap between $C^{\#}(\mathcal{F}_{2,2,d})$ and EDdegree($\mathcal{F}_{2,2,d}$) increases with d.

Example 4.5.6. Let $\mathcal{E}$ be the essential variety from (4.1.2). It is an irreducible variety of codimension three in $\mathbb{R}^{3\times 3}$ [19, Proposition 3.6]. By the transfer of smoothness (see the paragraph after the proof of Proposition 4.3.5), we know $\mathcal{E}^* = \mathcal{E} \setminus \{0\}$. Moreover, a straightforward computation verifies that any nonzero essential matrix is a regular point of $\mathcal{E}$. Thus $\mathcal{E}^{\text{reg}} = \mathcal{E}^* = \mathcal{E} \setminus \{0\}$, and by Theorem 4.5.2, we have

$$\text{EDdegree}(\mathcal{E}) \geq C^{\#}(\mathcal{E}) = 6. \tag{4.5.2}$$

With completely different tools, but entirely motivated by this work, one can prove the equality EDdegree($\mathcal{E}$) = 6. The details will be described in Chapter 5.

4.6 ED discriminants and data singular loci

The Cartan umbrella illustrates a yet unexplored feature of the EDdegree concerning the exceptional loci of points y at which $C^{\#\,\text{reg}}_{\mathcal{V}_{\mathbb{C}}}(y)$ can be different from EDdegree($\mathcal{V}$). We comment briefly on this topic which deserves further investigation.

For each $y \in \mathbb{C}^n$, the set $C^{\text{reg}}_{\mathcal{V}_{\mathbb{C}}}(y)$ is the variety of a polynomial ideal called the *critical ideal* of y (see (2.1) in [22]), and typically, this variety has EDdegree($\mathcal{V}$)-many distinct complex solutions. There are two ways in which the critical ideal of y can have fewer distinct roots; the first is because of roots with multiplicity, and the second is because a root may wander off into Sing($\mathcal{V}_{\mathbb{C}}$) due to closure issues. These situations create two exceptional loci in the space of data points y. The locus of $y \in \mathbb{C}^n$ for which the critical ideal has roots with multiplicity is called the *ED discriminant* of $\mathcal{V}_{\mathbb{C}}$, and is denoted as $\Sigma_{\mathcal{V}_{\mathbb{C}}}$. The ED discriminant is typically a hypersurface in $\mathbb{C}^n$ and can be computed from the equations of $\mathcal{V}$ (see Section 7 in [22] for algorithms and examples). We saw the ED discriminant of the parabola $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^2\}$ in Figure 4.1. It is the curve defined by

$$16y_2^3 - 27y_1^2 - 24y_2^2 + 12y_2 - 2 = 0.$$

The second type of exceptional locus has not been studied in [22] and was suggested to us by Bernd Sturmfels. To describe it, let us denote the copy of $\mathbb{C}^n$ that contains $\mathcal{V}_{\mathbb{C}}$ as $\mathbb{C}^n_x$ and the copy that contains the data points y as $\mathbb{C}^n_y$. The *ED correspondence* of $\mathcal{V}_{\mathbb{C}}$, denoted as $\mathbb{E}_{\mathcal{V}_{\mathbb{C}}}$ and described in [22, Section 4], is the Zariski closure of:

$$\left\{(x, y) \in \mathbb{C}^n_x \times \mathbb{C}^n_y \,:\, x \in \mathcal{V}^{\text{reg}}_{\mathbb{C}},\; y \in \mathbb{C}^n,\; x - y \in \mathcal{N}^{\text{alg}}_{\mathcal{V}_{\mathbb{C}}}(x)\right\}.$$

By [22, Theorem 4.1], the ED correspondence is an irreducible variety of dimension n in $\mathbb{C}^n_x \times \mathbb{C}^n_y$. It admits two natural projections π_x and π_y into $\mathbb{C}^n_x$ and $\mathbb{C}^n_y$ respectively. Over general $y \in \mathbb{C}^n$, the projection $\pi_y : \mathbb{E}_{\mathcal{V}_\mathbb{C}} \to \mathbb{C}^n_y$ has finite fibers of cardinality equal to EDdegree($\mathcal{V}$). The exceptional locus we are looking for is the set of data points $y \in \mathbb{C}^n$ that have critical points that fall into the singular locus $\mathrm{Sing}(\mathcal{V}_\mathbb{C})$. This is precisely the Zariski closure of the set

$$\pi_y \left(\mathbb{E}_{\mathcal{V}_\mathbb{C}} \cap (\mathrm{Sing}(\mathcal{V}_\mathbb{C}) \times \mathbb{C}^n_y)\right)$$

which we call the *ED data singular locus* of $\mathcal{V}_\mathbb{C}$, and denote by $D_{\mathcal{V}_\mathbb{C}}$. This affine scheme deserves further study, and we illustrate both $D_{\mathcal{V}_\mathbb{C}}$ and $\Sigma_{\mathcal{V}_\mathbb{C}}$ for the Cartan umbrella in Example 4.6.1. Note that $D_{\mathcal{V}_\mathbb{C}}$ is empty for the parabola in Figure 4.1 since its singular locus is empty. The upshot of the above discussion is that for all $y \in \mathbb{R}^n \setminus D_{\mathcal{V}_\mathbb{C}}$ but in a connected component of $\mathbb{R}^n_y \setminus \Sigma_{\mathcal{V}_\mathbb{C}}$, the number of regular critical points on $\mathcal{V}$, $C^{\#\,\mathrm{reg}}_{\mathcal{V}}(y)$, is a constant.

Example 4.6.1 (Cartan umbrella continued)**.** The ED discriminant of the Cartan umbrella in Example 4.5.1 is a degree 12 surface in the space of data points given by the following equation:

$$\begin{gathered}
256y_1^{12} - 35328y_1^{10}y_2^2 - 108984y_1^8y_2^4 - 111867y_1^6y_2^6 - 93975y_1^4y_2^8 - 9216y_1^2y_2^{10} \\
-2048y_2^{12} - 2304y_1^{11}y_3 - 2112y_1^9y_2^2y_3 - 149280y_1^7y_2^4y_3 - 116868y_1^5y_2^6y_3 \\
+53532y_1^3y_2^8y_3 + 34560y_1y_2^{10}y_3 + 6912y_1^{10}y_3^2 + 14016y_1^8y_2^2y_3^2 - 28764y_1^6y_2^4y_3^2 \\
+41502y_1^4y_2^6y_3^2 - 86430y_1^2y_2^8y_3^2 - 768y_2^{10}y_3^2 - 7936y_1^9y_3^3 + 150720y_1^7y_2^2y_3^3 \\
-200148y_1^5y_2^4y_3^3 - 411728y_1^3y_2^6y_3^3 + 1476y_1y_2^8y_3^3 + 9216y_1^8y_3^4 - 46656y_1^6y_2^2y_3^4 \\
+31908y_1^4y_2^4y_3^4 + 110817y_1^2y_2^6y_3^4 + 4953y_2^8y_3^4 - 27648y_1^7y_3^5 + 23808y_1^5y_2^2y_3^5 \\
+91236y_1^3y_2^4y_3^5 - 40284y_1y_2^6y_3^5 + 28672y_1^6y_3^6 - 196992y_1^4y_2^2y_3^6 - 240480y_1^2y_2^4y_3^6 \\
-2592y_2^6y_3^6 - 9216y_1^5y_3^7 + 14208y_1^3y_2^2y_3^7 + 28800y_1y_2^4y_3^7 + 27648y_1^4y_3^8 \\
+39168y_1^2y_2^2y_3^8 + 2304y_2^4y_3^8 - 27648y_1^3y_3^9 - 27648y_1y_2^2y_3^9 = 0.
\end{gathered}$$

The EDdegree of the Cartan umbrella is seven. The ED discriminant partitions $\mathbb{R}^3$ into two regions where the real regular critical points are either one or three, while the number of ED critical points is two or four. In particular, for all $y \in \mathbb{R}^3$ for which the polynomial defining the ED discriminant is positive, we get three real regular critical points.

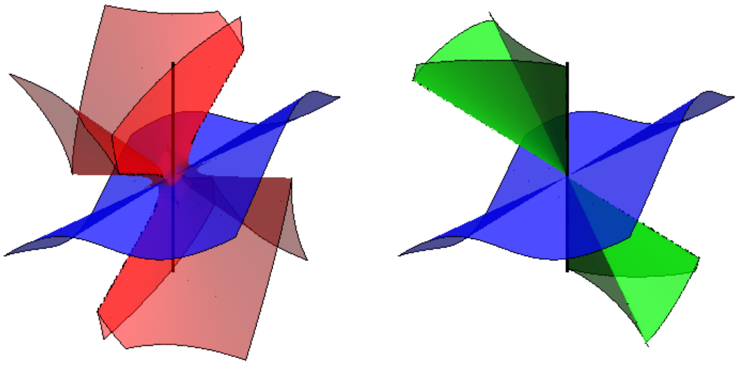

Figure 4.6: The Cartan umbrella with its ED discriminant on the left and ED data singular locus on the right.

The ED data singular locus, $D_{\mathcal{V}_{\mathbb{C}}}$, is the reducible surface in $\mathbb{C}^3$ defined by

$$(y_1^2 + y_2^2) \cdot (4y_1^4 + 8y_1^2 y_2^2 + 4y_2^4 + 4y_1^3 y_3 + 36 y_1 y_2^2 y_3 + 27 y_2^2 y_3^2) = 0$$

(see Figure 4.6). The data point $y = (-2, -1, 2)$ lies on $D_{\mathcal{V}_{\mathbb{C}}}$ but not on $\Sigma_{\mathcal{V}_{\mathbb{C}}}$. The critical ideal of this point has seven distinct complex roots but one of them is singular. Among the remaining six roots, two are real and $(-2, -1, 2)$ lies in the region where the polynomial defining $\Sigma_{\mathcal{V}_{\mathbb{C}}}$ is positive.

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Chapter 5

ED DEGREE OF ORTHOGONALLY INVARIANT MATRIX VARIETIES

5.1 Introduction

The problem of minimizing the Euclidean distance (ED) of an observed data point $y \in \mathbb{R}^n$ to a real algebraic variety $\mathcal{V} \subseteq \mathbb{R}^n$ arises frequently in applications, and amounts to solving the polynomial optimization problem

$$\text{minimize} \quad \sum_{i=1}^{n}(y_i - x_i)^2 \quad \text{subject to} \quad x \in \mathcal{V}.$$

The algebraic complexity of this problem is closely related to the number of complex regular critical points of y on the Zariski closure $\mathcal{V}_{\mathbb{C}}$ of $\mathcal{V}$. We will call such points the *ED critical points* of y with respect to $\mathcal{V}$; see Definition 5.4.1. The authors of [22] showed that the number of ED critical points of a general data point $y \in \mathbb{C}^n$ is a constant, and hence is an invariant of $\mathcal{V}$. This number is called the *Euclidean distance degree* (ED degree) of $\mathcal{V}$. As noted in [22], the computation of EDdegree($\mathcal{V}$) can be subtle, since it may change considerably under a linear transformation of $\mathcal{V}$.

In this work, we explore the ED degree of *orthogonally invariant* matrix varieties $\mathcal{M} \subseteq \mathbb{R}^{n\times t}$, meaning those varieties $\mathcal{M}$ satisfying

$$U\mathcal{M}V^{\top} = \mathcal{M} \quad \text{for all real orthogonal matrices} \quad U \in O(n), V \in O(t).$$

Without loss of generality, suppose $n \leq t$. Clearly, membership of a matrix M in such a variety $\mathcal{M}$ is fully determined by its vector of singular values $\sigma(M) = (\sigma_1(M), \ldots, \sigma_n(M))$, where we use the convention $\sigma_{i-1}(M) \geq \sigma_i(M)$ for each i. Indeed, we may associate with any orthogonally invariant matrix variety $\mathcal{M}$ its diagonal restriction $S = \{x : \mathrm{Diag}\,(x) \in \mathcal{M}\}$.

The variety S thus defined is *absolutely symmetric* (invariant under signed permutations) and satisfies the key relation $\mathcal{M} = \sigma^{-1}(S)$. Conversely, any absolutely symmetric set $S \subseteq \mathbb{R}^n$ yields the orthogonally invariant matrix variety $\sigma^{-1}(S)$; see e.g. [26, Theorem 3.4] and [17, Proposition 1.1].

In this chapter, we prove the elegant formula

$$\boxed{\text{EDdegree}(\mathcal{M}) = \text{EDdegree}(S).} \tag{$\star$}$$

In most interesting situations, the diagonal restriction $S \subseteq \mathbb{R}^n$ has simple geometry, as opposed to the matrix variety $\mathcal{M}$, and hence our main result ($\star$) provides elementary and transparent means to compute the ED degree of $\mathcal{M}$ by working with the simpler object S. Interesting consequences flow from there. For example, consider the *r-th rank variety*

$$\mathbb{R}^{n\times t}_r := \{X \in \mathbb{R}^{n\times t} : \operatorname{rank} X \leq r\},$$

and the *essential variety*

$$\mathcal{E} := \{X \in \mathbb{R}^{3\times 3} : \sigma_1(X) = \sigma_2(X),\, \sigma_3(X) = 0\}$$

from computer vision [2, 35, 58]; both $\mathbb{R}^{n\times t}_r$ and $\mathcal{E}$ are orthogonally invariant. The diagonal restrictions of $\mathbb{R}^{n\times t}$ and $\mathcal{E}$ are finite unions of linear subspaces and their ED degrees are trivial to compute. Moreover, our results readily imply that all ED critical points of a general real data matrix Y on $\mathbb{R}^{n\times t}_r$ and on $\mathcal{E}$ are real. This result has been previously shown for $\mathbb{R}^{n\times t}_r$ in [22] – a generalization of the Eckart-Young theorem – and is entirely new for the essential variety $\mathcal{E}$. A related further investigation of the essential variety appears in [30].

Our investigation of orthogonally invariant matrix varieties fits in a broader scope. The idea of studying orthogonally invariant matrix sets $\mathcal{M}$ via their diagonal restrictions S – the theme of this chapter – is not new, and goes back at least to von Neumann's theorem on unitarily invariant matrix norms [75]. In recent years, it has become clear that various analytic properties of $\mathcal{M}$ and S are in one-to-one correspondence, and this philosophy is sometimes called the "transfer principle"; see for instance, [17]. For example, $\mathcal{M}$ is C^p-smooth around a matrix X if and only if S is C^p-smooth around $\sigma(X)$ [15, 23, 50, 67, 73]. Other properties, such as convexity [18], positive reach [16], partial smoothness [15], and Whitney conditions [24] follow the same paradigm. In this sense, this chapter explores the transfer principle for the ED degree of algebraic varieties. To the best of our knowledge, this is the first result in this body of work that is rooted in algebraic geometry.

Though our main result ($\star$) is easy to state, the proof is subtle; moreover, the result itself is surprising in light of the discussion in [26, Section 5]. The outline of the chapter is as follows. In Section 5.2 we investigate invariance properties of the Zariski closure $\mathcal{M}_{\mathbb{C}} \subseteq \mathbb{C}^{n\times t}$ of an orthogonally invariant matrix variety $\mathcal{M} \subseteq \mathbb{R}^{n\times t}$, as well as the correspondence between irreducible components of S and those of $\mathcal{M}$. In Section 5.3, we discuss "algebraic singular value decompositions" for general matrices $Y \in \mathbb{C}^{n\times t}$, leading to Section 5.4 containing our main results. When S is a subspace arrangement, our results yield particularly nice consequences generalizing several classical facts in matrix theory – the content of Section 5.5.

5.2 Zariski closure, irreducibility, and dimension of matrix varieties

Setting the stage, we begin with some standard notation. For the fields $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, the symbol $\mathbb{F}[x] = \mathbb{F}[x_1, \ldots, x_n]$ will denote the ring of polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{F}$. Given polynomials $f_1, \ldots, f_s \in \mathbb{F}[x]$ the set $\mathcal{V} := \{x \in \mathbb{F}^n \,:\, f_1(x) = \cdots = f_s(x) = 0\}$ is called an (algebraic) *variety* over $\mathbb{F}$. The *Zariski closure* of an arbitrary set T in $\mathbb{C}^n$, denoted $\overline{T}$, is the smallest variety over $\mathbb{C}$ containing T. Unless otherwise specified, the topology on $\mathbb{C}^n$ is fixed to be the *Zariski topology*, obtained by defining the closed sets to be the varieties over $\mathbb{C}$. The topology on any subset of $\mathbb{C}^n$ will then always be the one induced by the Zariski topology.

Consider a real algebraic variety $\mathcal{V} \subseteq \mathbb{R}^n$. The *vanishing ideal* of $\mathcal{V}$ is defined to be $I(\mathcal{V}) := \{f \in \mathbb{R}[x] \,:\, f(x) = 0 \text{ for all } x \in \mathcal{V}\}$. Viewing $\mathcal{V}$ as a subset of $\mathbb{C}^n$, the Zariski closure of $\mathcal{V}$, denoted $\mathcal{V}_{\mathbb{C}}$, can be written as $\{x \in \mathbb{C}^n \,:\, f(x) = 0 \text{ for all } f \in I(\mathcal{V})\}$; see e.g. [76]. Note this notation is slightly redundant since by definition we have $\overline{\mathcal{V}} = \mathcal{V}_{\mathbb{C}}$. Nonetheless, we prefer to keep both symbols to ease notation when appropriate.

5.2.1 Invariance under closure

Consider a group G acting linearly on $\mathbb{C}^n$, which in turn induces a left action on $\mathbb{C}[x]$ via

$$(g \cdot f)(x) = f(g^{-1} \cdot x) \qquad \text{for any } g \in G,\ f \in \mathbb{C}[x].$$

A subset $T \subseteq \mathbb{C}^n$ is *G-invariant* if $g \cdot x$ lies in T for any $g \in G$ and $x \in T$. A polynomial $f \in \mathbb{C}[x]$ is *G-invariant* provided $g \cdot f = f$ for all $g \in G$.

We begin with the following elementary result.

Lemma 5.2.1. *If a set $T \subseteq \mathbb{C}^n$ is G-invariant, then its closure $\overline{T}$ is also G-invariant.*

Proof: Fixing $g \in G$, the map $\mu_g : \mathbb{C}^n \to \mathbb{C}^n$ given by $\mu_g(x) = g \cdot x$ is a linear isomorphism. Hence, assuming T is G-invariant, we deduce $\mu_g(\overline{T}) = \overline{\mu_g(T)} = \overline{T}$, as claimed. □

We now specialize the discussion to the main setting of the chapter. For a positive integer s, the symbol $O(s)$ will denote the set of all $s \times s$ real orthogonal matrices. This is both a group and a real variety and its Zariski closure $O_{\mathbb{C}}(s)$ is the set of all $s \times s$ complex orthogonal matrices — those satisfying $Q^\top Q = QQ^\top = I$. Henceforth, we fix two positive integers n and t with $n \leq t$, and consider the groups $O(n) \times O(t)$ and $O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t)$, along with the group $\Pi_n^{\pm}$ of all signed permutations of $\{1, \ldots, n\}$. Recall that we always consider the action of $O(n) \times O(t)$ on $\mathbb{R}^{n \times t}$ and the action of $O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t)$ on $\mathbb{C}^{n \times t}$ by conjugation $(U, V) \cdot X = UXV^\top$.

Now suppose $\mathcal{M} \subseteq \mathbb{R}^{n \times t}$ is a $O(n) \times O(t)$-invariant (orthogonally invariant) matrix variety. Then Lemma 5.2.1 shows that $\mathcal{M}_{\mathbb{C}}$ is $O(n) \times O(t)$-invariant. We now prove the stronger statement: $\mathcal{M}_{\mathbb{C}}$ is invariant under the larger group $O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t)$.

Proposition 5.2.2 (Closure invariance)**.** *A matrix variety $\mathcal{M} \subseteq \mathbb{R}^{n \times t}$ is $O(n) \times O(t)$-invariant if and only if $\mathcal{M}_{\mathbb{C}}$ is $O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t)$-invariant. Similarly, a variety $S \subseteq \mathbb{R}^n$ is $\Pi_n^{\pm}$-invariant if and only if $S_{\mathbb{C}}$ is $\Pi_n^{\pm}$-invariant.*

Proof: Since the proofs are similar, we only prove the first claim. Suppose first $\mathcal{M}_{\mathbb{C}}$ is $O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t)$-invariant. Then for every $X \in \mathcal{M}$ and $(U, V) \in O(n) \times O(t)$, the matrix

UXV^T lies in $\mathcal{M}_{\mathbb{C}} \cap \mathbb{R}^{n\times t} = \mathcal{M}$. Therefore $\mathcal{M}$ is $O(n) \times O(t)$-invariant, as claimed. Suppose conversely that $\mathcal{M}$ is $O(n) \times O(t)$-invariant. Let $X \in \mathcal{M}_{\mathbb{C}}$ be fixed. Then the map $\gamma_X : O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t) \to \mathbb{C}^{n\times t}$ defined by $\gamma(g) = g \cdot X$ is continuous. Lemma 5.2.1 yields the inclusion $O(n) \times O(t) \subseteq \gamma_X^{-1}(\mathcal{M}_{\mathbb{C}})$. Since $\gamma_X^{-1}(\mathcal{M}_{\mathbb{C}})$ is closed by continuity, we conclude $O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t) \subseteq \gamma_X^{-1}(\mathcal{M}_{\mathbb{C}})$. This completes the proof. $\square$

5.2.2 Irreducible components of orthogonally invariant varieties

For the rest of the section, fix a $\Pi_n^{\pm}$-invariant (absolutely symmetric) variety S in $\mathbb{R}^n$. Then the $O(n) \times O(t)$-invariant matrix set $\mathcal{M} := \sigma^{-1}(S)$ is a real variety in $\mathbb{R}^{n\times t}$; see [26, Theorem 3.4] or [17, Proposition 1.1]. Moreover, the diagonal restriction $\{x \in \mathbb{R}^n : \mathrm{Diag}\,(x) \in \mathcal{M}\}$ coincides with S. Here, we call a $n \times t$ matrix D diagonal if and only if $D_{ij} = 0$ whenever $i \neq j$, and for any vector $x \in \mathbb{R}^n$ the symbol $\mathrm{Diag}\,(x)$ denotes the diagonal matrix with $D_{ii} = x_i$ for each $i = 1, \ldots, n$.

In this section, we highlight the correspondence between the irreducible components of S and those of $\mathcal{M}$. Recall that a real or complex variety $\mathcal{V}$ is *irreducible* if it cannot be written as a union $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ of two proper subvarieties $\mathcal{V}_1$ and $\mathcal{V}_2$. Any variety $\mathcal{V}$ can be written as a union of finitely many irreducible subvarieties $\mathcal{V}_i$ satisfying $\mathcal{V}_i \nsubseteq \mathcal{V}_j$ for distinct indices i and j. The varieties $\mathcal{V}_i$ are called the *irreducible components* of $\mathcal{V}$, and are uniquely defined up to indexing.

Coming back to the aim of this section, let $\{S_i\}_{i=1}^k$ be the irreducible components of S. The varieties S_i are typically not absolutely symmetric. Hence we define their symmetrizations $S_i^\pi := \bigcup_{\pi\in\Pi_n^\pm} \pi S_i$ and the real varieties $\mathcal{M}_i := \sigma^{-1}(S_i^\pi)$. It is standard that a signed permutation maps an irreducible component of S to another irreducible component of S.

We record the following elementary observation for ease of reference.

Lemma 5.2.3. *For any pair of indices i, j, the following implications hold:*

$$\begin{aligned} S_i^\pi \subseteq S_j^\pi \quad &\Longrightarrow \quad S_i^\pi = S_j^\pi, \text{ and} \\ \mathcal{M}_i \subseteq \mathcal{M}_j \quad &\Longrightarrow \quad \mathcal{M}_i = \mathcal{M}_j \end{aligned}$$

Proof: If $S_i^\pi \subseteq S_j^\pi$, then we deduce that $S_i = \bigcup_{\pi \in \Pi_n^\pm}(S_i \cap \pi S_j)$. Hence for some $\pi \in \Pi_n^\pm$, the inclusion $S_i \subseteq \pi S_j$ holds. Since both S_i and πS_j are irreducible components of S, it must be that $S_i = \pi S_j$ and hence, $S_i^\pi = S_j^\pi$, as claimed. The second implication follows immediately. $\square$

For any $U \in O(n)$ and $V \in O(t)$, the map $X \mapsto UXV^\top$ is an automorphism of $\mathcal{M}_i$, and therefore maps an irreducible component of $\mathcal{M}_i$ to another irreducible component of $\mathcal{M}_i$. We now show that this action is transitive, just as the action of $\Pi_n^\pm$ on the components of S_i^π.

Lemma 5.2.4. *For any index i, the group $O(n) \times O(t)$ acts transitively on the irreducible components of $\mathcal{M}_i$. Consequently, the real variety $\mathcal{M}_i$ is equidimensional.*

Proof: Let $\mathcal{H}$ be an irreducible component of $\mathcal{M}_i$. Note that the set $\Gamma := \bigcup\{U\mathcal{H}V^\top : U \in O(n), V \in O(t)\}$ is a union of a nonempty collection of irreducible components of $\mathcal{M}_i$. Let Z be the union of the irreducible components of $\mathcal{M}_i$ not contained in Γ (if any). Observe that Z is an orthogonally invariant variety. Hence the two absolutely symmetric varieties $\{x : \mathrm{Diag}\,(x) \in \Gamma\}$ and $\{x : \mathrm{Diag}\,(x) \in Z\}$ cover S_i^π. Since S_i is irreducible, either Γ or Z coincides with all of $\mathcal{M}_i$. Since the latter is impossible by construction, we conclude that $\Gamma = \mathcal{M}_i$, as claimed. $\square$

We end with the following theorem, which will play a key role in the proof of Proposition 5.4.10, leading to the main result of the chapter.

Proposition 5.2.5. *Each variety $\mathcal{M}_i$ is a union of some irreducible components of $\mathcal{M}$.*

Proof:

Let $\{C_l\}$ be the set of irreducible components of $\mathcal{M}$. Then for any index l, we have $C_l = \bigcup_j (C_l \cap \mathcal{M}_j)$ which implies that C_l is contained in $\mathcal{M}_j$ for some index $j(l)$.

Fix a variety $\mathcal{M}_i$ and let $\mathcal{H}$ be an irreducible component of $\mathcal{M}_i$. From the equality $\mathcal{H} = \bigcup_l (C_l \cap \mathcal{H})$ we conclude that the inclusion $\mathcal{H} \subseteq C_l$ holds for some index l. This implies that $\mathcal{H} \subseteq C_l \subseteq \mathcal{M}_{j(l)}$, and hence by Lemma 5.2.4, $\mathcal{M}_i \subseteq \mathcal{M}_{j(l)}$. Lemma 5.2.3 then implies the equality $\mathcal{M}_i = \mathcal{M}_{j(l)}$, yielding $\mathcal{H} \subseteq C_l \subseteq \mathcal{M}_i$. Taking the union of this inclusion over all the irreducible components $\mathcal{H}$ of $\mathcal{M}_i$ and the corresponding C_l, we deduce that $\mathcal{M}_i$ is a union of some irreducible components of $\mathcal{M}$. $\square$

Since closures of irreducible components of a real variety $\mathcal{V}$ are the irreducible components of $\mathcal{V}_{\mathbb{C}}$, Proposition 5.2.5 immediately implies that $\overline{\mathcal{M}_i}$ is a union of some irreducible components of $\mathcal{M}_{\mathbb{C}}$.

5.2.3 Dimension of orthogonally invariant varieties

We next show how to read off the dimension of $\mathcal{M}_{\mathbb{C}}$ from the absolutely symmetric variety $S \subseteq \mathbb{R}^n$. To this end, note first that since the equality $\dim(\mathcal{M}_{\mathbb{C}}) = \dim(\mathcal{M})$ holds (see [76, Lemma 8]), it suffices to compute the dimension of the real variety $\mathcal{M}$ from S. We will assume that $\Pi_n^{\pm}$ acts transitively on the irreducible components of S, that is in the notation of Section 5.2.2 we have $S_i^{\pi} = S$ for all indices i. If this is not the case, we can treat each set S_i^{π} separately. With this simplification, both varieties S and $\mathcal{M}$ are equidimensional (Lemma 5.2.4).

The following recipe follows that in [15, Section 2.3] and [17] and hence we skip some of the explanations. The basic idea is to understand the dimension of the fiber $\sigma^{-1}(x^*)$ where $x^* \in S$ is (carefully) chosen so that the sum of the dimension of the fiber and the dimension of S equals $\dim(\mathcal{M})$.

Fixing notation, consider the convex cone

$$\mathbb{R}^n_{+,\geq} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}.$$

Observe that $\mathbb{R}^n_{+,\geq}$ is exactly the range of σ on $\mathbb{R}^{n\times t}$. Along with a point $x \in \mathbb{R}^n_{+,\geq}$, we associate the partition $\mathcal{P}_x = \{P_1, \ldots, P_{\rho_x}, P_0\}$ of the index set $\{1, \ldots, n\}$ so that $x_i = x_j$ if and only if $i, j \in P_l$, and $x_i > x_j$ for any $i \in P_q$ and $j \in P_r$ with $q > r$. We assume that P_0 contains the indices of the zero coordinates in x, and we define $p_l := |P_l|$. It could be that $p_0 = 0$ for a given x. On the other hand, we have $p_l > 0$ for all $l = 1, \ldots, \rho_x$. Recall the equality

$$\sigma^{-1}(x) = \{U \operatorname{Diag}(x) V^\top \,:\, U \in O(n), V \in O(t)\}.$$

Let

$$(O(n) \times O(t))_x := \{(U,V) \in O(n) \times O(t) \,:\, \operatorname{Diag}(x) = U \operatorname{Diag}(x) V^\top\}$$

denote the stabilizer of $\operatorname{Diag}(x)$, under the action of $O(n) \times O(t)$. Then one can check that (U,V) lies in the stabilizer $(O(n) \times O(t))_x$ if and only if U is block diagonal with blocks $U_i \in O(p_i)$ for $i = 0, \ldots, \rho_x$ and V is block diagonal with blocks $V_i \in O(p_i)$ for $i = 1, \ldots, \rho_x$, and a block $V_0 \in O(p_0 + (t-n))$. Further, $U_i V_i^\top = I$ for all $i = 1, \ldots, \rho_x$ which means that the U_i's determine the corresponding V_i's for all i except $i = 0$. This implies that the dimension of $(O(n) \times O(t))_x$ is

$$\dim((O(n) \times O(t))_x) = \sum_{l=0}^{\rho_x} \frac{p_l(p_l-1)}{2} + \frac{(p_0+t-n)(p_0+t-n-1)}{2}$$

yielding

$$\begin{aligned}\dim(\sigma^{-1}(x)) &= \dim(O(n) \times O(t)) - \dim((O(n) \times O(t))_x)\\ &= \frac{n(n-1)+t(t-1)}{2} - \sum_{l=0}^{\rho_x} \frac{p_l(p_l-1)}{2} - \frac{(p_0+t-n)(p_0+t-n-1)}{2}\\ &= \sum_{0 \le i < j \le \rho_x} p_i p_j + \frac{t(t-1)}{2} - \frac{(p_0+t-n)(p_0+t-n-1)}{2}. \end{aligned} \tag{5.2.1}$$

Here we used the observation

$$\frac{n(n-1)}{2} - \sum_{l=0}^{\rho_x} \frac{p_l(p_l-1)}{2} = \binom{\sum_{l=0}^{\rho_x} p_l}{2} - \sum_{l=0}^{\rho_x} \binom{p_l}{2} = \sum_{0 \le i < j \le \rho_x} p_i p_j.$$

For a partition $\mathcal{P}$ of $[n]$, define the set $\Delta_{\mathcal{P}} := \{x \in \mathbb{R}^n_{+,\ge} \,:\, \mathcal{P}_x = \mathcal{P}\}$. The set of all such Δ's defines an affine stratification of $\mathbb{R}^n_{+,\ge}$. Let $\mathcal{P}_*$ correspond to a stratum Δ_* in this stratification satisfying $S \cap \Delta_* \neq \emptyset$ and having maximal dimension among all strata that have a nonempty intersection with S. Then for any point $x^* \in S \cap \Delta_*$, we can choose a sufficiently small $\delta > 0$ satisfying $S \cap B_\delta(x^*) \subseteq \Delta_*$. Hence the fibers $\sigma^{-1}(x)$ have the same dimension for all $x \in S \cap B_\delta(x^*)$ and the preimage $\sigma^{-1}(S \cap B_\delta(x^*))$ is an open (in the Euclidean topology) subset of $\mathcal{M}$. Taking into account that both S and $\mathcal{M}$ are equidimensional, we deduce

$$\dim(\sigma^{-1}(S)) = \dim(S) + \dim(\sigma^{-1}(x^*)).$$

Appealing to (5.2.1), we arrive at the formula

$$\dim(\mathcal{M}) = \dim(S) + \left(\sum_{0\leq i<j\leq \rho^*} p_i^* p_j^*\right) + \frac{t(t-1)}{2} - \frac{(p_0^*+t-n)(p_0^*+t-n-1)}{2}. \qquad (5.2.2)$$

Example 5.2.6 (Rank variety)**.** Recall the rank variety $\mathbb{R}_r^{n\times t}$ of matrices of rank at most r. In this case, S is the union of all coordinate planes in $\mathbb{R}^n$ of dimension r and $S_\mathbb{C}$ is the set of all r-dimensional coordinate planes in $\mathbb{C}^n$. Also, $\mathcal{M}_\mathbb{C} = \mathbb{C}_r^{n\times t}$, the set of all matrices in $\mathbb{C}^{n\times t}$ of rank at most r.

Note that S is equidimensional. Then along with a point x^* we have $p_0^* = n - r$ and $p_i^* = 1$ for all $i = 1, \ldots, r$. Applying (5.2.2) we get that the dimension of $\mathbb{C}_r^{n\times t}$ is

$$r + \left(\binom{r}{2} + r(n-r)\right) + \frac{t(t-1)}{2} - \frac{(t-r)(t-r-1)}{2} = r(t+n-r).$$

Example 5.2.7 (Essential variety)**.** The essential variety is $\mathcal{E} = \{E \in \mathbb{R}^{3\times 3} : \sigma_1(E) = \sigma_2(E),\ \sigma_3(E) = 0\}$. Its Zariski closure $\mathcal{E}_\mathbb{C} \subseteq \mathbb{C}^{3\times 3}$ is known to be irreducible and of dimension six [19]. In this case, $S \subseteq \mathbb{R}^3$ consists of the six lines defined by $x_1 = \pm x_2$, $x_1 = \pm x_3$ and $x_2 = \pm x_3$ with the remaining coordinate set to zero in each case.

We can verify $\dim(\mathcal{E}_\mathbb{C}) = 6$ using (5.2.2). Indeed, picking a general point x^* on the line $x_1 = x_2$ in $\mathbb{R}_+^3$, we see that $\mathcal{P}_{x^*}$ has $p_0^* = 1$ and $p_1^* = 2$. Now applying the formula (5.2.2) we get $\dim(\mathcal{E}_\mathbb{C}) = 1 + 1\cdot 2 + 3 - 0 = 6$.

5.3 Algebraic Singular Value Decompositions and GIT quotients

In this section we fix a $\Pi_n^\pm$-invariant variety $S \subseteq \mathbb{R}^n$ and the induced $O(n)\times O(t)$-invariant matrix variety $\mathcal{M} := \sigma^{-1}(S)$. The description of $\mathcal{M}$ as the preimage $\sigma^{-1}(S)$ is not convenient when seeking to understand the algebraic geometric correspondences between $\mathcal{M}$ and S, since σ is not a polynomial map. Instead, we may equivalently write

$$\mathcal{M} = \{U \operatorname{Diag}(x) V^\top \ : \ U \in O(n),\ V \in O(t),\ x \in S\}. \tag{5.3.1}$$

In this notation, it is clear that $\mathcal{M}$ is obtained from S by an algebraic group action – a description that is more amenable to an algebraic analysis. Naturally then to understand geometric correspondences between the closures $\mathcal{M}_{\mathbb{C}}$ and $S_{\mathbb{C}}$, we search for a description analogous to (5.3.1), with $\mathcal{M}$, S, $O(n)$, and $O(t)$ replaced by their Zariski closures $\mathcal{M}_{\mathbb{C}}$, $S_{\mathbb{C}}$, $O_{\mathbb{C}}(n)$, and $O_{\mathbb{C}}(t)$. The difficulty is that an exact equality analogous to (5.3.1) usually fails to hold; instead, equality holds only in a certain generic sense that is sufficient for our purposes. We now make this precise.

5.3.1 Algebraic SVD

Our strategy revolves around an “algebraic singular value decomposition”, a notion to be made precise shortly. Note that the common extension of a singular value decomposition (SVD) from real to complex matrices using unitary matrices, their conjugates, and the Hermitian metric does not fit well in the algebraic setting because unitary matrices form a real (but not a complex) variety and conjugation is not an algebraic operation. In particular, it is not suitable for studying the EDdegree of a matrix variety. Hence we will need an algebraic analog of SVD that uses complex orthogonal matrices. For a recent geometric treatment of SVD rooted in algebraic geometry see the survey [62].

Definition 5.3.1 (Algebraic SVD)**.** We say that a matrix $A \in \mathbb{C}^{n\times t}$ admits an *algebraic SVD* if it can be factored as $A = UDV^\top$ for some orthogonal matrices $U \in O_{\mathbb{C}}(n)$ and $V \in O_{\mathbb{C}}(t)$, and a complex diagonal matrix $D \in \mathbb{C}^{n\times t}$.

Not all matrices admit an algebraic SVD; indeed, this is the main obstruction to an equality analogous to (5.3.1) in which the varieties $\mathcal{M}$, S, $O(n)$, and $O(t)$ are replaced by their closures. A simple example is the matrix $A = \left(\begin{smallmatrix} 1 & i \\ 0 & 0 \end{smallmatrix}\right)$, with $i = \sqrt{-1}$. Indeed, in light of the equality $AA^\top = 0$, if it were possible to write $A = UDV^\top$ for some $U, V \in O_{\mathbb{C}}(2)$ and a diagonal matrix D, then we would deduce that $UDD^\top U^\top = 0$ which implies that $A = 0$, a contradiction.

Fortunately, the existence question has been completely answered by Choudury and Horn [9, Theorem 2 & Corollary 3].

Theorem 5.3.2 (Existence of an algebraic SVD). *A matrix $A \in \mathbb{C}^{n\times t}$ admits an algebraic SVD, if and only if, $AA^\top$ is diagonalizable and* $\operatorname{rank}(A) = \operatorname{rank}(AA^\top)$.

Suppose A admits an algebraic SVD $A = U \operatorname{Diag}(d) V^\top$ for some orthogonal matrices $U \in O_{\mathbb{C}}(n)$ and $V \in O_{\mathbb{C}}(t)$, and a vector $d \in \mathbb{C}^n$. Then the numbers d_i^2 are eigenvalues of $A^\top A$ and $AA^\top$, and the columns of U are eigenvectors of $AA^\top$ and the columns of V are eigenvectors of $A^\top A$, arranged in the same order as d_i. We call the complex numbers d_i the *algebraic singular values* of A. They are determined up to sign.

We record the following immediate consequence of Theorem 5.3.2 for ease of reference.

Corollary 5.3.3. *Consider a matrix $A \in \mathbb{C}^{n\times t}$. If the eigenvalues of $AA^\top$ are nonzero and distinct, then A has an algebraic SVD.*

Suppose $\mathcal{V}$ is a variety over $\mathbb{R}$ or $\mathbb{C}$. We say that a property holds for a *general point* $x \in \mathcal{V}$ if the set of points $x \in \mathcal{V}$ for which the property holds contains an open dense subset of $\mathcal{V}$ (in Zariski topology). In this terminology, Theorem 5.3.2 implies that general complex matrices $A \in \mathbb{C}^{n\times t}$ do admit an algebraic SVD.

We can now prove the main result of this section (cf. equation (5.3.1)).

Theorem 5.3.4 (Generic description). *Suppose that a set $Q \subseteq S_{\mathbb{C}}$ contains an open dense subset of $S_{\mathbb{C}}$. Consider the set*

$$\mathcal{N}_Q := \{U \operatorname{Diag}(x) V^\top \ : \ U \in O_{\mathbb{C}}(n),\ V \in O_{\mathbb{C}}(t),\ x \in Q\}.$$

Then $\mathcal{N}_Q$ is a dense subset of $\mathcal{M}_{\mathbb{C}}$, and $\mathcal{N}_Q$ contains an open dense subset of $\mathcal{M}_{\mathbb{C}}$.

Proof: After we show $\mathcal{N}_Q$ is a dense subset of $\mathcal{M}_{\mathbb{C}}$, Chevalley's theorem [33, Theorem 3.16] will immediately imply that $\mathcal{N}_Q$ contains an open dense subset of $\mathcal{M}_{\mathbb{C}}$, as claimed.

We first argue the inclusion $\mathcal{N}_{S_{\mathbb{C}}} \subseteq \mathcal{M}_{\mathbb{C}}$ (and hence $\mathcal{N}_Q \subseteq \mathcal{M}_{\mathbb{C}}$). To this end, for any $f \in I(\mathcal{M}_{\mathbb{C}})$, note that the polynomial $q(x) := f(\mathrm{Diag}\,(x))$ vanishes on S and therefore on $S_{\mathbb{C}}$. Hence the inclusion $\{\mathrm{Diag}\,(x) \,:\, x \in S_{\mathbb{C}}\} \subseteq \mathcal{M}_{\mathbb{C}}$ holds. Since $\mathcal{M}_{\mathbb{C}}$ is $O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t)$-invariant (Proposition 5.2.2), we conclude that $\mathcal{N}_{S_{\mathbb{C}}} \subseteq \mathcal{M}_{\mathbb{C}}$, as claimed. Moreover, clearly $\mathcal{M}$ is a subset of $\mathcal{N}_{S_{\mathbb{C}}}$, and hence the inclusion $\mathcal{M}_{\mathbb{C}} \subseteq \overline{\mathcal{N}_{S_{\mathbb{C}}}}$ holds. We conclude the equality $\mathcal{M}_{\mathbb{C}} = \overline{\mathcal{N}_{S_{\mathbb{C}}}}$.

Now suppose that Q contains an open dense subset of $S_{\mathbb{C}}$ and consider the continuous polynomial map $P : O_{\mathbb{C}}(n) \times S_{\mathbb{C}} \times O_{\mathbb{C}}(t) \to \mathcal{M}_{\mathbb{C}}$ given by

$$P(U, x, V) := U\,\mathrm{Diag}\,(x)\,V^{\top}.$$

Noting the equations $\overline{Q} = S_{\mathbb{C}}$ and $\mathcal{N}_Q = P(O_{\mathbb{C}}(n) \times Q \times O_{\mathbb{C}}(t))$, we obtain

$$\begin{aligned}\overline{\mathcal{N}_Q} &= \overline{P(O_{\mathbb{C}}(n) \times Q \times O_{\mathbb{C}}(t))} = \overline{P(\overline{O_{\mathbb{C}}(n) \times Q \times O_{\mathbb{C}}(t)})} \\ &= \overline{P(O_{\mathbb{C}}(n) \times \overline{Q} \times O_{\mathbb{C}}(t))} = \overline{\mathcal{N}_{S_{\mathbb{C}}}} = \mathcal{M}_{\mathbb{C}}.\end{aligned}$$

Hence $\mathcal{N}_Q$ is a dense subset of $\mathcal{M}_{\mathbb{C}}$, as claimed. □

Remark 5.3.5. The variety $\mathcal{M}_{\mathbb{C}}$ may contain matrices that do not admit an algebraic SVD and hence the closure operation in Theorem 5.3.4 is not superfluous. For example the Zariski closure of $\mathbb{R}^{2\times 2}_1$ contains the matrix $\left(\begin{smallmatrix} 1 & i \\ 0 & 0 \end{smallmatrix}\right)$, which we saw earlier does not have an algebraic SVD.

Though in the notation of Theorem 5.3.4, the set $\mathcal{N}_{S_{\mathbb{C}}}$ coincides with $\mathcal{M}_{\mathbb{C}}$ only up to closure, we next show that equality does hold unconditionally when restricted to diagonal matrices. For any matrix $B \in \mathbb{C}^{n\times n}$ we define $e_1(B), \ldots, e_n(B)$ to be the n coefficients of the characteristic polynomial of B, that is $e_1(B), \ldots, e_n(B)$ satisfy

$$\det(\lambda I - B) = \lambda^n - e_1(B)\lambda^{n-1} + \cdots + (-1)^n e_n(B).$$

For any point $b \in \mathbb{C}^n$, we define $e_i(b) = e_i(\mathrm{Diag}\,(b))$ for every $i = 1, \ldots, n$. In other words, $e_1(b), \ldots, e_n(b)$ are the elementary symmetric polynomials in $b_1, \ldots, b_n$.

Theorem 5.3.6. *The equality,* $S_{\mathbb{C}} = \{x \in \mathbb{C}^n \colon \mathrm{Diag}\,(x) \in \mathcal{M}_{\mathbb{C}}\}$, *holds.*

Proof: The inclusion $\subseteq$ follows immediately from the inclusion $\mathcal{N}_{S_{\mathbb{C}}} \subseteq \mathcal{M}_{\mathbb{C}}$ established in Theorem 5.3.4. For the reverse inclusion, define the set

$$\Omega := \{y \in \mathbb{C}^n \ : \ y_i = e_i(x_1^2, \ldots, x_n^2) \quad \text{for some } x \in S_{\mathbb{C}}\}.$$

We first claim that Ω is a variety. To see this, by [72, Proposition 2.6.4], the variety $S_{\mathbb{C}}$ admits some $\Pi_n^{\pm}$-invariant defining polynomials $f_1, \ldots, f_k \in \mathbb{C}[x]$. Since f_j are invariant under coordinate sign changes, they are in fact symmetric polynomials in the squares $x_1^2, \ldots, x_n^2$. Then by the fundamental theorem of symmetric polynomials, we may write each f_j as some polynomial q_j in the quantities $e_i(x_1^2, \ldots, x_n^2)$. We claim that Ω is precisely the zero set of $\{q_1, \ldots, q_k\}$. By construction q_j vanish on Ω. Conversely, suppose $q_j(y) = 0$ for each j. Letting $x_1^2, \ldots, x_n^2$ be the roots of the polynomial $\lambda^n - y_1\lambda^{n-1} + \cdots + (-1)^n y_n$, we obtain a point $x \in \mathbb{C}^n$ satisfying $y_i = e_i(x_1^2, \ldots, x_n^2)$ for each i. We deduce then that x lies in $S_{\mathbb{C}}$ and hence y lies in Ω as claimed. We conclude that Ω is closed.

Observe the map $\pi \colon \mathcal{M}_{\mathbb{C}} \to \mathbb{C}^n$ defined by $\pi(X) = (e_1(XX^\top), \ldots, e_n(XX^\top))$ satisfies $\pi(\mathcal{N}_{S_{\mathbb{C}}}) \subseteq \Omega$, and so we deduce $\pi(\mathcal{M}_{\mathbb{C}}) = \pi(\overline{\mathcal{N}_{S_{\mathbb{C}}}}) \subseteq \overline{\pi(\mathcal{N}_{S_{\mathbb{C}}})} \subseteq \Omega$. Hence for any $y \in \mathbb{C}^n$ satisfying $\mathrm{Diag}\,(y) \in \mathcal{M}_{\mathbb{C}}$, there exists $x \in S_{\mathbb{C}}$ satisfying $e_i(x_1^2, \ldots, x_n^2) = e_i(y_1^2, \ldots, y_n^2)$ for each index $i = 1 \ldots, n$. We deduce that $x_1^2, \ldots, x_n^2$ and $y_1^2, \ldots, y_n^2$ are all roots of the same characteristic polynomial of degree n. Taking into account that $S_{\mathbb{C}}$ is $\Pi_n^{\pm}$-invariant, we conclude that y lies in $S_{\mathbb{C}}$. The result follows. $\square$

We conclude with the following two enlightening corollaries, which in particular characterize matrices in $\mathcal{M}_{\mathbb{C}}$ admitting an algebraic SVD.

Corollary 5.3.7 (SVD in the closure)**.** *A matrix* $X \in \mathcal{M}_{\mathbb{C}}$ *admits an algebraic SVD if and only if* $XX^\top$ *is diagonalizable,* $\mathrm{rank}(X) = \mathrm{rank}(XX^\top)$*, and the vector of algebraic singular values of* X *lies in* $S_{\mathbb{C}}$*.*

Proof: This is immediately from Proposition 5.2.2, and Theorems 5.3.2 and 5.3.6. $\square$

Corollary 5.3.8 (Eigenvalues in the closure). *If X is a matrix in $\mathcal{M}_{\mathbb{C}}$, then the vector of the square roots of the eigenvalues of $XX^\top$ lies in $S_{\mathbb{C}}$.*

Proof: Recall that, if $\mathcal{U}$ is an open dense subset of a variety $\mathcal{V}$, then $\mathcal{U}$ has nonempty intersection with any irreducible component of $\mathcal{V}$; see [6, 1.2 Proposition]. Hence the intersection of $\mathcal{U}$ with any irreducible component of $\mathcal{V}$ is open dense in that component, and is Euclidean dense in that component as well; see [60, page 60, Corollary 1]. Consequently, $\mathcal{U}$ is Euclidean dense in $\mathcal{V}$.

From Theorem 5.3.4 we know $\mathcal{N}_{S_{\mathbb{C}}}$ contains an open dense subset of $\mathcal{M}_{\mathbb{C}}$. It follows from the above discussion that $\mathcal{N}_{S_{\mathbb{C}}}$ is Euclidean dense in $\mathcal{M}_{\mathbb{C}}$. Given $X \in \mathcal{M}_{\mathbb{C}}$, we let x be the vector of the square roots of the eigenvalues of $XX^\top$, which is defined up to sign and order. We know there is a sequence $X_k := U_k \operatorname{Diag}(x^k) V_k^\top$, where $U_k \in O_{\mathbb{C}}(n)$, $V_k \in O_{\mathbb{C}}(t)$, and $x^k \in S_{\mathbb{C}}$ such that $X_k \to X$ as $k \to \infty$. Hence

$$(e_1(X_k X_k^\top), \ldots, e_n(X_k X_k^\top)) \to (e_1(XX^\top), \ldots, e_n(XX^\top)).$$

Since roots of polynomials are continuous with respect to the coefficients [77, Theorem 1], we deduce that the roots of the characteristic polynomial $\det(\lambda I - X_k X_k^\top)$, namely $((x_1^k)^2, \ldots, (x_n^k)^2)$, converge to $(x_1^2, \ldots, x_n^2)$ up to a coordinate reordering of x^k's and x. Passing to a subsequence, we deduce that x_k converge to x up to a signed permutation. Since $S_{\mathbb{C}}$ is closed, we conclude that x lies in $S_{\mathbb{C}}$, as claimed. $\square$

5.3.2 GIT perspective of algebraic SVD

The algebraic SVD can be viewed from the perspective of Geometric Invariant Theory (GIT) [20, Chapter 2]. Let G be the group $O_{\mathbb{C}}(n) \times O_{\mathbb{C}}(t)$ acting on $\mathbb{C}^{n\times t}$ via $(U,V)\cdot A = UAV^\top$. For any variety $\mathcal{V}$ over $\mathbb{C}$, let $\mathbb{C}[\mathcal{V}]$ be the ring of polynomial maps $\mathcal{V} \to \mathbb{C}$. Fix the G-invariant variety $\mathcal{M}_{\mathbb{C}}$ and define the *invariant ring*

$$\mathbb{C}[\mathcal{M}_{\mathbb{C}}]^G := \{f \in \mathbb{C}[\mathcal{M}_{\mathbb{C}}] \ : f \text{ is } G\text{-invariant}\}$$

as a subring of $\mathbb{C}[\mathcal{M}_{\mathbb{C}}]$. Consider a function $f \in \mathbb{C}[\mathcal{M}_{\mathbb{C}}]^G$. Since the map $q(x) := f \circ \mathrm{Diag}\,(x)$ lies in $\mathbb{C}[S_{\mathbb{C}}]$ and is $\Pi_n^{\pm}$-invariant, we may write q as a polynomial map in the values $e_i(x_1^2, \ldots, x_n^2)$. Hence by passing to the limit, f itself can be expressed as a polynomial over $\mathbb{C}$ in the ordered sequence of coefficients $e_1(XX^\top), \ldots, e_n(XX^\top)$. In other words, the following equality holds:

$$\mathbb{C}[\mathcal{M}_{\mathbb{C}}]^G = \mathbb{C}[e_1(XX^\top), \ldots, e_n(XX^\top)]$$

Observe that $\mathbb{C}[\mathcal{M}_{\mathbb{C}}]^G$ is a finitely generated reduced $\mathbb{C}$-algebra, and as such, there is a variety over $\mathbb{C}$ denoted by $\mathcal{M}_{\mathbb{C}}/\!\!/G$, such that $\mathbb{C}[\mathcal{M}_{\mathbb{C}}]^G$ is isomorphic to $\mathbb{C}[\mathcal{M}_{\mathbb{C}}/\!\!/G]$. This variety (up to isomorphism) is the categorical G-quotient constructed in the affine setting of GIT, and is denoted by $\mathcal{M}_{\mathbb{C}}/\!\!/G$. Concretely, we may write $\mathcal{M}_{\mathbb{C}}/\!\!/G$ as the variety corresponding to the ideal

$$\{f \in \mathbb{C}[x] : f(e_1(XX^\top), \ldots, e_n(XX^\top)) = 0 \quad \text{for all } X \in \mathcal{M}_{\mathbb{C}}\}.$$

A bit of thought shows that in our case, we may equivalently write

$$\mathcal{M}_{\mathbb{C}}/\!\!/G = \{y \in \mathbb{C}^n \ : \ y_i = e_i(x_1^2, \ldots, x_n^2) \quad \text{for some } x \in S_{\mathbb{C}}\}.$$

This was already implicitly shown in the proof of Theorem 5.3.6.

The *quotient map* $\pi : \mathcal{M}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}/\!\!/G$ is the surjective polynomial map associated to the inclusion $\mathbb{C}[\mathcal{M}_{\mathbb{C}}]^G \hookrightarrow \mathbb{C}[\mathcal{M}_{\mathbb{C}}]$. To be precise, in our case we have

$$\pi(X) = (e_1(XX^\top), \ldots, e_n(XX^\top))$$

Intuitively $\mathcal{M}_{\mathbb{C}}/\!\!/G$ can be “identified” with the space of closed orbits for the action of G on $\mathcal{M}_{\mathbb{C}}$, but not the orbit space. It can be proved that a G-orbit in $\mathcal{M}_{\mathbb{C}}$ is closed if and only if it is the orbit of a diagonal matrix. In other words, the orbit of a matrix X is closed if and only if X admits an algebraic SVD. By contrast, all $O(n) \times O(t)$-orbits in $\mathcal{M}$ are closed (compare these facts with [64, §16]).

5.4 ED critical points of an orthogonally invariant variety

We are now ready to prove our main results characterizing ED critical points of a data point $Y \in \mathbb{C}^{n\times t}$ with respect to an orthogonally invariant matrix variety $\mathcal{M} \subseteq \mathbb{R}^{n\times t}$. We first give the precise definition of an ED critical point; see [22, §2]. For any variety $\mathcal{V}$ over $\mathbb{R}$ or $\mathbb{C}$, we let $\mathcal{V}^{\mathrm{reg}}$ be the open dense subset of regular points in $\mathcal{V}$. Recall that if $\mathcal{V}$ is a union of irreducible varieties $\mathcal{V}_i$, then $\mathcal{V}^{\mathrm{reg}}$ is the union of $\mathcal{V}_i^{\mathrm{reg}}$ minus the points in the intersection of any two irreducible components. In what follows, for any two vectors $v, w \in \mathbb{C}^n$, the symbol $v \perp w$ means $v^\top w = 0$, and for any set $Q \subseteq \mathbb{C}^n$ we define $Q^\perp := \{v \in \mathbb{C}^n : v^\top w = 0 \text{ for all } w \in Q\}$.

Definition 5.4.1 (ED critical point, ED degree)**.** Let $\mathcal{V}$ be a real variety in $\mathbb{R}^n$ and consider a data point $y \in \mathbb{C}^n$. An *ED critical point* of y with respect to $\mathcal{V}$ is a point $x \in \mathcal{V}_{\mathbb{C}}^{\mathrm{reg}}$ such that $y - x \in \mathcal{T}_{\mathcal{V}_{\mathbb{C}}}(x)^\perp$, where $\mathcal{T}_{\mathcal{V}_{\mathbb{C}}}(x)$ is the tangent space of $\mathcal{V}_{\mathbb{C}}$ at x.

For any general point y in $\mathbb{C}^n$, the number of ED critical points of y with respect to $\mathcal{V}$ is a constant; see [22] called the *ED degree* of $\mathcal{V}$ and denoted by EDdegree($\mathcal{V}$).

Here is a basic fact that will be needed later.

Lemma 5.4.2. *Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a variety and let $\mathcal{W}$ be an open dense subset of $\mathcal{V}_{\mathbb{C}}$. Then all ED critical points of a general $y \in \mathbb{C}^n$ with respect to $\mathcal{V}$ lie in $\mathcal{W}$.*

Proof: The proof is a dimensional argument explained in [22]. Without loss of generality assume that $\mathcal{V}_{\mathbb{C}}$ is irreducible. Consider the ED correspondence $\mathcal{E}_{\mathcal{V}_{\mathbb{C}}}$, as defined in [22, §4], with its two projections π_1 on $\mathcal{V}_{\mathbb{C}}$ and π_2 on $\mathbb{C}^n$. Since π_1 is an affine vector bundle over $\mathcal{V}_{\mathbb{C}}^{reg}$ of rank $n - \dim \mathcal{V}_{\mathbb{C}}$, it follows that $\pi_2\left(\pi_1^{-1}(\mathcal{V}_{\mathbb{C}} \setminus \mathcal{W})\right)$ has dimension smaller than n. $\square$

Remark 5.4.3. We mention in passing, that the ED degree of a variety $\mathcal{V}$, as defined above equals the sum of the ED degrees of its irreducible components $\mathcal{V}_i$, which coincides with the original definition of ED degree in [22]. This follows from Lemma 5.4.2 by noting that the set $\mathcal{V}_{\mathbb{C}}^{\mathrm{reg}} \cap (\mathcal{V}_i)_{\mathbb{C}}$ is an open dense subset of $(\mathcal{V}_i)_{\mathbb{C}}$ for each i.

We say that two matrices X and Y admit a *simultaneous algebraic SVD* if there exist orthogonal matrices $U \in O_{\mathbb{C}}(n)$ and $V \in O_{\mathbb{C}}(t)$ so that both $U^\top XV$ and $U^\top YV$ are diagonal matrices. Our first main result is that every ED critical point X of a general matrix $Y \in \mathbb{C}^{n\times t}$ with respect to an orthogonally invariant variety $\mathcal{M}$ admits a simultaneous algebraic SVD with Y.

Theorem 5.4.4 (Simultaneous SVD)**.** *Fix an $O(n)\times O(t)$-invariant matrix variety $\mathcal{M} \subseteq \mathbb{R}^{n\times t}$. Consider a matrix $Y \in \mathbb{C}^{n\times t}$ so that the eigenvalues of $YY^\top$ are nonzero and distinct. Then any ED critical point X of Y with respect to $\mathcal{M}$ admits a simultaneous algebraic SVD with Y.*

The proof of this theorem relies on the following three lemmas.

Lemma 5.4.5. *The tangent space of $O_{\mathbb{C}}(n)$ at a point $U \in O_{\mathbb{C}}(n)$ is*

$$\begin{aligned}\mathcal{T}_{O_{\mathbb{C}}(n)}(U) &= \{ZU \ : \ Z \in \mathbb{C}^{n\times n} \text{ is skew-symmetric}\}\\ &= \{UZ \ : \ Z \in \mathbb{C}^{n\times n} \text{ is skew-symmetric}\}.\end{aligned}$$

Proof: Recall $O_{\mathbb{C}}(n) = \{W \in \mathbb{C}^{n\times n} \ : \ WW^\top = I\}$. Consider the map $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ given by $W \mapsto WW^\top$. Note that for any $W, B \in \mathbb{C}^{n\times n}$ and $t \in \mathbb{R}$, one has

$$(W + tB)(W + tB)^\top = WW^\top + t(WB^\top + BW^\top) + t^2BB^\top.$$

Hence given $U \in O_{\mathbb{C}}(n)$, we have $[\nabla F(U)](B) = UB^\top + BU^\top$. The tangent space $\mathcal{T}_{O_{\mathbb{C}}(n)}(U)$ is the kernel of the linear map $\nabla F(U)$. Consider the matrix $Z := BU^\top$. Then $[\nabla F(U)](B) = 0$ if and only if $Z^\top + Z = 0$ which means Z is skew-symmetric. This proves the first description of $\mathcal{T}_{O_{\mathbb{C}}(n)}(U)$. The second description follows by considering the map $W \mapsto W^\top W$ instead of F. $\square$

Lemma 5.4.6. *A matrix $A \in \mathbb{C}^{n\times n}$ is symmetric if and only if* $\operatorname{trace}(AZ) = 0$ *for any skew-symmetric matrix $Z \in \mathbb{C}^{n\times n}$.*

Proof: The "if" part follows because $A_{ij} - A_{ji} = \operatorname{trace}(A(E^{ij} - E^{ji}))$ where E^{ij} denotes the $n \times n$ matrix whose (i, j)-entry is one and all other entries are zero. The "only if" part follows by the same reasoning since $\{E^{ij} - E^{ji}\}$ is a basis for the space of skew-symmetric matrices. $\square$

Lemma 5.4.7. *Consider a matrix $A \in \mathbb{C}^{n\times t}$ and a diagonal matrix $D \in \mathbb{C}^{n\times t}$ with nonzero diagonal entries d_i such that the squares d_i^2 are distinct. Then if $AD^\top$ and $D^\top A$ are both symmetric, the matrix A must be diagonal.*

Proof: The symmetry of $AD^\top$ means $A_{ij}d_j = A_{ji}d_i$ for any $i, j = 1, \dots, n$. In addition, the symmetry of $D^\top A$ implies $A_{ij}d_i = A_{ji}d_j$ for all $i, j = 1, \dots, n$ and $A_{ij}d_i = 0$ for any $i = 1, \dots, n$ and $j > n$. Therefore for any i, j, one has

$$A_{ij}d_id_j = A_{ji}d_i^2 \qquad \text{and} \qquad A_{ij}d_id_j = A_{ji}d_j^2.$$

Since $d_i^2 \neq d_j^2$ for all $i \neq j$, we get $A_{ij} = 0$ for all $i \neq j$, $i, j = 1, \dots, n$. Since the d_i's are all nonzero and $A_{ij}d_i = 0$ for any $i = 1, \dots, n$ and $j > n$, we have $A_{ij} = 0$ for any $i = 1, \dots, n$ and $j > n$. Thus A is diagonal. $\square$

Remark 5.4.8. The assumption $d_i^2 \neq d_j^2$ for $i \neq j$ is necessary in Lemma 5.4.7. For example consider $D = I$ and the symmetric matrices

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \in O(2), \quad \theta \in \mathbb{R}$$

for which $AD^\top$ and $D^\top A$ are both symmetric. However, A is diagonal only when $\theta = k\pi$ with $k \in \mathbb{Z}$.

Proof of Theorem 5.4.4. By Corollary 5.3.3, we may write $Y = UDV^\top$ for some $U \in O_{\mathbb{C}}(n)$, $V \in O_{\mathbb{C}}(t)$, and a diagonal matrix $D \in \mathbb{C}^{n\times t}$. Let X be an ED critical point of Y with respect to $\mathcal{M}$. Then $A := U^\top XV$ lies in $\mathcal{M}_{\mathbb{C}}$ (Proposition 5.2.2). To prove the theorem, we need to show that $A \in \mathbb{C}^{n\times t}$ is diagonal.

Consider the map $F : O_{\mathbb{C}}(n) \to \mathcal{M}_{\mathbb{C}}$ given by $W \mapsto WAV^\top$. Then

$$[\nabla F(U)](B) = BAV^\top \in \mathcal{T}_{\mathcal{M}_{\mathbb{C}}}(X),$$

for any $B \in \mathcal{T}_{O_{\mathbb{C}}(n)}(U)$. By Lemma 5.4.5, we may write $B = UZ$ for a skew-symmetric Z, yielding $UZAV^\top \in \mathcal{T}_{\mathcal{M}_{\mathbb{C}}}(X)$. Varying B, we see that the tangent space of $\mathcal{M}_{\mathbb{C}}$ at X contains $\{UZAV^\top \ : \ Z^\top = -Z\}$. Then, by the definition of ED critical point we have $\operatorname{trace}((Y - X)(UZAV^\top)^\top) = 0$ for any skew-symmetric matrix Z, and hence

$$0 = \operatorname{trace}(U(D - A)V^\top V A^\top Z^\top U^\top) = \operatorname{trace}((D - A)A^\top Z^\top).$$

By Lemma 5.4.6, this means $(D - A)A^\top$ is symmetric. Since $AA^\top$ is symmetric, we have that $DA^\top$ is symmetric; therefore the transpose $AD^\top$ is symmetric.

By considering $F \ : \ O_{\mathbb{C}}(t) \to \mathcal{M}_{\mathbb{C}}$ given by $W \mapsto UAW^\top$, we get as above, that $\{UAZ^\top V^\top \ : \ Z^\top = -Z\} \subseteq \mathcal{T}_{\mathcal{M}_{\mathbb{C}}}(X)$. It follows that

$$0 = \operatorname{trace}((U(D - A)V^\top)^\top UAZ^\top V^\top) = \operatorname{trace}((D - A)^\top AZ^\top)$$

for any skew-symmetric matrix Z, and by Lemma 5.4.6, $(D - A)^\top A$ is symmetric. Again, since $A^\top A$ is symmetric, we get that $D^\top A$ is symmetric. Since $AD^\top$ and $D^\top A$ are both symmetric, we conclude A is diagonal by Lemma 5.4.7, as claimed. □

The next ingredient in our development is a version of Sard's Theorem in algebraic geometry (often called "generic smoothness" in textbooks); see [42, III, Corollary 10.7].

Theorem 5.4.9 (Generic smoothness on the target). *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties over $\mathbb{C}$. Consider a dominant polynomial map $f : \mathcal{V} \to \mathcal{W}$. Then there is an open dense subset $\mathcal{W}'$ of $\mathcal{W}^{\mathrm{reg}}$ (and hence of $\mathcal{W}$) such that for any $w \in \mathcal{W}'$ and any point $v \in \mathcal{V}^{\mathrm{reg}} \cap f^{-1}(w)$, the linear map $\nabla f(v) \colon \mathcal{T}_{\mathcal{V}}(v) \to \mathcal{T}_{\mathcal{W}}(w)$ is surjective.*

We now establish a key technical result: a representation of the tangent space of $\mathcal{M}_{\mathbb{C}}$ at a general matrix $X \in \mathcal{M}_{\mathbb{C}}$ in terms of the tangent space of $S_{\mathbb{C}}$ at the vector of algebraic singular values of X.

Proposition 5.4.10 (Transfer of tangent spaces). *Consider a $\Pi_n^{\pm}$-invariant variety $S \subseteq \mathbb{R}^n$ and the induced real variety $\mathcal{M} := \sigma^{-1}(S)$. Then the following statements hold.*

1. *A general point $X \in \mathcal{M}_{\mathbb{C}}$ lies in $\mathcal{M}_{\mathbb{C}}^{\mathrm{reg}}$, admits an algebraic SVD, and its vector of algebraic singular values lies in $S_{\mathbb{C}}^{\mathrm{reg}}$. Moreover, the tangent space $\mathcal{T}_{\mathcal{M}_{\mathbb{C}}}(X)$ admits the representation*

$$\mathcal{T}_{\mathcal{M}_{\mathbb{C}}}(X) = \left\{ \begin{array}{c} U Z_1 \operatorname{Diag}(x) V^\top + U \operatorname{Diag}(x) Z_2^\top V^\top + U \operatorname{Diag}(a) V^\top \ : \\ a \in \mathcal{T}_{S_{\mathbb{C}}}(x),\ Z_1,\ Z_2 \text{ are skew-symmetric} \end{array} \right\}, \quad (5.4.1)$$

for any $U \in O_{\mathbb{C}}(n)$, $V \in O_{\mathbb{C}}(t)$, and $x \in S_{\mathbb{C}}^{\mathrm{reg}}$ satisfying $X = U \operatorname{Diag}(x) V^\top$.

2. *A general point $x \in S_{\mathbb{C}}$ lies in $S_{\mathbb{C}}^{\mathrm{reg}}$. Moreover, for any $U \in O_{\mathbb{C}}(n), V \in O_{\mathbb{C}}(t)$, the point $X = U \operatorname{Diag}(x) V^\top$ lies in $\mathcal{M}_{\mathbb{C}}^{\mathrm{reg}}$, and satisfies* (5.4.1).

Proof: We begin by proving claim 1. By Theorem 5.3.4 with $Q = S_{\mathbb{C}}^{\mathrm{reg}}$, a general point $X \in \mathcal{M}_{\mathbb{C}}$ admits an algebraic SVD: $X = U' \operatorname{Diag}(x') V'^\top$ for some $x' \in S_{\mathbb{C}}^{\mathrm{reg}}$. As $\mathcal{M}_{\mathbb{C}}^{\mathrm{reg}}$ is an open dense subset of $\mathcal{M}_{\mathbb{C}}$, we can assume that X lies in $\mathcal{M}_{\mathbb{C}}^{\mathrm{reg}}$. Consider the dominant polynomial map $P : O_{\mathbb{C}}(n) \times S_{\mathbb{C}} \times O_{\mathbb{C}}(t) \to \mathcal{M}_{\mathbb{C}}$ given by

$$P(\widetilde{U}, \widetilde{x}, \widetilde{V}) := \widetilde{U} \operatorname{Diag}(\widetilde{x}) \widetilde{V}^\top.$$

By Theorem 5.4.9 we can assume that $\nabla P(U, x, V)$ is surjective whenever we can write $X = U \operatorname{Diag}(x) V^\top$ for some $U \in O_{\mathbb{C}}(n)$, $V \in O_{\mathbb{C}}(t)$ and $x \in S_{\mathbb{C}}^{\mathrm{reg}}$. Therefore the description of tangent space in (5.4.1) follows from Leibniz rule on P and Lemma 5.4.5. Hence claim 1 is proved.

Next, we argue claim 2. To this end, let Θ be the dense open subset of $\mathcal{M}_{\mathbb{C}}$ guaranteed to exist by 1. We claim that we can assume that Θ is in addition orthogonally invariant. To see this, observe that all the claimed properties in 1 continue to hold on the dense, orthogonally invariant subset $\Gamma := \bigcup \{U \Theta V^T : U \in O_{\mathbb{C}}(n),\ V \in O_{\mathbb{C}}(t)\}$ of $\mathcal{M}_{\mathbb{C}}$. By Lemma 5.2.1, the set $\overline{\mathcal{M}_{\mathbb{C}} \setminus \Gamma}$ is an orthogonally invariant variety. Note now the inclusions $\Theta \subseteq \mathcal{M}_{\mathbb{C}} \setminus (\overline{\mathcal{M}_{\mathbb{C}} \setminus \Gamma}) \subseteq \Gamma$. It follows that $\mathcal{M}_{\mathbb{C}} \setminus (\overline{\mathcal{M}_{\mathbb{C}} \setminus \Gamma})$ is an orthogonally invariant, open, dense variety in $\mathcal{M}_{\mathbb{C}}$ on which all the properties in 1 hold. Replacing Θ with $\mathcal{M}_{\mathbb{C}} \setminus (\overline{\mathcal{M}_{\mathbb{C}} \setminus \Gamma})$, we may assume that Θ is indeed orthogonally invariant in the first place.

Next, we appeal to some results of Section 5.2.2. Let $\{S_i\}_{i=1}^k$ be the irreducible components of S and define the symmetrizations $S_i^\pi := \bigcup_{\pi\in\Pi_n^\pm} \pi S_i$ and the varieties $\mathcal{M}_i := \sigma^{-1}(S_i^\pi)$. Observe that $\overline{S_i}$ are the irreducible components of $S_\mathbb{C}$ and we have $\overline{S_i^\pi} = \bigcup_{\pi\in\Pi_n^\pm} \pi\overline{S_i}$. Note that $\mathcal{M}_\mathbb{C}$ is the union of the varieties $\overline{\mathcal{M}_i}$.

By Proposition 5.2.5, each variety $\overline{\mathcal{M}_i}$ is a union of some irreducible components of $\mathcal{M}_\mathbb{C}$. Since the intersection of $\mathcal{M}_\mathbb{C}^{\text{reg}}$ with any irreducible component of $\mathcal{M}_\mathbb{C}$ is open and dense in that component, we deduce that the intersection $\overline{\mathcal{M}_i} \cap \mathcal{M}_\mathbb{C}^{\text{reg}}$ is an open dense subset of $\overline{\mathcal{M}_i}$ for each index i. Similarly, the intersection $\Theta \cap \overline{\mathcal{M}_i}$ is open and dense in each variety $\overline{\mathcal{M}_i}$. Then clearly Θ intersects $\mathcal{N}_{\overline{S_i^\pi}}$ for each index i, since by Theorem 5.3.4 the set $\mathcal{N}_{\overline{S_i^\pi}}$ contains an open dense subset of $\overline{\mathcal{M}_i}$. Therefore for each index i, the set Θ contains $\operatorname{Diag}(x_i)$ for some $x_i \in \overline{S_i^\pi}$.

We deduce that the diagonal restriction of Θ, namely the set

$$W := \{x \in \mathbb{C}^n : \operatorname{Diag}(x) \in \Theta\},$$

is an absolutely symmetric, open subset of $S_\mathbb{C}$ and it intersects each variety $\overline{S_i^\pi}$. In particular, W intersects each irreducible component $\overline{S_i}$. Since nonempty open subsets of irreducible varieties are dense, we deduce that W is dense in $S_\mathbb{C}$. Moreover, since for any point $x \in W$, the matrix $\operatorname{Diag}(x)$ lies in Θ, we conclude

- $\operatorname{Diag}(x)$ lies in $\mathcal{M}_\mathbb{C}^{\text{reg}}$ (and hence by orthogonal invariance so do all matrices $U \operatorname{Diag}(x) V^\top$ with $U \in O_\mathbb{C}(n), V \in O_\mathbb{C}(t)$) and x lies in $S_\mathbb{C}^{reg}$,

- equation (5.4.1) holds for $X = \operatorname{Diag}(x)$, and hence by orthogonal invariance of $\mathcal{M}_\mathbb{C}$ and of the description (5.4.1), the equation continues to hold for $X = U \operatorname{Diag}(x) V^\top$, where U and V arbitrary orthogonal matrices.

Thus all the desired conclusions hold for any x in the open dense subset W of $S_\mathbb{C}$. The result follows. $\square$

We are now ready to prove the main result of this chapter, equation ($\star$) from the introduction. As a byproduct, we will establish an explicit bijection between the ED critical points of a general matrix $Y = U\,\mathrm{Diag}\,(y)\,V^\top \in \mathbb{C}^{n\times t}$ on $\mathcal{M}$ and the ED critical points of y on $S_\mathbb{C}$.

Theorem 5.4.11 (ED degree)**.** *Consider a $\Pi_n^\pm$-invariant variety $S \subseteq \mathbb{R}^n$ and the induced real variety $\mathcal{M} := \sigma^{-1}(S)$. Then a general matrix $Y \in \mathbb{C}^{n\times t}$ admits a decomposition $Y = U\,\mathrm{Diag}\,(y)\,V^\top$, for some matrices $U \in O_\mathbb{C}(n)$, $V \in O_\mathbb{C}(t)$, and $y \in \mathbb{C}^n$. Moreover, then the set of ED critical points of Y with respect to $\mathcal{M}$ is*

$$\{U\,\mathrm{Diag}\,(x)\,V^\top \ :\ x \textit{ is an ED critical point of } y \textit{ with respect to } S\},$$

In particular, equality $\mathrm{EDdegree}(\mathcal{M}) = \mathrm{EDdegree}(S)$ *holds.*

Proof: For general $Y \in \mathbb{C}^{n\times t}$, the eigenvalues of $YY^\top$ are nonzero and distinct. Then by Corollary 5.3.3, we can be sure that Y admits an algebraic SVD. We fix such a decomposition $Y = U\,\mathrm{Diag}\,(y)\,V^\top$, for some $U \in O_\mathbb{C}(n)$, $V \in O_\mathbb{C}(t)$, and $y \in \mathbb{C}^n$.

Let X be an ED critical point of Y with respect to $\mathcal{M}$. By Theorem 5.4.4, we can assume that X and Y admit a simultaneous SVD, that is both $U'^\top X V'^\top$ and $U'^\top Y V'^\top$ are diagonal for some matrices $U' \in O_\mathbb{C}(n)$, $V' \in O_\mathbb{C}(t)$. Notice that the columns of U and U' are equal up to a sign change and a permutation. Similarly the first n columns of V and V' are equal up to a sign change and a permutation. Hence we may assume that X can be written as $X = U\,\mathrm{Diag}\,(x)\,V^\top$ for some $x \in S_\mathbb{C}$. By Lemma 5.4.2 and Proposition 5.4.10, we can further assume that X lies in $\mathcal{M}_\mathbb{C}^{\mathrm{reg}}$ and x lies in $S_\mathbb{C}^{\mathrm{reg}}$, and moreover the tangent space $\mathcal{T}_{\mathcal{M}_\mathbb{C}}(X)$ at $X = U\,\mathrm{Diag}\,(x)\,V^\top$ is given in (5.4.1).

We will now show that x is an ED critical point of y with respect to S. To see this, observe the inclusion

$$\{U\,\mathrm{Diag}\,(a)\,V^\top \ :\ a \in \mathcal{T}_{S_\mathbb{C}}(x)\} \subseteq \mathcal{T}_{\mathcal{M}_\mathbb{C}}(X).$$

and hence

$$0 = \mathrm{trace}(U\,\mathrm{Diag}\,(y-x)\,V^\top(U\,\mathrm{Diag}\,(a)\,V^\top)^\top) \quad \text{for any } a \in \mathcal{T}_{S_\mathbb{C}}(x).$$

Simplifying, we immediately conclude $(y-x)^\top a = 0$ for any $a \in \mathcal{T}_{S_\mathbb{C}}(x)$, and hence x is an ED critical point of y with respect to S.

Conversely, suppose $x \in S_\mathbb{C}^{\mathrm{reg}}$ is an ED critical point of y with respect to S. Applying Theorem 5.3.4, we deduce that if a set $Q \subseteq \mathbb{C}^n$ contains an open dense set in $\mathbb{C}^n$, then

$$\{\widehat{U}\,\mathrm{Diag}\,(z)\,\widehat{V}^\top : z \in Q,\ \widehat{U} \in O_\mathbb{C}(n),\ \widehat{V} \in O_\mathbb{C}(t)\}$$

contains an open dense subset of $\mathbb{C}^{n\times t}$. Define now the matrix $X := U\,\mathrm{Diag}\,(x)\,V^\top$. Then by Lemma 5.4.2 and Proposition 5.4.10, we may assume that X is regular and the tangent space of $\mathcal{M}_\mathbb{C}$ at X is generated by all matrices of the form

1. $UZ\,\mathrm{Diag}\,(x)\,V^\top$ with Z skew-symmetric,
2. $U\,\mathrm{Diag}\,(a)\,V^\top$ where a belongs to the tangent space of $S_\mathbb{C}$ at x,
3. $U\,\mathrm{Diag}\,(x)\,Z^\top V^\top$ with Z skew-symmetric.

We will show

$$Y - X \perp \mathcal{T}_{\mathcal{M}_\mathbb{C}}(X) \tag{5.4.2}$$

by dividing the proof according to the three cases 1,2, 3 above.

For 1, observe

$$\mathrm{trace}\,\big((X-Y)(UZ\,\mathrm{Diag}\,(x)\,V^\top)^\top\big) = \mathrm{trace}\,\big(\mathrm{Diag}\,(x-y)\,\mathrm{Diag}\,(x)^\top Z^\top\big) = 0,$$

where the last equality follows from Lemma 5.4.6. The computation for 3 is entirely analogous. For 2, we obtain

$$\mathrm{trace}\Big((X-Y)(U\,\mathrm{Diag}\,(a)\,V^\top)^\top\Big) = \mathrm{trace}\Big((\mathrm{Diag}\,(x-y)\mathrm{Diag}\,(a)^\top\Big) = 0,$$

where the last equation follows from the hypothesis that x is an ED critical point of y on $S_\mathbb{C}$. We conclude that X is an ED critical point of Y relative to $\mathcal{M}_\mathbb{C}$, as claimed. The equality, EDdegree($\mathcal{M}$) = EDdegree(S), quickly follows. □

Example 5.4.12. To illustrate Theorem 5.4.11, we now derive the ED degree of some notable orthogonally invariant varieties summarized in the following table. The pairs $(\mathcal{M}, S)$ in all these examples were also discussed in [26, Section 4]. The dimension of $\mathcal{M}$ or $\mathcal{M}_{\mathbb{C}}$ can be computed using (5.2.2).

orthogonally invariant variety $\mathcal{M}$	dimension	absolutely symmetric variety S	dimension	EDdegree
$\mathbb{R}^{n\times t}_r$	$r(n+t-r)$	$\mathbb{R}^n_r$	r	$\binom{n}{r}$
$\mathcal{E}$	6	$E_{3,2}$	1	6
$O(n)$	$\binom{n}{2}$	$\{(\pm 1, \ldots, \pm 1)\}$	0	2^n
$SL^{\pm}_n$	n^2-1	H_n	$n-1$	$n2^n$
$\mathcal{F}_{n,t,d}$ (d even)	$nt-1$	$F_{n,d}$	$n-1$	[48, Cor. 2.12]

In the first three examples, the set S is a subspace arrangement and hence its ED degree is the number of distinct maximal subspaces in the arrangement. We will elaborate on this situation in Section 5.5.

The matrix variety $SL^{\pm}_n$ consists of all matrices $A \in \mathbb{C}^{n\times n}$ satisfying $\det(A) = \pm 1$. The ED degree of $\mathrm{SL}^{\pm}_n$ was explicitly computed in [21]. We show below how our main theorem provides a simple alternate proof of their result.

The absolutely symmetric variety S in this case is $H_n := \{x \in \mathbb{R}^n : x_1 x_2 \cdots x_n = \pm 1\}$. To compute the ED degree of H_n, we add up the ED degrees of its two irreducible components

$$H^+_n := \{x \in \mathbb{R}^n : x_1 x_2 \cdots x_n = 1\}$$

and

$$H^-_n := \{x \in \mathbb{R}^n : x_1 x_2 \cdots x_n = -1\}.$$

To compute the ED degree of H^+_n, we begin with a point $y \in \mathbb{C}^n$. Then by a straightforward computation, x is an ED critical point of y with respect to H^+_n if and only if x solves the system

$$\begin{cases} x_i(x_i - y_i) = x_n(x_n - y_n) \quad \text{for all } i = 1, \ldots, n-1 \\ x_1 \cdots x_n = 1. \end{cases} \tag{5.4.3}$$

By Bézout's Theorem, we know $\mathrm{EDdegree}(H_n^+) \leq n2^{n-1}$. We now argue that the data point $y = 0$ has $n2^{n-1}$ ED critical points with respect to H_n^+ which proves that $\mathrm{EDdegree}(H_n^+) = n2^{n-1}$.

When $y = 0$, the system (5.4.3) is equivalent to

$$\begin{cases} x_1^2 = \cdots = x_n^2 \\ x_1 \cdots x_n = 1. \end{cases}$$

which has $n2^{n-1}$ solutions in $(H_n^+)_{\mathbb{C}}$. Indeed, choose x_1 such that $x_1^n = \pm 1$ ($2n$ choices); then choose x_i for $i = 2, \ldots, n-1$ such that $x_i^2 = x_1^2$ (2 choices for each i); finally set $x_n = \frac{1}{x_1 \cdots x_{n-1}}$. Hence $\mathrm{EDdegree}(H_n^+) = n2^{n-1}$. Similarly, $\mathrm{EDdegree}(H_n^-) = n2^{n-1}$, and therefore we conclude $\mathrm{EDdegree}(H_n) = n2^n$.

The variety $\mathcal{F}_{n,t,d} = \{X \in \mathbb{R}^{n \times t} \,:\, \|X\|_d = 1\}$ is the unit ball of the Schatten d-norm $\|X\|_d := \left[\sum_{i=1}^n \sigma_i(X)^d\right]^{\frac{1}{d}}$. When d is even, the corresponding absolute symmetric variety is the *affine Fermat hypersurface*

$$F_{n,d} := \left\{ x \in \mathbb{R}^n \;:\; \sum_{i=1}^n x_i^d = 1 \right\}.$$

The ED degree of a Fermat hypersurface was computed in [48].

5.5 Orthogonally invariant varieties from subspace arrangements

In this section, we augment the results of the previous section in the special (and important) case when S is a subspace arrangement. Many important matrix varieties, such as the rank varieties $\mathbb{R}_r^{n \times t}$ and the essential variety $\mathcal{E}$, fall in this category. Recall that S is a *subspace arrangement* if S can be written as a union of finitely many affine subspaces $\{S_i\}_{i=1}^k$ of $\mathbb{R}^n$. Assuming that the representation of S is chosen in such a way that S_i is not contained in

S_j for any distinct i, j, we call S_i the *affine components* of S. The following result follows directly from Theorem 5.4.11.

Corollary 5.5.1 (Affine arrangements). *Consider a* $\Pi_n^{\pm}$*-invariant subspace arrangement* $S \subseteq \mathbb{R}^n$ *with affine components* $\{S_i\}_{i=1}^k$*, and define the induced real variety* $\mathcal{M} := \sigma^{-1}(S)$*. Then the equality,* $\mathrm{EDdegree}(\mathcal{M}) = k$*, holds.*

Moreover, a general data point Y *in* $\mathbb{R}^{n\times t}$ *has exactly* k *ED critical points with respect to* $\mathcal{M}$*: for any decomposition* $Y = U \operatorname{Diag}(\sigma(Y)) V^\top$ *with orthogonal matrices* $U \in O(n)$ *and* $V \in O(t)$*, the set of ED critical points is precisely*

$$\{U \operatorname{Diag}(x) V^\top : x \textit{ is the orthogonal projection of } \sigma(Y) \textit{ onto } S_i\}.$$

In particular, all ED critical points of Y *with respect to* $\mathcal{M}$ *are real.*

Proof: Let Θ be the dense open subset of $\mathbb{C}^{n\times t}$ guaranteed to exist by Theorem 5.4.11. Clearly we can also assume that each matrix $Y \in \Theta$ has $\mathrm{EDdegree}(\mathcal{M})$ many ED critical points with respect to $\mathcal{M}$. A standard argument shows that the set $\Theta_{\mathbb{R}} := \{Y \in \mathbb{R}^{n\times t} : Y \in \Theta\}$ is a dense open subset of $\mathbb{R}^{n\times t}$. Fix a matrix $Y \in \Theta_{\mathbb{R}}$ and consider a singular value decomposition $Y = U \operatorname{Diag}(\sigma(Y)) V^\top$ with orthogonal matrices $U \in O(n)$ and $V \in O(t)$. By Theorem 5.4.11, the set of ED critical points of Y with respect to $\mathcal{M}$ is given by

$$\{U \operatorname{Diag}(x) V^\top \ : \ x \text{ is an ED critical point of } \sigma(Y) \text{ with respect to } S\}.$$

Since $\sigma(Y)$ is a real vector, the ED critical points of $\sigma(Y)$ with respect to S are precisely the orthogonal projections of $\sigma(Y)$ on each component S_i. Therefore we deduce $k = \mathrm{EDdegree}(S) = \mathrm{EDdegree}(\mathcal{M})$. □

The first three examples in Example 5.4.12 illustrate Corollary 5.5.1. Typically, as the data point $y \in \mathbb{R}^n$ varies, the number of real ED critical points of y with respect to a variety $\mathcal{V} \subseteq \mathbb{R}^n$ varies. Corollary 5.5.1 shows that when S is a subspace arrangement, all ED critical points of a real data point with respect to $\mathcal{M} = \sigma^{-1}(S)$ are again real and their number is constant. This unusual feature is easy to see using Theorem 5.4.11 that creates a bijection between the ED critical points of $\mathcal{M}$ and S, but is not at all obvious if S is not in the picture.

In common examples, outside of the subspace arrangement case, many ED critical points of a real data point may be non-real and the number of real critical points typically varies as the data point moves around. For instance, the hyperbola H_n in Example 5.4.12 can have complex ED critical points for a general $y \in \mathbb{R}^n$. The same is therefore true for $\mathrm{SL}_n^{\pm}$.

In a sense, Corollary 5.5.1 generalizes the fact that the pairs of singular vectors of a real matrix are real. Indeed, the pairs of singular vectors of a real matrix Y correspond to the ED critical points of Y with respect to the orthogonally invariant variety of rank one matrices; the corresponding absolutely symmetric variety is the union of all coordinate axes.

Remark 5.5.2. Results analogous to those in this chapter hold for symmetric matrices under the action of the orthogonal group $U \cdot A = UAU^{\top}$. More precisely, consider the space of real $n \times n$ symmetric matrices $\mathcal{S}^n$. A set $\mathcal{M} \subseteq \mathcal{S}^n$ is *orthogonally invariant* provided $U\mathcal{M}U^{\top} = \mathcal{M}$ for all matrices $U \in O(n)$. Such a set $\mathcal{M}$ can be written as $\lambda^{-1}(S)$ where $\lambda \colon \mathcal{S}^n \to \mathbb{R}^n$ assigns to each matrix X the vector of its eigenvalues in a nonincreasing order and S is the diagonal restriction $S = \{x \in \mathbb{R}^n : \mathrm{Diag}\,(x) \in \mathcal{M}\}$. Conversely any permutation invariant set $S \subseteq \mathbb{R}^n$ gives rise to the orthogonally invariant set $\lambda^{-1}(S)$. Similar techniques to the ones developed here can then be used to study the correspondence between ED critical points of algebraic varieties S and $\lambda^{-1}(S)$. This research direction deserves further investigation.

Chapter 6

CRITICAL POINTS FOR TWO-VIEW TRIANGULATION

6.1 Introduction

Two-view triangulation is the problem of estimating a point $X \in \mathbb{R}^3$ from two noisy image projections; see [35, Chapter 12] for its significance in structure from motion in computer vision. Assuming a Gaussian error distribution, one way to solve the problem is to compute the maximum likelihood estimates (MLE) for the true image point correspondences. After that the point $X \in \mathbb{R}^3$ can be recovered via linear algebra [35]. In this paper we study the above problem of finding the MLEs. According to the discussion in [3] or [35, Chapter 12], the problem is formulated as follows.

Consider a rank two matrix $F \in \mathbb{R}^{3\times 3}$ which is called a *fundamental matrix* in multi-view geometry. This matrix F encodes a pair of projective cameras [35, Chapter 9]. Given two points $u_1, u_2 \in \mathbb{R}^2$ which denote the noisy image projections, we solve the problem

$$\begin{aligned} &\min_{x_1, x_2 \in \mathbb{R}^2} \ \|x_1 - u_1\|_2^2 + \|x_2 - u_2\|_2^2 \\ &\text{subject to} \ \ \widehat{x}_2^\top F \widehat{x}_1 = 0 \end{aligned} \tag{6.1.1}$$

where $\widehat{x}_k := (x_k^\top \ 1)^\top \in \mathbb{R}^3$ for $k = 1, 2$. The equation $\widehat{x}_2^\top F \widehat{x}_1 = 0$ is called the *epipolar constraint*, which indicates that x_1 and x_2 are the true image projections under the projective cameras associated with F. The minimizers of (6.1.1) are the MLEs for the true image correspondences, assuming the error is Gaussian.

In [35, Chapter 12] (or [36]) there is a technique for finding the global minimizers of (6.1.1) using a non-iterative approach. They use multi-view geometry to reformulate the problem (6.1.1) as minimizing a fraction in a single real variable say t. Using the Fermat rule in elementary calculus, it turns out that the minimizers can be computed via finding the real roots of a polynomial in t of degree 6.

In this note, we view the problem (6.1.1) as minimizing a multivariate quadratic polynomial over one single equality constraint, and then employ the classical method of Lagrange multipliers to locate the potential local minimizers. These candidates are called *critical points*. For general rank two matrices F and general points u_1, u_2, there are six critical points. They can be computed via finding the roots of a polynomial of degree 6 in the Lagrange multiplier. Assuming that a global minimizer exists, the minimizer of (6.1.1) can be obtained from the critical points.

6.2 Six critical points for two-view triangulation

6.2.1 Reformulation of the problem (6.1.1)

Given a fundamental matrix $F \in \mathbb{R}^{3\times 3}$ and $u_1 = \begin{pmatrix} u_{11} & u_{12} \end{pmatrix}^\top$, $u_2 = \begin{pmatrix} u_{21} & u_{22} \end{pmatrix}^\top \in \mathbb{R}^2$, consider the invertible matrices $W_1 := \left(\begin{smallmatrix} 1 & 0 & -u_{11} \\ 0 & 1 & -u_{12} \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $W_2 := \left(\begin{smallmatrix} 1 & 0 & -u_{21} \\ 0 & 1 & -u_{22} \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Note that $\|x_k - u_k\|^2 = \|\widehat{x}_k - \widehat{u}_k\|^2$. and that problem (6.1.1) is equivalent to the problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}^2} \quad & \|W_1 \widehat{x}_1\|_2^2 + \|W_2 \widehat{x}_2\|_2^2 \\ \text{subject to} \quad & \widehat{x}_2^\top F \widehat{x}_1 = 0 \end{aligned}$$

For all $k = 1, 2$, the last coordinate of $W_k \widehat{x}_i$ equals one. As a result, we let $y_k \in \mathbb{R}^2$ be such that $\widehat{y}_k = W_k \widehat{x}_k$. Then (6.1.1) is further equivalent to the problem

$$\begin{aligned} \min_{y_1, y_2 \in \mathbb{R}^2} \quad & \frac{1}{2}\left(\|\widehat{y}_1\|_2^2 + \|\widehat{y}_2\|_2^2\right) \\ \text{subject to} \quad & \widehat{y}_2^\top F' \widehat{y}_1 = 0 \end{aligned} \tag{6.2.1}$$

where $F' := W_2^{-\top} F W_1^{-1} = \left(\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & i \end{smallmatrix}\right)$ is another fundamental matrix.

6.2.2 Derivation of a six degree polynomial

Let $G(y_1, y_2) := \frac{1}{2}\left(\|\widehat{y}_1\|_2^2 + \|\widehat{y}_2\|_2^2\right)$ and $H(y_1, y_2) := \widehat{y}_2^\top F' \widehat{y}_1$. The Karush-Kuhn-Tucker (KKT) equation for (6.2.1) is $\nabla G + \lambda \nabla H = 0$ for some $\lambda \in \mathbb{C}$ called the Lagrange multiplier; see any nonlinear programming text e.g. [4]. Unwinding this equation we obtain a linear system in four variables, namely,

$$\begin{pmatrix} 1 & 0 & \lambda a & \lambda b \\ 0 & 1 & \lambda d & \lambda e \\ \lambda a & \lambda d & 1 & 0 \\ \lambda b & \lambda e & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{21} \\ y_{22} \\ y_{11} \\ y_{12} \end{pmatrix} = -\lambda \begin{pmatrix} c \\ f \\ g \\ h \end{pmatrix} \tag{6.2.2}$$

where $y_k = \begin{pmatrix} y_{k1} & y_{k2} \end{pmatrix}^\top$ for $k = 1, 2$, and λ is the Lagrange multiplier. To acquire the critical points we derive a polynomial equation in λ. It comes from first expressing y_k, $k = 1, 2$, in terms of u_1, u_2, F and then substituting these expressions into the epipolar constraint $\widehat{y}_2^\top F' \widehat{y}_1 = 0$. Let A_λ be the 4×4 coefficient matrix of the above system. One has

$$\det(A_\lambda) = (bd - ae)^2 \lambda^4 - (a^2 + b^2 + d^2 + e^2)\lambda^2 + 1.$$

Define $p_{kl} := \det(A_\lambda) y_{kl}$ for $k, l = 1, 2$. By Cramer's rule one has

$$\begin{aligned} p_{21} &= \lambda[(bd - ae)(eg - dh)\lambda^3 + (d^2c + e^2c - adf - bef)\lambda^2 + (ag + bh)\lambda - c] \\ p_{22} &= \lambda[(bd - ae)(ah - bg)\lambda^3 + (a^2f + b^2f - acd - bce)\lambda^2 + (dg + eh)\lambda - f] \\ p_{11} &= \lambda[(bd - ae)(ce - bf)\lambda^3 + (b^2g + e^2g - abh - deh)\lambda^2 + (ac + df)\lambda - g] \\ p_{12} &= \lambda[(bd - ae)(af - cd)\lambda^3 + (a^2h + d^2h - abg - deg)\lambda^2 + (bc + ef)\lambda - h]. \end{aligned}$$

Consider the polynomial

$$T := -\det(A_\lambda)^2 \widehat{y}_2^\top F' \widehat{y}_1 = -p_2^\top F' p_1$$

where $p_k := \begin{pmatrix} p_{k1} & p_{k2} & \det(A_\lambda) \end{pmatrix}^\top$ for $k = 1, 2$. Since $\det(A_\lambda)$ is a quartic in λ, and p_{kl} is also a quartic in λ for $k, l = 1, 2$, we know T is a polynomial in λ of degree at most 8. By a careful and slightly tedious computation without using any machines, or by using the following `Macaulay2` [32] code:

```
R = QQ[a,b,c,d,e,f,g,h,i,L];
A = matrix{{1,0,L*a,L*b},{0,1,L*d,L*e},{L*a,L*d,1,0},{L*b,L*e,0,1}};
detA = det A;
p21 = det matrix{{-L*c,0,L*a,L*b},{-L*f,1,L*d,L*e},{-L*g,L*d,1,0},{-L*h,L*e,0,1}};
p22 = det matrix{{1,-L*c,L*a,L*b},{0,-L*f,L*d,L*e},{L*a,-L*g,1,0},{L*b,-L*h,0,1}};
p11 = det matrix{{1,0,-L*c,L*b},{0,1,-L*f,L*e},{L*a,L*d,-L*g,0},{L*b,L*e,-L*h,1}};
p12 = det matrix{{1,0,L*a,-L*c},{0,1,L*d,-L*f},{L*a,L*d,1,-L*g},{L*b,L*e,0,-L*h}};
T = -(a*p11*p21+b*p12*p21+c*p21*detA+d*p11*p22+
      e*p12*p22+f*p22*detA+g*p11*detA+h*p12*detA+i*detA*detA);
```

we know the coefficient of λ^7 is zero. The coefficient of λ^8 is

$$\begin{aligned}
&-(bd-ae)^2(eg-dh)(ace-abf+baf-bcd+cbd-cae)+\\
&-(bd-ae)^2(ah-bg)(dce-dbf+eaf-ecd+fbd-fae)+\\
&-(bd-ae)^3(gce-gbf+haf-hcd+ibd-iae)=(bd-ae)^3\det(F)=0
\end{aligned}$$

since F has rank two. This implies T is a polynomial in λ of degree at most six.

6.2.3 The six critical points

By solving $T=0$ for λ, we get six (complex) solutions (counting multiplicities) for λ, say $\lambda_1,\ldots,\lambda_6$. Plugging in these six values of λ into the linear system (6.2.2), solving the linear system for y_1 and y_2, and computing x_1 and x_2, one obtains the critical points for two-view triangulation. If $\det(A_{\lambda_k})\neq 0$ for every $k=1,\ldots,6$ then there are precisely six critical points counting multiplicities.

Now we claim that for general fundamental matrices F and points $u_1,u_2\in\mathbb{R}^2$, there are six distinct critical points for two-view triangulation. The claim is false if and only if the discriminant of T or the resultant of T and $\det(A_\lambda)$ are zero polynomials. Instead of computing the desired discriminant and resultant which depend on u_1,u_2 and F, one can find an example of (u_1,u_2,F) such that the discriminant of T and the resultant of T and $\det(A_\lambda)$

x_{21}	x_{22}	x_{11}	x_{12}
0.0596	-0.0321	-0.312	-0.891
-0.0843	-2.06	-0.438	-0.0259
$-2.42+0.0137i$	$-1.02-1.56i$	$-1.57+0.714i$	$-1.246-1.51i$
$-2.42-0.0137i$	$-1.02+1.56i$	$-1.57-0.714i$	$-1.246+1.51i$
$-1.69+0.0226i$	$-0.935+0.414i$	$0.748+0.169i$	$-0.279-0.574i$
$-1.69-0.0226i$	$-0.935-0.414i$	$0.748-0.169i$	$-0.279+0.574i$

Table 6.1: Six critical points for (6.1.1) when $u_1 = (0\ 0)^\top$, $u_2 = u_1$ and $F = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 3 \end{smallmatrix}\right)$.

take a nonzero value, that is, $\det(A_\lambda) \neq 0$ for every solution λ of T, and the six critical points obtained are distinct. If we consider the data $u_1 = \begin{pmatrix} 0 & 0 \end{pmatrix}^\top$, $u_2 = u_1$ and $F = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 3 \end{smallmatrix}\right)$, then the polynomial T becomes $-2\lambda^6 + 6\lambda^5 + 3\lambda^4 - 12\lambda^3 - 3\lambda^2 + 12\lambda - 3$, and there are six distinct complex critical points for the problem (6.1.1); see Table 6.1.

We summarize the discussion in the following theorem.

Theorem 6.2.1. *For general points $u_1, u_2 \in \mathbb{R}^2$ and fundamental matrices F, there are six complex critical points for the problem* (6.1.1)*.*

6.3 Discussion

One can make sense of the critical points for n-view triangulation where n is greater than two. The authors in [69] (cf. [41]) computed the number of critical points for 2 to 7 view triangulation are 6, 47, 148, 336, 638, 1081. Draisma et al. [22] call this list of numbers the *Euclidean distance degrees* of the multi-view variety associated to 2 to 7 cameras. They conjecture that the general term of this sequence is

$$C(n) := \frac{9}{2}n^3 - \frac{21}{2}n^2 + 8n - 4.$$

One can apply the Bézout's theorem to conclude that $C(n)$ has order n^3, and our paper verified $C(2) = 6$. However a proof of the above general formula is still unknown.

Chapter 7

ON THE EXISTENCE OF A PROJECTIVE RECONSTRUCTION

7.1 Introduction

Let a set of point correspondences $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$ $(i = 1, \dots, m)$ be given. Consider the following three statements:

(A) (x_i, y_i) are the images of m points in $\mathbb{R}^3$ in two uncalibrated cameras.

(B) (x_i, y_i) are the images of m points in $\mathbb{P}^3$ in two uncalibrated cameras.

(C) There exists a fundamental matrix F such that the (x_i, y_i) satisfy the epipolar constraints.

The goal of this note is to understand the connection among these three statements. In the following we summarize our contribution. All the results are proved using just linear algebra.

1. A standard result in two-view geometry [35, §9.2] states that (A) implies (C). In [35] this result was proved by classical projective geometry and drawing pictures. We offer a modern, more rigorous, and linear algebraic proof; see Theorem 7.4.1.

2. It is clear that (A) implies (B). We will show (A) and (B) are indeed equivalent; see Theorem 7.3.1. The proof is based on constructing an appropriate projective transformation.

3. We show that (C) implies (A) after making an additional assumption about the point pairs (x_i, y_i). Indeed, if (C) holds, one can construct a pair of uncalibrated cameras

P_1, P_2 associated to F. If we assume that x_i is an epipole of P_1 if and only if y_i is an epipole of P_2, then (A) holds. This assumption is also necessary for (A) to hold. As a result, we know (A) holds if and only if (C) and this assumption hold. This is the main theorem of this note; see Theorem 7.4.6.

In Section 7.2 we introduce the notation and definitions that will be used. In Section 7.3 we discuss projective reconstruction using finite, infinite, coincident and non-coincident cameras. Finally we provide a proof of the main theorem using linear algebra, in Section 7.4.

7.2 Notation and definitions

To begin with, we introduce the notation and definitions that will be used in this note; see [35].

Denote the n-dimensional real projective space by $\mathbb{P}^n$. For any $x, y \in \mathbb{P}^n$, we say $x \sim y$ if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. The set of $m \times n$ matrices with entries in $\mathbb{R}$ is denoted by $\mathbb{R}^{m\times n}$, and by $\mathbb{P}^{m\times n}$ if the matrices are only up to scale. For $v \in \mathbb{R}^3$,

$$[v]_\times := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

is a skew-symmetric matrix whose rank is two unless $v = 0$. Also, $[v]_\times w = v \times w$, where $\times$ denotes the vector cross product. For any $x \in \mathbb{R}^n$ the symbol $\widehat{x}$ denotes $(x, 1)^\top$ in $\mathbb{P}^n$. A point in $\mathbb{P}^n$ is called *finite* if it can be identified with $(x, 1)^\top$ for some $x \in \mathbb{R}^n$.

A *(projective) camera* can be modeled by a matrix $P \in \mathbb{P}^{3\times 4}$ with $\operatorname{rank}(P) = 3$. Partitioning a camera as $P = \begin{pmatrix} A & b \end{pmatrix}$ where $A \in \mathbb{R}^{3\times 3}$, we say that P is a *finite camera* if A is nonsingular. The camera center of P is $(-A^{-1}b, 1)^\top$ if P is finite; and $(w, 0)^\top$ otherwise, where w lies in the kernel of A. Two cameras P_1, P_2 with camera centers c_1, c_2 are *coincident* if $c_1 \sim c_2$. A tuple $(P_1, P_2, \{w_i\}_{i=1}^m)$ is called a *(projective) reconstruction* of $\{(x_i, y_i)\}_{i=1}^m \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ if P_1 and P_2 are projective cameras, $w_i \in \mathbb{P}^3$ and

$$P_1 w_i \sim \widehat{x}_i, \;\; P_2 w_i \sim \widehat{y}_i \;\; \text{for all } i = 1, \ldots, m.$$

If in addition, P_1, P_2 are finite cameras and w_i are finite points for all i, then $(P_1, P_2, \{w_i\})$ is called a *finite (projective) reconstruction.*

A real 3×3 matrix F is a *fundamental matrix* associated to $\{(x_i, y_i)\}$ if F has rank two and the following *epipolar constraints* hold:

$$\widehat{y_i}^\top F \widehat{x_i} = 0 \text{ for any } i.$$

7.3 Projective reconstruction

Given point correspondences $\{(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2,\ i = 1, \ldots, m\}$, the *projective reconstruction problem* is to decide if there is a projective reconstruction of these point pairs, and the *finite projective reconstruction problem* is to determine if the pairs admit a finite projective reconstruction. We first show that these two problems, as well as two others that naturally interpolate between them, are all equivalent.

Theorem 7.3.1. *Let $\{(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2,\ i = 1, \ldots, m\}$ be given. Then the following statements are equivalent:*

1. *There are cameras P_1, P_2 and points $w_i \in \mathbb{P}^3,\ i = 1, \ldots, m$, such that $(P_1, P_2, \{w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$.*

2. *There are* FINITE *cameras P_1, P_2 and points $w_i \in \mathbb{P}^3,\ i = 1, \ldots, m$, such that $(P_1, P_2, \{w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$.*

3. *There are* FINITE *cameras P_1, P_2 and* FINITE *points $w_i \in \mathbb{P}^3,\ i = 1, \ldots, m$, such that $(P_1, P_2, \{w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$.*

4. *There is a* FINITE *camera P_2 and* FINITE *points $w_i \in \mathbb{P}^3,\ i = 1, \ldots, m$, such that $(P_1, P_2, \{w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$, with the first camera $P_1 := \begin{pmatrix} I & 0 \end{pmatrix}$ where I is the 3×3 identity matrix.*

If P is a camera matrix, there is a nonsingular matrix $H \in \mathbb{R}^{4\times 4}$ such that $PH^{-1} = \begin{pmatrix} I & 0 \end{pmatrix}$. For instance, take H to be the nonsingular 4×4 matrix obtained by adding an appropriately chosen additional row to P. In order to prove Theorem 7.3.1, we will first need the following simple fact that for any finite collection of nonzero points in $\mathbb{R}^n$, there is always a hyperplane through the origin that avoids all of them.

Lemma 7.3.2. *Given $v_1, \dots, v_m \in \mathbb{R}^n \setminus \{0\}$, there exists $a \in \mathbb{R}^n$ such that $a^\top v_i \neq 0$ for all i.*

Proof: Let $S := \{v_1, \dots, v_m\}$. We want to show that there exists $a \in \mathbb{R}^n$ such that $a^\perp \cap S = \emptyset$. Suppose to the contrary, for any $a \in \mathbb{R}^n$ one has $a^\perp \cap S \neq \emptyset$. Then $a \in v_i^\perp$ for some i. Thus $\mathbb{R}^n = v_1^\perp \cup \cdots \cup v_m^\perp$ which implies that $\mathbb{R}^n = v_i^\perp$ for some i, and hence, this $v_i = 0$. This contradicts our assumption. □

We now come to the key ingredient in the proof of Theorem 7.3.1 which allows us to always replace a projective reconstruction with a finite projective reconstruction whenever the first camera is of the form $\begin{pmatrix} I & 0 \end{pmatrix}$.

Lemma 7.3.3. *Given point pairs $\{(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2,\ i = 1, \dots, m\}$, suppose we have cameras $P_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} A & b \end{pmatrix}$, a set $\sigma \subseteq \{1, \dots, m\}$, and points $v_i \in \mathbb{R}^3$, $i = 1, \dots, m$ such that:*

$$\forall\ i \in \sigma, \qquad v_i \neq 0, \qquad P_1 \begin{pmatrix} v_i \\ 0 \end{pmatrix} \sim \widehat{x}_i \quad \text{and} \quad P_2 \begin{pmatrix} v_i \\ 0 \end{pmatrix} \sim \widehat{y}_i;$$

$$\forall\ i \notin \sigma, \qquad P_1 \widehat{v}_i \sim \widehat{x}_i \qquad \text{and} \quad P_2 \widehat{v}_i \sim \widehat{y}_i.$$

Then there exists a finite camera P_2' and points $v_i' \in \mathbb{R}^3$, $i = 1, \dots, m$ such that $(P_1, P_2', \{\widehat{v}_i'\})$ is a finite reconstruction of $\{(x_i, y_i)\}$. In addition, if $b \neq 0$, then P_1 and P_2' are non-coincident cameras.

Proof: Let the camera centers of P_1 and P_2 be represented by $c_1 = \widehat{0}$ and c_2 respectively. Since $c_1, c_2, (v_i^\top, 0)^\top,\ i \in \sigma$ and $\widehat{v}_i,\ i \notin \sigma$ are all nonzero points in $\mathbb{R}^4$, by Lemma 7.3.2 there is a vector $a \in \mathbb{R}^3$ and a scalar $\alpha \in \mathbb{R}$ such that

$$(a^\top \ \ \alpha)\, c_i \neq 0,\ i = 1, 2,\ \ (a^\top \ \ \alpha) \begin{pmatrix} v_i \\ 0 \end{pmatrix} \neq 0 \ (i \in \sigma),\ \ (a^\top \ \ \alpha)\, \widehat{v}_i \neq 0 \ (i \notin \sigma). \tag{7.3.1}$$

Since $(a^\top \ \ \alpha)\, c_1 \neq 0$, we have that $\alpha \neq 0$. So by scaling, we may assume that $\alpha = 1$ in (7.3.1).

Consider the invertible matrix $H := \begin{pmatrix} I & 0 \\ a^\top & 1 \end{pmatrix}$. Then $H^{-1} := \begin{pmatrix} I & 0 \\ -a^\top & 1 \end{pmatrix}$, and $P_1 H^{-1} = P_1$ and $P_2 H^{-1} = \begin{pmatrix} A - ba^\top & b \end{pmatrix}$. Furthermore,

$$Hc_2 = \begin{pmatrix} * \\ \begin{pmatrix} a^\top & 1 \end{pmatrix} c_2 \end{pmatrix},\ \ H \begin{pmatrix} v_i \\ 0 \end{pmatrix} = \begin{pmatrix} v_i \\ \begin{pmatrix} a^\top & 1 \end{pmatrix} \begin{pmatrix} v_i \\ 0 \end{pmatrix} \end{pmatrix},\ \ H\widehat{v}_i = \begin{pmatrix} v_i \\ \begin{pmatrix} a^\top & 1 \end{pmatrix} \widehat{v}_i \end{pmatrix}$$

which are all finite by (7.3.1). In particular, $P_2 H^{-1}$ is a finite camera as its center Hc_2 is finite. The proof is completed by taking $P_2' = P_2 H^{-1}$, $\widehat{v}_i' \sim H \begin{pmatrix} v_i \\ 0 \end{pmatrix}$ $(i \in \sigma)$ and $\widehat{v}_i' \sim H\widehat{v}_i$ $(i \notin \sigma)$.

If we further assume $b \neq 0$, then P_1 and P_2 are non-coincident cameras. Hence $P_1 = P_1 H^{-1}$ and $P_2' = P_2 H^{-1}$ are also non-coincident.

□

Proof of Theorem 7.3.1: Clearly, (4) ⇒ (3) ⇒ (2) ⇒ (1). For (1) ⇒ (4), let H be a homography so that $P_1' := P_1 H^{-1} = \begin{pmatrix} I & 0 \end{pmatrix}$ and let $P_2' := P_2 H^{-1} = \begin{pmatrix} A & b \end{pmatrix}$. Then $(P_1', P_2', \{Hw_i\})$ is a reconstruction of $\{(x_i, y_i)\}$. We can now use Lemma 7.3.3 to turn this into a finite reconstruction where the first camera is still $\begin{pmatrix} I & 0 \end{pmatrix}$. Therefore, we conclude that all four statements in the theorem are equivalent. □

We now prove that the equivalences in Theorem 7.3.1 also hold if we further require that the cameras are non-coincident (coincident) in each statement.

Theorem 7.3.4. *The four statements in Theorem 7.3.1 are equivalent if we replace "cameras P_1, P_2" in each statement with "non-coincident cameras P_1, P_2".*

Proof: As before, we only need to show that (1) $\Rightarrow$ (4). Let $P_1' = P_1 H^{-1} = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2' = P_2 H^{-1} = \begin{pmatrix} A & b \end{pmatrix}$ as in the proof of this direction in Theorem 7.3.1. If P_1 and P_2 are non-coincident in (1), then P_1' and P_2' are also non-coincident. If A is nonsingular then $b \neq 0$. If A is singular, then $b \neq 0$ because $\mathrm{rank}(P_2') = 3$. Now using the last part of Lemma 7.3.3, we can turn the reconstruction $(P_1', P_2', \{Hw_i\})$ into a finite reconstruction with non-coincident cameras with the first camera equal to $\begin{pmatrix} I & 0 \end{pmatrix}$. This is the statement in (4). ☐

Theorem 7.3.5. *The four statements in Theorem 7.3.1 are equivalent if we replace "cameras P_1, P_2" in each statement with "coincident cameras P_1, P_2".*

Proof: Again, we only need to prove that (1) $\Rightarrow$ (4). If P_1, P_2 are coincident cameras in (1), then $P_1' = P_1 H^{-1} = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2' = P_2 H^{-1} = \begin{pmatrix} A & b \end{pmatrix}$ are also coincident. Therefore, $\widehat{0}$ is their common center and hence $b = 0$. This implies that A is nonsingular since otherwise $\mathrm{rank}(P_2') < 3$. Now consider the points w_i' obtained by setting the last coordinate of each w_i from the reconstruction in (1) to 1. Then $(P_1', P_2', \{w_i'\})$ is a finite reconstruction of $\{(x_i, y_i)\}$. ☐

By the above results we can always obtain a finite projective reconstruction whenever a projective reconstruction exists. Also, if the projective reconstruction was with non-coincident (coincident) cameras there is also a finite reconstruction with non-coincident (coincident) cameras. Further, in each case the first camera can be assumed to be $\begin{pmatrix} I & 0 \end{pmatrix}$. This understanding will be useful in the next section.

We end this section by discussing the geometry of the point pairs for which a projective reconstruction with coincident cameras exists.

Definition 7.3.6. Given $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$, $i = 1, \ldots, m$, we say that $\{x_i\}$ is *projectively equivalent* to $\{y_i\}$ if there is a nonsingular matrix $H \in \mathbb{R}^{3\times 3}$ such that $H\widehat{x}_i \sim \widehat{y}_i$ for all $1 \leq i \leq m$.

The following result captures the close relationship between projectively equivalent point sets and projective reconstruction with coincident cameras.

Theorem 7.3.7. *Let* $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$*,* $i = 1, \ldots, m$ *be given. Then there exists a finite reconstruction* $(P_1 = \begin{pmatrix} I & 0 \end{pmatrix}, P_2, \{\widehat{w}_i\}_{i=1}^m)$ *of* $\{(x_i, y_i)\}$ *where* P_1 *and* P_2 *are coincident cameras if and only if* $\{x_i\}$ *is projectively equivalent to* $\{y_i\}$*.*

Proof: Suppose $P_2 = \begin{pmatrix} A & b \end{pmatrix}$. If P_1 and P_2 are coincident, then their common camera center is $\widehat{0}$ which is finite. Hence P_2 is a finite camera and $b = 0$. Unwinding $P_1\widehat{w}_i \sim \widehat{x}_i$ and $P_2\widehat{w}_i \sim \widehat{y}_i$ we obtain $A\widehat{x}_i \sim \widehat{y}_i$ for all $i = 1, \ldots, m$.

For the converse, suppose there exists a nonsingular matrix $H \in \mathbb{R}^{3\times 3}$ such that $H\widehat{x}_i \sim \widehat{y}_i$ for all $i = 1, \ldots, m$. Then setting $P_1 := \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 := \begin{pmatrix} H & 0 \end{pmatrix}$, and using the notation $\widehat{\widehat{a}}$ for $(\widehat{a}^\top, 1)^\top$ where $a \in \mathbb{R}^2$, we see that $(P_1, P_2, \{\widehat{\widehat{x}}_i\}_{i=1}^m)$ is a projective reconstruction of $\{(x_i, y_i)\}$ with two coincident cameras. $\square$

7.4 Main theorem

We now come to the more general situation of reconstruction. In this case, there is a distinguished fundamental matrix associated to the point pairs coming from the cameras in the reconstruction. We remark that the some results in this section are formally or informally stated in [35], but we reprove all results using linear algebra instead of classical projective geometry.

Theorem 7.4.1. *Let* $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$*,* $i = 1, \ldots, m$ *be given. Consider two finite cameras* $P_1 := \begin{pmatrix} I & 0 \end{pmatrix}$ *and* $P_2 := \begin{pmatrix} A & b \end{pmatrix}$*. Suppose that there exist* $w_i \in \mathbb{R}^3$ $(1 \le i \le m)$ *such that* $(P_1, P_2, \{\widehat{w}_i\})$ *is a reconstruction of* $\{(x_i, y_i)\}$*. Then there is a fundamental matrix associated to* $\{(x_i, y_i)\}$*.*

Proof: Suppose that P_1 and P_2 are non-coincident cameras. Since A is nonsingular one has $b \neq 0$. Define $F := [b]_\times A$. Since $b \neq 0$, $\operatorname{rank}([b]_\times) = 2$ and $\operatorname{rank}(F) = 2$. For a fixed i, the relations $P_1\widehat{w}_i \sim \widehat{x}_i$ and $P_2\widehat{w}_i \sim \widehat{y}_i$ imply that $\lambda_i A\widehat{x}_i + b = \mu_i\widehat{y}_i$ for some $\lambda_i \neq 0$, $\mu_i \neq 0$. Hence, F satisfies the epipolar constraints involving x_i and y_i:

$$\widehat{y}_i^\top F\widehat{x}_i \sim (\lambda_i(A\widehat{x}_i)^\top + b^\top)[b]_\times A\widehat{x}_i \sim (A\widehat{x}_i)^\top[b]_\times A\widehat{x}_i = (A\widehat{x}_i)^\top(b \times A\widehat{x}_i) = 0.$$

If P_1 and P_2 are coincident, then there is a nonsingular matrix H such that $Hx_i \sim y_i$ for all $1 \leq i \leq m$, by Theorem 7.3.7. Let t be any nonzero vector in $\mathbb{R}^3$. It follows that for any $i = 1, \ldots, m$,

$$y_i^\top [t]_\times Hx_i = x_i^\top [t]_\times y_i = 0.$$

Thus $[t]_\times H$ is a fundamental matrix associated to $\{(x_i, y_i)\}$. $\square$

We now introduce a regularity condition on $\{(x_i, y_i)\}_{i=1}^m$ that is necessary for the existence of a projective reconstruction with non-coincident cameras. We will see that when the point pairs (x_i, y_i) are regular, a reconstruction with non-coincident cameras exists if and only if a fundamental matrix exists.

Definition 7.4.2. *Let $A \in \mathbb{R}^{3\times3}$ and $b \in \mathbb{R}^3$. We say that $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ is* (A, b)-irregular *if one of the following mutually exclusive conditions hold:*

$$([b]_\times A\widehat{x} = 0 \textit{ and } \widehat{y}^\top [b]_\times \neq 0) \quad \text{or} \quad ([b]_\times A\widehat{x} \neq 0 \textit{ and } \widehat{y}^\top [b]_\times = 0). \tag{7.4.1}$$

Say (x, y) is (A, b)-regular *if it is not (A, b)-irregular.*

If (x, y) is an (A, b)-irregular pair then $\widehat{y}^\top [b]_\times A\widehat{x} = 0$. This implies that if $P_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} A & b \end{pmatrix}$ are non-coincident finite cameras then (x, y) satisfies the epipolar constraint $\widehat{y}^\top F\widehat{x} = 0$ (where $F = [b]_\times A$) whether or not there is a reconstruction $w \in \mathbb{P}^3$ of (x, y). In fact, more is true.

Since $P_2 = \begin{pmatrix} A & b \end{pmatrix}$ is non-coincident with P_1, one has $b \neq 0$. Since the fundamental matrix $F := [b]_\times A$ has rank two, both the left and right kernel of F are one-dimensional. Let $e_1, e_2 \in \mathbb{R}^3 \setminus \{0\}$ be a basis vector of the right and left kernel of F respectively. Then e_1 is called an *epipole* of P_1 while e_2 is called an epipole of P_2. It is known that $P_1 c_2 \sim e_1$ and $P_2 c_1 \sim e_2$, where $c_1 = \widehat{0}$ and $c_2 = (-A^{-1}b^\top, 1)^\top$ are the camera centres of P_1 and P_2 respectively. This implies we can take $e_1 := A^{-1}b$ and $e_2 := b$.

Suppose (x, y) is (A, b)-irregular. Then as we saw earlier, $\widehat{y}^\top F\widehat{x} = 0$ holds. If $[b]_\times A\widehat{x} = 0$ and $\widehat{y}^\top [b]_\times \neq 0$ then $\widehat{x}$ is an epipole of P_1 but $\widehat{y}$ is not an epipole of P_2. If $[b]_\times A\widehat{x} \neq 0$ and $\widehat{y}^\top [b]_\times = 0$ holds then $\widehat{x}$ is not an epipole of P_1 but $\widehat{y}$ is an epipole of P_2. On the

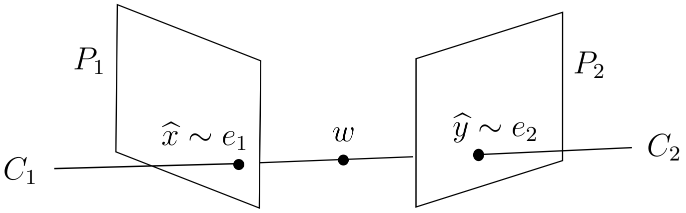

Figure 7.1: The tuple $(P_1, P_2, \widehat{w})$ reconstructs (x, y).

other hand, we see from Figure 7.1 that if $(P_1, P_2, \widehat{w})$ is a reconstruction of (x, y) for some $w \in \mathbb{R}^3$, and if $\widehat{x}$ is the epipole of P_1, then $\widehat{y}$ has to the epipole of P_2 (the epipoles of the two cameras lie on the line connecting the centers of the two cameras.) This means if (x, y) is (A, b)-irregular, then there is no finite reconstruction for (x, y) using P_1, P_2, even though the epipolar constraint is trivially satisfied. This proves the following lemma.

Lemma 7.4.3. *Suppose that* $P_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ *and* $P_2 = \begin{pmatrix} A & b \end{pmatrix}$ *are two non-coincident finite cameras. Then, if* (x, y) *is* (A, b)*-irregular, then there is no* $w \in \mathbb{R}^3$ *such that* $(P_1, P_2, \widehat{w})$ *is a reconstruction of* (x, y)*.*

Notice that Lemma 7.4.3 can also be verified using a simple algebraic computation, without using the notion of an epipole and the help of Figure 7.1.

The following two lemmas will be used to prove the main theorem of this section.

Lemma 7.4.4. *Suppose that* $P_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ *and* $P_2 = \begin{pmatrix} A & b \end{pmatrix}$ *are two non-coincident finite cameras. Then, if* (x, y) *is* (A, b)*-regular and* $\widehat{y}^\top [b]_\times A\widehat{x} = 0$*, then there exists* $w \in \mathbb{P}^3$ *such that* (P_1, P_2, w) *is a reconstruction of* (x, y)*.*

Proof: The assumptions about P_1 and P_2, and the equation $\widehat{y}^\top [b]_\times A\widehat{x} = 0$ imply $\widehat{y}, b, A\widehat{x}$ are nonzero linearly dependent vectors in $\mathbb{R}^3$. Thus there are scalars $\gamma, \beta, \alpha \in \mathbb{R}$, not all zero, such that

$$\gamma A\widehat{x} = \beta\widehat{y} - \alpha b. \tag{7.4.2}$$

For a scalar δ, define $w_\delta := \begin{pmatrix} \widehat{x} \\ \delta \end{pmatrix}$. Then we obtain

$$P_1 w_\delta = \widehat{x}, \quad \text{and } P_2 w_\delta = A\widehat{x} + \delta b.$$

There are three cases to consider.

Case 1: $\gamma = 0$.

Then $\widehat{y} \sim b$. If $A\widehat{x} = 0$, then $P_2 w_\alpha = \beta \widehat{y} \sim \widehat{y}$ so (P_1, P_2, w_α) is a reconstruction of (x, y). If $A\widehat{x} \neq 0$, then $\widehat{y} \sim A\widehat{x}$ by the regularity of (x, y). Thus $P_2 w_0 = A\widehat{x} \sim \widehat{y}$ so (P_1, P_2, w_0) is a reconstruction of (x, y).

Case 2: $\gamma \neq 0$ and $\beta = 0$.

In this case (7.4.2) gives $A\widehat{x} = -\alpha b$ after scaling. If $\alpha = 0$ then $A\widehat{x} = 0$ and $\widehat{y} \sim b$ by the regularity of (x, y). Thus $P_2 w_1 = b \sim \widehat{y}$ which means (P_1, P_2, w_1) is a reconstruction of (x, y). If $\alpha \neq 0$ then $A\widehat{x} \neq 0$ and $A\widehat{x} \sim b$. By the regularity of (x, y), one has $\widehat{y} \sim A\widehat{x}$. Thus (P_1, P_2, w_0) is a reconstruction of (x, y).

Case 3: $\gamma \neq 0$ and $\beta \neq 0$.

(7.4.2) implies $A\widehat{x} = \beta \widehat{y} - \alpha b$ after scaling. Hence $P_2 w_\alpha = A\widehat{x} + \alpha b = \beta \widehat{y} \sim \widehat{y}$ which concludes that (P_1, P_2, w_α) is a reconstruction of (x, y). ☐

Lemma 7.4.5. *Let F be a fundamental matrix and let $e_2 \in \ker(F^\top) \setminus \{0\}$. Define $P := \begin{pmatrix} [e_2]_\times F & e_2 \end{pmatrix}$. Then P has rank three.*

Proof: The proof can be found in [35, page 256], but we rewrite it here for the self-containedness of this note. Since $e_2 \in \ker(F^\top) \setminus \{0\}$, we have $\operatorname{rank}([e_2]_\times F) = 2$. It implies that the column space of $[e_2]_\times F$ is a plane in $\mathbb{R}^3$. Since e_2 is a nonzero vector orthogonal to any vector in this plane, we know $\operatorname{rank}(P) = 3$. ☐

We are now ready to prove the main theorem [35, Theorem 10.1].

Theorem 7.4.6. *Let $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$, $i = 1, \ldots, m$ be given. Then the following statements are equivalent:*

1. *There exists a finite reconstruction of* $\{(x_i, y_i)\}$ *where one of the cameras is* $\begin{pmatrix} I & 0 \end{pmatrix}$ *and the two cameras are non-coincident.*

2. *There is a fundamental matrix* F *associated to* $\{(x_i, y_i)\}$ *such that* (x_i, y_i) *is* $([e_2]_\times F, e_2)$*-regular for all* i*, where* $e_2 \in \ker(F^\top) \setminus \{0\}$*.*

Proof: First we show (2) $\Rightarrow$ (1). Let the matrix F stated in (2) be given. Notice that $\begin{pmatrix} [e_2]_\times F & e_2 \end{pmatrix}$ is a camera matrix by Lemma 7.4.5. Then, take $a \in \mathbb{R}^3$ so that $A := [e_2]_\times F - e_2 a^\top$ is nonsingular and $P_1 := \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 := \begin{pmatrix} A & e_2 \end{pmatrix}$ are non-coincident finite cameras; see the proof of Lemma 7.3.3 for how a is chosen. As $[e_2]_\times A = -e_2^\top e_2 F$, one has $y_i^\top [e_2]_\times A x_i = 0$ for all i. Then (1) holds by Theorem 7.3.1 and Lemma 7.4.4.

Next we show the converse. Assume (1) holds. Then there is a finite camera P_2 so that $P_1 := \begin{pmatrix} I & 0 \end{pmatrix}$, P_2 are non-coincident cameras, and there are $w_i \in \mathbb{R}^3$ $(1 \leq i \leq m)$ such that $(P_1, P_2, \{\widehat{w}_i\})$ is a reconstruction of $\{(x_i, y_i)\}$. We let $P_2 := \begin{pmatrix} A & b \end{pmatrix}$ where $A \in \mathbb{R}^{3\times 3}$ is nonsingular and $b \in \mathbb{R}^3 \setminus \{0\}$. Consider the fundamental matrix $F := [b]_\times A$. By Theorem 7.4.1 and Lemma 7.4.3, the epipolar constraints are satisfied and each (x_i, y_i) is (A, b)-regular. Since $F^\top = -A^\top [b]_\times$, we have $b \in \ker(F^\top) \setminus \{0\}$. Moreover, as $[b]_\times F = -b^\top b A$, we know each (x_i, y_i) is $([b]_\times F, b)$-regular. Thus the statement (2) follows. $\square$

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