Density degree sets and the geometry of curves

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Abstract

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Abstract: The arithmetic of an algebraic curve X over a field F is captured by it's scheme-theoretic closed points. Roughly, a closed point of degree d is a rational point over a degree-d extension of F. We investigate how Zariski dense degree-d points place restrictions on the geometry of X. For a curve over a discretely valued Henselian field K which is a cyclic cover of $\mathbb{P}^1$, we describe when the degree-d points are Zariski dense in terms of the ramification of the cyclic cover. For curves over number fields k, we relate having Zariski dense degree-d points over a finite extension of k to the Ueno locus of the degree-d effective divisors, and compute the Ueno locus of an arithmetically interesting class of curves.

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0.1 Attribution of joint work

This work is two separate projects which have been joined together. Both projects, which can be found in Chapter 3 and Chapter 5, represent original work. While Chapter 5 is attributed solely to the author, the results in Chapter 3 originate in a long and fruitful collaboration with Alexander Wang, and have also been included in his dissertation. Although work done collaboratively is equally shared, for the purposes of their dissertations, Wang and the author must split their work. We have chosen to attribute the results in Section 3.1, Section 3.3, and Section 3.4 to Alexander Wang, while the results in Section 3.2, together with Theorem 3.5.4, are attributed to the author.

Chapter 1

Introduction

The Mordell conjecture, now Faltings's curve theorem, established that the set of rational points of a curve X of genus $g \geq 2$ over a number field k is finite. Faltings's curve theorem is an example of a surprising principle in arithmetic geometry: geometry controls arithmetic. Following this principle, we seek to describe the arithmetic of X/k in terms of the geometry of $X/\bar{k}$.

The set of rational points is a small subset of the set of closed points of X/k. A scheme-theoretic closed point $P \in X/k$ naturally has a residue field $\mathbf{k}(P)$ which is a finite extension of k, and the degree of P is the degree of the extension $\mathbf{k}(P)/k$, see Section 1.0.1. Collecting all possible degrees of closed points into a single set, we define the *degree set* of X/k by

$$\mathcal{D}(X/k) \coloneqq \{[\mathbf{k}(P) : k] \mid P \in X \text{ closed}\}.$$

Studying $\mathcal{D}(X/k)$ may require studying Zariski closed subsets, as for example, by Faltings's curve theorem, the set of degree 1 points of a curve over a number field is Zariski closed. Understanding Zariski closed subsets is a difficult undertaking, so following [VV26], we define the *density degree set*

$$\delta(X/k) \coloneqq \{d \mid \text{degree-}d \text{ points are Zariski dense}\}.$$

Work of Bourdon, Ejder, Liu, Odumodu, and Viray [BEL+19], building on results of Faltings, describes the density degree set almost entirely in terms of the geometry of Pic_{X}/k. Their result shows that Zariski dense degree-d points arise for one of two reasons: they are $\mathbb{P}^1$-parametrized or AV-parametrized [VV26].

The $\mathbb{P}^1$-parametrized points are well-understood: they are fibers of a degree-d map $X \to \mathbb{P}^1$. The AV-parametrized points are more difficult to understand. Examples in [DF93, Section 4] show that AV-parametrized points do not necessarily arise from a morphism of smooth curves.

The description of AV-parametrized points provided by [BEL+19] or [VV26] is not purely geometric. Let W_X^d denote the locus of effective divisors in Pic_X^d. A degree-d point $x \in X$ is AV-parametrized if there exists a positive rank abelian variety $A \subset \mathrm{Pic}_X$ such that $[x] + A \subset W_X^d$. The condition that $[x] + A \subset W_X^d$ is a geometric one, however, we are left with the question: how do we determine, in terms of the geometry of Pic_X, if an abelian translate contained in W_X^d has a single degree-d point? We give an answer in Theorem 1.0.2.

In order to state our main theorem on AV-parametrized points, we make a definition:

Definition 1.0.1. Let $z \in W_X^d(k)$ and let $A \subset \mathrm{Pic}_X^0$ be an abelian variety such that $z + A \subset W_X^d$. The abelian translate $z + A$ is *decomposable* if there exists $e < d$ and abelian translates $u + B \subset W_X^e$, $v + G \subset W_X^{d-e}$ such that $z + A \subset u + v + B + G$. Otherwise, $z + A$ is *indecomposable*.

The decomposable translates should be thought of as accounted for by lower degree translates, while the indecomposable translates are "new". For a discussion on how Definition 1.0.1 arises from considering degree-d points, see Section 5.1.2. A degree-d point which is not $\mathbb{P}^1$-parametrized is $\mathbb{P}^1$-isolated.

Theorem 1.0.2. *Let $x \in X$ be a degree-d point and let $A \subset \mathrm{Pic}_X^0$ be an abelian variety such that $[x] + A \subset W_X^d$. If x is $\mathbb{P}^1$-isolated, then $[x] + A$ is indecomposable.*

Thus, a decomposable abelian translate does not contain any degree-d points, unless those points are the fibers of a morphism $X \to \mathbb{P}^1$. There is a partial converse to Theorem 1.0.2: one should expect that an indecomposable abelian translate contains many degree-d points, see Proposition 5.1.4 for a precise statement. As a consequence of Theorem 1.0.2 and Proposition 5.1.4, we describe when a curve has infinitely many degree-d points over some extension k'/k.

Theorem 1.0.3. *The following are equivalent:*

1. *There exists a finite extension k'/k such that X/k' has infinitely many degree-d points.*
2. *There exists a degree-d map $X/\overline{k} \to \mathbb{P}^1/\overline{k}$ or there exists a positive dimensional indecomposable abelian translate in $W_X^d/\overline{k}$.*

Applying Theorem 1.0.3 relies on determining if $W_X^d/\overline{k}$ contain a positive dimensional indecomposable translate. Existing results in the literature, such as [HS91], [Tuc02, Appendix A], [Der25], [KV25], or [VV26, Theorem 6.0.1(2)], can describe the abelian translates in $W_X^d/\overline{k}$ for d roughly in the interval $[1, \sqrt{g}]$. We construct genus-7 curves C for which we can show that $W_C^5/\overline{k}$ contains only decomposable translates.

Theorem 1.0.4. *Let E/k and F/k be non-isogenous elliptic curves, let $F \to \mathbb{P}^1$ be a degree-2 map, let $E \to \mathbb{P}^1$ be a degree-3 map, and let C/k be the curve $E \times_{\mathbb{P}^1} F$. If C is smooth and $J(C) \sim E \times F \times A$, where $A/\bar{k}$ is simple, then $W^5_C/\bar{k}$ contains only decomposable translates.*

In the text, we describe every positive dimensional abelian translate in $W^5_C/\bar{k}$ explicitly, see Theorem 5.3.12. Theorem 1.0.4 is proved in Section 5.2 and Section 5.3. In Section 5.2, we investigate curves with a degree-2 map $\phi\colon X \to E$ to an elliptic curve. We show that for $d < g$, the only indecomposable translate of $\phi^* \operatorname{Pic}^0_E$ in W^d_X is $\phi^* \operatorname{Pic}^1_E$. In Section 5.3, we analyze translates of $\theta^*_C \operatorname{Pic}^0_F$, heavily relying on the geometry of C.

We apply Theorem 1.0.3 and Theorem 1.0.4 to a question of Viray and Vogt [VV26, Question 7.1.1]: if X/k has infinitely many degree-d and degree-e points, does there exist some finite extension k'/k such that X/k' has infinitely many degree-$d+e$ points? We give a negative answer:

Corollary 1.0.5. *There exists a curve $C/\mathbb{Q}$ such that for any number field k, C/k has infinitely many degree-2 and degree-3 points, but only finitely many degree-5 points.*

Although so far we have worked over a number field, the degree set and the density degree set can be defined over any field. Over a discretely valued Henselian field K, the set of rational points of a curve X/K is either empty or Zariski dense. Thus, $\delta(X/K)$ equals $\mathcal{D}(X/K)$, and so we study $\mathcal{D}(X/K)$.

Define the *index* $\operatorname{Ind}(X/K)$ to be the greatest common divisor of the elements of the degree set. By construction, $\mathcal{D}(X/K)$ will be contained in $\operatorname{Ind}(X/K)\mathbb{N}$. A nice curve X over any field F will admit maps to $\mathbb{P}^1$ of all large enough degrees which are multiples of the index, and so over a finitely generated, infinite field k, Hilbert's Irreducibility implies that $\mathcal{D}(X/k)$ contains all sufficiently large multiples of the index. In other words, $\mathcal{D}(X/k)$ is *cofinite* in $\operatorname{Ind}(X/k)\mathbb{N}$. When referring to the complement of $\mathcal{D}(X/K)$, we always do so in $\operatorname{Ind}(X/K)\mathbb{N}$. Over a discretely valued Henselian field K, however, Hilbert's Irreducibility fails (see Example 2.3.2 for an explicit example), and thus the degree need not contain sufficiently large multiples of the index. For example, there exist curves over $\mathbb{Q}_p$ of index 1 whose degree set is $2\mathbb{N} \cup 3\mathbb{N}$, see [CV25, Equation 1.2]. In other words, $\mathcal{D}(X/K)$ is not cofinite.

Following [DDMM19], we characterize when the degree set of a superelliptic curve $y^q = F(x)$ is not cofinite in terms of subsets of the roots of $F(x)$ contained in an open disk; such subsets are called *clusters*. As the absolute Galois group $\operatorname{Gal}(\bar{K}/K)$ acts on the roots of $F(x)$, there is a natural action of $\operatorname{Gal}(\bar{K}/K)$ on clusters (see Definition 3.1.1). Let $O(\mathfrak{s})$ denote the orbit of a cluster $\mathfrak{s}$ under $\operatorname{Gal}(\bar{K}/K)$. In the special case where the ramification of the cyclic cover $C \to \mathbb{P}^1$ forms a single Galois orbit, our criterion simplifies to the following:

Theorem 1.0.6. *Suppose that the residue field of K is algebraically closed, that $F(x)$ is irreducible, and that $F(x)$ has a root of non-integer valuation, and that the splitting field of $F(x)$ is tamely ramified. Let $F(x) = \sum_{i=0}^{d} a_i x^i$. Then, $\mathcal{D}(C/K)$ is not cofinite if and only if*

1. $\deg F(x) \equiv 0 \pmod{q}$,
2. $v(a_d) \not\equiv 0 \pmod{q}$, *and*
3. $v(a_0) \not\equiv 0 \pmod{q}$.

When these conditions are satisfied,

$$\mathcal{D}(C/K) \subseteq q\mathbb{N} \cup \bigcup_{\mathfrak{s}} |O(\mathfrak{s})|\mathbb{N},$$

where the union is taken over all $\mathfrak{s}$ which are not Galois invariant.

In the text, at the cost of added notation, we prove a significantly stronger version of Theorem 1.0.6, see Theorem 3.5.1.

The two main tools used in the proof of Theorem 1.0.6 are Lemma 3.1.4 and Lemma 3.3.1. Lemma 3.1.4 provides a formula to compute $v(F(x_0))$ as a function of $x_0 \in \bar{K}$ and the nearest cluster, and as such may be of independent interest. Lemma 3.3.1 shows that there is an obstruction arising from the valuation to points of arbitrary degree.

Our methods allow us to produce examples of superelliptic curves with prescribed degree set.

Theorem 1.0.7. *Let K be a discretely valued Henselian field with algebraically closed residue field of characteristic p. Let q be a prime not equal to p, and let $n_1, \dots, n_\ell$ be positive integers. Then, there exists a superelliptic curve C/K such that*

$$\mathcal{D}(C/K) = q\mathbb{N} \cup n_1\mathbb{N} \cup \dots \cup n_\ell\mathbb{N}.$$

We prove a more refined version of Theorem 1.0.7, which allows for control over the genus of C/K, in Theorem 3.5.4.

As a special case of Theorem 1.0.7, we find that for every pair of odd primes p and q, there exists a curve C/K such that $\mathcal{D}(C/K) = p\mathbb{N} \cup q\mathbb{N}$. Thus, the density of $\mathcal{D}(C/K)$ in $\mathrm{Ind}(C/K)\mathbb{N}$ can be arbitrarily small.

1.0.1 Degree-d points

Throughout, we work with the scheme-theoretic closed points of X/k, of which the rational points $X(k)$ are a small subset. Over an algebraically closed field, there is a bijection from the

set of closed points of X to $X(\bar{k})$, but this is not true if k is not algebraically closed:

Example 1.0.8. Consider $\mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[x]$. Points of $\operatorname{Spec} \mathbb{Q}[x]$ are prime ideals in $\mathbb{Q}[x]$, and as $\mathbb{Q}$ is a field, every prime ideal is generated by an irreducible polynomial. Specifically, (x^2-2) is a point of $\mathbb{A}^1_{\mathbb{Q}}$, and in the Zariski topology, the set $\{(x^2-2)\}$ is closed. Every irreducible degree-d polynomial generates prime ideal which is a degree-d closed point of $\mathbb{A}^1_{\mathbb{Q}}$.

Definition 1.0.9. The *degree* of a closed point $x \in X/k$ is the degree of the extension $\mathbf{k}(x)/k$.

The set of closed points captures the arithmetic over any finite extension k'/k without a choice of algebraic closure $\bar{k}/k$, as for a closed point $x \in X$, the residue field $\mathbf{k}(x)$ is independent of choice of $\bar{k}$. Although not necessary, we often fix a choice of algebraic closure $\bar{k}/k$, so that a closed point $x \in X$ can be thought of as a $\operatorname{Gal}(\bar{k}/k)$-orbit of a geometric point $P \in X(\bar{k})$. The residue field $\mathbf{k}(x)$ is then (isomorphic to) the minimal field of definition k' of P.

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Chapter 2

Degree sets over discretely valued Henselian fields

2.1 Preliminaries

We will assume throughout that the reader is familiar with local rings. For our local ring $(A, \mathfrak{m})$, we will fix a valuation $v : A \to \mathbb{Z}$, and let ℓ denote $A/\mathfrak{m}$. For any unit u in A, we will let $\bar{u}$ be the reduction modulo $\mathfrak{m}$. We will assume that A is integral, and let K denote the field of fractions of A.

By definition, a Henselian ring is a ring which satisfies Hensel's lemma, and a Henselian field is the field of fractions of such a ring. How exactly to state Hensel's lemma, however, can vary. We use the following version.

Definition 2.1.1. A local ring $(A, \mathfrak{m})$ is *Henselian* if the following holds: Let $f(x) \in A[x]$ be a monic polynomial, and let $\bar{f}(x)$ denote the reduction of $f(x)$ modulo $\mathfrak{m}A[x]$. If $\bar{f}$ factors as $\bar{g}\bar{h}$, where $\bar{g}$ and $\bar{h}$ are monic and relatively prime, then f factors as gh, where g and h are monic and reduce to $\bar{g}$ and $\bar{h}$.

From now on, K will denote a Henselian field. Any complete local ring is Henselian, see [Mil20, Theorem 7.33]. We can classify n-th powers in a Henselian field K, provided the residue field has characteristic not dividing n.

Lemma 2.1.2. *Let K be a discretely valued Henselian field with uniformizer π and residue field ℓ, and let n be a natural number. Suppose that the characteristic of the residue field does not divide n. For $a \in K$,*

$$a \in K^{\times n} \quad \text{if and only if} \quad v(a) \equiv 0 \pmod{n} \quad \text{and} \quad \frac{\bar{a}}{\pi^{v(a)}} \in \ell^{\times n}.$$

Proof. If there exists $b \in K$ such that $b^n = a$, then $v(a) = v(b^n) = nv(a)$, and hence the valuation of a is a multiple of n in $\mathbb{Z}$. Similarly,

$$\frac{a}{\pi^{v(a)}} = \frac{b^n}{\pi^{nv(b)}} \equiv \left(\frac{b}{\pi^{v(b)}}\right)^n \pmod{\pi}$$

and hence $\frac{\bar{a}}{\pi^{v(a)}} \in \ell^{\times n}$.

For the converse, consider the polynomial $f(x) := x^n - a \in K[x]$. We show that if $v(a) \equiv 0 \pmod{n}$ and $\frac{a}{\pi^{v(a)}} \in \ell^{\times n}$, then f has a root in K. Let $m = v(a)/n \in \mathbb{Z}$. Consider $f(\pi^m x) = \pi^{mn}\left(x^n - \frac{a}{\pi^{mn}}\right) = \pi^{v(a)}\left(x^n - \frac{a}{\pi^{v(a)}}\right)$. It suffices to show that $g(x) = \frac{1}{\pi^{v(a)}} f(\pi^m x)$ has a root in K, as then f does as well. By assumption, $\frac{\bar{a}}{\pi^{v(a)}}$ is an n-th power in ℓ, and further does not equal zero by construction. Hence, $\bar{g}$ has a linear factor in $\ell[x]$. The derivative of $\bar{g}$ equals nx^{n-1}, and so has a unique root $x = 0$ as the characteristic of ℓ does not divide n. Thus, the $\bar{g}$ has no repeated roots, as a repeated root is a root of both $\bar{g}$ and $\bar{g}'$. By Hensel's lemma, g has a linear factor in $A[x]$, and hence f has a root in K as desired. □

2.2 Structure of degree sets of general schemes over Henselian fields

Let K be a Henselian field. We establish two lemmas that are crucial to the computation of degree sets. Both are due to Liu and Lorenzini [LL18]. We state them here in the form we require.

Lemma 2.2.1. *Let X/K be a geometrically integral and smooth scheme of finite type and positive dimension. Let $m \in \mathbb{N}$. If $m \in \mathcal{D}(X/K)$, then $m\mathbb{N} \subseteq \mathcal{D}(X/K)$.*

Proof. Since K is Henselian, K is a large field [Pop10, Theorem 1.1]. If $m \in \mathcal{D}(X/K)$, there exists some extension L/K of degree m such that $X(L)$ is nonempty, and hence contains a smooth point. From [LL18, Proposition 8.3], we find that any separable extension M/L contains infinitely many new points of X. We can then pick a separable extension M_n/L of degree n for each $n \in \mathbb{N}$, as for example, if π is a uniformizer for K, then $X^n + \pi X + \pi$ will be an irreducible and separable polynomial. Thus, we find that $m\mathbb{N} \subseteq \mathcal{D}(X/K)$. □

Lemma 2.2.2. *Let X/K be a geometrically integral and smooth scheme of finite type and positive dimension. Let W be a Zariski open subset of X. Then, $\mathcal{D}(W/K) = \mathcal{D}(X/K)$.*

Proof. As W embeds into X, $\mathcal{D}(W/K) \subseteq \mathcal{D}(X/K)$. If $m \in \mathcal{D}(X/K)$, then there exists some L/K of degree m such that $X(L)$ is nonempty. Since K is Henselian, L is also Henselian, and hence

is a large field [Pop10, Theorem 1.1], so that $X(L)$ is Zariski dense. Thus, $W(L)$ is nonempty, and hence there exists some point of W having degree dividing m. Applying Lemma 2.2.1, we find that $m \in \mathcal{D}(W/K)$. □

2.3 Degree Sets and Maps Between Varieties

We would like to study the degree sets of varieties by studying the maps between them.

Lemma 2.3.1. *Let X/K and Y/K be geometrically integral and smooth schemes of finite type, and suppose that X is positive dimensional and that there exists a map $X \to Y$. Then, $\mathcal{D}(X/K) \subseteq \mathcal{D}(Y/K)$.*

Proof. Let $P \in X(\bar{K})$ be a point of degree d and let $Q \in Y(\bar{K})$ be the image of P. Let d' be the degree of Q. We have an inclusion of residue fields $\mathbf{k}(Q) \to \mathbf{k}(P)$, which shows that d' divides d. By Lemma 2.2.1, $\mathcal{D}(Y/K)$ contains all multiples of d', and hence contains d. □

Consider the case where X and Y are nice curves. A general curve X/K has simple Jacobian, and hence admits no maps $X \to Y$ to a curve Y/K of positive genus. When Y/K has genus 0, $\mathcal{D}(Y/K)$ is either $\mathbb{N}$ or $2\mathbb{N}$. Hence for a general curve X/K, Lemma 2.3.1 is not very helpful for studying $\mathcal{D}(X/K)$.

The containment in Lemma 2.3.1 can be an equality. We give an example which takes this equality to the extreme: we construct a map $X \to \mathbb{P}^1$ such that for *all* $t \in \mathbb{P}^1(K)$, the fiber over t consists *only* of K-rational points. Note that our example strongly depends on K being Henselian: over a finitely generated field k, Hilbert's irreducibility states that for many $t \in \mathbb{P}^1(k)$, the fiber over t is a closed point of degree equal to the degree of the map.

Example 2.3.2. Let $X/\mathbb{Q}_5$ be the hyperelliptic curve given by $y^2 = x^6 - 5^2$, and let $\pi : X \to \mathbb{P}^1$ be the x-coordinate map. Let $u : \operatorname{Spec} \mathbb{Q}_5 \to \mathbb{P}^1$ be a point, and let $U := \operatorname{Spec} \mathbb{Q}_5 \times_{\mathbb{P}^1} X$ be the fiber product with π. We show that $U(\mathbb{Q}_5)$ has exactly two distinct elements, so that $U/\mathbb{Q}_5$ is not integral.

To find a smooth projective variety isomorphic to X, we consider the weighted projective space $\mathbb{P} = \mathbb{P}(1,3,1)$ with variables V, W, Z. Then, X is isomorphic to the variety $W^2 = V^6 - 5^2 Z^6$ in $\mathbb{P}$, see [Gal12, Section 10.1.1] for details. The x-coordinate map is given by $[V : W : Z] \mapsto [V : Z]$.

First suppose that the image of u is contained in the image of $Z = 0$. The image of u is then $[1 : 0] \in \mathbb{P}^1$ and $U(\mathbb{Q}_5)$ has two points mapping to $[1 : 1 : 0]$ and $[1 : -1 : 0]$ in $\mathbb{P}$.

Now suppose that u is contained in the image under the x-coordinate map of the affine open $Z \neq 0$. Then, u lies in the image of the affine open in X isomorphic to $y^2 = x^6 - 5^2$ in $\mathbb{A}^2$, the

x-coordinate map restricts to a map to $\mathbb{A}^1$ given by $(x, y) \mapsto x$, and there exists $u_0 \in \mathbb{Q}_5$ such that the image of u is $u_0 \in \mathbb{A}^1(\mathbb{Q}_5)$. The elements of $U(\overline{\mathbb{Q}_5})$ are $\pm(u_0^6 + 5^2)^{1/2}$. We show that, in fact, $(u_0^6 + 5^2)^{1/2} \in \mathbb{Q}_5$. By the strong triangle inequality,

$$v_5(u_0^6 + 5^2) \leq \min\{6v_5(u_0), 2\}.$$

As $u_0 \in \mathbb{Q}_5$, $v_5(u_0^6) = 6v_5(u_0) \in 6\mathbb{Z}$, and as $6 \nmid 2$, the minimum is unique and the inequality becomes an equality. If $v_5(u_0^6 + 5^2) = 2$, then $v_5(u_0^6) > 2$ and

$$\frac{u_0^6 + 5^2}{5^2} \equiv \frac{5^2}{5^2} \equiv 1 \pmod 5$$

so that by Lemma 2.1.2, $u_0^6 + 5^2$ is a square in $\mathbb{Q}_5$. Similarly, if $v_5(u_0^6 + 5^2) = 6v_5(u_0)$, then $6v_5(u_0) < 2 = v_5(5^2)$ and hence

$$\frac{u_0^6 + 5^2}{5^{6v_5(u_0)}} \equiv \frac{u_0^6}{5^{6v_5(u_0)}} \equiv \left(\frac{u_0}{5^{v_5(u_0)}}\right)^6 \pmod 5$$

so that by Lemma 2.1.2, $u_0^6 + 5^2$ is a square in $\mathbb{Q}_5$. Thus, $U(\mathbb{Q}_5)$ consists of two distinct elements, namely the positive and negative square roots of $u_0^6 + 5^2$.

As we intend to study when $\mathcal{D}(X/K)$ is not cofinite, we study curves with maps whose behavior differs from Example 2.3.2 as much as possible. We add further structure by studying two maps simultaneously:

Lemma 2.3.3. *Let X/K, Y/K, and Z/K be varieties with maps $\varphi : X \to Y$ and $\theta : X \to Z$. Let $P \in X(\bar{K})$. Then,* $\operatorname{lcm}\{\deg_K \varphi(P), \deg_K \theta(P)\}$ *divides the degree of P over K.*

Proof. We have inclusions of residue fields $\mathbf{k}(\varphi(P))$ into $\mathbf{k}(P)$ and $\mathbf{k}(\theta(P))$ into $\mathbf{k}(P)$. Thus, $\deg_K P$ is divisible by both $\deg_K \varphi(P)$ and $\deg_K \theta(P)$, and hence is divisible by the least common multiple. □

For a very general curve X/K, both Y/K and Z/K must have genus 0. We thus specialize further, studying the case where both Y/K and Z/K are $\mathbb{P}^1_K$, and use the structure of $\bar{K}$ to study points on $\mathbb{P}^1_K$. As K is a Henselian discretely valued field, we have a unique extension of the valuation to a map $v : \bar{K} \to \mathbb{Q} \cup \{\infty\}$. Thus, v induces a Galois equivariant map $v : \mathbb{A}^1(\bar{K}) \to \mathbb{Q} \cup \{\infty\}$. Fix an affine open $W \subseteq X$ not including poles of φ and θ, so that $\varphi|_W$ and $\theta|_W$ map to $\mathbb{A}^1$. We study the degrees of points $P \in W(\bar{K})$ by studying their image under the compositions $v \circ \varphi|_W$ and $v \circ \theta|_W$. By Lemma 2.2.2, $\mathcal{D}(W/K)$ equals $\mathcal{D}(X/K)$, and hence it

suffices to only study points in W. Let $\operatorname{denom}\left(\frac{a}{b}\right)$ denote the denominator of $\frac{a}{b}$ when written in lowest terms.

Lemma 2.3.4. *Let X/K be a variety with two maps $\varphi, \theta\colon X \to \mathbb{P}^1$, and let $W \subseteq X$ be an affine open away from the poles of φ and θ. Let $P \in W(\bar{K})$. Define*

$$\Delta_{P,\varphi,\theta} \coloneqq \operatorname{lcm}\{\operatorname{denom}(v(\varphi(P))), \operatorname{denom}(v(\theta(P)))\}.$$

Then, $\Delta_{P,\varphi,\theta}$ divides the degree of P over K.

Proof. We first show that $\operatorname{denom}(v(\varphi(P)))$ divides $\deg_K \varphi(P)$. As $\varphi(P) \in \mathbb{A}^1(\bar{K})$, there exists some element $\gamma \in \bar{K}$ such that $\varphi(P)$ is the prime ideal $(x - \gamma)$, so $\mathbf{k}(\varphi(P)) = K(\gamma)$ and $v(\varphi(P)) = v(\gamma)$. As $v(\gamma)$ is an element of the value group of $K(\gamma)$, the denominator of $v(\gamma)$ divides the ramification degree of $K(\gamma)/K$, which divides the degree of $K(\gamma)/K$, as desired.

The statement is symmetric in φ and θ, and thus $\operatorname{denom}(v(\theta(P))$ divides $\deg_K \theta(P)$. Hence, $\operatorname{denom}(v(\varphi(P))$ and $\operatorname{denom}(v(\theta(P))$ both divide $\operatorname{lcm}\{\deg_K \varphi(P), \deg_K \theta(P)\}$, and hence $\Delta_{P,\varphi,\theta}$ divides the least common multiple of $\deg_K \varphi(P)$ and $\deg_K \theta(P)$ as well. Finally, Lemma 2.3.3 shows that $\operatorname{lcm}\{\deg_K \varphi(P), \deg_K \theta(P)\}$ divides $\deg_K P$, and hence $\Delta_{P,\varphi,\theta}$ does as well. □

Lemma 2.3.5. *Let X/K be a variety with two maps $\varphi, \theta\colon X \to \mathbb{P}^1$, and let $W \subseteq X$ be an affine open away from the poles of φ and θ. Suppose that there exists a finite set $\{r_1, \dots, r_n\} \subseteq \mathbb{N}$ such that for all $P \in W(\bar{K})$, there exists r_i (depending on P) such that r_i divides $\Delta_{P,\varphi,\theta}$. Then,*

$$\mathcal{D}(X/K) \subseteq \bigcup_i r_i\mathbb{N}.$$

Proof. By Lemma 2.2.2, $\mathcal{D}(X/K) = \mathcal{D}(W/K)$, thus it suffices to bound $\mathcal{D}(W/K)$. For all $P \in W(\bar{K})$, $\Delta_{P,\varphi,\theta}$ divides $\deg_K P$ by Lemma 2.3.4, and thus by hypothesis there exists r_i such that r_i divides $\deg_K P$. Hence, $\deg_K P \in r_i\mathbb{N}$ and the result follows. □

It follows from Theorem 3.5.1 that if C/K is a superelliptic curve with equation $y^q = F(x)$ such that $F(x)$ has no roots of integer valuation, then in fact $\mathcal{D}(C/K)$ is cofinite if and only if there exists a set $\{r_1, \dots, r_n\}$ as in Lemma 2.3.5.

In general, it is not reasonable to expect one pair of maps to completely explain why the degree set is not cofinite. We can weaken the conditions on Lemma 2.3.5 by letting the pair of maps vary. As part of the proof of Theorem 3.5.1, we show that if C/K is a superelliptic curve with equation $y^q = F(x)$ and the roots of $F(x)$ are tamely ramified, then in fact $\mathcal{D}(C/K)$ is cofinite if and only if there exists a finite collection of maps (θ_i, φ_i) obstructing the degrees of points.

Lemma 2.3.6. *Let X/K be a variety with a finite set of pairs of maps $\{(\varphi_1, \theta_1), \dots, (\varphi_m, \theta_m)\}$ to $\mathbb{P}^1$, and let $W \subseteq X$ be an affine open away from the poles of $\varphi_1, \theta_1, \dots, \varphi_m, \theta_m$. Suppose there exists a finite set $\{r_1, \dots, r_n\} \subseteq \mathbb{N}$ such that for all $P \in W(\bar{K})$, there exists i, j such that r_i divides $\Delta_{P, \varphi_j, \theta_j}$. Then,*

$$\mathcal{D}(X/K) \subseteq \bigcup_i r_i \mathbb{N}.$$

Proof. By Lemma 2.2.2, $\mathcal{D}(X/K) = \mathcal{D}(W/K)$, thus it suffices to bound $\mathcal{D}(W/K)$. By hypothesis, for all $P \in W$, there exists some pair of maps $\varphi_j : W \to \mathbb{P}^1$ and $\theta_j : W \to \mathbb{P}^1$ such that r_i divides $\Delta_{P, \varphi_j, \theta_j}$, and hence we can apply Lemma 2.3.4. The result follows. □

2.4 General lemmas

In this subsection we collect several lemmas for later use.

Lemma 2.4.1. *Let p be a prime, let k be a field, and let $a \in k^*$. The polynomial $h(y) = y^p - a$ is either irreducible or $a \in k^{\times p}$.*

Proof. This follows from [Lan02, Chapter VI, Theorem 9.1]. □

Lemma 2.4.2. *Let $\rho\colon \frac{1}{r}\mathbb{Z} \to \frac{1}{rb}\mathbb{Z}$ be an affine map defined by*

$$\rho(x) = \frac{a}{b}x + \frac{c}{b}.$$

The image of ρ intersects $\frac{1}{r}\mathbb{Z}$ if and only if $\gcd(a, b)$ divides rc. If this condition holds, then for any open interval I of length greater than $\frac{b+1}{r}$, $im(\rho)$ restricted to $I \cap \frac{1}{r}\mathbb{Z}$ intersects $\frac{1}{r}\mathbb{Z}$.

Proof. We construct a map $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ from ρ. Explicitly, let $\theta\colon \mathbb{Z} \to \frac{1}{r}\mathbb{Z}$ be division by r, and let $\psi\colon \frac{1}{rb}\mathbb{Z} \to \mathbb{Z}$ be multiplication by rb. Let $\varphi(x) := (\psi \circ \rho \circ \theta)(x) = ax + rc$, so $\mathrm{im}(\rho)$ intersects $\frac{1}{r}\mathbb{Z}$ if and only if $\mathrm{im}(\varphi)$ intersects $b\mathbb{Z}$, as $b\mathbb{Z}$ is the image of $\frac{1}{r}\mathbb{Z}$ under ψ. The image of φ intersects $b\mathbb{Z}$ precisely when there is a solution to $ax \equiv -rc \pmod{b}$, which exists if and only if $\gcd(a, b)$ divides rc. For an interval I, if $|I| > \frac{b+1}{r}$, by the pigeonhole principle, the numerators of $I \cap \frac{1}{r}\mathbb{Z}$ contain a complete set of representatives of congruence classes modulo b. □

The following is a strengthening of Krasner's lemma, provided that one works with an element of K^t, the tamely ramified closure.

Lemma 2.4.3. *Let B be an open ball in $\bar{K}$ such that $B \cap K^t$ is nonempty. There exists $\beta \in B$ such that for all $\alpha \in B$, $K(\beta) \subseteq K(\alpha)$. Further, $\deg_K(\beta)$ equals the size of the orbit of B under $\mathrm{Gal}(K^t/K)$.*

Proof. This is the one dimensional case of the main theorem of [Duc13]. □

Lemma 2.4.4. *Assume that the residue field of K is algebraically closed. If r is coprime to the residue characteristic of K, then $K(\sqrt[r]{\pi})$ is the unique extension of K of degree r.*

Proof. This follows from [Lan13, Chapter II, Proposition 12]. □

Chapter 3

Degree sets of superelliptic curves over Henselian fields

We now begin working towards proving Theorem 3.5.1. We work with a superelliptic curve C over a discretely valued Henselian field K with algebraically closed residue field such that the cyclic cover $C \to \mathbb{P}^1$ has prime degree q, and fix an equation $y^q = F(x)$ such that the variety $V(y^q - F(x))$ is isomorphic to an affine open U/K of C/K. The key ingredient, which may be of wider interest, is a good understanding of $v(F(x_0))$ for any $x_0 \in \overline{K}$ in terms of the roots of $F(x)$, see Lemma 3.1.4. We then consider a point $(x_0, y_0) \in U(\overline{K})$ in Section 3.3. As $v(y_0) = v(F(x_0))/q$, we can determine if $v(y_0)$ is an integer in terms of $v(F(x_0))$, a condition that can be expressed in terms of the roots of $F(x)$. If $v(y_0)$ is not an integer, then $K(y_0)/K$ must have degree a multiple of q, thus "obstructing" points of degree relatively prime to q. Conversely, we use Hensel's lemma to show that if the roots of $F(x)$ satisfy certain properties, we can construct points in $U(\overline{K})$ of large degree, see Lemma 3.4.1.

3.1 The valuation of a polynomial

In this section we address the question of determining, for any $x_0 \in \bar{K}$, the valuation of $F(x_0)$ in terms of the valuation of x_0. We give a complete answer in terms of the roots $\{\alpha_i\}$ of the polynomial $F(x)$. The following definition is [DDMM19, Definition 1.1].

Definition 3.1.1. A **cluster** is a nonempty subset $\mathfrak{s} \subseteq \{\alpha_i\}$ of the form $\mathfrak{s} = D \cap \{\alpha_i\}$ for some disk $D_{z,d} = \{x \in \bar{K} \mid |x - z| \leq d\}$ for some $z \in \bar{K}$ and $d \in \mathbb{R}$. Let $O(\mathfrak{s})$ denote the orbit of $\mathfrak{s}$ as a set under $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, and let $\mathfrak{s}^c$ denote the complement of $\mathfrak{s}$ in $\{\alpha_i\}$.

Lemma 3.1.2. *Let $x_0 \in \bar{K}$. Let $R_{x_0} = \min_i |x_0 - \alpha_i|$ and let S be the set of roots that achieve the minimum. Then, S is a cluster.*

Proof. Choose $z = x_0$ and $d = R_{x_0}$. We then have that $S = D_{z,d} \cap \{\alpha_i\}$. □

Definition 3.1.3. Let $X_{\mathfrak{s}} \subseteq \bar{K}$ be the set

$$X_{\mathfrak{s}} = \{x \in \bar{K} \mid \forall \alpha \in \mathfrak{s}, |x - \alpha| = R_x, \forall \alpha' \in \mathfrak{s}^c, |x - \alpha'| > R_x\}.$$

We think of $X_{\mathfrak{s}}$ as partitioning $\bar{K}$ by the roots of minimal distance. By construction, the collection of $\{X_{\mathfrak{s}}\}$ covers $\bar{K}$ as $\mathfrak{s}$ varies over all clusters.

Lemma 3.1.4. *Fix a cluster $\mathfrak{s}$. There exists $c_{\mathfrak{s}} \in \mathbb{Q}$ such that for all $x_0 \in X_{\mathfrak{s}}$ and any $\alpha \in \mathfrak{s}$,*

$$v(F(x_0)) = |\mathfrak{s}| \cdot v(x_0 - \alpha) + c_{\mathfrak{s}},$$

where $c_{\mathfrak{s}}$ is given by

$$c_{\mathfrak{s}} = v(a_d) + \sum_{\beta \in \mathfrak{s}^c} v(\alpha - \beta)$$

and all expressions above are independent of the choice of $\alpha \in \mathfrak{s}$.

Note that for any choice of $\alpha \in \mathfrak{s}$, the expression $|\mathfrak{s}|v(x_0 - \alpha)$ is equivalent to $\sum_{\alpha' \in \mathfrak{s}} v(x_0 - \alpha')$, since all roots in $\mathfrak{s}$ are equidistant from x_0. This definition is more symmetric in $\mathfrak{s}$ but less useful computationally.

Remark 3.1.5. By Lemma 3.1.4, $|\mathfrak{s}|$ and $c_{\mathfrak{s}}$ are the coefficients of the piecewise affine linear function given by the slope formula, see [BPR13, Theorem 5.15].

Proof of Lemma 3.1.4. For $\beta \in \mathfrak{s}^c$,

$$R_{x_0} < |x_0 - \beta| = |x_0 - \alpha + \alpha - \beta| \leq \max\{|x_0 - \alpha|, |\alpha - \beta|\} = \max\{R_{x_0}, |\alpha - \beta|\}.$$

We must have that $|\alpha - \beta| > R_{x_0}$ and thus by strong triangle inequality, $|x_0 - \beta| = |\alpha - \beta|$ and we have the same equality on valuations. Thus, we have

$$v(F(x_0)) = v(a_d) + \sum_{\alpha' \in \mathfrak{s}} v(x_0 - \alpha') + \sum_{\beta \in \mathfrak{s}^c} v(x_0 - \beta)$$

$$= v(a_d) + \sum_{\alpha' \in \mathfrak{s}} v(x_0 - \alpha) + \sum_{\beta \in \mathfrak{s}^c} v(\alpha - \beta)$$

$$= |\mathfrak{s}| \cdot v(x_0 - \alpha) + \left(v(a_d) + \sum_{\beta \in \mathfrak{s}^c} v(\alpha - \beta) \right)$$

which gives the desired formula for $x_0 \in X_\mathfrak{s}$ (note that the choice of α was arbitrary). □

3.2 Properties of clusters

Lemma 3.2.1. *Let $\mathfrak{s}$ be a cluster. Then, at least one of the following is true:*

1. *there exists some root $\alpha_{\min}$ of $\mathfrak{s}$ such that $\mathfrak{s} = \{\alpha \mid v(\alpha) \geq v(\alpha_{\min})\}$, or*
2. *all roots of $\mathfrak{s}$ are contained in $B_{|\alpha|}(\alpha)$ for any $\alpha \in \mathfrak{s}$, and further $X_\mathfrak{s} \subseteq B_{|\alpha|}(\alpha)$.*

Proof. Let $\alpha_{\min}$ denote a root of $\mathfrak{s}$ of minimal valuation. Suppose that case 1 does not hold, that is, there exists some $\beta \in \mathfrak{s}^c$ such that $|\beta| \leq |\alpha_{\min}|$. Let $x \in X_\mathfrak{s}$. As $\beta \in \mathfrak{s}^c$, by definition of $X_\mathfrak{s}$, $|x - \beta| > |x - \alpha_{\min}|$. If $|x| \neq |\alpha_{\min}|$, then $|x - \alpha_{\min}| = \max\{|x|, |\alpha_{\min}|\} \geq \max\{|x|, |\beta|\} \geq |x - \beta|$, a contradiction, and thus $|x| = |\alpha_{\min}|$. As $|\beta| \leq |\alpha_{\min}| = |x|$, the strong triangle inequality gives that $|x - \beta| \leq |x|$, and hence $|x - \alpha_{\min}| < |x - \beta| \leq |x| = |\alpha_{\min}|$, so that $x \in B_{|\alpha_{\min}|}(\alpha_{\min})$. For all $\alpha \in \mathfrak{s}$, by the definition of $X_\mathfrak{s}$, $|x - \alpha| = |x - \alpha_{\min}| < |\alpha_{\min}|$. Hence,

$$|\alpha_{\min} - \alpha| = |\alpha_{\min} - x + x - \alpha| \leq \max\{|x - \alpha_{\min}|, |x - \alpha|\} < |\alpha_{\min}|.$$

As all elements of $B_{|\alpha_{\min}|}(\alpha_{\min})$ have the same absolute value and any element of the ball can be chosen as the center, the ball $B_{|\alpha|}(\alpha)$ does not depend on choice of $\alpha \in \mathfrak{s}$. □

Lemma 3.2.2. *Suppose that the residue field of K is algebraically closed and that the roots of $F(x)$ are tamely ramified. Let $\mathfrak{s}$ be a Galois-invariant cluster. There exists $\gamma_\mathfrak{s} \in K$ and $\alpha_{\min} \in \mathfrak{s}$ such that $\mathfrak{s} = \{\alpha \mid v(\alpha - \gamma_\mathfrak{s}) \geq v(\alpha_{\min} - \gamma_\mathfrak{s})\}$.*

Proof. If $|\mathfrak{s}| = 1$, then we can take $\gamma_\mathfrak{s} = \alpha$ for the unique $\alpha \in \mathfrak{s}$, which is an element of K as $\mathfrak{s}$ is fixed by the Galois action. Now suppose $|\mathfrak{s}| > 1$, and fix some $\alpha \in \mathfrak{s}$. Define

$$\epsilon_\mathfrak{s} := \max_{\alpha' \in \mathfrak{s}} \{|\alpha - \alpha'|\} \qquad \delta_\mathfrak{s} := \min_{\beta \in \mathfrak{s}^c} \{|\alpha - \beta|\}.$$

For the remainder of the proof, let $\alpha' \in \mathfrak{s}$ be such that $|\alpha - \alpha'| = \epsilon_\mathfrak{s}$, and let $\beta \in \mathfrak{s}^c$ be such that $|\alpha - \beta| = \delta_\mathfrak{s}$. Let $x \in X_\mathfrak{s}$, so that by the definition of $X_\mathfrak{s}$,

$$\delta_\mathfrak{s} = |\alpha - \beta| = |\alpha - x + x - \beta| \leq \max\{|\alpha - x|, |x - \beta|\} = |x - \beta|,$$
$$\epsilon_\mathfrak{s} = |\alpha - \alpha'| = |\alpha - x + x - \alpha'| \leq \max\{R_x, R_x\} < |x - \beta| = \delta_\mathfrak{s}.$$

Now, let L/K be the splitting field of $F(x)$. As all roots of $F(x)$ are tamely ramified, by Lemma 2.4.4, there exists $r \in \mathbb{N}$ and a uniformizer τ for L such that $\tau^r = \pi$ and $L = K(\tau)$. Expanding α, α' and β in τ,

$$\alpha = \sum_i b_i \tau^i, \qquad \alpha' = \sum_i b_i' \tau^i, \quad \text{and} \quad \beta = \sum_i c_i \tau^i.$$

Let N be the smallest integer such that $b_N \neq c_N$, so that $\delta_{\mathfrak{s}} = |\alpha - \beta| = |\tau^N|$, and let N' be the smallest integer such that $b_{N'} \neq b'_{N'}$, so that $\epsilon_{\mathfrak{s}} = |\alpha - \alpha'| = |\tau^{N'}|$. As $|\tau^{N'}| < |\tau^N|$, $N < N'$. Let $\gamma_{\mathfrak{s}} := \sum_{-\infty<i}^{N'-1} b_i \tau^i$. Note that by construction of $\gamma_{\mathfrak{s}}$ and definition of β, for all $\beta' \in \mathfrak{s}^c$ and all $\alpha_\ell \in \mathfrak{s}$,

$$v(\beta' - \gamma_{\mathfrak{s}}) \leq v(\beta - \gamma_{\mathfrak{s}}) = Nv(\tau) < N'v(\tau) = v(\alpha' - \gamma_{\mathfrak{s}}) \leq v(\alpha_\ell - \gamma_{\mathfrak{s}}). \tag{3.1}$$

Thus, $\gamma_{\mathfrak{s}}$ has the desired property. Finally, we show that $\gamma_{\mathfrak{s}} \in K$. As L/K is tamely ramified and the residue field of K is algebraically closed, $\operatorname{char} K \nmid [L : K]$, and hence L/K is separable and thus Galois. Suppose for the sake of contradiction that $\gamma_{\mathfrak{s}} \notin K$, so that there exists $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\gamma_{\mathfrak{s}}) \neq \gamma_{\mathfrak{s}}$. As the residue field of K is algebraically closed, for all i, $\sigma(b_i) = b_i$. Thus, we see there must exist some $M < N'$ such that $\sigma(\tau^M) \neq \tau^M$ and $b_M \neq 0$. However, this shows that $\sigma(\alpha) \neq \alpha'$, and hence as $\mathfrak{s}$ is Galois invariant, there exists $\alpha_{\min} \in \mathfrak{s}$ such that $|\alpha - \alpha_{\min}| = |\tau^M| > |\tau^{N'}| = |\alpha - \alpha'| = \epsilon_{\mathfrak{s}}$, contradicting the definition of $\epsilon_{\mathfrak{s}}$. □

Remark 3.2.3. For a cluster $\mathfrak{s}$, there must exist some cluster $\mathfrak{s}'$ such that $\gamma_{\mathfrak{s}} \in X_{\mathfrak{s}'}$. From Equation (3.1), for any $\alpha \in \mathfrak{s}$ and $\beta' \in \mathfrak{s}^c$, $|\beta' - \gamma_{\mathfrak{s}}| > |\alpha - \gamma_{\mathfrak{s}}|$, and thus the collection of roots closest to $\gamma_{\mathfrak{s}}$ is a subset of $\mathfrak{s}$, that is, $\mathfrak{s}' \subseteq \mathfrak{s}$. When $\gamma_{\mathfrak{s}}$ is closest to some root α, then by Galois-invariance of the valuation, $v(\gamma_{\mathfrak{s}} - \alpha')$ is constant for every conjugate α' of α. We therefore conclude that $\mathfrak{s}' \subseteq \mathfrak{s}$ must be Galois-invariant, and furthermore if $\mathfrak{s}$ contains no Galois-invariant subclusters, then $\gamma_{\mathfrak{s}} \in X_{\mathfrak{s}}$.

Lemma 3.2.4. *Suppose that the residue field of K is algebraically closed and that the roots of $F(x)$ are tamely ramified. Let $\mathfrak{s}$ be a Galois-invariant cluster. Then $c_{\mathfrak{s}} \in \mathbb{Z}$.*

Proof. Note that we can apply any rational change of coordinates: if $\gamma \in K$, consider $F(x + \gamma)$. The set $\mathfrak{s}_\gamma = \{\alpha - \gamma \mid \alpha \in \mathfrak{s}\}$ is a Galois-invariant cluster for $F(x+\gamma)$ and $c_{\mathfrak{s}_\gamma} = c_{\mathfrak{s}}$. Thus, letting $\gamma = \gamma_{\mathfrak{s}}$ from Lemma 3.2.2, we can assume that there exists $\alpha_{\min}$ such that $\mathfrak{s} = \{\alpha \mid v(\alpha) \geq v(\alpha_{\min})\}$.

As $v(a_d) \in \mathbb{Z}$, we must show that $\sum_{\beta \in \mathfrak{s}^c} v(\alpha_{\min} - \beta) \in \mathbb{Z}$. Both $\mathfrak{s}$ and the set of roots $\{\alpha_i\}$ are Galois invariant, $\mathfrak{s}^c$ is as well. For each Galois orbit contained in $\mathfrak{s}^c$, pick some representative

η with minimal polynomial f_η, and let $\mathcal{M} \subseteq \mathfrak{s}^c$ be the set of all η. The sum $\sum_{\beta \in \mathfrak{s}^c} v(\alpha_{\min} - \beta)$ equals $\sum_{\eta \in \mathcal{M}} v(f_\eta(\alpha_{\min}))$. We show that for all $\eta \in \mathcal{M}$, $v(f_\eta(\alpha_{\min})) \in \mathbb{Z}$.

For all $\beta \in \mathfrak{s}^c$, $v(\beta) < v(\alpha_{\min})$, specifically, for all $\eta \in \mathcal{M}$, $v(\eta) < v(\alpha_{\min})$. As f_η is irreducible, all roots have the same valuation. Thus, by the strong triangle inequality,

$$\begin{aligned} v(f_\eta(\alpha_{\min})) &= v\left(\prod_{\beta | f_\eta(\beta)=0} (\alpha_{\min} - \beta)\right) \\ &= \sum_{\beta | f_\eta(\beta)=0} v(\alpha_{\min} - \beta) = \sum_{\beta | f_\eta(\beta)=0} v(\beta) = (\deg f_\eta) \cdot v(\eta). \end{aligned}$$

As f_η is the minimal polynomial of η, a Newton polygon argument shows that the denominator of $v(\eta)$ divides $\deg f_\eta$, and hence $v(f_\eta(\alpha_{\min})) \in \mathbb{Z}$ as desired. □

Remark 3.2.5. Neither Lemma 3.2.2 nor Lemma 3.2.4 necessarily hold if the roots of $F(x)$ are wildly ramified. Consider $F(x) = (x^3 - 3)(x^2 - 3)$ over $\mathbb{Q}_3$. Let $\zeta_3 = \frac{-1+\sqrt{-3}}{2}$ be a root of the second cyclotomic polynomial, x^2+x+1, so that the roots of $F(x)$ are $\{\sqrt{3}, -\sqrt{3}, \sqrt[3]{3}, \zeta_3\sqrt[3]{3}, \zeta_3^2\sqrt[3]{3}\}$, and let $\mathfrak{s} = \{\sqrt[3]{3}, \zeta_3\sqrt[3]{3}, \zeta_3^2\sqrt[3]{3}\}$. As $\mathfrak{s} = B_{|\sqrt[3]{3}|}(\sqrt[3]{3}) \cap \{\alpha_i\}$, $\mathfrak{s}$ is a Galois-invariant cluster. For all $\gamma \in K$, either $v(\gamma) \geq 1$, in which case $v(\alpha - \gamma) = v(\alpha)$ for all α, or $v(\gamma) \leq 0$, in which case $v(\alpha - \gamma) = v(\gamma)$ for all α. Therefore, no γ can satisfy the conclusions of Lemma 3.2.2 for $\mathfrak{s}$. Further,

$$c_\mathfrak{s} = v(1) + v\left(\sqrt[3]{3} - \sqrt{3}\right) + v\left(\sqrt[3]{3} + \sqrt{3}\right) = \frac{2}{3}.$$

Remark 3.2.6. The proof of Lemma 3.2.4 shows that if there exists $\alpha_{\min} \in \mathfrak{s}$ such that $\mathfrak{s} = \{\alpha \mid v(\alpha) \geq v(\alpha_{\min})\}$, then $c_\mathfrak{s}$ is an integer.

3.3 Obstructing points of arbitrary degree

We study the degree set of a superelliptic curve C/K via the affine open $U \subseteq C$ such that $U \simeq V(y^q - F(x)) \setminus \{(\alpha_i, 0)\}$. We consider the x-coordinate map $x\colon C \to \mathbb{P}^1$. We first use the valuation to show that under certain modular arithmetic conditions, we cannot obtain points of arbitrary degree.

Lemma 3.3.1. *Suppose that the residue field of K is algebraically closed and that every root of $F(x)$ is tamely ramified. Fix a Galois-invariant cluster $\mathfrak{s}$. Let $\gamma_\mathfrak{s}$ be constructed from $\mathfrak{s}$ as in Lemma 3.2.2. Let r be a natural number not divisible by q. If there exists $P \in U(\bar{K})$ of degree r with $x(P) \in X_\mathfrak{s}$, then one of the following holds:*

1. $|\mathfrak{s}| \not\equiv 0 \pmod{q}$, *or*

2. $|\mathfrak{s}| \equiv 0 \pmod{q}$ *and* $c_{\mathfrak{s}} \equiv 0 \pmod{q}$, *or*

3. $0 \in X_{\mathfrak{s}}$ *and* $v(F(0)) \equiv 0 \pmod{q}$, *or*

4. $\gamma_{\mathfrak{s}} \in X_{\mathfrak{s}}$ *and* $v(F(\gamma_{\mathfrak{s}})) \equiv 0 \pmod{q}$.

Remark 3.3.2. The quantities appearing in conditions 1 and 2 in Lemma 3.3.1 are the coefficients appearing in the slope formula, see Remark 3.1.5.

Proof of Lemma 3.3.1. Let $(x_0, y_0) \in U(\bar{K})$ with $[K(x_0, y_0) : K] = r$. By Lemma 2.4.1, either $K(x_0, y_0)$ is a degree q extension of $K(x_0)$, or $F(x_0) \in K(x_0)^{\times q}$. The first case is not possible by assumption, so we must have that $F(x_0) \in K(x_0)^{\times q}$ and that $K(x_0, y_0) = K(x_0)$. As the residue field of K is algebraically closed, $K(x_0)$ is totally ramified with value group $\frac{1}{r}\mathbb{Z}$, so that $v(F(x_0))$ is a multiple of q in $\frac{1}{r}\mathbb{Z}$. As $x_0 \in X_{\mathfrak{s}}$, by Lemma 3.1.4, for any $\alpha \in \mathfrak{s}$,

$$\frac{v(F(x_0))}{q} = \frac{|\mathfrak{s}|}{q} v(x_0 - \alpha) + \frac{c_{\mathfrak{s}}}{q}.$$

We consider the possibilities for $v(x_0 - \alpha)$, which by strong triangle inequality, are as follows:

1. $v(x_0 - \alpha) = v(x_0) \leq v(\alpha)$

2. $v(x_0 - \alpha) = v(\alpha) < v(x_0)$

3. $v(x_0 - \alpha) > v(\alpha) = v(x_0)$.

Case 1: First suppose that $v(x_0 - \alpha) = v(x_0)$. Then, as $\frac{v(F(x_0))}{q} \in \frac{1}{r}\mathbb{Z}$, the map $\frac{1}{r}\mathbb{Z} \to \frac{1}{rq}\mathbb{Z}$ given by $x \mapsto \frac{|\mathfrak{s}|}{q}x + \frac{c_{\mathfrak{s}}}{q}$ satisfies the conditions of Lemma 2.4.2, so that $|\mathfrak{s}| \not\equiv 0 \pmod{q}$ or $|\mathfrak{s}| \equiv 0 \pmod{q}$ and $c_{\mathfrak{s}} \equiv 0 \pmod{q}$.

Case 2: Now assume that $v(x_0 - \alpha) = v(\alpha)$ and $v(x_0) > v(\alpha)$. Note that $x_0 \in X_{\mathfrak{s}}$ but $x_0 \notin B_{|\alpha|}(\alpha)$ as $v(x_0 - \alpha) = v(\alpha)$, so that by Lemma 3.2.1, there exists $\alpha_{\min}$ such that $\mathfrak{s} = \{\alpha' \mid v(\alpha') \geq v(\alpha_{\min})\}$. As $v(x_0 - \alpha)$ is independent of the choice of $\alpha \in \mathfrak{s}$, all roots of $\mathfrak{s}$ must have the same valuation, which further must be the maximal valuation among all roots of $F(x)$. Thus, for any root α' of $F(x)$, $v(x_0) > v(\alpha')$. We see that $0 \in X_{\mathfrak{s}}$ as 0 is closest to the roots of maximal valuation, so

$$v(F(x_0)) = v(a_d) + \sum_i v(x_0 - \alpha_i) = v(a_d) + \sum_i v(\alpha_i) = v(F(0)).$$

Thus, $v(F(x_0)) \in \mathbb{Z}$. As $v(F(x_0))$ is a multiple of q in $\frac{1}{r}\mathbb{Z}$ and q does not divide r, $v(F(x_0)) = v(F(0)) = v(F(\gamma_{\mathfrak{s}}))$ is a multiple of q in $\mathbb{Z}$.

Case 3: Finally, suppose that $v(x_0-\alpha) > v(\alpha) = v(x_0)$. By Lemma 3.2.2, there exists $\gamma_{\mathfrak{s}} \in K$ and $\alpha_{\min} \in \mathfrak{s}$ such that $\mathfrak{s} = \{\alpha \mid v(\alpha - \gamma_{\mathfrak{s}}) \geq v(\alpha_{\min} - \gamma_{\mathfrak{s}})\}$. Let $G(x) = F(x + \gamma_{\mathfrak{s}})$, consider the isomorphic curve $y^q = G(x)$, and let $\mathfrak{s}'$ be the $G(x)$-cluster $\{\alpha - \gamma_S \mid \alpha \in \mathfrak{s}\}$.

We show that for all $\alpha \in \mathfrak{s}$, $x_0 - \gamma_{\mathfrak{s}}$ is not contained in $B_{|\alpha-\gamma_{\mathfrak{s}}|}(\alpha - \gamma_{\mathfrak{s}})$. If $|\mathfrak{s}| = 1$, then $\gamma_{\mathfrak{s}} = \alpha$ and as $P \in U(\bar{K})$, $x(P)$ does not equal α, and hence $x_0 - \gamma_{\mathfrak{s}} \notin B_0(0)$. Now suppose that $|\mathfrak{s}| > 1$, and let $\alpha' \in \mathfrak{s}$ be such that $|\alpha - \alpha'|$ is maximal as in the proof of Lemma 3.2.2. By construction of $\gamma_{\mathfrak{s}}$, $|\alpha - \gamma_{\mathfrak{s}}| = |\alpha - \alpha'|$. As $x_0 \in X_{\mathfrak{s}}$,

$$|x_0 - \gamma_{\mathfrak{s}} - (\alpha - \gamma_{\mathfrak{s}})| = |x_0 - \alpha| = |x_0 - \alpha' + \alpha' - \alpha|$$

$$\leq \max\{|x_0 - \alpha'|, |\alpha' - \alpha|\} = \max\{|x_0 - \alpha|, |\alpha - \gamma_{\mathfrak{s}}|\}.$$

If $|\alpha - \gamma_{\mathfrak{s}}| > |x_0 - \alpha|$, then the strong inequality gives a contradiction. Thus, $|\alpha - \gamma_{\mathfrak{s}}| \leq |x_0 - \alpha|$, and hence $x_0 - \gamma_{\mathfrak{s}} \notin B_{|\alpha-\gamma_{\mathfrak{s}}|}(\alpha - \gamma_{\mathfrak{s}})$ as desired. Then, $v((x_0 - \gamma_{\mathfrak{s}}) - (\alpha - \gamma_{\mathfrak{s}})) \leq v(\alpha - \gamma_{\mathfrak{s}})$, and hence $x_0 - \gamma_{\mathfrak{s}}$ must fall in either case 1 or case 2 when considering the $G(x)$-cluster $\mathfrak{s}'$. As $|\mathfrak{s}'| = |\mathfrak{s}|$ and $c_{\mathfrak{s}'} = c_{\mathfrak{s}}$, in case 1 the congruence conditions hold. Finally, $G(0) = F(\gamma_{\mathfrak{s}})$, and thus in case 2, $v(F(\gamma_{\mathfrak{s}})) \in q\mathbb{Z}$. □

Remark 3.3.3. If there exist $\alpha, \alpha' \in \mathfrak{s}$ such that $v(\alpha) \neq v(\alpha')$, then $\gamma_{\mathfrak{s}} = 0$. Thus, for many Galois-invariant $\mathfrak{s}$, $\gamma_{\mathfrak{s}}$ is not contained in $X_{\mathfrak{s}}$.

Remark 3.3.4. Define the **depth** of $\mathfrak{s}$ by $d_{\mathfrak{s}} := \min_{\alpha,\alpha' \in \mathfrak{s}} v(\alpha - \alpha')$. When $|\mathfrak{s}| > 1$ and $\gamma_{\mathfrak{s}} \in X_{\mathfrak{s}}$, $v(\gamma_{\mathfrak{s}} - \alpha_j) = d_{\mathfrak{s}}$, and hence $v(F(\gamma_{\mathfrak{s}})) = |\mathfrak{s}|d_{\mathfrak{s}} + c_{\mathfrak{s}}$.

3.4 Constructing points of large degree

We prove a strengthening of the converse of Lemma 3.3.1. Using Hensel's lemma, we show that if there exists a point $P \in U(\bar{K})$ with degree not a multiple of q, then the degree set contains points of all sufficiently large degrees.

Lemma 3.4.1. *Suppose that the residue field of K is algebraically closed and that every root of $F(x)$ is tamely ramified. Fix a Galois-invariant cluster $\mathfrak{s}$. If one of the following holds:*

1. $|\mathfrak{s}| \not\equiv 0 \pmod{q}$, *or*
2. $|\mathfrak{s}| \equiv 0 \pmod{q}$ *and* $c_{\mathfrak{s}} \equiv 0 \pmod{q}$, *or*
3. $0 \in X_{\mathfrak{s}}$ *and* $v(F(0)) \equiv 0 \pmod{q}$, *or*

4. $\gamma_{\mathfrak{s}} \in X_{\mathfrak{s}}$ *and* $v(F(\gamma_{\mathfrak{s}})) \equiv 0 \pmod{q}$,

then there exists $r_0 \in \mathbb{N}$ *such that for all* $r \geq r_0$ *with* $\gcd(r,q) = 1$, *there exists a point* $P \in U(\bar{K})$ *of degree* r *with* $x(P) \in X_{\mathfrak{s}}$.

Proof. Following the argument in the proof of case 3 of Lemma 3.3.1, without loss of generality we can take $\gamma_{\mathfrak{s}} = 0$, so condition 4 reduces to condition 3. Suppose that condition 3 holds, so that $0 \in X_{\mathfrak{s}}$ and hence $\mathfrak{s}$ is the set of roots of maximal valuation. We let $r_0 = 1$ and construct points of all degrees $r \geq r_0$. Let N be an integer greater than $\max\{v(\alpha)\}$, and let $x_0 = \pi^{N+\frac{1}{r}}$. By construction, $x_0 \in X_{\mathfrak{s}}$, and hence by Lemma 3.1.4, for any $\alpha \in \mathfrak{s}$,

$$v(F(x_0)) = |\mathfrak{s}| v(x_0 - \alpha) + c_{\mathfrak{s}} = |\mathfrak{s}| v(\alpha) + c_{\mathfrak{s}} = v(F(0)) \equiv 0 \pmod{q}.$$

Thus, $(x_0, \sqrt[q]{F(x_0)})$ is a $\bar{K}$-point of degree r over K.

Now suppose that either condition 1 or condition 2 holds. As $\gamma_{\mathfrak{s}} = 0$, by Lemma 3.2.2, there exists $\alpha_{\min}$ such that $\mathfrak{s}$ indexes the set $\{\alpha \mid v(\alpha) \geq v(\alpha_{\min})\}$. Let $N = v(\alpha_{\min})$ and let $M = \max_{\beta \in \mathfrak{s}^c}\{v(\beta)\}$. Let r_0 be the smallest integer greater than $\frac{q+1}{N-M} + 1$. We construct a point of degree r for all $r \geq r_0$. Consider the affine map $\rho\colon (M,N) \cap \frac{1}{r}\mathbb{Z} \to \frac{1}{rq}\mathbb{Z}$ given by $\rho(x) = \frac{|\mathfrak{s}|}{q}x + \frac{c_{\mathfrak{s}}}{q}$. By Lemma 2.4.2, there exists $\frac{a}{r} \in (M,N)$ such that $\rho\left(\frac{a}{r}\right) \in \frac{1}{r}\mathbb{Z}$. Write $\frac{a}{r} = \frac{su}{st}$ with $\gcd(u,t) = 1$, and let x_0' be a root of $x^t - \pi^u$, so that x_0' has degree t. Let τ be a uniformizer for $K(x_0')$ and let R be an integer larger than $s \cdot \frac{u}{t}$ relatively prime to s. Let ε be a root of $x^s - \tau^R$. A Newton polygon argument shows that the degree of ε over $K(x_0')$ is s and that $v(\varepsilon) = \frac{R}{s} > v(x_0')$. Let $x_0 = x_0' + \varepsilon$, so that $K(x_0)$ has degree $st = r$ over K, and $v(x_0) = v(x_0') = \frac{a}{r}$. As $M < v(x_0) < N$, $x_0 \in X_{\mathfrak{s}}$, and further

$$\frac{v(F(x_0))}{q} = \frac{|\mathfrak{s}|}{q} v(x_0 - \alpha) + \frac{c_{\mathfrak{s}}}{q} = \frac{|\mathfrak{s}|}{q} v(x_0) + \frac{c_{\mathfrak{s}}}{q} = \rho\left(\frac{a}{r}\right) \in \frac{1}{r}\mathbb{Z}.$$

Thus, $v(F(x_0))$ is a multiple of q in $\frac{1}{r}\mathbb{Z}$, and hence $(x_0, \sqrt[q]{F(x_0)})$ has degree r over K. □

Lemma 3.4.2. *The degree set* $\mathcal{D}(C/K)$ *contains* $q\mathbb{N}$.

Proof. We show that either $1 \in \mathcal{D}(C/K)$ (and hence $\mathcal{D}(C/K)$ equals $\mathbb{N}$) or that $q \in \mathcal{D}(C/K)$, which is sufficient by Lemma 2.2.1. Set $x = 0$ and consider $y^q = F(0)$. By Lemma 2.4.1, either $y^q - F(0)$ is irreducible, or there exists $b \in K$ such that $b^q = F(0)$. If $y^q - F(0)$ is irreducible, $(0, F(0)^{1/q})$ has degree q, otherwise, the point $(0,b)$ has degree 1. □

Lemma 3.4.3. *Suppose* F *has a separable irreducible factor of degree* r. *Then,* $r\mathbb{N} \subseteq \mathcal{D}(C/K)$.

Proof. Let α be a separable root of degree r and let ℓ be the multiplicity of α as a root of $F(x)$. Since F is assumed to be free of q-th powers, let i be a lift of $\ell^{-1} \pmod q$. Consider the isomorphic curve given by $y^q = (F(x))^i$. After eliminating q-th powers from $(F(x))^i$, the resulting isomorphic curve is smooth at $(\alpha, 0)$, showing that $\mathcal{D}(C/K)$ contains $\deg \alpha$. □

3.5 Proofs of main theorems on superelliptic curves

Theorem 3.5.1. *Suppose that the residue field of K is algebraically closed and that every root of $F(x)$ is tamely ramified. Then, $\mathcal{D}(C/K)$ is not cofinite if and only if $v(F(0)) \not\equiv 0 \pmod q$, and for every Galois-invariant cluster $\mathfrak{s}$ of roots of $F(x)$,*

1. *$|\mathfrak{s}| \equiv 0 \pmod q$ and $c_{\mathfrak{s}} \not\equiv 0 \pmod q$, and*
2. *if $\gamma_{\mathfrak{s}} \in X_{\mathfrak{s}}$, then $v(F(\gamma_{\mathfrak{s}})) \not\equiv 0 \pmod q$.*

When these conditions are satisfied,

$$\mathcal{D}(C/K) \subseteq q\mathbb{N} \cup \bigcup_{\mathfrak{s}} |O(\mathfrak{s})|\mathbb{N},$$

where the union is taken over all $\mathfrak{s}$ which are not Galois invariant.

Remark 3.5.2. In [Dok21], Dokchitser constructed a regular model from the Newton polytopes associated to an affine equation $f(x, y) = 0$, under the assumption that f is Δ_v-regular. Thus, if $y^q = F(x)$ is Δ_v-regular, one could compute $\mathcal{D}(C/K)$ by first computing the special fiber of a regular model using [Dok21, Theorem 1.1], and if the special fiber has simple normal crossings, applying [CV25, Theorem 1.1]. We analyze how such an approach relates to Theorem 3.5.1

Following the construction of [Dok21], for many Galois invariant clusters $\mathfrak{s}$ (specifically, for those that fall in case 1 of Lemma 3.2.1), there exists a v-edge $L_{\mathfrak{s}}$, and that $\delta_{L_{\mathfrak{s}}}$ equals 1 if and only if the conditions of Theorem 3.5.1 are not satisfied. When $\delta_{L_{\mathfrak{s}}}$ equals 1, there are intersecting copies of $\mathbb{P}^1$ with coprime multiplicity, and thus [CV25, Theorem 1.1] shows that for some r, $\mathbb{N}_{>r} \subseteq \mathcal{D}(C/K)$. We compute the degrees of all points reducing to the $\mathbb{P}^1$'s arising from $L_{\mathfrak{s}}$ simultaneously in Lemma 3.4.1. When C/K is not Δ_v-regular, the methods of [Dok21] do not give a regular model, see Remark 3.6.2. As in Example 3.6.1, we can often compute the degree set even when C/K is not Δ_v-regular.

Remark 3.5.3. When C/K is a hyperelliptic curve, [FN20, Theorem 1.2] shows that, if the curve has potential semistable reduction (which is equivalent to every root of $F(x)$ being tamely ramified), the special fiber can be computed from a *cluster diagram* (see [DDMM23] or the

overview [BBB+22]) and thus the degree set depends only on the cluster diagram. Using an additional quantity $c_{\mathfrak{s}}$, both $|\mathfrak{s}|$ and $v(F(\gamma_{\mathfrak{s}}))$ can be computed from the cluster diagram (see Remark 3.3.4). We do not know how $c_{\mathfrak{s}}$ relates to the cluster diagram.

Proof. By Lemma 2.2.2, $\mathcal{D}(C/K) = \mathcal{D}(U/K)$. Let $P \in U(\bar{K})$ and consider $x(P)$. As the collection $\{X_{\mathfrak{s}}\}$ covers $\mathbb{A}^1(\bar{K})$, there exists some $\mathfrak{s}$ such that $x(P) \in X_{\mathfrak{s}}$. If $\mathfrak{s}$ is Galois invariant, then Lemma 3.3.1 shows that either q divides the degree of $x(P)$, or that one of the conditions 1-4 holds. By Lemma 3.4.1, if one of the conditions 1-4 hold, then the degree set contains $\mathbb{N}_{\geq r_0}$ and is cofinite. Thus, $\mathcal{D}(C/K)$ is not cofinite if and only if the conditions 1-4 do not hold. If $\mathfrak{s}$ is not Galois invariant, the degree of $x(P)$ is a multiple of $|O(\mathfrak{s})|$, and $\deg_K(P)$ is a multiple of the degree of $x(P)$, so that $\deg_K(P) \in |O(\mathfrak{s})|\mathbb{N}$. □

Proof of Theorem 1.0.6. As $F(x)$ is irreducible, there is exactly one Galois invariant cluster: the set $\mathfrak{s}$ of all roots of $F(x)$. Then, $|\mathfrak{s}| = \deg F(x)$ and $c_{\mathfrak{s}} = v(a_d)$.

We show that $\gamma_{\mathfrak{s}} = 0$. Let L/K be the splitting field of $F(x)$ and let τ be a uniformizer for L such that $\tau^n = \pi$ for some $n \in \mathbb{N}$. Let α be the root of $F(x)$ of non-integer valuation, and let τ^i be the first non-zero power of τ appearing in the expansion of α in terms of τ. As $v(\alpha)$ is not an integer, i is not a multiple of n, and τ^i is not fixed by the action of $\mathrm{Gal}(L/K)$. Thus, there is a root $\alpha' \in \mathfrak{s}$ whose expansion in terms of τ has a different multiple of τ^i, and as $\gamma_{\mathfrak{s}}$ is the finite sum of powers of τ such that all roots in $\mathfrak{s}$ have the same multiple of the power, $\gamma_{\mathfrak{s}} = 0$. Now apply Theorem 3.5.1 with $|\mathfrak{s}| = \deg F(x)$, $c_{\mathfrak{s}} = v(a_d)$, and $\gamma_{\mathfrak{s}} = 0$. □

Theorem 3.5.4. *Let D be a positive integer divisible by q, and let $\{n_k\} \subseteq \mathbb{N}$ be a finite sequence such that $q\left(\sum_k n_k\right) = D$. Suppose that q is odd, or that the number of times 1 occurs in the sequence $\{n_k\}$ is even. Then, there exists a superelliptic curve C/K with equation $y^q = F(x)$, where $D = \deg F(x)$, such that*

$$\mathcal{D}(C/K) = q\mathbb{N} \cup \bigcup_{n_k > 1} n_k\mathbb{N}.$$

The added condition when q is even is necessary, as [CV25, Lemma 5.8] shows that when the residue field of K is algebraically closed, there does not exist a curve $y^2 = F(x)$ such that $\mathcal{D}(C/K) = 2\mathbb{N}$ with $\deg F(x) \equiv 2 \pmod 4$.

Proof. We begin with the case where q is odd. Let m_k denote the subsequence of n_k where all elements are greater than 1, and reorder so that both n_k and m_k are weakly increasing. We

construct a superelliptic curve C/K with equation $y^q = F(x)$ such that

$$\mathcal{D}(C/K) = q\mathbb{N} \cup \bigcup_{n_k>1} n_k\mathbb{N} = q\mathbb{N} \cup \bigcup_k m_k\mathbb{N}.$$

Define $c := D - q\sum_k m_k$ and let a be relatively prime to c, congruent to 1 (mod q), and less than $-c$. As D is a multiple of q, so is c. Let $\tau_1, \ldots, \tau_q$ denote lifts of distinct elements of the residue field (which exist as the residue field is algebraically closed). Define $F(x)$ by

$$F(x) := \pi(x^c - \pi^a) \prod_{n_k>1} \prod_{i=1}^{q} (x^{n_k} - \tau_i\pi) \tag{3.2}$$

and consider the curve C/K defined by $y^q = F(x)$. By Lemma 3.4.2 and Lemma 3.4.3, $q\mathbb{N} \cup \bigcup_{n_k>1} n_k\mathbb{N} \subseteq \mathcal{D}(C/K)$. We show that $\mathcal{D}(C/K) \subseteq q\mathbb{N} \cup \bigcup_{n_k>1} n_k\mathbb{N}$. Let $P \in U(\bar{K})$ and let $\mathfrak{s}$ be such that $x(P) \in X_{\mathfrak{s}}$. By Lemma 3.2.1, $X_{\mathfrak{s}} \subseteq B_{|\alpha|}(\alpha)$ or there exists $\alpha_{\min}$ such that $\mathfrak{s}$ indexes the set $\{\alpha \mid v(\alpha) \geq v(\alpha_{\min})\}$.

Consider the first case. By construction, the roots of $F(x)$ have valuations $\frac{a}{c}$ or $\frac{1}{n_k}$, and as $x(P) \in B_{|\alpha|}(\alpha)$, $v(x(P)) = \frac{a}{c}$ or $v(x(P)) = \frac{1}{n_k}$. Let d denote the denominator of $v(x(P))$. We have a chain of divisibility conditions: d divides $[K(x(P)) : K]$, which divides $[K(P) : K]$, and as $d = n_k$ or $q \mid c = d$, $[K(P) : K] \in q\mathbb{N} \cup \bigcup_k n_k\mathbb{N}$.

In the second case, there exists $\alpha_{\min}$ such that $\mathfrak{s}$ indexes the set $\{\alpha \mid v(\alpha) \geq v(\alpha_{\min})\}$. We see that $\mathfrak{s}$ is Galois invariant. Note that $F(x)$ may have wildly ramified roots, so that the assumptions of Lemma 3.3.1 fail. For this specific $\mathfrak{s}$, however, all $P \in U(\bar{K})$ fall into case 1 or case 2, and the proof proceeds as written provided we verify the conclusion of Lemma 3.2.4 (that $c_{\mathfrak{s}}$ is an integer). Let k_0 be the largest index such that $\frac{1}{m_{k_0}} < v(\alpha_{\min})$. We compute

$$\begin{aligned} c_{\mathfrak{s}} &= v(\pi) + \sum_{\beta \in \mathfrak{s}^c} v(\alpha_{\min} - \beta) = 1 + \sum_{\beta \in \mathfrak{s}^c} v(\beta) \\ &= 1 + c\left(\frac{a}{c}\right) + qm_1\left(\frac{1}{m_1}\right) + \ldots + qm_{k_0}\left(\frac{1}{m_{k_0}}\right) \in \mathbb{Z}. \end{aligned}$$

As $a \equiv 1 \pmod q$, $c_{\mathfrak{s}} \equiv 2 \pmod q$, and as $q \geq 3$, $c_{\mathfrak{s}} \not\equiv 0 \pmod q$. We compute $|\mathfrak{s}|$. The roots of $F(x)$ have valuations $\frac{a}{c} < \frac{1}{m_1} \leq \frac{1}{m_2} \leq \cdots$. The number of roots of each valuation is a multiple of q, and thus the number of roots α such that $v(\alpha_{\min}) \leq v(\alpha)$ is a multiple of q. Thus, $|\mathfrak{s}| \equiv 0 \pmod q$. Finally, we check that $v(F(0)) \not\equiv 0 \pmod q$, which is necessary for case 2. Let k_1 be the largest index in the sequence $\{m_k\}$. By construction, $v(F(0)) = v(\pi^{a+1}\pi^{qk_1}) \equiv a+1 \pmod q$ and as before, $1+a \not\equiv 0 \pmod q$. As $|\mathfrak{s}| \equiv 0 \pmod q$, $c_{\mathfrak{s}} \not\equiv 0 \pmod q$, and $v(F(0)) \not\equiv 0 \pmod q$, we conclude that for all $P \in U(\bar{K})$ such that $x(P) \in X_{\mathfrak{s}}$, $q \mid \deg P$, as in (the contrapositive of)

Lemma 3.3.1.

When q is 2, we construct $F(x)$ as before, with slight modifications. Note that $c_{\mathfrak{s}} \equiv 1 + a \pmod 2$, and thus if we want $2 \nmid c_{\mathfrak{s}}$, we must have $a \equiv 0 \pmod 2$. Thus, choose a such that $a \equiv 2 \pmod 4$ and is smaller than $-cn_k$ for all k. Then, as

$$c = D - 2\sum_{n_k>1} n_k = 2\sum_{n_k=1} n_k$$

and the number of k such that $n_k = 1$ is even by assumption, $c \equiv 0 \pmod 4$, so that $\frac{a}{c}$ (in lowest terms) has even denominator. The rest of the proof is similar to when q is odd. □

Proof of Theorem 1.0.7. If there exists k such that $n_k = 1$, pick a superelliptic curve C/K with a rational point. Otherwise, the proof is immediate from Theorem 3.5.4. □

3.6 Examples

Our results also provide a method for computing degree sets. Although our lemmas assume algebraically closed residue field, this example is computed over $\mathbb{Q}_7$, as the points constructed of minimal degree (i.e., do not arise from having degree a multiple of a degree already in the degree set) are totally ramified, so the degree set is unchanged over $\mathbb{Q}_7^{\mathrm{unr}}$.

Example 3.6.1. Let $\alpha = 7^{\frac{1}{2}} + 7^{\frac{3}{5}}$, $\beta = 2 \cdot 7^{\frac{1}{2}} + 7^{\frac{3}{5}}$, and $\gamma = 3 \cdot 7^{\frac{1}{2}} + 7^{\frac{3}{5}}$ as elements of $\bar{\mathbb{Q}}_7$, and let $f_1(x)$, $f_2(x)$, and $f_3(x)$ be the minimal polynomials of α, β, and γ respectively over $\mathbb{Q}_7$. Let $F(x) = 7f_1(x)f_2(x)f_3(x)$, and let $C/\mathbb{Q}_7$ be the curve given by $y^3 = F(x)$. Then,

$$\mathcal{D}(C/\mathbb{Q}_7) = 3\mathbb{N} \cup 2\left(\{8, 11, 13, 14\} \cup \mathbb{N}_{>15}\right) \cup 10\mathbb{N} \subseteq 3\mathbb{N} \cup 2\mathbb{N}.$$

Proof. Fix a primitive 5th root of unity ζ_5 and let $\mathfrak{s}$ be a cluster for $F(x)$. By Lemma 3.2.1, all roots of $\mathfrak{s}$ are contained in $B_{|\alpha|}(\alpha)$, or there exists $\alpha_{\min}$ such that $\mathfrak{s} = \{\alpha \mid v(\alpha) \geq v(\alpha_{\min})\}$.

We first consider the case where there exists $\alpha_{\min}$ such that $\mathfrak{s} = \{\alpha \mid v(\alpha) \geq v(\alpha_{\min})\}$. As all roots of $F(x)$ have the same valuation, $\mathfrak{s}$ must contain all roots of $F(x)$. Then, $|\mathfrak{s}| = 30$ and $c_{\mathfrak{s}} = 1$, and hence

$$v(y(P)) = \frac{30}{3}v(x(P) - \alpha) + \frac{1}{3}$$

so that by Lemma 2.4.2, 3 divides the denominator of $v(y(P))$, and hence $\deg P \in 3\mathbb{N}$. Thus, we can suppose that all roots of $\mathfrak{s}$ are contained in $B_{|\alpha|}(\alpha)$. Each $B_{|\alpha|}(\alpha)$ contains exactly 5 roots of $F(x)$, and as all roots in $B_{|\alpha|}(\alpha)$ are equidistant, either $|\mathfrak{s}| = 5$ or $|\mathfrak{s}| = 1$.

If $|\mathfrak{s}| = 1$, then there exists $\alpha \in \mathfrak{s}$ such that $x(P)$ is closer to α than all Galois conjugates of α, and Krasner's lemma implies that $10 \mid \deg x(P) \mid \deg P$, showing that $\deg P \in 10\mathbb{N}$. As the roots of $F(x)$ have degree 10, by Lemma 3.4.3, $\mathcal{D}(C/K)$ contains $10\mathbb{N}$.

Suppose $|\mathfrak{s}| = 5$. We detail only the case where $\mathfrak{s} = \{7^{\frac{1}{2}} + \zeta_5^i 7^{\frac{3}{5}}\}_{i=1}^5$, as the other cases are similar. Thus, $c_{\mathfrak{s}} = \frac{27}{2}$. Fix $\alpha \in \mathfrak{s}$. By Lemma 3.2.1, $X_{\mathfrak{s}} \subseteq B_{|\alpha|}(\alpha)$, and applying Lemma 2.4.3, we find that for all $\eta \in B_{|\alpha|}(\alpha)$, $K(7^{\frac{1}{2}}) \subseteq K(\eta)$, so that if $x(P) \in X_{\mathfrak{s}}$, then $K(7^{\frac{1}{2}}) \subseteq \mathbf{k}(x(P))$. We extend to $K(7^{\frac{1}{2}})$ and renormalize the valuation, remembering that we have divided the degree of our point by 2. Following Lemma 3.4.1 over $K(7^{\frac{1}{2}})$, $\gamma_{\mathfrak{s}} = 7^{\frac{1}{2}}$ and $(M, N) = (1, \frac{6}{5})$. If $v(x(P) - \gamma_{\mathfrak{s}}) = \frac{a}{r}$, then

$$v(y(P)) = \frac{5}{3} \cdot \frac{a}{r} + \frac{27}{3} = \frac{5a + 27r}{3r}$$

and so 3 divides the denominator of $v(y(P))$ unless $5a + 27r \equiv 0 \pmod 3$, which occurs if and only if $a \equiv 0 \pmod 3$. We search for $r \in \mathbb{N}$ that produce $\frac{a}{r} \in (1, \frac{6}{5})$ such that $a \equiv 0 \pmod 3$. For r large enough, by Lemma 2.4.2, this condition is always satisfied, and a computer finds the condition is satisfied for 8, 11, 13, 14, and all integers greater than 15. □

Remark 3.6.2. The methods of [Dok21] are unable to fully compute the special fiber of Example 3.6.1, and thus is insufficient for computing the degree set in this case. Using the main result of [CV25] and the fact that Example 3.6.1 will have the same degree set over $\mathbb{Q}_7^{\mathrm{unr}}$, the methods of [Dok21] show that every component of the special fiber must have multiplicity divisible by 2 or 3.

Chapter 4

Curves over number fields and their density degree sets

In this chapter we provide some background for Chapter 5. Following the expository paper [VV26], we define $\mathbb{P}^1$-parametrized and AV-parametrized points, and show how a curve over a number field has infinitely many closed points of degree d if and only if there exists a parametrized point of degree d. The key observations are that a degree-d point of a nice curve X over a number field k is a rational point of Sym_X^d and that there is a natural map (the Abel or Abel-Jacobi map) $\mathrm{Sym}_X^d \to \mathrm{Pic}_X^d$. Thus, to study degree-$d$ points on X/k, we can study the fibers of the Abel-Jacobi map and the image W_X^d in Pic_X^d.

4.1 Sym_X^d, $\mathrm{Pic}^d X$, Pic_X^d, and W_X^d

We begin by defining some spaces associated to X/k which capture much of the geometry of our curve. We begin with Sym_X^d, which will allow us to turn our study of degree-d points into the study of rational points. We then define the abelian group $\mathrm{Div}\, X$ of divisors on X/k and construct the Picard group $\mathrm{Pic}\, X$ as a quotient of $\mathrm{Div}\, X$. In order to give ourselves more structure, we work with the Picard *scheme* Pic_X, which is a variety over k which admits an injective map of abelian groups $\mathrm{Pic}\, X \to \mathrm{Pic}_X(k)$.

Definition 4.1.1. Let X be a nice curve, and let X^d be the d-th fold product of X with itself. The symmetric group S_d acts on X^d by permuting the factors. Let Sym_X^d be the quotient X^d/S_d.

As X is one dimensional and smooth, Sym_X^d is smooth as well, see [EH24, Chapter 5.1]. We denote elements of $X^d(\bar{k})$ as tuples $(x_1, \dots, x_d)$, where $x_i \in X(\bar{k})$, while elements of $\mathrm{Sym}_X^d(\bar{k})$ are unordered multisets $\{x_1, \dots, x_d\}$. Our notation differs from convention as elements of $\mathrm{Sym}_X^d(\bar{k})$

are more commonly denoted $x_1 + \ldots + x_d$. We forgo this notation as $\mathrm{Sym}^d_X(\bar{k})$ does not have a group structure, making the use of the addition sign confusing.

A closed point $x \in X/k$ of degree d gives a point of $\mathrm{Sym}^d_X(k)$ as follows (see Section 1.0.1 for the definition of a degree-d point). As x is a degree-d point, there exists some $x_1 \in X(\bar{k})$ such that x is the $\mathrm{Gal}(\bar{k}/k)$-orbit of x_1. Let $x_1, \ldots, x_d$ denote the Galois conjugates of x_1. Then, $\{x_1, \ldots, x_d\}$ is a point of $\mathrm{Sym}^d_X(\bar{k})$, and as the set $\{x_1, \ldots, x_d\}$ is a complete $\mathrm{Gal}(\bar{k}/k)$-orbit, $\{x_1, \ldots, x_d\}$ is fixed by $\mathrm{Gal}(\bar{k}/k)$. As Sym^d_X is a variety, $\mathrm{Sym}^d_X(k)$ is precisely the subset of $\mathrm{Sym}^d_X(\bar{k})$ fixed by $\mathrm{Gal}(\bar{k}/k)$, and hence $\{x_1, \ldots, x_d\}$ is a point of $\mathrm{Sym}^d_X(k)$. Thus, if X/k has infinitely many closed points of degree d, then $\mathrm{Sym}^d_X(k)$ is infinite.

We now define a divisor. As we work only with curves X/k, we can significantly simply our definition.

Definition 4.1.2. Let $\mathrm{Div}\, X$ denote the free abelian group on closed points $x \in X/k$. A divisor $D = \sum_x n_x x$ on X is an element of $\mathrm{Div}\, X$. The degree of D is the integer $\sum_x n_x \deg x$. A divisor is effective if for all x, $n_x \geq 0$.

We can relate elements of $\mathrm{Sym}^d_X(k)$ and $\mathrm{Div}\, X$. Note that the map $X^{n+m} \to \mathrm{Sym}^{n+m} X$ factors through the product $\mathrm{Sym}^n X \times \mathrm{Sym}^m X$, as $\mathrm{Sym}^n X \times \mathrm{Sym}^m X$ corresponds to the subgroup $S_n \times S_m$ inside S_{n+m}.

Lemma 4.1.3. *Let $x \in (\mathrm{Sym}^d_X)(k)$. There exists a unique multiset $\{y_1, \ldots, y_n\}$ of closed points of X/k of degrees $e_1, \ldots, e_n$ such that x is the image of $(y_1, \ldots, y_n)$ under the map $\mathrm{Sym}^{e_1} X \times \ldots \times \mathrm{Sym}^{e_n} X \to \mathrm{Sym}^d_X$.*

Proof. As $x \in (\mathrm{Sym}^d_X)(k)$, we can write $x = \{x_1, \ldots, x_d\}$ for some $x_i \in X(\bar{k})$, and as a multiset, $\{x_1, \ldots, x_d\}$ is fixed by $\mathrm{Gal}(\bar{k}/k)$. Each complete $\mathrm{Gal}(\bar{k}/k)$-orbit contained in $\{x_1, \ldots, x_d\}$ is by definition a closed point of X/k. Let $y_1, \ldots, y_n$ be the complete $\mathrm{Gal}(\bar{k}/k)$-orbits contained in $\{x_1, \ldots, x_d\}$. Then, we see that $\sum_i e_i = d$ as the y_i partition the multiset $\{x_1, \ldots, x_d\}$, and that x is the image of $(y_1, \ldots, y_d)$ as the map is the union of multisets.

If there exists a second multiset $\{y'_1, \ldots, y'_m\}$ of closed points of X/k such $(y'_1, \ldots, y'_m)$ maps to x, then each y'_i is supported on some x_j, and hence y'_i is the complete $\mathrm{Gal}(\bar{k}/k)$ orbit of x_j. Thus, we see that the multisets $\{y'_1, \ldots, y'_m\}$ and $\{y_1, \ldots, y_n\}$ must be equal. □

Remark 4.1.4. Over an imperfect field, Lemma 4.1.3 may not hold as the action of the absolute Galois group is not transitive, see [Ma21, Example 4.4 and Proposition 4.5].

Thus, there is a bijection of sets between $\bigcup_{d \geq 1} (\mathrm{Sym}^d_X)(k)$ and the effective divisors in $\mathrm{Div}\, X$. Explicitly, we can map $x \in (\mathrm{Sym}^d_X)(k)$ to the sum $y_1 + \ldots + y_n$. The map is well-defined as the multiset $\{y_1, \ldots, y_n\}$ is unique as a multiset.

In order to make the group $\operatorname{Div} X$ more tractable, we quotient by a large subgroup. Given a map $f\colon X \to Y$ of curves, there exists a natural homomorphism $f^* : \operatorname{Div} Y \to \operatorname{Div} X$ defined as follows. For a closed point $y \in Y/k$, we let $f^*(y) = \sum_{x_i \in f^{-1}(y)} e_f(x_i)x_i$, where $e_f(x_i)$ is the ramification index of f at the closed point x_i. As $\operatorname{Div} Y$ is generated by the closed points of Y, we can extend f^* to a homomorphism on all of $\operatorname{Div} Y$. A divisor D is *principal* if there exists a morphism $f\colon X \to \mathbb{P}^1$ such that $D = f^*(0 - \infty)$. As the degree of $f^*(0)$ equals the degree of $f^*(\infty)$, every principal divisor has degree zero. The subset of principal divisors forms a subgroup of $\operatorname{Div} X$. Indeed, suppose that D_1 and D_2 are principal. We show that $D_1 - D_2$ is principal. Let f_1, f_2 be the maps to $\mathbb{P}^1$ such that $f_i^*(0 - \infty) = D_i$. We can write f_1 in coordinates on $\mathbb{P}^1$ as $[f_{11}, f_{12}]$ and f_2 as $[f_{21}, f_{22}]$. Then, one can verify that the morphism $g\colon X \to \mathbb{P}^1$ defined by in coordinates by $[f_{11}f_{22}, f_{12}f_{21}]$ satisfies $\operatorname{div} g = D_1 - D_2$.

Definition 4.1.5. The abelian group $\operatorname{Pic} X$ is the quotient of $\operatorname{Div} X$ by the subgroup of principal divisors. We denote elements of $\operatorname{Pic} X$ by $[D]$, where $D \in \operatorname{Div} X$ is some choice of lift. The degree of $[D]$ is the degree of D and is independent of choice of D. We let $\operatorname{Pic}^d X$ denote the subset of $\operatorname{Pic} X$ of degree d.

Over an algebraically closed field, $\operatorname{Pic}^d X$ is isomorphic to the abelian variety $\operatorname{Pic}^0 X$. One would hope that the construction of $\operatorname{Pic} X$ would respect the structure of field extensions. Let $X_{\bar{k}}$ denote the base change of X to $\bar{k}$. The action of $\mathfrak{G}_k = \operatorname{Gal}(\bar{k}/k)$ on $X(\bar{k})$ induces an action of $\mathfrak{G}_k$ on $\operatorname{Pic} X_{\bar{k}}$. Concretely, for any divisor class $[D] \in \operatorname{Pic} X_{\bar{k}}$, we can write $[D] = [n_1p_1 + \ldots + n_dp_d]$ for some $p_i \in X(\bar{k})$ and $n_i \in \mathbb{Z}$, and then $\sigma \in \mathfrak{G}_k$ acts on $[D]$ by $\sigma([D]) = [n_1\sigma(p_1) + \ldots + n_d\sigma(p_d)]$. If $[D]$ is a closed point defined over k, then σ will permute the $\bar{k}$-points in the support of D but leave the sum fixed. Thus, any divisor class defined over k will be fixed by the action $\mathfrak{G}_k$. Ideally, $\operatorname{Pic} X$ would equal the fixed points of $\operatorname{Pic} X_{\bar{k}}$ under the action of $\mathfrak{G}_k$. As it turns out, this is too much to ask for.

Remark 4.1.6. The abelian group $\operatorname{Pic} X$ does not necessarily equal the points of $\operatorname{Pic} X_{\bar{k}}$ fixed by the Galois action. There exists an injective map of abelian groups $\operatorname{Pic} X \to (\operatorname{Pic} X_{\bar{k}})^{\mathfrak{G}_k}$ with cokernel contained in $\operatorname{Br} k$, the Brauer group of k, see [Poo17, Corollary 6.7.8].

The difficulties in Remark 4.1.6 are due to the fact that the relative Picard functor can fail to be representable. Thankfully, Grothendieck proved that for many schemes (including every nice curve) the relative Picard functor is representable in the étale topology [Gro62]. We let Pic_X denote this representing scheme. For our purposes, the fact that Pic_X exists together with some the basic properties described in the next two paragraphs are sufficient. For details and proofs, see [Kle14].

The abelian group $\operatorname{Pic} X_{\bar{k}}$ is isomorphic to $\operatorname{Pic}_X(\bar{k})$. As Pic_X is a scheme over k, $\operatorname{Pic}_X(k) = (\operatorname{Pic} X_{\bar{k}})^{\mathfrak{G}_k}$, and hence we have a map $\operatorname{Pic} X \to \operatorname{Pic}_X(k)$, which can fail to be surjective as in Remark 4.1.6. See [Kle14, Exercise 2.4] for an example of such a curve.

Just as $\operatorname{Pic} X$ and $\operatorname{Div} X$ have a notion of degree, so does Pic_X. We let Pic_X^d denote the subset of Pic_X of degree d. The map $\operatorname{Pic} X \to \operatorname{Pic}_X(k)$ respects degree, and thus we get a map $\operatorname{Pic}^d X \to \operatorname{Pic}_X^d(k)$.

Definition 4.1.7. The *Abel-Jacobi map* is the map $\operatorname{Sym}_X^d \to \operatorname{Pic}_X^d$ of varieties defined by mapping $\{x_1, \dots, x_d\} \in \operatorname{Sym}_X^d(\bar{k})$ to the image of the divisor $x_1 + \dots + x_d$ in $\operatorname{Pic}_X^d(\bar{k})$.

One can show that the Abel-Jacobi map is defined over k by checking that it is Galois-equivariant. The fibers of the Abel-Jacobi map are complete linear systems $|x|$ and hence are geometrically isomorphic to $\mathbb{P}^n$ for some $n \geq 0$. A variety geometrically isomorphic to $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^n$ over k if and only if it has a rational point, so if a fiber of the Abel-Jacobi map is non-empty, it is the k-points of $\mathbb{P}^n$ for some $n \geq 0$.

A degree-d point $x \in X/k$ maps into $\operatorname{Pic}_X^d(k)$, as the image of x under the quotient map $\operatorname{Div} X \to \operatorname{Pic} X$ is contained in $\operatorname{Pic}^d X$, which maps into $\operatorname{Pic}_X^d(k)$.

Definition 4.1.8. The locus of effective divisors of degree d, denoted by W_X^d, is the scheme-theoretic image of $\operatorname{Sym}_X^d \to \operatorname{Pic}_X^d$.

As the map $\operatorname{Sym}_X^d \to \operatorname{Pic}_X^d$ is proper, W_X^d is a closed subvariety of Pic_X^d. The dimension of W_X^d is $\min\{d, g(X)\}$ by the Jacobi inversion theorem, see [EH24, Corollary 5.10]. The image of $\operatorname{Sym}_X^d(k)$ is contained in $W_X^d(k)$, however, the containment can be strict, as the image of $\operatorname{Sym}_X^d(k)$ equals $W_X^d(k) \cap \operatorname{Pic} X$. By identifying $\operatorname{Sym}_X^d(k)$ and the degree-d divisors (as in Lemma 4.1.3), we think of points in $W_X^d(k) \cap \operatorname{Pic} X$ as the image of a divisor under the map $\operatorname{Div} X \to \operatorname{Pic} X$.

4.2 $\mathbb{P}^1$-paramterized points

In order to motivate $\mathbb{P}^1$-parametrized points, we recall the following consequence of Hilbert's Irreducibility from [VV26, Proposition 3.3.1]:

Proposition 4.2.1. *Let $\pi : X \to \mathbb{P}^1_k$ be a degree-d morphism. Then there exists a Zariski dense subset of points $z \in \mathbb{P}^1(k)$ such that $\pi^{-1}(z)$ is a degree-d point on X. In particular, $d \in \delta(X/k)$.*

Remark 4.2.2. Over a local field K (or more generally, over not finitely generated fields), Hilbert's Irreducibility fails. Indeed, one can construct morphisms $\varphi : X \to \mathbb{P}^1$ such that for all $t_0 \in \mathbb{P}^1(K)$, $\varphi^{-1}(t_0)$ is not integral, see Example 2.3.2.

From Proposition 4.2.1 we see that one reason we may have $d \in \delta(X/k)$ is if there exists a map $X \to \mathbb{P}^1$ of degree d. Points arising in this way are referred to as $\mathbb{P}^1$-parametrized.

Definition 4.2.3 ([VV26])**.** A closed point $x \in X/k$ is $\mathbb{P}^1$-parametrized if there exists a morphism $\pi : X \to \mathbb{P}^1$ with $\deg(\pi) = \deg(x)$ and $\pi(x) \in \mathbb{P}^1(k)$.

Lemma 4.2.4 ([VV26])**.** *Let $x \in X$ be a closed point of degree d. The following are equivalent:*

1. *The point x is $\mathbb{P}^1$-parameterized.*
2. *There exists a nonconstant map $\mathbb{P}^1 \to \mathrm{Sym}^d_X$ whose image contains $x \in (\mathrm{Sym}^d_X)(k)$.*
3. *The dimension $h^0([x])$ is at least 2, where $[x]$ denotes the divisor class of x.*
4. *The linear system $|x|$ is basepoint free.*

Definition 4.2.5. Let $\delta_{\mathbb{P}^1}(X/k) := \{\deg(x) \mid x \text{ is } \mathbb{P}^1\text{-parametrized}\}$.

4.3 AV-parametrized points

In Proposition 4.2.1, we constructed infinitely many degree-d points of X/k using a map $X \to \mathbb{P}^1$. One key feature of the construction is that the set $\mathbb{P}^1(k)$ is infinite. Thus, given a map $X \to Y$ of degree d between nice curves such that $Y(k)$ is infinite, one may wonder if there are necessarily infinitely many closed points of degree d.

Note first that if $Y(k)$ is infinite, then by Faltings's theorem (also known as the Mordell conjecture), $g(Y) \leq 1$. We thus consider the case where there exists a map $X \to E$, where E/k is a positive rank elliptic curve. This question was studied by Thomas Tucker. From [Tuc02, Remark 2.4], there exist maps $\varphi : X \to E$ of degree d such that for all $z \in E(k)$, $\varphi^{-1}(z)$ is not a degree-d point. Tucker proves, however, that after some finite extension k'/k, $d \in \delta(X/k')$, see [Tuc02, Theorem 2.5]. Thus, maps of degree d to positive rank elliptic curves can give infinitely many closed points of degree d, provided some additional information about the fibers. In general, these points do not arise from a map $X \to \mathbb{P}^1$, as for example there exist curves which are double covers of elliptic curves but are not double covers of $\mathbb{P}^1$.

The notion of an AV-parametrized point, which was first hinted at in [BEL+19] and defined precisely in [VV26], accounts for degree-d points arising from elliptic curves, but is much more general.

Definition 4.3.1. A degree-d point $x \in X$ is *AV-parametrized* if there exists an abelian subvariety $A \leq \mathrm{Pic}\, X_{\bar{k}}$ defined over k of positive k-rank such that $[x] + A \subset W^d_X$. A degree-d point $x \in X$ is *isolated* if it is neither $\mathbb{P}^1$ or AV-parametrized.

Definition 4.3.2. Let $\delta_{AV}(X/k) := \{\deg(x) \mid x \text{ is } AV\text{-parametrized}\}$.

The utility of $\mathbb{P}^1$ and AV-parametrized points become apparent from the following theorem, due to [BEL+19, Theorem 4.2].

Theorem 4.3.3 ([BEL+19])**.** *Let X be a nice curve over a number field.*

1. *There are finitely many isolated points in X (regardless of degree).*
2. *There are infinitely many degree-d points if and only if there exists a degree-d $\mathbb{P}^1$ or AV-parametrized point.*

We give some ideas as to the proof. If d is large enough, then Riemann-Roch implies that if there exists a degree-d point x, $h^0([x]) \geq 2$, and then Lemma 4.2.4 implies that x is $\mathbb{P}^1$-parametrized and there are infinitely many degree-d points. Thus to prove 1 and 2, it suffices to only consider the cases where d is small. We show only that if there are infinitely many degree-d points, there is a parametrized point. Suppose that there are infinitely many degree-d points. Then $(\mathrm{Sym}^d_X)(k)$ is infinite. Let π denote the Abel-Jacobi map and let $z \in \mathrm{Pic}^d_X(k)$. Consider the fiber $\pi^{-1}(z)$. As the fibers of the Abel-Jacobi map are the complete linear systems, if $\pi^{-1}(z)$ has infinitely many points, we have a map to $\mathbb{P}^n$ for some $n \geq 1$, and hence (after taking a sub-linear system) a $\mathbb{P}^1$-parametrized point. Thus, if there does not exist a $\mathbb{P}^1$-parametrized point we can assume that the fibers of π are finite, so that the image in $\mathrm{Pic}^d_X(k)$ is infinite. As the composition $\mathrm{Sym}^d_X(k) \to \mathrm{Pic}\, X \to \mathrm{Pic}^d_X(k)$ maps into $W^d_X(k)$ and $\mathrm{Sym}^d_X(k)$ is infinite, W^d_X has infinitely many rational points. We now apply Faltings's Theorem on the Subvarieties of an Abelian Variety [Fal91] [Fal94], also known as the Mordell-Lang conjecture.

Theorem 4.3.4 (Faltings's Subvarieties of AV Theorem)**.** *Let $W \subset P$ be a closed subvariety of an abelian variety over a number field k. Then*

$$W(k) = \bigcup_{i=1}^{N} (w_i + A_i(k)),$$

where $w_i \in W(k)$ and A_i are abelian subvarieties $A_i \subset P$ such that $w_i + A_i \subset W$.

By Faltings Subvarieties of AV Theorem, for all but finitely many points $p \in W^d_X(k)$, there exists a positive rank abelian variety A_i such that $w_i + A_i \subset W^d_X$ and $p \in w_i + A_i(k)$. As there are infinitely many degree-d points in $W^d_X(k)$, we see there must be an AV-parametrized point.

Chapter 5

Indecomposable translates and the density degree set

5.1 The geometry of the density degree set

5.1.1 Decomposable and indecomposable translates

Let $z \in W_X^d(k)$ and suppose there exists a positive dimensional abelian variety $A \leq \mathrm{Pic}_X^0$ such that $z + A \subset W_X^d$. We describe when there exists a degree-d point $x \in X$ such that $[x] \in z + A$.

We begin by considering which divisors are *not* degree-d points. By [VV26, Lemma 2.1.3], a degree-d divisor D which is not a closed point is contained in R, the set of sums of lower degree effective divisors. We will apply Faltings's Subvarieties of an Abelian Variety Theorem By Faltings's Theorem on the Subvarieties of an Abelian Variety Theorem 4.3.4, for any $e < d$, $\overline{W_X^e(k)}$ can be written as finite union of abelian translates with dense k points. Thus,

$$\overline{R} = \bigcup_{e<d} \overline{W_X^{d-e}(k) + W_X^e(k)} = \bigcup_{e<d} \overline{W_X^{d-e}(k)} + \overline{W_X^e(k)} = \bigcup_{e<d} \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (z_i + z_j + B_i^{d-e} + B_j^e),$$

where $z_i + B_i^{d-e}$ are the subvarieties of W_X^{d-e} arising from Faltings's theorem, and similarly $z_j + B_j^e$ are subvarieties of W_X^e. As $z + A$ is irreducible and the unions are finite, either there exists some e, i, j such that $z + A \subset z_i + z_j + B_i^{d-e} + B_j^e$ or $(z + A) \setminus \overline{R}$ is a non-empty dense open set in $z + A$. This dichotomy is captured by Definition 1.0.1. If $(z + A) \setminus \overline{R}$ is non-empty, then $(z + A) \setminus R$ is as well, and if $(z + A) \setminus R$ contains a k-point, it is a degree-d point. For a sufficient condition on an indecomposable translate containing a degree-d point, see Proposition 5.1.4. We make an additional definition for the base change to $\overline{k}$.

Definition 5.1.1. An abelian translate is *geometrically decomposable* if the base change to $\overline{k}$ is

decomposable, and *geometrically indecomposable* otherwise.

Remark 5.1.2. In general, determining if $z + A$ is indecomposable can be challenging. One sufficient condition is: If for all $e < d$, W_X^e does not contain any abelian varieties of dimension greater than or equal to $(\dim A)/2$, then $z + A$ is indecomposable. Thus, by the dimension bounds of [DF93, Proposition 3.3], Pic_X^g is a geometrically indecomposable abelian translate.

Remark 5.1.3. If A is zero dimensional, then $z + A \subset W_X^d$ is geometrically indecomposable if and only if $d = 1$.

5.1.2 Degree-d points in indecomposable translates

An indecomposable translate may not contain a degree-d point. Indecomposable translates with enough points, however, will necessarily contain many degree-d points:

Proposition 5.1.4. *If $z + A \subset W_X^d$ is indecomposable, $h^0(X, z) = 1$, and $A(k)$ is dense in A, then there exists an open set $U \subset A$ such that $U(k)$ is Zariski dense in A and every element of $z + U(k)$ is uniquely represented by a degree-d point over k.*

Remark 5.1.5. If A has positive dimension, then the set $z+U(k)$ in Proposition 5.1.4 is infinite, so $d \in \delta_{AV}(X/k)$. Note we do not assume that z is represented by a degree-d point, only that $h^0(X, z) = 1$, which by [VV26, Lemma 3.2.1] implies that there exists a unique effective divisor D over k such that $[D]$ equals z.

We prove three lemmas from which Proposition 5.1.4 follows quickly.

Lemma 5.1.6. *Let $z \in W_X^d(k)$ be such that $h^0(X, z) = 1$. The unique effective representative D for z is a degree-d point if and only if z is not in the image of the sum map $W_X^e(k) \times W_X^{d-e}(k)$ for any $0 < e < d$.*

Proof. The "if" statement is [VV26, Lemma 4.3.7]. For the converse, we prove the contrapositive. Suppose there exists some $e < d$ and $z_1 \in W_X^e(k)$, $z_2 \in W_X^{d-e}(k)$ such that $z = z_1 + z_2$. As tensoring induces an injective map on global sections, for $1 \leq i \leq 2$, $h^0(X, z_i) \leq h^0(X, z)$, and hence $h^0(X, z_i) = 1$. By [VV26, Lemma 3.2.1], z_1 and z_2 are represented by effective divisors D_1 and D_2 over k. Thus, $D = D_1 + D_2$ is the unique effective representative for z, which is not a degree-d point. □

Lemma 5.1.7. *If $z + A \subset W_X^d$ is indecomposable and $h^0(X, z) = 1$, then there is a Zariski open $U \subset A$ such that every element of $z + U(k)$ has a unique effective representative over k which is a degree-d point.*

Proof. For $e < d$, consider the image of the sum map $W_X^e(k) \times W_X^{d-e}(k) \to W_X^d(k)$. By Faltings' Subvarieties of Abelian Varieties Theorem,

$$\bigcup_{e<d} W_X^{d-e}(k) + W_X^e(k) = \bigcup_{e<d} \bigcup_i \bigcup_j z_i + z_j + B_i^{d-e}(k) + B_j^e(k).$$

As $x + A$ is indecomposable, for all e, i, and j, $(z + A) \cap (z_i + z_j + B_i^{d-e} + B_j^e)$ must be a proper closed subset of $z + A$. Hence, $R \cap (z + A)$ is contained in a finite union of proper closed subsets of $z + A$, which is a closed set Z. By uppersemicontinuity, there exists an open set V such that for all $v \in z + V$, $h^0(X, v) = 1$, and hence the effective representative for any $v \in z + V(k)$ must be defined over k. Let $U := Z^c \cap V$. Then, every point of $z + U(k)$ is uniquely represented by a degree-d point by Lemma 5.1.6. □

Lemma 5.1.8. *Let $\mathcal{X}$ be an irreducible topological space, $U \subset \mathcal{X}$ an open set, and $S \subset \mathcal{X}$ a dense subset. Then, $U \cap S$ is dense in $\mathcal{X}$.*

Proof. Let $V \subset \mathcal{X}$ be an open set. As $\mathcal{X}$ is irreducible, every open set is dense, and so $U \cap V$ is non-empty. As $U \cap V$ is open and S is dense, $(S \cap U) \cap V = S \cap (U \cap V)$ is non-empty. □

Proof of Proposition 5.1.4. Let U be the open set of Lemma 5.1.7. As $A(k)$ is dense in A, by Lemma 5.1.8, $U(k)$ is dense as well. □

We state the following corollary of Proposition 5.1.4 for later use.

Corollary 5.1.9. *If D is an effective divisor on X/k, $[D] + A \subset W_X^d$ is indecomposable, A has positive dimension, and $A(k)$ is dense in A, then $d \in \delta_{AV}(X/k)$.*

Proof. If $h^0(X, D) \geq 2$, then by Hilbert's Irreducibility, there exists a degree-d divisor linearly equivalent to D, and then $d \in \delta_{AV}(X/k)$. If $h^0(X, D) = 1$, apply Proposition 5.1.4. □

5.1.3 Proof of the main theorem

In this subsection we prove our main theorem relating degree-d points and indecomposable translates, Theorem 1.0.2. Let $W_X^{d,r} := \{\mathcal{L} \in \mathrm{Pic}_X^d \mid h^0(X, \mathcal{L}) \geq r + 1\}$.

Lemma 5.1.10. *Let $w_1 + H_1, \ldots, w_n + H_n$ be effective, geometrically indecomposable, positive dimensional abelian translates defined over $\bar{k}$, and let $f = \deg(\sum_{i=1}^n w_i)$. If the sum map $\theta\colon \prod_{i=1}^n H_i \to \sum_{i=1}^n H_i$ has finite kernel, then $\sum_{i=1}^n w_i + H_i \subset W_X^{f,1}$.*

Proof. Let ℓ/k be a finite extension such that for all $1 \leq i \leq n$, $w_i + H_i$ is defined over ℓ and $H_i(\ell)$ is Zariski dense in H_i, and further every element of the kernel of θ is defined over ℓ. Let

$1 \leq i \leq n$. Let $\pi_i \colon \prod_{j=1}^n H_j \to H_i$ denote the i-th projection. If for any i, $w_i + H_i \subset W_X^{\deg w_i, 1}$, then the conclusion holds, so we can assume that a generic element of $w_i + H_i$ does not move, in particular that $h^0(X, w_i) = 1$. Then, by Lemma 5.1.7 there exists a non-empty open set $W_i \subset w_i + H_i$ such that every element of $W_i(\ell)$ is uniquely represented by a closed point over ℓ. The intersection $w_i + H_i \cap w_j + H_j$ is closed, and if $w_i + H_i \subset w_j + H_j$, then an abelian variety isogenous to H_i is contained in the kernel of θ, a contradiction as the kernel of θ is not positive dimensional. So, $w_i + H_i \cap w_j + H_j$ is a proper closed subset of $w_i + H_i$. As $\ker \theta$ is finite, $Z_i \coloneqq \bigcup_{a \in \ker \theta} \pi_i(a) + (W_i^c \cup (\bigcup_{j=1}^n (w_i + H_i \cap w_j + H_j)))$ is also a proper closed subset of $w_i + H_i$. Let U_i be the complement of Z_i in $w_i + H_i$. As Z_i is closed under translation by $\pi_i(a)$ for any $a \in \ker \theta$, so is U_i.

As $U_i \subset W_i$, any element of $U_i(\ell)$ has a unique effective representative, integral over ℓ. Let $[x_i] \in U_i(\ell)$. Fix some non-trivial $a \in \ker \theta$. For any $[x_i] \in U_i(\ell)$, let x_i^a be the effective representative for $[x_i] + \pi_i(a) \in U_i(\ell)$. For any $([x_i]) \in \prod_{i=1}^n U_i(\ell)$, consider the two sums of divisors $y \coloneqq \sum_{i=1}^n x_i$ and $y^a \coloneqq \sum_{i=1}^n x_i^a$. As $\sum_{i=1}^n \pi_i(a) = \theta(a) = 0$, $[y]$ and $[y^a]$ are equal. We show that y and y^a are not equal *as divisors*, so $h^0(X, [y]) \geq 2$. As a is not the identity, there exists some i such that $\pi_i(a)$ is not the identity, and hence x_i and x_i^a are not equal. As $[x_i] \in U_i$, for any j, $[x_i] \notin (w_i + H_i) \cap (w_j + H_j)$, and hence x_i does not equal x_j^a for any j. So, y^a is not supported on x_i, and $y \neq y^a$. Thus, if $y \in \theta(\prod_{i=1}^n U_i(\ell))$, then $h^0(X, y) \geq 2$. By Lemma 5.1.8, $U_i(\ell)$ is dense in $w_i + H_i$, and hence $\prod_{i=1}^n U_i(\ell)$ is dense in $\prod_{i=1}^n w_i + H_i$. As θ is continuous, $\theta(\prod_{i=1}^n U_i(\ell))$ is dense in $\sum_{i=1}^n w_i + H_i$, showing that $\sum_{i=1}^n w_i + H_i \subset W_X^{f,1}$. □

Proposition 5.1.11. *Let $u + B \subset W_X^e$ and $v + G \subset W_X^{d-e}$ be abelian translates. If the sum map $B \times G \to B + G$ is not injective, then $u + v + B + G \subset W_X^{d,1}$.*

Proof. We first consider the case where the kernel of the map $B \times G \to B + G$ has positive dimensional kernel, so that $\dim B + G < \dim B \times G$. Let $\epsilon_f \colon \mathrm{Sym}_X^f \to W_X^f$ be the natural map. If $u + B \subset W_X^{e,1}$ or $v + G \subset W_X^{d-e,1}$, then $u + v + B + G \subset W_X^{d,1}$, so we can assume ϵ_e and ϵ_{d-e} are birational over $u + B$ and $v + G$. Let $\tilde{B}$ and $\tilde{G}$ be the strict transforms of $u + B$ and $v + G$ under ϵ_e and ϵ_{d-e}. Recall that the map $\sigma \colon \mathrm{Sym}_X^e \times \mathrm{Sym}_X^{d-e} \to \mathrm{Sym}_X^d$ has finite fibers, and let $J = \sigma(\tilde{B} \times \tilde{G})$. As $\dim J = \dim(\tilde{B} \times \tilde{G}) \geq \dim(B \times G) > \dim(B + G)$, we conclude that the map $\epsilon_d|_J \colon J \to u + v + B + G$ has positive dimensional fibers, which implies that $u + v + B + G \subset W_X^{d,1}$.

We now consider the case where the kernel is finite. Let $w_1 + H_1, \ldots, w_m + H_m$ be geometrically indecomposable abelian translates, ordered such that for $1 \leq i \leq n$, H_i has positive dimension, while for $n + 1 \leq i \leq m$, H_i has dimension zero, such that there exists some $I \subset \{1, \ldots, m\}$ such that $u + B \subset \sum_{i \in I} w_i + H_i$ and $v + G \subset \sum_{i \in I^c} w_i + H_i$. The sum map $\prod_{i=1}^m H_i \to \sum_{i=1}^m H_i$ factors through the partial sum $\prod_{i=1}^m H_i \to \sum_{i \in I} H_i \times \sum_{i \in I^c} H_i \to \sum_{i=1}^m H_i$. As $B \times G \subset \sum_{i \in I} H_i \times \sum_{i \in I^c} H_i$

and $B \times G \to B + G$ has non-trivial kernel, so does $\sum_{i\in I} H_i \times \sum_{i\in I^c} H_i \to \sum_{i=1}^m H_i$, and hence so does $\theta\colon \prod_{i=1}^m H_i \to \sum_{i=1}^m H_i$. If $\ker\theta$ has positive dimension, then we proceed as in the positive dimensional case, so we can assume that θ has finite kernel. Zero dimensional abelian varieties are points, and hence $\prod_{i=1}^m H_i \simeq \prod_{i=1}^n H_i$ and the map $\prod_{i=1}^n H_i \to \sum_{i=1}^n H_i$ has non-trivial finite kernel. Let $f = \deg\left(\sum_{i=1}^n w_i\right)$. By Lemma 5.1.10, $\sum_{i=1}^n w_i + H_i \subset W_X^{f,1}$. Translation defines an injective map on global sections, so $u + v + B + G \subset \sum_{i=1}^m w_i + H_i \subset W_X^{d,1}$. □

Proof of Theorem 1.0.2. Suppose for contradiction that $[x] + A$ is decomposable. Let $u + B \subset W_X^e$ and $v + G \subset W_X^{d-e}$ be abelian translates such that $[x] + A \subset u + v + B + G$. If the sum map $B \times G \to B + G$ is not injective, by Proposition 5.1.11, $h^0(X, x) \geq 2$, a contradiction as x is $\mathbb{P}^1$-isolated. Thus we can suppose the sum map $B \times G \to B + G$ is injective. Then, there exist $z \in u + B(k)$ and $y \in v + G(k)$ such that $z + y = [x]$, contradicting Lemma 5.1.6. □

As a corollary of Theorem 1.0.2, we give an improvement to [VV26, Proposition 4.3.5] (see also [BEL+19, Theorem 4.2]).

Corollary 5.1.12. *Let $x \in X$ be a degree-d point and suppose that there exists a positive rank abelian variety A such that $[x] + A \subset W_X^d$. Either x is $\mathbb{P}^1$-parametrized or there exists an open set $U \subset A$ such that $U(k)$ is infinite and every element of $[x] + U(k)$ is uniquely represented by a degree-d point over k.*

Proof. Let $B \leq A$ be the maximal abelian subvariety such that $B(k)$ is dense in B. Then, the Zariski closure of $[x] + A(k)$ is contained in a finite union $\bigcup_i z_i + B$. Re-indexing if necessary, we can suppose that $[x] + B = z_1 + B$. By Theorem 1.0.2, either x is $\mathbb{P}^1$-parametrized or $[x] + B$ is indecomposable. Thus, we can suppose that $[x] + B$ is indecomposable. By Proposition 5.1.4, there exists an open set $V \subset B$ such that $[x] + V(k)$ has the desired properties. Let $Z \subset B$ be the complement of V in B. As $B \subset A$ is closed, $Z \subset A$ is closed. Let $[x] + U$ be the complement of $[x] + Z \cup \bigcup_{i>1} z_i + B$ in $[x] + A$. Then, $U \subset A$ is open and $[x] + U(k) = [x] + V(k)$. As $[x] + V(k)$ has the desired properties, $[x] + U(k)$ does as well. □

5.1.4 Indecomposable but geometrically decomposable translates

We collect several results on indecomposable but geometrically decomposable translates. For an example of such an abelian translate, see [VV26, Example 5.5.3]. Abelian translates which are indecomposable but geometrically decomposable exhibit interesting arithmetic.

Corollary 5.1.13. *Let $x \in X$ be a degree-d, $\mathbb{P}^1$-isolated point. If A is positive rank and $[x] + A \subset W_X^d$ is geometrically decomposable, then there exists a non-trivial finite extension ℓ/k*

and infinitely many degree-d points $x_i \in X$ such that $[x_i] \in [x] + A(k)$ and $\mathbf{k}(x_i)$ contains a proper subfield isomorphic to ℓ.

Proof. Let $U \subset A$ be the open set of Corollary 5.1.12. Let $\tilde{k}/k$ be a Galois extension such that $[x] + A$ is decomposable over $\tilde{k}$. By Theorem 1.0.2, no element of $[x] + U(k)$ is represented by a degree-d point over $\tilde{k}$. Thus, for all $x_i \in [x] + U(k)$, the fields $\mathbf{k}(x_i)$ and $\tilde{k}$ are not linearly disjoint, so as $\tilde{k}/k$ is Galois, $\mathbf{k}(x_i)$ must contain a subfield isomorphic to a subfield of $\tilde{k}$, see [DF04, Corollary 14.4.20] for instance. As there are finitely many subfields of $\tilde{k}$, by the pigeonhole principle, there exists some subfield ℓ such that for infinitely many i, $\mathbf{k}(x_i)$ contains a subfield isomorphic to ℓ, which must a proper subfield by Faltings's proof of the Mordell Conjecture. □

Remark 5.1.14. The proof of Corollary 5.1.13 gives some control over the field ℓ. Let $u + B$ and $v + G$ be abelian translates over $\overline{k}$ such that $[x] + A \subset u + v + B + G$, and let k_1 and k_2 be the fields of definition of $u + B$ and $v + G$, respectively. Let $\tilde{k}/k$ be a Galois extension such that $\tilde{k}$ contains k_1 and k_2. Then, ℓ is some subfield of $\tilde{k}$.

Remark 5.1.15. If d is prime and A has positive rank, then Corollary 5.1.13 shows that any geometrically decomposable $[x] + A \subset W_X^d$ is decomposable, as prime degree extensions contain no non-trivial subfields.

Lemma 5.1.16. *If $z + A \subset W_X^d$ is indecomposable but geometrically decomposable, $h^0(X, z) = 1$, and $A(k)$ is dense in A, then there exists some e dividing d such that W_X^e and W_X^{d-e} contain abelian translates $u + B$ and $v + G$, defined over $\overline{k}$, such that $z + A_{\overline{k}} \subset u + v + B + G$.*

Proof. Let $U \subset A$ be the open set of Proposition 5.1.4, and let $\{x_i\}_{i \in \mathbb{N}}$ be the set of degree-d points representing elements of $[x] + U(k)$. Let ℓ/k be a Galois extension such that $[x] + A$ is decomposable over ℓ. By Proposition 5.1.11, the sum map for the decomposition is injective, and hence every x_i is not a closed point over ℓ as it is a sum of lower degree divisors. By [VV26, Lemma 2.2.1(1)], for all i, every irreducible component of x_i/ℓ has the same degree f_i which divides d. Thus, using Faltings' Theorem on the Subvarieties of an Abelian Variety,

$$[x] + U(k) \subset \bigcup_{f \mid d} W_X^f(\ell) + W_X^{d-f}(\ell) = \bigcup_{f \mid d} \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} z_i + z_j + B_i^{d-f}(k) + B_j^f(k).$$

As the union is finite, $[x] + A$ is irreducible, and $[x] + U(k)$ is dense in $[x] + A$, there exists some $e \mid d$ and some i and j such that $[x] + A \subset z_i + z_j + B_i^{d-e} + B_j^e$. □

Although a geometrically decomposable abelian translate may not be decomposable, we show that certain geometrically decomposable translates do not contain $\mathbb{P}^1$-isolated points.

Corollary 5.1.17. *Let $x \in X$ be a degree-d point, let A be a positive rank abelian variety, and let $[x] + A \subset W_X^d$ be geometrically decomposable. Let $u + B \subset W_X^e/\bar{k}$ and $v + G \subset W_X^{d-e}/\bar{k}$ be such that $[x] + A_{\bar{k}} \subset u + v + B + G$. If G is zero dimensional, then x is $\mathbb{P}^1$-parametrized.*

Proof. Suppose for contradiction that x is $\mathbb{P}^1$-isolated. By Corollary 5.1.12, there exist infinitely many degree-d points x_i such that $[x_i] \in [x] + A$. Note that $G(\bar{k})$ is a point, so $v + G = v$ and the sum map $B \times G \to B + G$ is injective. Hence, every $x_i/\bar{k}$ is supported on v, a contradiction as degree-d points have disjoint support. □

5.1.5 The geometry of the potential density degree set

Define the potential density degree set

$$\wp(X/k) := \bigcup_{k'/k \text{ finite}} \delta(X/k').$$

Analogous to $\wp(X/k)$, we define the potential AV-parametrized density degree set and potential $\mathbb{P}^1$-parametrized density degree set by

$$\wp_{AV}(X/k) := \bigcup_{k'/k \text{ finite}} \delta_{AV}(X/k') \quad \text{and} \quad \wp_{\mathbb{P}^1}(X/k) := \bigcup_{k'/k \text{ finite}} \delta_{\mathbb{P}^1}(X/k').$$

The set $\wp_{\mathbb{P}^1}(X/k)$ can be described in terms of the geometry of X as

$$\wp_{\mathbb{P}^1}(X/k) = \{\deg(\mathcal{L}) \mid \text{there exists a base-point free } \mathcal{L} \in W_X^d(\bar{k})\}.$$

In this subsection, we describe $\wp_{AV}(X/k)$ in terms of the geometry of Pic_X.

Definition 5.1.18. Let $\wp_{ind}(X/k)$ denote the *geometrically indecomposable degree set*,

$$\wp_{ind}(X/k) := \{d \mid W_X^d \text{ contains an indecomposable abelian translate of positive dimension}\}.$$

In contrast to $\delta(X/k)$ and $\wp(X/k)$, $\wp_{ind}(X/k)$ is a finite set, as when $d \geq g$, W_X^d is isomorphic to Pic_X^d, and so any coset is geometrically decomposable for $d > g$. Let Z_X^d denote the Ueno locus of W_X^d, which is defined to be the union of all positive dimensional abelian translates contained in W_X^d.

Theorem 5.1.19.

$$\wp_{AV}(X/k) = \wp_{ind}(X/k) \cup \left\{\deg(\mathcal{L}) \mid \text{there exists base-point free } \mathcal{L} \in Z_X^d(\bar{k})\right\}$$

We can describe $\wp(X/k)$ in terms of only $\wp_{\mathbb{P}^1}(X/k)$ and $\wp_{ind}(X/k)$.

Corollary 5.1.20 (Theorem 1.0.2)**.** $\wp(X/k) = \wp_{\mathbb{P}^1}(X/k) \cup \wp_{ind}(X/k)$.

Proof of Theorem 5.1.19. We first show that $\wp_{ind}(X/k) \subset \wp_{AV}(X/k)$. Let $z+A$ be a positive dimensional geometrically indecomposable abelian translate, and let ℓ/k be a finite extension such that $A(\ell)$ is Zariski dense in A and z has an effective representative over ℓ. By Corollary 5.1.9, $d \in \delta_{AV}(X/\ell) \subset \wp_{AV}(X/\ell)$.

Let $\mathcal{S}$ be the set of degrees d such that there exists a base-point free $\mathcal{L} \in Z^d_X(\bar{k})$. We show that $\mathcal{S} \subset \wp_{AV}(X/k)$. Suppose that $x \in Z^d_X(\bar{k})$ is base-point free, and let $A/\bar{k}$ be an abelian variety such that $x + A \subset W^d_X$. Let ℓ/k be a finite extension such that x and A are defined over ℓ, $h^0(X, x) \geq 2$ over ℓ, and $A(\ell)$ has positive rank. By Hilbert's Irreducibility, there exists a degree-d point y/ℓ such that $[y]$ equals x, and y is AV-parametrized, so that $d \in \wp_{AV}(X/k)$.

Finally, we show that $\wp(X/k) \smallsetminus \wp_{ind}(X/k) \subset \mathcal{S}$. Let $x \in X$ be a degree-d AV-parametrized point. By definition, there exists a positive rank abelian variety A such that $[x] + A \subset W^d_X$. Taking some abelian subvariety if necessary, we may assume that $A(k)$ is dense in A. As $d \notin \wp_{ind}(X/k)$, $[x] + A$ is geometrically decomposable. If x is $\mathbb{P}^1$-parametrized, then $d \in \mathcal{S}$, so we can assume that x is $\mathbb{P}^1$-isolated. By Lemma 5.1.16, there exists an abelian translate $u + B \subset W^e_X$ such that e divides d. Let $n = d/e$. Then, by (the proof of) [VV26, Proposition 5.2.1], $|nu| \in nu + B$ is base-point free, so that $d \in \mathcal{S}$. □

5.2 Bielliptic curves and effective translates

In this section we investigate bielliptic curves generally. For the remainder of this section, we will work over an algebraically closed field, and assume that there exists a degree-2 map $\phi\colon X \to E$ to an elliptic curve. As every element of Pic^1_E is effective, $\phi^* \mathrm{Pic}^1_E \subset W^2_X$, and so for any $y \in W^{d-2}_X$, $y + \phi^* \mathrm{Pic}^1_E \subset W^d_X$. Note that $\phi^* \mathrm{Pic}^1_E$ is a coset of $\phi^* \mathrm{Pic}^0_E$ in Pic_X: let $a \in \mathrm{Pic}^1_E$, and consider $\phi^* a + \phi^* \mathrm{Pic}^0_E \subset \mathrm{Pic}^2_X$. The main result of this section, Proposition 5.2.1, shows that, for $d < g$, every coset of $\phi^* \mathrm{Pic}^0_E$ except for $\phi^* \mathrm{Pic}^1_E$ is decomposable.

Proposition 5.2.1. *Let $2 < d < g$ and let $x \in W^d_X$. If $x + \phi^* \mathrm{Pic}^0_E \subset W^d_X$, then there exists $y \in W^{d-2}_X$ such that $x + \phi^* \mathrm{Pic}^0_E = y + \phi^* \mathrm{Pic}^1_E$.*

Proposition 5.2.1 improves on [DF93, Lemma 3.4] in the bielliptic case: their results apply only when $d \leq g/2$. The Debarre-Fahlaoui curves (see [DF93, Bottom of p. 242 and Proposition 5.14]) give counterexamples to the analogous statement of Proposition 5.2.1 when the degree of the map $X \to E$ is greater than or equal to 5, and similar constructions yield counterexamples

when the degree of $X \to E$ is 3 or 4. We do not expect statements analogous to Proposition 5.2.1 without strong control of the geometry of X: both the proofs of [DF93, Lemma 3.4] and Proposition 5.2.1 rely on a complete description of $\mathbb{P}(H^0(X, \mathcal{L}))$ for some suitable choices of $\mathcal{L}$ in terms of E.

We now begin working towards the proof of Proposition 5.2.1. We give an outline as the method of proof will be used heavily in Section 5.3. First, we find an appropriate coset $z+\phi^* \operatorname{Pic}^0_E$ such that for a general $\mathcal{L} \in z + \phi^* \operatorname{Pic}^0_E$, we can describe $\mathbb{P}(H^0(X, \mathcal{L}))$. We then use the fact that cosets are disjoint or equal to show that an "unexpected" coset $x + \phi^* \operatorname{Pic}^0_E \subset W^d_X$ would give elements of $\mathbb{P}(H^0(X, \mathcal{L}))$ not matching our description, a contradiction.

Lemma 5.2.2. *Let $d < g$ and let $\mathcal{L} \in \phi^* \operatorname{Pic}^d_E$ be different from ω_X. Every effective representative of $\mathcal{L}$ is the pullback of an effective divisor on E.*

Proof. Let $\mathcal{L}_E \in \operatorname{Pic}^d_E$ be such that $\phi^* \mathcal{L}_E = \mathcal{L}$. Suppose first that $d = g-1$. By Riemann-Roch, $h^0(E, \mathcal{L}_E) = g - 1$. Pullback induces an injective map $H^0(E, \mathcal{L}_E) \to H^0(X, \phi^* \mathcal{L}_E)$. We show that this map is surjective by dimension counting. Riemann-Roch on X shows that

$$h^0(X, \phi^* \mathcal{L}_E) = 2g - 2 - g + 1 + h^0(\omega_X \otimes \mathcal{L}^{-1}) = g - 1$$

as $\omega_X \otimes \mathcal{L}^{-1}$ is degree 0 and $\mathcal{L} \neq \omega_X$ by assumption. As every section of $\mathcal{L}$ is a pullback of a section of $\mathcal{L}_E$, the divisor of zeroes is a pullback as well.

Now suppose that $d < g-1$. Let D be an effective representative of $\mathcal{L}$. Let $\mathcal{L}' \in \phi^* \operatorname{Pic}^{g-1-d}_E$ be different from $\omega_X \otimes \mathcal{L}^{-1}$, and let $\mathcal{L}'_E \in \operatorname{Pic}^{g-1-d}_E$ be such that $\phi^* \mathcal{L}'_E = \mathcal{L}'$. Choose an effective representative D'_E for $\mathcal{L}'_E$, so that $\phi^* D'_E$ is an effective representative for $\mathcal{L}'$. Now consider $D + \phi^* D'_E$, which is an effective representative for $\mathcal{L} \otimes \mathcal{L}' \in \phi^* \operatorname{Pic}^{g-1}_E$. As the case $d = g - 1$ has been proved, any effective representative of $\mathcal{L} \otimes \mathcal{L}'$ is a pullback from E, so that there exists some effective divisor D''_E of degree $g - 1$ on E such that $D + \phi^* D'_E = \phi^* D''_E$. (Note that this is *equality* of effective divisors, not linear equivalence.) Comparing supports, we find that D is the pullback of the (effective) divisor $D''_E - D'_E$. □

Proof of Proposition 5.2.1. As $x \in W^d_X$, there exist $p_1, \ldots, p_d \in X$ such that $D = p_1 + \ldots + p_d$ is a representative for x. It suffices to show that D contains a pullback under ϕ. Suppose for contradiction that D does not contain a pullback under ϕ. As ϕ has degree two, the corresponding extension of function fields is Galois, and there exists $\sigma \in \operatorname{Aut}(X)$ such that for all $p \in X$, $\phi(p) = \phi(\sigma(p))$. Extend σ to act on divisors. As D does not contain a pullback under ϕ, neither does $\sigma(D)$. Consider $[\sigma(D) + D] + \phi^* \operatorname{Pic}^0_E$. As $[\sigma(D) + D] \in \phi^* \operatorname{Pic}^d_E$ and cosets are disjoint or equal, $[\sigma(D) + D] + \phi^* \operatorname{Pic}^0_E = \phi^* \operatorname{Pic}^d_E$. Let D' be an effective divisor on X

such that $[D'] \in x + \phi^* \operatorname{Pic}^0_E$ is different from $[D]$ and $[K_X - D]$. Then, $D' + \sigma(D)$ is an effective representative of $\mathcal{L}(D' + \sigma(D)) \in \phi^* \operatorname{Pic}^d_E$, and hence by Lemma 5.2.2, there exists an effective divisor D_E on E such that $D' + \sigma(D) = \phi^*(D_E)$. (Note that this is *equality* of divisors, not linear equivalence.) As $\sigma(D)$ does not contain a pullback under ϕ and has degree d, any collection of d pullbacks under ϕ whose support contains $\sigma(D)$ is exactly $D + \sigma(D)$, so $D' + \sigma(D) = D + \sigma(D)$, a contradiction as $[D'] \neq [D]$. □

5.3 A fiber product of elliptic curves

Let Z^d_X denote Ueno locus of W^d_X, which is defined to be the union of all positive dimensional abelian translates contained in W^d_X. We study Z^5_C for some special genus-7 curves. In this section we work over an algebraically closed field.

5.3.1 Construction and preliminaries

Let E and F be non-isogenous elliptic curves. Fix a degree-2 map $\phi\colon F \to \mathbb{P}^1$ and a degree-3 map $\theta\colon E \to \mathbb{P}^1$, and let C be the fiber product $E \times_{\mathbb{P}^1} F$. Let $\phi_C\colon C \to E$ be the base change of ϕ to C, and similarly let $\theta_C\colon C \to F$ denote the base change of θ. For a map of curves $\gamma\colon X \to Y$, let B_γ denote the branch divisor of γ.

Lemma 5.3.1. *Let $p \in C$. Suppose that B_ϕ and B_θ have disjoint support. Then,*

1. *the image of C under the inclusion into $E \times F$ is smooth,*
2. *ϕ_C is ramified at p if and only if ϕ is ramified at $\theta_C(p)$,*
3. *θ_C is ramified at p if and only if θ is ramified at $\phi_C(p)$, and*
4. *$\theta_C^*(\theta_C(p)) \cap \phi_C^*(\phi_C(p)) = p$.*

Proof. We show that the image of the inclusion $C \to E \times F$ is smooth. Let $q \in C$ and let $x = \theta(\phi_C(q))$. The tangent space $T_q(E \times_{\mathbb{P}^1} F)$ is isomorphic to $V = T_{\phi_C(q)}E \times_{T_x\mathbb{P}^1} T_{\theta_C(q)}F$. As E and F are smooth, $T_{\phi_C(q)}E$ and $T_{\theta_C(q)}F$ are 1-dimensional, and hence V is 2-dimensional if and only if the maps $T_{\phi_C(q)}E \to T_x\mathbb{P}^1$ and $T_{\theta_C(q)}F \to T_x\mathbb{P}^1$ are both identically zero. Since the branch divisors are disjoint, the maps are never both identically zero, so V is 1-dimensional and C is smooth. Both (2) and (3) follow from general properties of base change. Finally, (4) follows as at most one of ϕ_C and θ_C is ramified at p, and a generic fiber of $\theta \circ \phi_C$ contains 6 distinct points with fibers of ϕ_C and θ_C intersecting at a point.

□

We assume that B_ϕ and B_θ have disjoint support so that C is smooth.

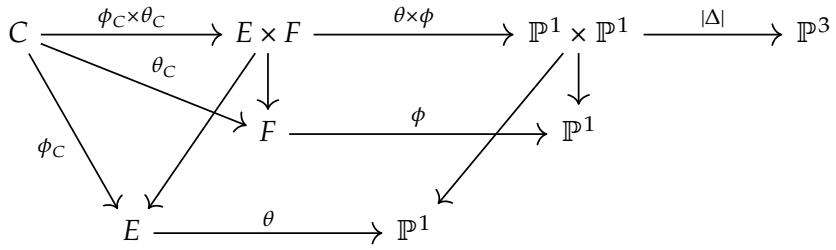

Figure 5.1: A diagram showing C maps onto Δ in $\mathbb{P}^1 \times \mathbb{P}^1$

5.3.2 Low degree maps and the canonical divisor

Having established the class of curves we will consider, we compute ω_C and the low degree maps to $\mathbb{P}^n$. Let $W_C^{d,r} \subset \mathrm{Pic}_C^d$ denote the space of line bundles with at least $r+1$ sections.

Lemma 5.3.2. *The following hold:*

1. *There does not exist a map $C \to \mathbb{P}^1$ of degree 2, 3, or 5.*
2. *Every degree 4 map $C \to \mathbb{P}^1$ factors through ϕ_C, that is, $W_C^{4,1} = \phi_C^* \mathrm{Pic}_E^2$.*
3. $W_C^{5,1} = W_C^1 + \phi_C^* \mathrm{Pic}_E^2$.

Proof. Suppose that there exists a map $\gamma\colon C \to \mathbb{P}^1$ of degree d. If γ does not factor through ϕ_C, then by the Castelnuovo-Severi inequality [ACGH85, p.366], $7 \le d+1$, so that either $d \ge 6$ or γ factors through ϕ_C. This gives statements (1) and (2). For statement (3), note that every element of $W_C^{5,1}$ must have a base point. □

Lemma 5.3.3. *The genus of C is 7 and $\omega_C = \phi_C^* \theta^* O_{\mathbb{P}^1}(1) \otimes \theta_C^* \phi^* O_{\mathbb{P}^1}(1) = \phi_C^* \theta^* O_{\mathbb{P}^1}(2)$.*

Proof. Consider Figure 5.1. Let Δ denote the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, so that $(\theta \times \phi)^*(\Delta) = C$. The divisor Δ determines the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, and as $\mathcal{L}(C) = (|\Delta| \circ \phi \times \theta)^*(O_{\mathbb{P}^3}(1)) = (\phi \times \theta)^*(\pi_1^* O_{\mathbb{P}^1}(1) \otimes \pi_2^* O_{\mathbb{P}^1}(1))$. By Lemma 5.3.1, C is a smooth curve in $E \times F$, so by adjunction, $\omega_C \simeq \omega_{E\times F} \otimes \mathcal{L}(C) \otimes O_C$. As $E \times F$ is an abelian variety, $\omega_{E\times F}$ is trivial, so ω_C is $(\phi_C \times \theta_C)^*(\mathcal{L}(C))$. By commutativity of the diagram,

$$\begin{aligned} \omega_C &= (\phi_C \times \theta_C)^*((\phi \times \theta)^*(\pi_1^* O_{\mathbb{P}^1}(1) \otimes \pi_2^* O_{\mathbb{P}^1}(1))) \\ &= \phi_C^* \theta^* O_{\mathbb{P}^1}(1) \otimes \theta_C^* \phi^* O_{\mathbb{P}^1}(1). \end{aligned}$$

The second equality follows as $\phi_C^* \theta^* = \theta_C^* \phi^*$. Finally, C has genus 7 as ω_C has degree 12. □

5.3.3 The geometry of Z_C^d

We now begin investigating Z_C^d directly. Having classified all translates of $\phi_C^* \operatorname{Pic}_E^0$ contained in W_C^d for $d < 7$ by Proposition 5.2.1, we turn to translates of $\theta_C^* \operatorname{Pic}_F^0$. Let $x = p_1 + ... + p_5$ be a degree-5 divisor on C, and let $p_6, p_7 \in C$. We begin by applying Riemann-Roch in Lemma 5.3.6 to show that for most p_6 and p_7, for a general $\mathcal{L} \in [p_6 + p_7] + \theta_C^* \operatorname{Pic}_F^2$, any element of $\mathbb{P}(H^0(C, \mathcal{L}))$ is a pullback along θ_C. Then in Section 5.3.3, we show that under some assumptions on the divisor x, an "unexpected" translate $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_X^5$ gives effective representatives of some line bundle in $[p_6 + p_7] + \theta_C^* \operatorname{Pic}_F^2$ which are not pullbacks, a contradiction. The final step of the proof is to remove the assumptions on x, which is done by leveraging $\mathcal{L} = O(p_1+p_2+p_3) \otimes \phi_C^* \theta^* O_{\mathbb{P}^1}(1)$, which is contained in both $[p_1+p_2+p_3] + \phi_C^* \operatorname{Pic}_E^3$ and $[p_1 + p_2 + p_3] + \theta_C^* \operatorname{Pic}_F^2$. Elements of $\mathbb{P}(H^0(C, \mathcal{L}))$ are pullbacks from E plus $p_1 + p_2 + p_3$, and as before, an "unexpected" translate $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_X^5$ gives effective representatives of $\mathcal{L}$ which are not pullbacks plus base points.

We define two functions on C. As the map ϕ_C has degree 2, there exists an automorphism $\iota \colon C \to C$ such that for all $p \in C$, $\phi_C(\iota(p)) = \phi_C(p)$. We extend ι to act on divisors as a group homomorphism. Let $\psi \colon C \to \operatorname{Sym}_C^2$ be the map $p \mapsto \theta_C^*(\theta(p)) \setminus \{p\}$, which is well-defined as $p \in \theta_C^*(\theta(p))$.

Remark 5.3.4. The map ψ has the following property. For $p, q \in C$, if $p \leq \psi(q)$, then there exists $p' \in C$ such that $\theta_C^*(\theta_C(p)) = \{p, q, p'\}$ and $\psi(p') = \{p, q\}$.

Translates of $\theta_C^* \operatorname{Pic}_F^2$ in W_C^8

We study translates of $\theta_C^* \operatorname{Pic}_F^2$ in W_C^8. Our goal is Lemma 5.3.7, which relates cosets $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^5$ to cosets in W_C^8. Let $p, q \in C$.

Lemma 5.3.5. *Let $\mathcal{L} \in [p + q] + \theta_C^* \operatorname{Pic}_F^1$ be general. Then $h^0(C, \mathcal{L}) = 1$.*

Proof. By Lemma 5.3.2, statement (3), $h^0(C, \mathcal{L}) > 1$ for general $\mathcal{L}$ if and only if $[p + q] + \theta_C^* \operatorname{Pic}_F^1 \subset W_C^1 + \phi_C^* \operatorname{Pic}_E^2$. Suppose for the sake of contradiction that $[p + q] + \theta_C^* \operatorname{Pic}_F^1 \subset W_C^1 + \phi_C^* \operatorname{Pic}_E^2$. By Lemma 5.3.2, every line bundle $M \in W_C^1 + \phi_C^* \operatorname{Pic}_E^2$ has a (necessarily unique) basepoint $p_M \in C$. Thus, we have a map $W_C^1 + \phi_C^* \operatorname{Pic}_E^2 \to C \times \phi_C^* \operatorname{Pic}_E^2$, which is injective as if two distinct M_1 and M_2 in $W_C^1 + \phi_C^* \operatorname{Pic}_E^2$ have the same basepoint $p \in C$, then $M_1(-p)$ and $M_2(-p)$ are distinct in $\phi_C^* \operatorname{Pic}_E^2$. Restricting to $[p + q] + \theta_C^* \operatorname{Pic}_F^1$ and composing with the projection to C gives a map $F \to C$, which must be constant, but the fibers of the projection are isomorphic to E, a contradiction as E and F are not isogenous. □

Lemma 5.3.6. *Let $\mathcal{L} \in [p+q] + \theta_C^* \operatorname{Pic}_F^2$ be general. If $p+q \notin \psi(C)$, then $h^0(C, \mathcal{L}) = 2$ and every effective representative for $\mathcal{L}$ is a pullback from F plus $p+q$.*

Proof. By Riemann-Roch, $h^0(C, \mathcal{L}) = 2 + h^0(\omega_C \otimes \mathcal{L}^{-1})$. As $\omega_C \in \theta_C^* \operatorname{Pic}_F^4$ by Lemma 5.3.3, the set $\omega_C - \theta_C^* \operatorname{Pic}_F^2$ equals $\theta_C^* \operatorname{Pic}_F^2$. Consider $\mathcal{M} = \omega_C \otimes \mathcal{L}^{-1} \otimes \mathcal{O}(q)$, which is a general element of $[\psi(p)] + \theta_C^* \operatorname{Pic}_F^1$. As $\omega_C \otimes \mathcal{L}^{-1} = \mathcal{M}(-q)$, $h^0(C, \omega_C \otimes \mathcal{L}^{-1}) = h^0(C, \mathcal{M}) - 1$ unless q is a basepoint of $\mathcal{M}$. By Lemma 5.3.5, $h^0(C, \mathcal{M}) = 1$, so it suffices to show that q is not a basepoint of $\mathcal{M}$, which is equivalent to $q \leq \psi(p)$. If $q \leq \psi(p)$, then by Remark 5.3.4, there exists $p' \in C$ such that $\psi(p') = p+q$, contradicting our assumption. Thus, a general $\mathcal{M}$ has a unique effective representative not supported on q. The result follows. □

Lemma 5.3.7. *Let x be a degree-5 divisor such that $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^5$. If there exist $p_1, p_2, p_3, q_1, q_2 \in C$ such that $[p_1+p_2+p_3+x] + \theta_C^* \operatorname{Pic}_F^0$ equals $[q_1+q_2] + \theta_C^* \operatorname{Pic}_F^2$ and $q_1+q_2 \notin \psi(C)$, then $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^2 + \theta_C^* \operatorname{Pic}_F^1$.*

Proof. Let y be an effective representative for a general $[y] \in [x] + \theta_C^* \operatorname{Pic}_F^0$. By Lemma 5.3.6, there exists $a \in \operatorname{Sym}_F^2$ such that $p_1+p_2+p_3+y = q_1+q_2+\theta_C^*(a)$. If y contains a pullback under θ_C, then $[y] + \theta_C^* \operatorname{Pic}_F^0 = [x] + \theta_C^* \operatorname{Pic}_F^0$ is contained in $W_C^2 + \theta_C^* \operatorname{Pic}_F^1$ as desired. Suppose for contradiction that y does not contain a pullback under θ_C. Write $a = e_1 + e_2$ and let $\theta_C^*(e_1) = p_4+p_5+p_6$, $\theta_C^*(e_2) = p_7+p_8+p_9$. As y does not contain a pullback under θ_C, without loss of generality, there are two possible cases:

$$y = q_1 + p_4 + p_5 + p_7 + p_8 \qquad \text{or} \qquad y = q_1 + q_2 + p_4 + p_5 + p_7.$$

Either $p_1+p_2+p_3 = q_2+p_6+p_8$ or $p_1+p_2+p_3 = p_6+p_8+p_9$. In either case, $\{\theta_C(p_6), \theta_C(p_8)\}$ is contained in $\{\theta_C(p_1), \theta_C(p_2), \theta_C(p_3)\}$. As $a = \theta_C(p_6) + \theta_C(p_8)$, there are finitely many choices of a, a contradiction as $[y]$ is general and a determines $[y]$. □

Translates of $\theta_C^* \operatorname{Pic}_F^0$ in W_C^5

Equipped with Lemma 5.3.7, we give sufficient conditions for a translate $[x] + \theta_C^* \operatorname{Pic}_F^0$ to be contained in $W_C^2 + \theta_C^* \operatorname{Pic}_F^1$ in terms of x.

Lemma 5.3.8. *If there exists $q \in C$ such that $\psi(q) \leq x$, then $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^2 + \theta_C^* \operatorname{Pic}_F^1$.*

Proof. Let $\psi(q) = p_1 + p_2$ and write $x = p_1 + \ldots + p_5$. First assume that $p_4 + p_5 \notin \psi(C)$. Consider the translate $[q + \psi(p_3) + x] + \theta_C^* \operatorname{Pic}_F^0$, which equals $[p_4+p_5] + \theta_C^* \operatorname{Pic}_F^2$ as the two cosets have non-empty intersection, and apply Lemma 5.3.7. Thus, we can assume there exists

$p_6 \in C$ such that $\psi(p_6) = p_4 + p_5$. Let $p_7 \in C$ be a point not contained in the support of $\psi(p_3)$. The translate $[q + p_6 + p_7 + x] + \theta_C^* \operatorname{Pic}_F^0$ equals $[p_3 + p_7] + \theta_C^* \operatorname{Pic}_F^2$, and apply Lemma 5.3.7. □

Lemma 5.3.9. *If there exists $a \in E$ such that $\phi_C^*(a) \leq x$, then $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^2 + \theta_C^* \operatorname{Pic}_F^1$.*

Proof. Let y be a degree-3 divisor such that $x = \phi_C^*(a) + y$. Consider the translate $[\iota(y) + x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^8$. Note that $[\iota(y) + x] \in \phi_C^* \operatorname{Pic}_E^4$. By the commutativity of the fiber square, $\phi_C^* \operatorname{Pic}_E^3$ intersects $\theta_C^* \operatorname{Pic}_F^2$ in W_C^6. The difference $O(\iota(y) + x) \otimes (\phi_C^* \theta^* O_{\mathbb{P}^1}(1))^{-1}$ is an element of $\phi_C^* \operatorname{Pic}_E^1$, and so has an effective representative of the form $p_6 + \iota(p_6)$ for some $p_6 \in C$. So, $O(\iota(y) + x) = (\theta_C^* \phi^* O_{\mathbb{P}^1}(1))(p_6 + \iota(p_6))$, and hence the cosets $[\iota(y) + x] + \theta_C^* \operatorname{Pic}_F^0$ and $[p_6 + \iota(p_6)] + \theta_C^* \operatorname{Pic}_F^2$ must be equal. By Lemma 5.3.2, statement (4), $\theta_C^*(\theta_C(p_6))$ is supported away from $\iota(p_6)$, so we conclude with Lemma 5.3.7. □

Lemma 5.3.10. *If there exists a degree-3 divisor $y \leq x$ such that $(\phi_C)_*(O_C(y))$ equals $\theta^* O_{\mathbb{P}^1}(1)$, then $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^2 + \theta_C^* \operatorname{Pic}_F^1$.*

Proof. Write $x = y + z$. By assumption, $\phi_C^*(\phi_C)_*(O(y))$ equals $\phi_C^* \theta^* O_{\mathbb{P}^1}(1) = \theta_C^* \phi^* O_{\mathbb{P}^1}(1) \in \theta_C^* \operatorname{Pic}_F^2$, so the cosets $[\iota(y) + x] + \theta_C^* \operatorname{Pic}_F^0$ and $[z] + \theta_C^* \operatorname{Pic}_F^2$ must be equal. If $z \notin \psi(C)$, we conclude with Lemma 5.3.7, while if $z \in \psi(C)$, we conclude with Lemma 5.3.8. □

We now prove the main tool used to prove Theorem 5.3.12. We show that any "unexpected" translate $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^5$ must obey strong conditions on the points in the support of x.

Proposition 5.3.11. *Let x be an effective degree-5 divisor such that $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^5$. For any $p, q \in C$ such that $p + q \leq x$, $\theta_C(\iota(p)) = \theta_C(q)$ or $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^2 + \theta_C^* \operatorname{Pic}_F^1$.*

Proof. Let y be a degree-3 divisor such that $x = p + q + y$. Consider the translate $[\psi(p) + \psi(q) + x] + \theta_C^* \operatorname{Pic}_F^0$. As $[\psi(p) + \psi(q) + x] + \theta_C^* \operatorname{Pic}_F^0$ has non-empty intersection with $[y] + \theta_C^* \operatorname{Pic}_F^2$, the two cosets are equal. By the commutativity of the fiber square, $[y] + \theta_C^* \operatorname{Pic}_F^2$ intersects $[y] + \phi_C^* \operatorname{Pic}_E^3$ at $\mathcal{L} = O(y) \otimes \phi_C^* \theta^* O_{\mathbb{P}^1}(1)$. By Riemann-Roch and Lemma 5.3.3, $h^0(C, \mathcal{L}) = 3 + h^0(\omega_C \otimes \mathcal{L}^{-1})$. We show that if $h^0(C, \omega_C \otimes \mathcal{L}^{-1}) > 0$, then $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^5$. If $h^0(C, \omega_C \otimes \mathcal{L}^{-1}) > 0$, then as $\omega_C \otimes \mathcal{L}^{-1} = \phi_C^* \theta^* O_{\mathbb{P}^1}(1) \otimes O(y)^{-1}$, $\phi_C^* \theta^* O_{\mathbb{P}^1}(1)$ has an effective representative supported on y. By Lemma 5.2.2, every effective representative for $\phi_C^* \theta^* O_{\mathbb{P}^1}(1)$ is a pullback from E. If y contains a pullback of ϕ_C, then Lemma 5.3.9 implies that $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^2 + \theta_C^* \operatorname{Pic}_F^1$, so we can assume that y does not contain a pullback under ϕ_C. Thus $\phi_C^*(\phi_C)_*(y)$ is the unique pullback under ϕ_C supported on y, and hence $\phi_C^* \theta^* O_{\mathbb{P}^1}(1)$ equals $\phi_C^*(\phi_C)_* O(y)$. As ϕ is ramified, the map $\phi_C^* \colon \operatorname{Pic}_E^3 \to \operatorname{Pic}_C^6$ is injective by [BL04, Proposition 11.4.3], so $\theta^* O_{\mathbb{P}^1}(1)$ equals $(\phi_C)_* O(y)$. Applying Lemma 5.3.10 shows that $[x] + \theta_C^* \operatorname{Pic}_F^0$.

Thus, we can assume that $h^0(C, \mathcal{L}) = 3$ and every effective representative of $\mathcal{L}$ is a pullback from E plus y, so that there exists an effective degree-5 divisor z with $[z] \in [x] + \theta_C^* \operatorname{Pic}_F^0$ and $a \in \operatorname{Sym}_E^3$ such that $\psi(p) + \psi(q) + z = y + \phi_C^*(a)$. (Note that this is *equality* of divisors, not linear equivalence.) If the support of y intersects the support of $\psi(p)$ or $\psi(q)$, then by Remark 5.3.4 and Lemma 5.3.8, $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^2 + \theta_C^* \operatorname{Pic}_F^1$. Thus, there exist $p', q' \in C$ such that $z = y + p' + q'$ and $\psi(p) + \psi(q) + p' + q' = \phi_C^*(a)$. Consider a point r in the support of $\psi(p) + \psi(q)$, and note that by Lemma 5.3.1, statement (4), for all $c \in C$, the degree of $\phi_C^*(\phi_C(r)) \cap \psi(c)$ is at most one. Thus, there must exist some r in the support of $\psi(p) + \psi(q)$ such that $\phi_C^*(\phi_C(r)) \leq \psi(p) + \psi(q)$. Then, $\phi_C^*(\theta^*(\theta(\phi_C(r))))$ must intersect both $\psi(p)$ and $\psi(q)$, and by the commutativity of the fiber square, must equal $p + \psi(p) + q + \psi(q)$. By Lemma 5.3.1, statement (4), $\iota(p) \notin \psi(p)$ and hence $\iota(p) \leq q + \psi(q)$, so that $\theta_C(\iota(p)) = \theta_C(q)$ as desired. □

Theorem 5.3.12. *Let E/k and F/k be non-isogenous elliptic curves, let $\phi\colon F \to \mathbb{P}^1$ be a degree-2 map, let $\theta\colon E \to \mathbb{P}^1$ be a degree-3 map, let C/k be the curve $E \times_{\mathbb{P}^1} F$, and let $\phi_C : C \to E$ and $\theta_C : C \to F$ denote the base changes of ϕ and θ. If C is smooth and $J(C) \sim E \times F \times A$, where $A/\bar{k}$ is simple, then*

$$Z_C^5 = \left(\phi_C^* \operatorname{Pic}_E^1 + W_C^3\right) \bigcup \left(\theta_C^* \operatorname{Pic}_F^1 + W_C^2\right).$$

Proof. By assumption, $\operatorname{Pic}_C^0 \simeq E \times F \times A$, where A is simple of dimension 5. By [DF93, Proposition 3.3], any simple, positive-dimensional abelian translate contained in W_C^5 must be a translate of $\phi_C^* \operatorname{Pic}_E^0$ or $\theta_C^* \operatorname{Pic}_F^0$. As $\phi_C^* \operatorname{Pic}_E^1$ and $\theta_C^* \operatorname{Pic}_F^1$ are effective, $W_C^3 + \phi_C^* \operatorname{Pic}_E^1$ and $W_C^2 + \theta_C^* \operatorname{Pic}_F^1$ are contained in W_C^5. By Proposition 5.2.1, every translate of $\phi_C^* \operatorname{Pic}_E^0$ contained in W_C^5 is a translate of $\phi_C^* \operatorname{Pic}_E^1$. We show that any translate of $\theta_C^* \operatorname{Pic}_F^0$ contained in W_C^5 is a translate of $\theta_C^* \operatorname{Pic}_F^1$.

Let $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^5$ and suppose for contradiction that $[x] + \theta_C^* \operatorname{Pic}_F^0$ is not contained in $W_C^2 + \theta_C^* \operatorname{Pic}_F^1$. Let z be an effective degree-5 divisor such that $[z] \in [x] + \theta_C^* \operatorname{Pic}_F^0$, and let $p, q \in C$ be points such that $p + q \leq z$. By Proposition 5.3.11, for any $q' \in C$ such that $p + q' \leq z$, $\theta_C(\iota(p)) = \theta_C(q')$, and hence $(\theta_C)_*(z) = \theta_C(p) + 4\theta_C(\iota(p))$. If $q' \leq \psi(q)$, then by Remark 5.3.4 and Lemma 5.3.8, $[x] + \theta_C^* \operatorname{Pic}_F^0 \subset W_C^2 + \theta_C^* \operatorname{Pic}_F^1$, and hence $q' = q$ and $z = p + 4q$. Switching the roles of p and q, an identical argument shows that $y = 4p + q$, and so $y = 5p$. Thus $[x] + \theta_C^* \operatorname{Pic}_F^0$ is contained in the image of the diagonal map $\Delta : C \to \operatorname{Sym}_C^5 \to W_C^5$, a contradiction as by Lemma 5.3.2, statement (3), Δ is birational onto its image. □

We now prove Corollary 1.0.5. By Corollary 5.1.20 and Theorem 5.3.12, it essentially suffices to exhibit a curve $C/\mathbb{Q}$ such that $C/\overline{\mathbb{Q}}$ satisfies the conditions of Theorem 5.3.12.

Proof of Corollary 1.0.5. Let $E/\mathbb{Q}$ be the positive rank elliptic curve $Y^2Z = X^3+2XZ^2+Z^3$ and $F/\mathbb{Q}$ be the non-isogenous positive rank elliptic curve $U^2W = V^3 + VW^2 + W^3$. Let $\theta : E \to \mathbb{P}^1$ be the map induced by the rational map $[X : Y : Z] \mapsto [Y : Z]$, let $\phi : F \to \mathbb{P}^1$ be the map induced by the rational map $[V : U : W] \mapsto [W : V]$, and let $C = E \times_{\mathbb{P}^1} F$. The branch divisor B_θ is supported on $[1 : 0]$ and is supported on $t \in \mathbb{A}^1(\overline{\mathbb{Q}})$ if and only $x^3 + 2x + 1 - t^2$ shares a root with its derivative $3x^2 + 2$. Similarly, B_ϕ is supported on $\{1/\alpha_1, 1/\alpha_2, 1/\alpha_3, 0\}$, where the α_i are the roots of $v^3 + v + 1$. As B_ϕ and B_θ have disjoint support, C is a smooth genus-7 curve by Lemma 5.3.1.

We show that $J(C) \sim E \times F \times A$, where A is absolutely simple. For $p \in \{5, 7\}$, we compute in Magma [BCP97] to find that the genus of $C/\mathbb{F}_p$ is 7, so by [Liu02, Corollary 10.1.25], both are primes of good reduction. For both primes, the numerator of the Zeta function of $C/\mathbb{F}_p$ factors into two quadratics and an irreducible factor $Q_p(t)$ of degree 10, which is the characteristic polynomial of some abelian variety $A/\mathbb{Q}$ of dimension 5. We verify in Magma that $Q_5(t)$ and $Q_7(t)$ satisfy the conditions of [Sto08, Lemma 3] (which can be applied to any abelian variety with $g = \dim A$, not just Jacobians), so that $\mathrm{End}_{\overline{\mathbb{Q}}} A \otimes \mathbb{Q}$ embeds into the number field generated by $Q_p(t)$. The number fields generated by Q_5 and Q_7, however, are linearly disjoint, so $\mathrm{End}_{\overline{\mathbb{Q}}} A$ must be $\mathbb{Z}$, which implies that $A/\overline{\mathbb{Q}}$ is simple.

By Corollary 5.1.20, $\wp(C/\mathbb{Q}) = \wp_{ind}(C/\mathbb{Q}) \cup \wp_{\mathbb{P}^1}(C/\mathbb{Q})$. Riemman-Roch implies that $\mathbb{N}_{>7} \subset \wp_{\mathbb{P}^1}(C/\mathbb{Q})$, and as $C/\mathbb{Q}$ admits a degree-2 map $C \to E$ to a positive rank elliptic curve, $\{4, 6\} \subset \wp_{\mathbb{P}^1}(C/\mathbb{Q})$. From Lemma 5.3.2, $2, 3, 5 \notin \wp_{\mathbb{P}^1}(C/k)$. By Remark 5.1.2, $7 \in \wp_{ind}(C/\mathbb{Q})$ We now determine, for $d \in \{2, 3, 5\}$, if $d \in \wp_{ind}(C/k)$. By Theorem 5.3.12, every abelian translate in W^5_C is geometrically decomposible, while both $\phi^*_C \mathrm{Pic}^1_E$ and $\theta^*_C \mathrm{Pic}^1_F$ are geometrically indecomposable. Thus, $\wp(C/\mathbb{Q}) = \mathbb{N} \setminus \{1, 5\}$. □

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