Twistor Spaces for Supersingular K3 Surfaces

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A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2018

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Program Authorized to Offer Degree:
Mathematics

University of Washington

Abstract

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We develop a theory of twistor spaces for supersingular K3 surfaces, extending Artin's analogy between supersingular K3 surfaces and complex analytic K3 surfaces. Our twistor spaces are families of twisted supersingular K3 surfaces over the affine line, and are obtained as relative moduli spaces of twisted sheaves on universal gerbes associated to supersingular K3 surfaces. To study these families, we develop a theory of crystals for twisted supersingular K3 surfaces, and study the resulting period morphism from the moduli space of twisted supersingular K3 surfaces to the space of crystals.

As applications of this theory, we give a new proof of the Ogus's crystalline Torelli theorem, inspired by Verbitsky's proof in the complex analytic setting. We also obtain a new proof of the result of Rudakov-Shafarevich that supersingular K3 surfaces have potentially good reduction. Finally, we apply our twistor spaces to study elliptic fibrations. Using results of Max Lieblich, we show that every elliptic fibration on a supersingular K3 surface admits a purely inseparable multisection. As a consequence of this result, we give a new proof of the unirationality of supersingular K3 surfaces. Our techniques work uniformly in odd characteristic, and in particular we are able to extend all of these results to characteristic 3, where they were not previously known.

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ACKNOWLEDGMENTS

I would like to thank Max Lieblich for his constant support, and for sharing his ideas, his energy, and his enthusiasm with me.

During the development of this thesis I had helpful conversations with Bhargav Bhatt, Aise Johan de Jong, Sándor Kovács, David Benjamin Lim, Keerthi Madapusi Pera, and Martin Olsson. I thank Josh Swanson for the proof of Lemma 3.0.9.

I am grateful to my fellow students at the University of Washington for making the math department a welcoming and friendly place. I enjoyed learning algebraic geometry alongside Kristin DeVleming, Gabriel Dorfsman-Hopkins, Charlie Godfrey, Siddharth Mathur, Travis Scholl, and Lucas Van Meter.

I would not be where I am today without my family. I am endlessly grateful to Dominique for her support, and for sharing this experience with me.

Finally, I would like to thank Lucky, Ted, Twinky, Toph, Lucy, and Lilith.

DEDICATION

To Lilith

Chapter 1

INTRODUCTION

A K3 surface over an algebraically closed field is said to be *supersingular* if the rank of its Picard group is 22. This is the maximal value possible among K3 surfaces, and only occurs in positive characteristic. It turns out that this condition forces a variety of interesting behaviors that are radically different from those of algebraic K3 surfaces in characteristic 0. In some ways, supersingular K3 surfaces are the most difficult class of K3 surfaces in positive characteristic to study, as they have the weakest connection to characteristic zero phenomenon. In particular, results that classically follow from Hodge theory, such as the vanishing of $\mathrm{H}^0(X, T^1_X)$ and the Tate conjecture, are most difficult to prove in the supersingular case. In return, however, supersingular K3 surfaces have a much tighter connection with special positive characteristic structures related to the Frobenius morphism.

The systematic study of supersingular K3 surfaces was initiated by Artin [1], who described a surprising series of analogies between supersingular K3 surfaces over a fixed algebraically closed field of positive characteristic and analytic K3 surfaces over the complex numbers. In this thesis, we continue this philosophy by developing a positive characteristic analog of *twistor families*. Our construction begins with a result of Artin: if X is a supersingular K3 surface, then the flat cohomology group $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$ can be naturally viewed as the k-points of a group scheme whose connected component is isomorphic to $\mathbf{A}^1$. This results in a *universal twistor family* $\mathscr{X} \to \mathbf{A}^1$ of twisted K3 surfaces (see Section 2.1), whose central fiber is the trivial $\boldsymbol{\mu}_p$-gerbe over X. Although the underlying K3 surface of each fiber is constant, the gerbe structure is varying. This situation is only possible in positive characteristic: if k has characteristic zero, then the groups $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ are all finite, and so there do not exist such continuously varying families of $\boldsymbol{\mu}_n$-gerbes. By taking relative moduli spaces of twisted sheaves on $\mathscr{X} \to \mathbf{A}^1$ with suitably chosen topological invariants, we can produce new families $\mathscr{X}(v) \to \mathbf{A}^1$ of twisted supersingular K3 surfaces. We call such families *(supersingular) twistor families.* (Note that the original universal family of gerbes

$\mathscr{X} \to \mathbf{A}^1$ is the stack of $\mathscr{X}$-twisted sheaves of length 1, so it is itself a family of moduli spaces of $\mathscr{X}$-twisted sheaves.) While the universal twistor family has constant coarse moduli space, the coarse moduli family of a general twistor family will typically be non-isotrivial.

This construction (in the special case when the Mukai vector v has rank 0) was first described by Lieblich in [31]. Moreover, certain special cases of these twistor families have been studied before by various authors over the years, and their connection to the Brauer group was noticed already by Artin (see the footnote on page 552 of [1]). The novelty in our approach lies in realizing that they are most naturally viewed as families of $\boldsymbol{\mu}_p$-gerbes over supersingular K3 surfaces (that is, as families of twisted K3 surfaces). To develop this viewpoint, we take as our basic object of study a $\boldsymbol{\mu}_p$-gerbe $\mathscr{X} \to X$ over a supersingular K3 surface, and seek to extend the material of [40, 41] to the twisted setting. To define the crystalline periods of a twisted K3 surface, we take our inspiration from the Hodge theoretic constructions in [23]. A key technical tool is a certain crystalline analog of a B-field. This allows us to twist the total crystalline cohomology of X by a factor depending on a given Brauer class. The resulting object is analogous to the twisted Hodge structures of [23]. We remark that, while these twisted constructions have found many applications over the complex numbers, the study of twisted K3 surfaces is particularly interesting in the supersingular case, where there exist positive dimensional families of Brauer classes. We then introduce an appropriate moduli space of twisted supersingular K3 surface, and study the corresponding period morphism.

By working throughout with twisted K3 surfaces, we are able to treat the universal twistor families $\mathscr{X} \to \mathbf{A}^1$ on the same footing as any other family of supersingular K3 surfaces. This allows us to give a uniform construction of twistor families, and to obtain precise control over their periods.

The theory we develop is sufficiently strong to enable the translation of certain twistor-theoretic proofs over the complex numbers to the supersingular setting. Section 5.1 is devoted to a supersingular form of Verbitsky's proof of the Torelli theorem: the existence of the twistor lines gives a relatively easy proof that the twisted period morphism is an isomorphism (after adding ample cones). This is a generalization of Ogus's crystalline Torelli theorem. In particular, we obtain a new proof of the main result of [41]. As a corollary, we also obtain a new proof of a result of Rudakov and Shafarevich on degenerations of supersingular K3 surfaces (Theorem 3 of

[44]). These new proofs work uniformly in all odd characteristics, and so we are able to extend both of these results to characteristic $p = 3$, where they were not previously known. In section 5.2, we derive some geometric consequences for twistor families arising from an elliptic fibration. Building on ideas of Liedtke [33], we show that every elliptic fibration on a supersingular K3 surface admits a purely inseparable multisection. We apply this to construct purely inseparable isogenies between K3 surfaces connected by Artin-Tate twistor lines, and show inductively that all supersingular K3 surfaces are unirational in any positive characteristic, verifying a conjecture of Artin [1]. Among other things, this fills in the cases that were not handled in [33].

It is our hope that similar constructions will be useful in the study of other classes of varieties in positive characteristic. In particular, it seems reasonable that some the constructions in this paper might extend to supersingular holomorphic symplectic varieties.

1.1 Supersingular K3 surfaces

In this section we give a brief explanation of certain invariants of K3 surfaces in positive characteristic. We discuss different notions of supersingularity, and the relationships between them. A nice survey of these topics, including examples and extensive references to the literature, is contained in chapter 18 of [21].

Let X be a smooth projective variety over an algebraically closed field k. It is known that the Néron-Severi group $N(X) = \operatorname{Pic}(X)/\operatorname{Pic}^0(X)$ is a finitely generated abelian group, and we set $\rho(X) = \operatorname{rank} N(X)$. If l is a prime not equal to the characteristic of k, then the l-adic étale Chern class map

$$c_1\colon \operatorname{Pic}(X) \to \mathrm{H}^2(X, \mathbf{Z}_l(1))$$

descends to an injection $N(X) \hookrightarrow \mathrm{H}^2(X, \mathbf{Z}_l(1))$, and so we find the bound $\rho(X) \leq b_2$, where $b_2 \stackrel{\text{def}}{=} \operatorname{rank}_{\mathbf{Z}_l} \mathrm{H}^2(X, \mathbf{Z}_l(1))$ is the second Betti number of X. We may also consider the Hodge Chern class map

$$c_1\colon \operatorname{Pic}(X) \to \mathrm{H}^1(X, \Omega^1_X)$$

which descends to a map $N(X) \to \mathrm{H}^1(X, \Omega^1_X)$. If k has characteristic 0, then this map is known to be injective, and so we obtain the stronger bound $\rho(X) \leq h^{1,1} \stackrel{\text{def}}{=} \dim_k \mathrm{H}^1(X, \Omega^1_X)$. If the characteristic of k is positive, however, then this map is rarely injective, as $\mathrm{H}^1(X, \Omega^1_X)$ is torsion.

We now suppose that X is a K3 surface. As $\mathrm{Pic}^0(X) = 0$, we have $\mathrm{Pic}(X) = N(X)$, and we will use the two interchangeably. If k has characteristic 0, then the Hodge Chern class map shows that $\rho(X) \leq h^{1,1} = 20$. In positive characteristic, only the weaker bound $\rho(X) \leq 22$ is true in general. Those K3 surfaces satisfying $\rho(X) = 22$ are known as ***Shioda supersingular***. The following example is well known, and may be found for instance in [21].

Example 1.1.1. If A is a supersingular abelian surface, then the Kummer surface $\mathrm{Km}(A)$ associated to A is Shioda supersingular.

It was noticed by Artin [1] that the discriminant of the quadratic form on the Néron-Severi group $N(X)$ of a Shioda supersingular K3 surface is of the form $-p^{2\sigma_0}$ for some integer $1 \leq \sigma_0 \leq 10$. The integer σ_0 is now known as the ***Artin invariant*** of X.

For the remainder of this section, we will fix an algebraically closed field k of positive characteristic. Another notion of supersingularity was studied by Artin in his seminal paper [1]. We recall the following definition.

Definition 1.1.2. Suppose that E is a sheaf of abelian groups on the big fppf site of $\mathrm{Spec}\, k$. We define a functor on the category of local Artinian k-algebras with residue field k by

$$\widehat{E}(A) = \ker\left(E(\mathrm{Spec}\, A) \to E(\mathrm{Spec}\, k)\right)$$

which we call the *completion of E at the identity section.*

In particular, we define the ***formal Brauer group*** $\widehat{\mathrm{Br}(X)}$ of X to be the completion at the identity section of the functor $S \mapsto \mathrm{H}^2(X_S, \mathbf{G}_m)$. This is an example of a formal group of dimension 1. Such objects are classified by a discrete invariant h known as the ***height***, which assumes the values $h = 1, 2, 3, \ldots$ or $h = \infty$. It is known that, for X a K3 surface, $h(\widehat{\mathrm{Br}(X)})$ takes the values $h = 1, \ldots, 10$ or $h = \infty$ (see chapter 18 of [21] for details). We say that X is ***Artin supersingular*** if $h(\widehat{\mathrm{Br}(X)}) = \infty$. It is known that this condition is equivalent to $\widehat{\mathrm{Br}(X)} \cong \mathbf{G}_a$.

A third definition of supersingularity arises from crystalline cohomology. Let $W = W(k)$ be the ring of Witt vectors of k, let $F_W\colon W \to W$ be the canonical lift of the Frobenius of k, and let $K = W[\frac{1}{p}]$ be the field of fractions of W.

Definition 1.1.3. An *F-crystal* is a pair (H, Φ), where H is a free W-module of finite rank, and $\Phi\colon H \to H$ is an F_W-linear endomorphism.[1]

A morphism of F-crystals is a map of W-modules commuting with the respective endomorphisms. A morphism $(H, \Phi) \to (H', \Phi')$ of F-crystals is an *isogeny* if the map $H \otimes_W K \to H' \otimes_W K$ is an isomorphism of K-vector spaces.

There is a well known classification of F-crystals up to isogeny due to Dieudonné (see chapter 18 of [21], and the included references). If X is a K3 surface, then its second crystalline cohomology group $\mathrm{H}^2(X/W)$ is a free W-module of rank 22, which is equipped with a F_W-semilinear endomorphism

$$\Phi\colon \mathrm{H}^2(X/W) \to \mathrm{H}^2(X/W)$$

called the Frobenius. The pair $(\mathrm{H}^2(X/W), \Phi)$ is an F-crystal, and its isogeny type gives an invariant of X. We refer to chapter 18 of [21] for a discussion of this invariant Consider the crystal $M = W^{\oplus 22}$, with endomorphism given by pF, where $F\colon W^{\oplus 22} \to W^{\oplus 22}$ is the map induced by the Frobenius on W. We say that a K3 surface X is *Ogus supersingular* if the crystal $(\mathrm{H}^2(X/W), \Phi)$ is isogenous to the crystal (M, pF). In terms of Dieudonné's classification, this condition says that the slopes of the crystal $\mathrm{H}^2(X/W)$ are all equal to 1.

Theorem 1.1.4. *Let X be a K3 surface over an algebraically closed field k of characteristic $p > 0$. The following are equivalent.*

1. $\rho(X) = 22$ *(X is Shioda supersingular),*
2. $h(\widehat{\mathrm{Br}(X)}) = \infty$ *(X is Artin supersingular), and*
3. *the slopes of the crystal* $\mathrm{H}^2(X/W)$ *are all equal to 1 (X is Ogus supersingular).*

The implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are relatively straightforward, as is $2 \Leftrightarrow 3$ (see chapter 18 of [21] and the included references). That 2 and 3 imply 1 is very difficult, and is known to be equivalent to the statement that the Tate conjecture holds for Artin supersingular K3 surfaces

[1] That is, Φ is a map of W modules satisfying $\Phi(\lambda x) = F_W(\lambda)\Phi(x)$ for all $\lambda \in W$ and $x \in H$.

over finite fields. This is now known in all characteristics by recent work of various authors [11, 12, 29, 34, 35].

In light of this theorem, we will not distinguish between the various notions of supersingularity, and refer to a K3 surface satisfying any of the equivalent conditions of Theorem 1.1.4 as supersingular.

We end this section by mentioning a few important results in the theory of supersingular K3 surfaces. Of course, the literature on this topic is very large, and so we necessarily omit a great many interesting topics. In his groundbreaking papers [40, 41], Ogus stated and proved a Torelli theorem for supersingular K3 surfaces using crystalline cohomology (building on work of Rudakov and Shafarevich). In its weaker form, it asserts the following.

Theorem 1.1.5 (Ogus [40, 41]). *If X and Y be supersingular K3 surfaces over an algebraically closed field of characteristic $p \geq 5$, then X isomorphic to Y if and only if there is an isomorphism of F-crystals $\mathrm{H}^2(X/W) \cong \mathrm{H}^2(Y/W)$ that preserves the respective bilinear forms.*

This result is quite remarkable, as nothing comparable holds for general K3 surfaces in positive characteristic. As an application of our methods, we will give an alternative proof of this theorem. We mention that our method works in characteristic $p \geq 3$, while Ogus's proof was restricted to $p \geq 5$ (see section 5.1 for further discussion).

Finally, we mention a fourth characterization of supersingularity among K3 surfaces. Recall that a variety X is said to be *unirational* if there exists a dominant rational map $\mathbf{P}^n \dashrightarrow X$ from a projective space.

Theorem 1.1.6. *A K3 surface over an algebraically closed field k of characteristic $p > 0$ is Shioda supersingular if and only if it is unirational.*

Thus, in light of Theorem 1.1.4, a K3 surface is unirational if and only if it is supersingular, in any of the equivalent senses. Let us briefly discuss the history of this result, and our contribution to it. By a theorem of Shioda, a unirational surface must have Picard number equal to its second Betti number. Therefore, a unirational K3 surface has $\rho(x) = 22$, and therefore is Shioda supersingular. The converse was conjectured by various authors, and many special cases were proven over the years. In particular, Rudakov and Shafarevich showed in [43] that any Shioda supersingular K3 surface X is unirational if $p = 2$, or if $p = 3$ and $\sigma_0(X) \leq 6$. The problem was then essentially settled by Liedtke [33], who showed that every supersingular K3 surface in characteristic $p \geq 5$ is unirational. After this work, the only cases left open were $p = 3$ and $7 \leq \sigma_0(X) \leq 10$. As an application of our methods, we present in section 5.2 a variant of Liedtke's method that works in this range, thus completing the proof of this result. We refer to Section 5.2 and the discussion therein for more details.

1.2 Complex analytic vs supersingular

Let us make explicit some of the analogies between supersingular and complex analytic K3 surfaces. Recall that in the complex setting, a twistor family associated to a K3 surface X is given by fixing a Kähler class $\alpha \in \mathrm{H}^2(X, \mathbf{R})$ and varying the complex structure of X using a sphere spanned by the hyper-Kähler structure $I, J, K \in \mathrm{H}^2(X, \mathbf{R})$ associated to α. There results a non-algebraic family $X(\alpha) \to \mathbf{P}^1$ of analytic K3 surfaces. If we take the 3-plane W in $\mathrm{H}^2(X, \mathbf{R})$ spanned by I, J, K, we can describe the periods of the line $X(\alpha)$ by intersecting the complexification $W_{\mathbf{C}}$ with the period domain D, viewed as a subvariety of $\mathbf{P}(\mathrm{H}^2(X, \mathbf{C}))$ (see, for example, Section 6.1 and Section 7.3 of [21]).

In place of the Kähler class in $\mathrm{H}^2(X, \mathbf{R})$, our supersingular twistor spaces are determined by a choice of isotropic vector v in the quadratic $\mathbf{F}_p$-space $pNS(X)^*/pNS(X)$. In place of the sphere of complex structures in $\mathrm{H}^2(X, \mathbf{R})$, we use continuous families of classes in the cohomology group $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$. In place of a fixed differentiable surface carrying a varying complex structure, we end up with a fixed algebraic surface carrying a varying Brauer class (which is an algebraic avatar of transcendental structure). The group $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$ may be naturally viewed as a subvariety of $\mathrm{H}^2_{dR}(X/k)$, and just as in the classical setting, this family of cohomology classes traces out a line in an appropriate period domain classifying supersingular K3 crystals.

We also compare discrete invariants, and the behavior of their corresponding strata in the moduli space. The moduli space of supersingular K3 surfaces is stratified by the *Artin invariant*, which takes integral values $1 \leq \sigma_0 \leq 10$. The dimension of the locus of Artin invariant at most σ is $\sigma - 1$. In particular, the moduli space of supersingular K3 surfaces has dimension 9. In this work, we extend the Artin invariant to a twisted supersingular K3 surface, and find that σ_0 now goes up to 11. Accordingly, the moduli space of twisted supersingular K3 surfaces has dimension

10, and the moduli space of non-twisted surfaces is naturally embedded in it as a divisor. This bears some resemblance to the situation over the complex numbers, where loci of algebraic K3 surfaces are embedded in the space of analytic K3 surfaces as divisors.

Here is a rough dictionary translating between the classical analytic situation and the supersingular one, extending Artin's Table 4.12 [1]. For a K3 surface over the complex numbers, we let ρ be the Picard number and ρ_0 be the rank of the transcendental lattice. For a supersingular K3 surface with Artin invariant σ_0 we will write $\sigma = 11 - \sigma_0$.

Complex	**Supersingular**
Complex structure, class in $\mathrm{H}^2(X, \mathbf{R})$	Gerbe, class in $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$
Twistor line $\mathbf{P}^1 \subset \mathbf{P}(\mathrm{H}^2_{dR}(X/\mathbb{C}))$	Supersingular twistor line $\mathbf{A}^1 \subset \mathrm{H}^2_{dR}(X/k)$
Algebraic moduli space	Moduli space of supersingular K3 surfaces
Analytic moduli space	Moduli space of twisted supersingular K3 surfaces
Period domain	Space of supersingular K3 crystals
$\rho_0 \geq 2$	$\sigma_0 \geq 1$
$\rho + \rho_0 = 22$	$\sigma + \sigma_0 = 11$
Generic analytic K3 has $\rho = 0$	Generic twisted supersingular K3 has $\sigma = 0$

Particularly interesting are the aspect of this dictionary where the analogy breaks down. Notably, while a classical twistor line gives a $\mathbf{P}^1$ in the analytic moduli space, our construction only yields an $\mathbf{A}^1$ in the moduli space of twisted supersingular K3 surfaces. We know in retrospect from the crystalline Torelli theorem that the underlying family of surfaces of a twistor family extends to a family over $\mathbf{P}^1$. However, the theory developed in this paper only yields a direct description of such a family over the open locus $\mathbf{A}^1 \subset \mathbf{P}^1$. It would be very interesting to find a natural interpretation of the fiber over infinity. There are also aspects of the classical twistor space theory for which we have yet to find analogs on the supersingular side, such as the theory of hyperholomorphic sheaves.

1.3 A brief outline

Chapter 2 contains some results on various cohomology groups that will be needed later in the work. Section 2.1 is a collection of facts on the Brauer group, gerbes, and flat cohomology of $\boldsymbol{\mu}_p$, all of which are more or less well known. In section 2.2 we define the relative étale site. This is a convenient technical tool that will be useful for certain calculations later in the paper. Finally, in section 2.3, we formulate some properties of de-Rham cohomology in the context of the relative étale site.

In chapter 3, we define and study the periods of twisted supersingular K3 surfaces, building on work of Ogus [40, 41]. In section 3.0.1 we recall Ogus's moduli space of characteristic subspaces, which is a crystalline analog of the classical period domain, and study the relationship between the spaces classifying K3 crystals of different Artin invariants. Twistor lines in this period domain are defined in section 3.0.2. In section 3.1 we relate the flat cohomology of $\boldsymbol{\mu}_p$ to de Rham cohomology. In section 3.2 we develop a crystalline analog of the classical B-fields associated to a Brauer class, which we use to construct a K3 crystal associated to a twisted supersingular K3 surface. In section 3.3 we study the moduli space of marked twisted supersingular K3 surfaces. Using the twisted K3 crystals of the previous section, we define a period morphism on this space, and compute its differential.

In chapter 4 we construct supersingular twistor spaces. We begin in section 4.1 by discussing moduli spaces of twisted sheaves. We define an appropriate twisted Chern character, and extend various results which are well known in characteristic 0 to our setting. We then use moduli spaces of twisted sheaves to construct our twistor families. These naturally fall into two types, which we describe separately. The main result of section 4.2 is a construction of twistor families of positive rank in the moduli space of twisted supersingular K3 surfaces. In section 4.3, we construct Artin-Tate twistor families, which is somewhat more involved than the positive rank case. We show how in the presence of an elliptic fibration this construction yields a geometric form of the Artin-Tate isomophism (Proposition 4.3.8).

Chapter 5 contains some applications of our theory. Section 5.1 is devoted to a supersingular form of Verbitsky's proof of the Torelli theorem: the existence of the twistor lines gives a relatively easy proof that the twisted period morphism is an isomorphism (after adding ample cones), which is a twisted version of Ogus's crystalline Torelli theorem. This implies immediately that Ogus's period morphism is an isomorphism, giving a new proof of the main result of [41]. As a corollary, we obtain a new proof of a result of Rudakov and Shafarevich on degenerations of supersingular K3 surfaces (Theorem 3 of [44]). These new proofs work uniformly in all odd characteristics, and so we are able to extend both of these results to characteristic $p = 3$, where they were not previously known. In section 5.2 we apply the material of section 4.3 to show that every elliptic fibration on a supersingular K3 surface admits a purely inseparable multisection. We use this result to construct purely inseparable isogenies between K3 surfaces connected by Artin-Tate twistor lines (a construction inspired by the ideas of [33]). Combining this with results of Rudakov and Shafarevich on quasi-elliptic fibrations, we show that all supersingular K3 surfaces are unirational in any positive characteristic, verifying a conjecture of Artin [1]. Among other things, this fills in the cases that were not handled in [33].

1.4 Conventions

We fix throughout a prime number p. In certain places we will assume that p is odd. The main reason for this restriction will be our use of bilinear forms. We think it is likely that our techniques should yield similar results in characteristic $p = 2$ as well, with suitable modifications to certain definitions, but we will not pursue this here.

For general background on K3 surfaces, we will refer to the wonderful book on the topic by Huybrechts [21].

Chapter 2

COHOMOLOGY

2.1 The Brauer group

Let X be a scheme. Fix a prime number p, and consider the group scheme $\boldsymbol{\mu}_p$ over X. This group is related to the sheaf of units $\mathscr{O}_X^\times = \mathbf{G}_{m,X}$ by the Kummer sequence

$$1 \to \boldsymbol{\mu}_p \to \mathscr{O}_X^\times \xrightarrow{x \mapsto x^p} \mathscr{O}_X^\times \to 1$$

which is exact in the flat topology (if X has characteristic nowhere equal to p, then it is also exact in the étale topology). There is an induced long exact sequence

$$\ldots \to \mathrm{H}^1(X, \mathscr{O}_X^\times) \xrightarrow{\cdot p} \mathrm{H}^1(X, \mathscr{O}_X^\times) \xrightarrow{\delta} \mathrm{H}^2(X_{\mathrm{fl}}, \boldsymbol{\mu}_p) \to \mathrm{H}^2(X, \mathscr{O}_X^\times) \xrightarrow{\cdot p} \mathrm{H}^2(X, \mathscr{O}_X^\times) \to \ldots$$

There are a variety of geometric interpretations of the elements of the cohomology groups $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$ and $\mathrm{Br}(X) = \mathrm{H}^2(X, \mathscr{O}_X^\times)$. In this work, we will use the language of gerbes.

Definition 2.1.1. A *twisted K3 surface* is a $\boldsymbol{\mu}_n$ or $\mathscr{O}_X^\times$ gerbe over a K3 surface.

By general results of Giraud [16], if A is any sheaf of abelian groups on X, then there is a canonical bijection between $\mathrm{H}^2(X, A)$ and isomorphism classes of A-gerbes on X. We will denote the cohomology class of an A-gerbe $\mathscr{X} \to X$ by $[\mathscr{X}] \in \mathrm{H}^2(X, A)$[1] We begin this section by collecting some results related to gerbes banded by the sheaves $\boldsymbol{\mu}_p$ and $\mathscr{O}_X^\times$. We refer to Section 2 of [30] for background, and for the definition of twisted and n-twisted sheaves. If $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$ or $\mathscr{O}_X^\times$-gerbe and n is an integer, we will write $\mathbf{Qcoh}^{(1)}(\mathscr{X})$ for the category of n-twisted

[1] The reader unfamiliar with this language will not lose much by replacing every instance of a twisted K3 surface with the associated pair (X, α), where $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_n)$ or $\mathrm{H}^2(X, \mathscr{O}_X^\times)$. In fact, in almost all of this work our constructions relating to a twisted K3 surface will depend canonically only on the associated cohomology class. The exception is in chapter 4, where we will need to use the category of twisted sheaves on a twisted K3 surface. This category is not determined canonically by the pair (X, α). Choosing a gerbe corresponding to α gives one way of realizing this category. It may also be approached by picking a cocycle representing α, or by choosing an associated Azumaya algebra or Brauer-Severi variety. We refer to Chapter 1 of [10] for a discussion of the relation between these approaches.

quasicoherent sheaves on $\mathscr{X}$, and $\mathbf{Coh}^{(1)}(\mathscr{X})$ for the category of n-twisted coherent sheaves on $\mathscr{X}$.

Definition 2.1.2. If $\mathscr{L} \in \mathrm{Pic}(X)$ is a line bundle, then the *gerbe of p-th roots* of $\mathscr{L}$ is the stack $\{\mathscr{L}^{1/p}\}$ over X whose objects over an X-scheme $T \to X$ are line bundles $\mathscr{M}$ on T equipped with an isomorphism $\mathscr{M}^{\otimes p} \xrightarrow{\sim} \mathscr{L}_T$.

The stack $\{\mathscr{L}^{1/p}\}$ is canonically banded by $\boldsymbol{\mu}_p$, and $\{\mathscr{L}^{1/p}\} \to X$ is a $\boldsymbol{\mu}_p$-gerbe. Its cohomology class is the image of $\mathscr{L}$ under the boundary map δ.

Lemma 2.1.3. *Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe with cohomology class $\alpha \in \mathrm{H}^2(X_{\mathrm{fl}}, \boldsymbol{\mu}_p)$. The following are equivalent.*

1. *There exists an invertible twisted sheaf on $\mathscr{X}$,*
2. *there is an isomorphism $\mathscr{X} \cong \{\mathscr{L}^{1/p}\}$ of $\boldsymbol{\mu}_p$-gerbes for some line bundle $\mathscr{L}$ on X,*
3. *the cohomology class α is in the image of the boundary map δ, and*
4. *the associated $\mathscr{O}_X^\times$-gerbe is trivial.*

Proof. We will show $(1) \Leftrightarrow (2)$. Suppose that there exists an invertible twisted sheaf $\mathscr{M}$ on $\mathscr{X}$. The $\boldsymbol{\mu}_p$-action on $\mathscr{M}^{\otimes p}$ is trivial, so the adjunction map $p^*p_*\mathscr{M}^{\otimes p} \xrightarrow{\sim} \mathscr{M}^{\otimes p}$ is an isomorphism. We obtain by descent an isomorphism $\mathscr{X} \cong \{(p_*\mathscr{M}^{\otimes p})^{1/p}\}$ of $\boldsymbol{\mu}_p$-gerbes. Conversely, on $\{\mathscr{L}^{1/p}\}$ there is a universal invertible twisted sheaf $\mathscr{M}$. The image of a line bundle $\mathscr{L}$ under the boundary map δ in non-abelian cohomology is $\{\mathscr{L}^{1/p}\}$, so $(2) \Leftrightarrow (3)$. Finally, $(3) \Leftrightarrow (4)$ by the exactness of the long exact sequence on cohomology. $\square$

We say that a $\boldsymbol{\mu}_p$-gerbe satisfying the conditions of Lemma 2.1.3 is *essentially trivial.* We will also need the following slightly more general construction. Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe and $\mathscr{L} \in \mathrm{Pic}(X)$ a line bundle.

Definition 2.1.4. Let $\mathscr{X}\{\mathscr{L}^{1/p}\}$ be the stack on X whose objects over an X-scheme $T \to X$ are invertible sheaves $\mathscr{M}$ on $\mathscr{X}_T$ that are (-1)-twisted, together with an isomorphism $\mathscr{M}^{\otimes p} \xrightarrow{\sim} \mathscr{L}_{\mathscr{X}_T}$.

The stack $\mathscr{X}\{\mathscr{L}^{1/p}\} \to X$ is canonically banded by $\boldsymbol{\mu}_p$, and is again a $\boldsymbol{\mu}_p$-gerbe over X. In the special case where $\mathscr{X} = \mathbf{B}\boldsymbol{\mu}_p$ is the trivial gerbe, there is a canonical isomorphism $\mathscr{X}\{\mathscr{L}^{1/p}\} \cong \mathscr{L}^{1/p}$ of $\boldsymbol{\mu}_p$-gerbes over X. Furthermore, there is a universal invertible twisted sheaf

$$\mathscr{M} \in \mathrm{Pic}^{(-1,1)}(\mathscr{X} \times_X \mathscr{X}\{\mathscr{L}^{1/p}\}).$$

Lemma 2.1.5. *The cohomology class $\mathscr{X}\{\mathscr{L}^{1/p}\} \in \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ is equal to $[\mathscr{X}] + [\{\mathscr{L}^{1/p}\}]$.*

Proof. The group structure on $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$ is induced by the contracted product of $\boldsymbol{\mu}_p$-gerbes along the anti-diagonal $\boldsymbol{\mu}_p \to \boldsymbol{\mu}_p \times \boldsymbol{\mu}_p$. That is, given two $\boldsymbol{\mu}_p$-gerbes $\mathscr{A} \to X$ and $\mathscr{B} \to X$, a $\boldsymbol{\mu}_p$-gerbe $\mathscr{C} \to X$ represents the class $[\mathscr{A}] + [\mathscr{B}]$ if and only if there is a morphism of X-stacks $\mathscr{A} \times_X \mathscr{B} \to \mathscr{C}$ such that the induced morphism of bands $\boldsymbol{\mu}_p \times \boldsymbol{\mu}_p \to \boldsymbol{\mu}_p$ is the multiplication map. (The reader is referred to Section IV.3.4 and the associated internal references of [16] for details.)

Consider the $\boldsymbol{\mu}_p$-gerbe $\mathscr{Y} \to X$ whose sections over $T \to X$ are given by the groupoid of pairs (M, τ) where M is an invertible $\mathscr{X}_T$-twisted sheaf and $\tau : M^{\otimes p} \xrightarrow{\sim} \mathscr{O}_{\mathscr{X}_T}$ is a trivialization; these are "$\mathscr{X}_T$-twisted $\boldsymbol{\mu}_p$-torsors". Such a pair (M, τ) corresponds precisely to a morphism $\mathscr{X}_T \to \mathbf{B}\boldsymbol{\mu}_{p,T}$ that induces the identity map $\boldsymbol{\mu}_p \to \boldsymbol{\mu}_p$ on bands. There results a morphism $\mathscr{X} \times \mathscr{Y} \to \mathbf{B}\boldsymbol{\mu}_p$ that induces the multiplication map $\boldsymbol{\mu}_p \times \boldsymbol{\mu}_p \to \boldsymbol{\mu}_p$ on bands. Thus, $\mathscr{Y}$ represents $-[\mathscr{X}] \in \mathrm{H}^2(X, \boldsymbol{\mu}_p)$. Similarly, $\mathscr{X}$ is isomorphic to the stack of $\mathscr{Y}$-twisted $\boldsymbol{\mu}_p$-torsors. Since (-1)-twisted sheaves on $\mathscr{X}$ correspond to $\mathscr{Y}$-twisted sheaves, we conclude that $\mathscr{X}$ is itself isomorphic to the X-stack of pairs (N, γ) with N an invertible (-1)-twisted sheaf on $\mathscr{X}$ and $\gamma : N^{\otimes p} \xrightarrow{\sim} \mathscr{O}$ a trivialization of the pth tensor power. We will use this identification in what follows.

Now consider the morphism $\mathscr{X} \times_X \{\mathscr{L}^{1/p}\} \to \mathscr{X}\{\mathscr{L}^{1/p}\}$ defined as follows. As above, an object of $\mathscr{X}$ over $T \to X$ can be thought of as a pair (N, γ) with N a (-1)-twisted invertible sheaf on $\mathscr{X}_T$ and $\gamma : N^{\otimes p} \xrightarrow{\sim} \mathscr{O}_{\mathscr{X}_T}$. An object of $\{\mathscr{L}^{1/p}\}$ over T is a pair $(\mathscr{L}, \chi)$ with $\mathscr{L}$ an invertible sheaf on T and $\chi : \mathscr{L}^{\otimes p} \xrightarrow{\sim} L$ an isomorphism. We get an object of $\mathscr{X}\{\mathscr{L}^{1/p}\}$ by sending $(N, \gamma), (\mathscr{L}, \chi)$ to $(N \otimes \mathscr{L}, \gamma \otimes \chi)$. This induces the multiplication map on bands, giving the desired result. $\square$

Let $\pi \colon X \to S$ be a morphism. Consider the group scheme $\boldsymbol{\mu}_p$ as a sheaf of abelian groups on the big fppf site X_{fl} of X. The morphism π induces a map $\pi^{\mathrm{fl}}\colon X_{\mathrm{fl}} \to S_{\mathrm{fl}}$ of sites. We will study the sheaf $\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p$ on S_{fl}. This sheaf is of formation compatible with arbitrary base change, in the sense that for any S-scheme $T \to S$ there is a canonical isomorphism

$$\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p(T) = \mathrm{H}^0(T, \mathbf{R}^2\pi^{\mathrm{fl}}_{T*}\boldsymbol{\mu}_p)$$

The sheaf $\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p$ may be described as the sheafification in the flat topology of the functor

$$T \mapsto \mathrm{H}^2(X_{T\,\mathrm{fl}}, \boldsymbol{\mu}_p)$$

In this paper, we will only consider the cohomology of $\boldsymbol{\mu}_p$ with respect to the flat topology, so we will when convenient omit the subscript indicating the topology.

Lemma 2.1.6. *Suppose that $\pi\colon X \to S$ is a morphism such that $\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p = \boldsymbol{\mu}_p$ and $\mathbf{R}^1\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p = 0$ (for instance, a relative K3 surface). For any $T \to S$, there is an exact sequence*

$$0 \to \mathrm{H}^2(T, \boldsymbol{\mu}_p) \to \mathrm{H}^2(X_T, \boldsymbol{\mu}_p) \to \mathrm{H}^0(T, \mathbf{R}^2\pi^{\mathrm{fl}}_{T*}\boldsymbol{\mu}_p) \to \mathrm{H}^3(T, \boldsymbol{\mu}_p)$$

Proof. This follows from the Leray spectral sequence for the sheaf $\boldsymbol{\mu}_p$ on $X_T \to T$. □

In particular, if $S = \operatorname{Spec} k$ is the spectrum of an algebraically closed field, then there is a canonical identification

$$\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p(k) = \mathrm{H}^2(X_{\mathrm{fl}}, \boldsymbol{\mu}_p)$$

The following is a relative form of a result of Artin, published as Theorem 3.1 of [1].

Theorem 2.1.7. *If $\pi\colon X \to S$ is a relative K3 surface, then the sheaf $\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p$ on the big flat site of S is representable by a group algebraic space locally of finite presentation over S.*

Note that we make no assumptions as to the characteristic of S, although we will be most interested in the case when S has characteristic p. For a proof of this result, we refer to section 4 of [31] (although the result is stated there in the special case where $S = \operatorname{Spec} k$, the proof works without change in the relative situation).

Let us now suppose that $\pi\colon X \to S$ is a morphism of schemes over $\mathbf{F}_p$. We have a diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"] \arrow[rr, bend left, "F_X"] \arrow[dr, "\pi"'] & X^{(p/S)} \arrow[r, "W_{X/S}"] \arrow[d, "\pi^{(p/S)}"] & X \arrow[d, "\pi"] \\ & S \arrow[r, "F_S"] & S \end{tikzcd} \tag{2.1.7.1}$$

where F_X and F_S are the absolute Frobenius morphisms, and the square is Cartesian. If there is no risk of confusion, we may write $F = F_X$, $X^{(p/S)} = X^{(p)}$, $\pi^{(p/S)} = \pi^{(p)}$, and $W_{X/S} = W$. Given a morphism $T \to X^{(p)}$, we consider the diagram

$$\begin{tikzcd} T \arrow[r, "F_{T/X^{(p)}}"] \arrow[dr] & T^{(p/X^{(p)})} \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & T \arrow[d] \\ & X^{(p)} \arrow[r, "W_{X/S}"] & X \arrow[r, "F_{X/S}"] & X^{(p)} \end{tikzcd}$$

where the squares are Cartesian. Pulling back along $T \to T'$ and $T' \to T$ induces maps

$$F_{X/S*}\mathscr{O}_X^\times \to \mathscr{O}_{X^{(p)}}^\times, \qquad \mathscr{O}_{X^{(p)}}^\times \xrightarrow{F_{X/S}^*} F_{X/S*}\mathscr{O}_X^\times \tag{2.1.7.2}$$

of big étale sheaves whose composition (in either order) is given by the p-th power map $x \mapsto x^p$.

We now further specialize to the case that $S = \operatorname{Spec} k$ is the spectrum of an algebraically closed field of characteristic p, and X is a K3 surface over k. We will record some facts about the Brauer group of X and the flat cohomology group $\mathscr{S} = \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$. Taking cohomology and completing at the identity section, the maps (2.1.7.2) induce maps

$$\widehat{\operatorname{Br}(X)} \xrightarrow{F} \widehat{\operatorname{Br}(X)}^{(p)}, \qquad \widehat{\operatorname{Br}(X)}^{(p)} \xrightarrow{V} \widehat{\operatorname{Br}(X)}$$

whose composition in either order is equal to multiplication by p. The map F may be described concretely as the relative Frobenius of the formal group $\widehat{\operatorname{Br}(X)}$ over $\operatorname{Spec} k$. The map V is in fact uniquely determine by F, as we record here.

Lemma 2.1.8. *There is a unique map $V\colon \widehat{\operatorname{Br}(X)}^{(p)} \to \widehat{\operatorname{Br}(X)}$ such that the composition*

$$\widehat{\operatorname{Br}(X)}^{(p)} \xrightarrow{V} \widehat{\operatorname{Br}(X)} \xrightarrow{F} \widehat{\operatorname{Br}(X)}^{(p)}$$

is equal to multiplication by p.

Proof. If V and V' are two such maps, then $v = V - V'$ satisfies $Fv = 0$. Thus, v factors through the kernel of F. But, the kernel of F is a finite group scheme, and any map from a smooth formal group to a finite group scheme is constant. $\square$

The association $E \mapsto \widehat{E}$ gives a functor from the category of sheaves of abelian groups on $(\operatorname{Spec} k)_{\mathrm{fl}}$ to the category of presheaves on the opposite of the category of local Artinian k-algebras. It is immediate that this functor is left exact.

Lemma 2.1.9. *Let* $0 \to H \to G \to K \to 0$ *be a short exact sequence of sheaves of abelian groups on* $(\operatorname{Spec} k)_{\mathrm{fl}}$. *If* H *is representable by a smooth group scheme over* $\operatorname{Spec} k$, *then the induced sequence*

$$0 \to \widehat{H} \to \widehat{G} \to \widehat{K} \to 0$$

of presheaves is exact.

Proof. Let A be a local Artinian k-algebra with residue field k. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & H(A) \arrow[r] \arrow[d] & G(A) \arrow[r] \arrow[d] & K(A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H(k) \arrow[r] & G(k) \arrow[r] & K(k) \arrow[r] & 0 \end{tikzcd}$$

We claim that the rows are exact. It will suffice to show that the flat cohomology groups $\mathrm{H}^1(\operatorname{Spec} A_{\mathrm{fl}}, H|_{\operatorname{Spec} A})$ and $\mathrm{H}^1(\operatorname{Spec} k_{\mathrm{fl}}, H)$ vanish. For the latter, note that by the Nullstellensatz every cover in $\operatorname{Spec} k_{\mathrm{fl}}$ admits a section, and hence the functor of global sections on the category of sheaves of abelian groups is exact. For the former, our assumption that H is smooth implies by a theorem of Grothendieck (Theorem 2.2.10) that $\mathrm{H}^1(\operatorname{Spec} A_{\mathrm{fl}}, H|_{\operatorname{Spec} A}) = \mathrm{H}^1(\operatorname{Spec} A_{\text{ét}}, H|_{\operatorname{Spec} A})$. As A is a strictly Henselian local ring, the latter group vanishes.

Finally, as H is smooth, the map $H(A) \to H(k)$ is surjective. The result therefore follows from the snake lemma. $\square$

Let us now suppose that X is supersingular. We then have $\widehat{\mathrm{Br}(X)} \cong \widehat{\mathbf{G}_a}$. In particular, multiplication by p on $\widehat{\mathrm{Br}(X)}$ is equal to 0. By Lemma 2.1.8, the map V is also equal to 0.

Lemma 2.1.10. *If* $\pi\colon X \to \operatorname{Spec} k$ *is a supersingular K3 surface, then the natural map* $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^2\pi_*^{\acute{E}t}\mathscr{O}_X^\times$ *induces an isomorphism*

$$\widehat{\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p} \xrightarrow{\sim} \widehat{\mathrm{Br}(X)}$$

on completions at the identity section.

Proof. The flat Kummer sequence induces a short exact sequence

$$0 \to \operatorname{Pic}_X / p \operatorname{Pic}_X \to \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^2\pi_*^{\text{Ét}}\mathscr{O}_X^\times[p] \to 0$$

The quotient $\operatorname{Pic}_X / p \operatorname{Pic}_X$ is smooth, so by Lemma 2.1.9 the induced sequence on completions is exact. Moreover, it is discrete, and hence has trivial completion, so obtain an isomorphism

$$\widehat{\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p} \xrightarrow{\sim} \widehat{\mathbf{R}^2\pi_*^{\text{Ét}}\mathscr{O}_X^\times[p]}$$

(with no assumptions on the height of X). We also have an exact sequence

$$0 \to \mathbf{R}^2\pi_*^{\text{Ét}}\mathscr{O}_X^\times[p] \to \mathbf{R}^2\pi_*^{\text{Ét}}\mathscr{O}_X^\times \xrightarrow{\cdot p} \mathbf{R}^2\pi_*^{\text{Ét}}\mathscr{O}_X^\times$$

Because X is supersingular, multiplication by p induces the zero map on the formal Brauer group. Taking completions, we get an isomorphism

$$\mathbf{R}^2\widehat{\pi_*^{\text{Ét}}\mathscr{O}_X^\times}[p] \xrightarrow{\sim} \widehat{\mathrm{Br}(X)}$$

This completes the proof. ☐

Proposition 2.1.11. *If $\pi\colon X \to S$ is a supersingular K3 surface, then $\mathscr{S} = \mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p$ is a smooth group scheme over* $\operatorname{Spec} k$ *of dimension 1 with connected component isomorphic to* $\mathbf{G}_a$.

Proof. By Lemma 2.1.10, the completion of $\mathscr{S}$ at the identity section is isomorphic to $\widehat{\mathbf{G}_a}$, which is formally smooth and p-torsion. It follows that $\mathscr{S}$ is smooth over k with 1-dimensional connected component. The only 1-dimensional connected group scheme over $\operatorname{Spec} k$ that is smooth and p-torsion is $\mathbf{G}_a$, so the identity component of $\mathscr{S}$ is isomorphic to $\mathbf{G}_a$. ☐

Lemma 2.1.12. *If $S = \operatorname{Spec} A$ is an affine scheme of characteristic p, then $\mathrm{H}^i(S, \boldsymbol{\mu}_p) = 0$ for all $i \geq 3$.*

Proof. This follows from the exact sequence of Lemma 2.3.5 and Milne's isomorphisms

$$\mathrm{H}^i(S_{\text{ét}}, \nu(1)) \xrightarrow{\sim} \mathrm{H}^{i+1}(S_{\text{fl}}, \boldsymbol{\mu}_p)$$

(see the proof of Corollary 1.10 of [36]), together with the vanishing of the higher cohomology of quasi-coherent sheaves on affine schemes. ☐

As a scheme, $\mathscr{S} = \mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p$ is a disjoint union of finitely many copies of $\mathbf{A}^1$. Let $\mathbf{A}^1 \subset \mathscr{S}$ be a connected component. By Tsen's theorem, $\mathrm{H}^2(\mathbf{A}^1, \boldsymbol{\mu}_p) = 0$, and by Lemma 2.1.12, $\mathrm{H}^3(\mathbf{A}^1, \boldsymbol{\mu}_p) = 0$. So, the exact sequence of Lemma 2.1.6 gives an isomorphism

$$\mathrm{H}^2(X_{\mathscr{S}}, \boldsymbol{\mu}_p) \xrightarrow{\sim} \mathrm{H}^0(\mathscr{S}, \mathbf{R}^2\pi_{\mathscr{S}*}^{\text{fl}}\boldsymbol{\mu}_p)$$

In particular, the universal cohomology class $\alpha \in \mathrm{H}^0(\mathscr{S}, \mathbf{R}^2\pi_{\mathscr{S}*}^{\text{fl}}\boldsymbol{\mu}_p)$ is the image of a unique element of $\mathrm{H}^2(X_{\mathscr{S}}, \boldsymbol{\mu}_p)$. This element corresponds to an isomorphism class of $\boldsymbol{\mu}_p$-gerbes over

$X \times \mathscr{S}$. Let $\mathscr{X} \to X \times \mathscr{S}$ be such a $\boldsymbol{\mu}_p$-gerbe. We will think of $\mathscr{X}$ as a family of twisted K3 surfaces over $\mathscr{S}$ via the composition

$$\mathscr{X} \to X \times \mathscr{S} \to \mathscr{S}$$

of the coarse space morphism with the projection, and we will refer to such a gerbe $\mathscr{X}$ as a ***tautological*** or ***universal*** family of $\boldsymbol{\mu}_p$-gerbes on X. Restricting $\mathscr{X} \to \mathscr{S}$ to a connected component $\mathbf{A}^1 \subset \mathscr{S}$, we find a family

$$\mathscr{X}' \to \mathbf{A}^1$$

of twisted supersingular K3 surfaces. This is the basic example of a twistor family.

To obtain more precise information regarding the groups $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$ and $\mathrm{Br}(X)$, we recall some consequences of flat duality for supersingular K3 surfaces. By Lemma 2.1.6, there is a canonical identification

$$\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k) = \mathrm{H}^2(X_{\mathrm{fl}}, \boldsymbol{\mu}_p) \tag{2.1.12.1}$$

In [1], Artin identifies a certain subgroup $\mathrm{U}^2(X, \boldsymbol{\mu}_p) \subset \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ and a short exact sequence

$$0 \to \mathrm{U}^2(X, \boldsymbol{\mu}_p) \to \mathrm{H}^2(X, \boldsymbol{\mu}_p) \to \mathrm{D}^2(X, \boldsymbol{\mu}_p) \to 0 \tag{2.1.12.2}$$

(see also Milne [36]). It is an immediate consequence of the definitions that under the identification (2.1.12.1) the subgroup $\mathrm{U}^2(X, \boldsymbol{\mu}_p) \subset \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ is equal to the subgroup of $\mathscr{S}(k)$ consisting of the k-points of the connected component of the identity. Thus, as a group $\mathrm{U}^2(X, \boldsymbol{\mu}_p)$ is isomorphic to the underlying additive group of the field k. We will use the following result.

Theorem 2.1.13. *Write* $\Lambda = \operatorname{Pic}(X)$. *The Kummer sequence induces a diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \frac{p\Lambda^*}{p\Lambda} \arrow[r] \arrow[d] & \frac{\Lambda}{p\Lambda} \arrow[r] \arrow[d] & \frac{\Lambda}{p\Lambda^*} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathrm{U}^2(X, \boldsymbol{\mu}_p) \arrow[r] \arrow[d] & \mathrm{H}^2(X, \boldsymbol{\mu}_p) \arrow[r] \arrow[d] & \mathrm{D}^2(X, \boldsymbol{\mu}_p) \arrow[r] & 0 \\
 & \operatorname{Br}(X) \arrow[r, equal] \arrow[d] & \operatorname{Br}(X) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{2.1.13.1}$$

In particular, the natural map $\mathrm{U}^2(X, \boldsymbol{\mu}_p) \to \operatorname{Br}(X)$ *is surjective, and the Brauer group* $\operatorname{Br}(X)$ *is* p*-torsion.*

Proof. In Theorem 4.2 of [1], this result is shown under the assumptions that X admits an elliptic fibration, which is now known to always be the case by the Tate conjecture, and under the then-conjectural existence of a certain flat duality theory, which was subsequently developed by Milne [36]. $\square$

2.2 The relative étale site

If X is a scheme, we write $X_{\text{Ét}}$ for the big étale site of X and $X_{\text{ét}}$ for the small étale site of X. Given a morphism $\pi\colon X \to S$ of schemes (or algebraic spaces), we define a site $(X/S)_{\text{ét}}$, which is halfway between $X_{\text{ét}}$ and $S_{\text{Ét}}$. A closely related construction is briefly discussed in Section 3 of [2].

Definition 2.2.1. The *relative étale site* of $X \to S$ is the category $(X/S)_{\text{ét}}$ whose objects are pairs (U, T) where T is a scheme over S, and U is a scheme étale over $X \times_S T$. An arrow $(U', T') \to (U, T)$ consists of a morphism $U' \to U$ and a morphism $T' \to T$ over S such that the diagram

$$\begin{tikzcd}
U' \arrow[r] \arrow[d] & U \arrow[d] \\
T' \arrow[r] & T
\end{tikzcd}$$

commutes. Note that this data is equivalent to giving a morphism $T' \to T$ over S and a morphism $U' \to U \times_{X_T} X_{T'}$ over $X_{T'}$. A family of morphisms $\{(U_i, T_i) \to (U, T)\}_{i \in I}$ is a covering if both $\{U_i \to U\}_{i \in I}$ and $\{T_i \to T\}_{i \in I}$ are étale covers.

The small, relative, and big étale sites are related by various obvious functors, which induce a diagram of sites

$$\begin{tikzcd}
X_{\text{Ét}} \arrow[r] \arrow[rr, bend left, "\alpha_X"] \arrow[dr, "\pi^{\text{Ét}}"'] & (X/S)_{\text{ét}} \arrow[r] \arrow[d, "\pi^{\text{ét}}"] & X_{\text{ét}} \arrow[d] \\
& S_{\text{Ét}} \arrow[r, "\alpha_S"] & S_{\text{ét}}
\end{tikzcd}$$

Remark 2.2.2. The relative étale site is well suited to studying structures that are compatible with arbitrary base change on S and étale base change on X, such as sheaves of relative differentials and the relative Frobenius. In this work, we will be concerned with various representable functors on $S_{\text{Ét}}$ obtained as higher pushforwards of sheaves along the map $\pi^{\text{ét}}\colon (X/S)_{\text{ét}} \to S_{\text{Ét}}$. The relative étale site will be useful to us because it is more constrained than $X_{\text{Ét}}$, while still being large enough to admit a map to the big étale site of S.

If $\pi'\colon X' \to S'$ is a morphism, and $g\colon S' \to S$ and $f\colon X' \to X$ are morphisms such that $\pi \circ f = g \circ \pi'$, then there is an induced map of sites

$$(f, g)\colon (X'/S')_{\text{ét}} \to (X/S)_{\text{ét}}$$

On underlying categories, this map sends a pair (U, T) to $(U \times_X X', T \times_S S')$. We obtain a corresponding pushforward functor $(f, g)_*$, which has an exact left adjoint $(f, g)^{-1}$, and together give a morphism of topoi.

Lemma 2.2.3. *If f and g are universal homeomorphisms, then the pushforward*

$$(f, g)_*\colon \mathrm{Sh}_{(X'/S')_{\text{ét}}} \to \mathrm{Sh}_{(X/S)_{\text{ét}}}$$

between the respective categories of sheaves of abelian groups on the relative étale sites is exact.

Proof. Being the right adjoint to the pullback, $(f, g)_*$ is left exact. That it is also right exact follows from the topological invariance of the small étale site under universal homeomorphisms

[47, 05ZG]. Let $\mathscr{E} \to \mathscr{F}$ be a surjective map of sheaves of abelian groups on $(X'/S')_{\text{ét}}$. We will show that

$$(f,g)_*\mathscr{E} \to (f,g)_*\mathscr{F}$$

remains surjective. That is, we will show that for any object $(U,T) \in (X/S)_{\text{ét}}$, any section $x \in (f,g)_*\mathscr{F}(U,T)$ is in the image after taking a cover of (U,T). By a change of notation, we reduce to the case when $T = S$. Let $U_{S'} = U \times_S S'$, and choose a cover $\{(U_i', T_i') \to (U_{S'}, S')\}_{i\in I}$ such that the restriction of $x \in (f,g)_*\mathscr{F}(U,T) = \mathscr{F}(U_{S'}, S')$ is in the image of each $\mathscr{F}(U_i', T_i') \to \mathscr{E}(U_i', T_i')$. By the topological invariance of the étale site applied to $g\colon S' \to S$, we may find morphisms $T_i \to S$ such that $T_i \times_S S' = T_i'$. Applying the same to the universal homeomorphisms $X'_{T_i'} \to X_{T_i}$, we find morphisms $U_i \to X_{T_i}$ such that $U_i \times_S S' = U_i'$. The restriction of x to the cover $\{(U_i, T_i) \to (U,S)\}_{i\in I}$ is then in the image by construction. $\square$

There is a sheaf of rings $\mathscr{O}_X$ on $(X/S)_{\text{ét}}$ given by

$$\mathscr{O}_X(U,T) = \Gamma(U, \mathscr{O}_U)$$

The pair $((X/S)_{\text{ét}}, \mathscr{O}_X)$ is a ringed site, in the sense of [47, Tag 03AD].

Definition 2.2.4. A sheaf $\mathscr{E}$ of $\mathscr{O}_X$-modules on $(X/S)_{\text{ét}}$ is *quasi-coherent* if

1. for every object (U,T) of $(X/S)_{\text{ét}}$ there exists an open cover $U_i \to U$ such that for each i the restriction $\mathscr{E}|_{U_{i\text{zar}}}$ of $\mathscr{E}$ to the small Zariski site of U_i is a quasi-coherent sheaf (in the usual sense), and

2. if $(f,g)\colon (U',T') \to (U,T)$ is a morphism in $(X/S)_{\text{ét}}$ then the natural comparison map

$$f^*(\mathscr{E}|_{U_{\text{zar}}}) \to \mathscr{E}|_{U'_{\text{zar}}}$$

is an isomorphism.

The sheaf $\mathscr{E}$ is ***locally free of finite rank*** if it is quasi-coherent, and for each object (U,T) the restriction of $\mathscr{E}$ to the small Zariski site of U is locally free of finite rank in the usual sense.

Remark 2.2.5. This definition is similar to [47, Tag 06WK]. In [47, Tag 03DL] there is given a general definition of quasi-coherent sheaf and locally free sheaf on a ringed site. One can show that these agree with the above definitions in our special case.

Our main interest in this paper is when X and S are schemes over $\mathbf{F}_p$, which we will now assume. We record a few facts about the relative Frobenius $F_{X/S}\colon X \to X^{(p/S)}$.

Lemma 2.2.6. *If π is smooth, then $F_{X/S}$ is finite and flat. If π is locally of finite presentation, then the relative Frobenius $F_{X/S}$ is an isomorphism if and only if π is étale.*

Proof. See Proposition 3.2 of [5]. The second claim is Proposition 2 of Exposé XV 1 in [24]. $\square$

Lemma 2.2.7. *The relative Frobenius is compatible with arbitrary base change on S in the following sense. For any morphism $T \to S$, there is a canonical identification*

$$(X^{(p/S)}) \times_S T = (X \times_S T)^{(p/T)}$$

and a commutative diagram

$$\begin{tikzcd}
X_T \arrow[r, "F_{X_T/T}"'] \arrow[d, "\pi_X"'] \arrow[rr, bend left, "\pi_T"] & (X_T)^{(p/T)} \arrow[r, "\pi_T^{(p/T)}"'] \arrow[d] & T \arrow[d] \\
X \arrow[r, "F_{X/S}"] \arrow[rr, bend right, "\pi"'] & X^{(p/S)} \arrow[r, "\pi^{(p/S)}"] & S
\end{tikzcd}$$

where both squares are Cartesian.

Lemma 2.2.8. *The relative Frobenius is compatible with étale base change on $X^{(p)}$, in the following sense. Suppose that $U \to X^{(p)}$ is étale, and define $U_X = X \times_{X^{(p)}} U$ by the Cartesian diagram*

$$\begin{tikzcd}
U_X \arrow[r] \arrow[d] & U \arrow[d] \\
X \arrow[r, "F_{X/S}"] & X^{(p)}
\end{tikzcd}$$

The relative Frobenius $F_{U/X^{(p)}}\colon U \to U^{(p/X^{(p)})} = (U_X)^{(p/S)}$ is an isomorphism, and there is a diagram

$$\begin{tikzcd}
U_X \arrow[r, "F_{U_X/S}"] \arrow[dr] & (U_X)^{(p/S)} \\
 & U \arrow[u, "F_{U/X^{(p)}}"', "\wr"]
\end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd}
U_X \arrow[r] \arrow[d, "F_{U_X/X}"', "\wr"] & U \arrow[d, "F_{U/X^{(p)}}", "\wr"'] & & \\
(U_X)^{(p/X)} \arrow[r] \arrow[d] & (U_X)^{(p/S)} \arrow[r] \arrow[d] & U_X \arrow[r] \arrow[d] & U \arrow[d] \\
X \arrow[r, "F_{X/S}"] \arrow[dr] & X^{(p)} \arrow[r, "W_{X/S}"] \arrow[d] & X \arrow[r, "F_{X/S}"] \arrow[d] & X^{(p)} \\
& S \arrow[r, "F_S"] & S &
\end{tikzcd}$$

where the squares are Cartesian. By Lemma 2.2.6, $F_{U/X^{(p)}}$ is an isomorphism, and the composition $U_X \to (U_X)^{(p/X)} \to (U_X)^{(p/S)}$ is equal to $F_{U_X/S}$. □

The diagram (2.1.7.1) induces various maps on the relative étale sites. Where appropriate, we shall use the notations $F_X = (F_X, F_S)$, $F_{X/S} = (F_{X/S}, \mathrm{id}_S)$, and $W_{X/S} = (W_{X/S}, F_S)$.

Consider the sheaf of units $\mathscr{O}_X^\times$ on $X_{\text{Ét}}$ given by $\mathscr{O}_X^\times(T) = \Gamma(T, \mathscr{O}_T)^\times$. Its pushforward to $(X/S)_{\text{ét}}$, which we will also denote by $\mathscr{O}_X^\times$, is given by

$$\mathscr{O}_X^\times(U, T) = \Gamma(U, \mathscr{O}_U)^\times$$

The morphisms (2.1.7.2) give a commuting square

$$\begin{tikzcd}
F_{X/S*}\mathscr{O}_X^\times \arrow[r, "x \mapsto x^p"] \arrow[d] & F_{X/S*}\mathscr{O}_X^\times \arrow[d, equal] \\
\mathscr{O}_{X^{(p)}}^\times \arrow[r, "F_{X/S}^*"] & F_{X/S*}\mathscr{O}_X^\times
\end{tikzcd} \tag{2.2.8.1}$$

of sheaves on $X_{\text{Ét}}^{(p)}$, and by restriction on $(X^{(p)}/S)_{\text{ét}}$. It is shown in Lemma 2.1.18 of [25] that the restriction of the map $F_{X/S}^*$ to the small étale site is injective. By Lemma 2.2.7 and Lemma 2.2.8, it is also injective on the relative étale site.

Definition 2.2.9. We define a sheaf $\nu(1)$ on $(X^{(p)}/S)_{\text{ét}}$ by the cokernel of the pullback map $F_{X/S}^*\colon \mathscr{O}_{X^{(p)}}^\times \to F_{X/S*}\mathscr{O}_X^\times$ If $X \to S$ is smooth, then this map is injective, and we have a short exact sequence

$$1 \to \mathscr{O}_{X^{(p)}}^\times \to F_{X/S*}\mathscr{O}_X^\times \to \nu(1) \to 1 \tag{2.2.9.1}$$

If X is a scheme, we let $\varepsilon_X\colon X_{\mathrm{fl}} \to X_{\text{Ét}}$ and $\varepsilon_{X/S}\colon X_{\mathrm{fl}} \to (X/S)_{\text{ét}}$ denote the natural maps of sites. These are related by a diagram

$$\begin{tikzcd} X_{\mathrm{fl}} \arrow[r, "\varepsilon_{X/S}"] \arrow[d, "\pi^{\mathrm{fl}}"'] \arrow[dr, "\Theta"] & (X/S)_{\text{ét}} \arrow[d, "\pi^{\text{ét}}"] \\ S_{\mathrm{fl}} \arrow[r, "\varepsilon_S"] & S_{\text{Ét}} \end{tikzcd} \tag{2.2.9.2}$$

We recall the following theorem of Grothendieck.

Theorem 2.2.10 ([17], Théorème 11.7). *If X is a scheme and A is a smooth group scheme over X, then $\mathbf{R}^p\varepsilon_{X*}A = 0$ for $p > 0$.*

We consider the long exact sequence induced by applying $\varepsilon_{X/S*}$ to the short exact sequence

$$1 \to \boldsymbol{\mu}_p \to \mathscr{O}_X^\times \xrightarrow{x\mapsto x^p} \mathscr{O}_X^\times \to 1 \tag{2.2.10.1}$$

of sheaves on X_{fl}. Because pushforward along the map $X_{\text{Ét}} \to (X/S)_{\text{ét}}$ is exact, Grothendieck's Theorem implies that $\mathbf{R}^p\varepsilon_{X/S*}\mathscr{O}_X^\times = 0$ for $p > 0$. Thus, we obtain an exact sequence

$$1 \to \boldsymbol{\mu}_p \to \mathscr{O}_X^\times \xrightarrow{x\mapsto x^p} \mathscr{O}_X^\times \to \mathbf{R}^1\varepsilon_{X/S*}\boldsymbol{\mu}_p \to 1$$

of sheaves on $(X/S)_{\text{ét}}$. Let us write

$$\mathbf{R}^1\varepsilon_{X/S*}\boldsymbol{\mu}_p = \mathscr{O}_X^\times/\mathscr{O}_X^{\times p}$$

where it is understood that the quotient is taken in the étale topology. By Lemma 2.2.3, this sequence remains exact after applying $F_{X/S*}$, and so the square (2.2.8.1) induces a morphism

$$F_{X/S*}(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p}) \to \nu(1) \tag{2.2.10.2}$$

of sheaves on $(X^{(p)}/S)_{\text{ét}}$.

Proposition 2.2.11. *If $\pi\colon X \to S$ is a smooth morphism of $\mathbf{F}_p$-schemes such that*

1. *The adjunction $\mathscr{O}_S \to \pi_*\mathscr{O}_X$ is universally an isomorphism, and*

2. *$\mathbf{R}^1\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p = 0$,*

then there is a natural map

$$\Upsilon\colon \varepsilon_{S*}\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$$

of big étale sheaves. For any perfect scheme T over S, this map is a bijection on T-points.

Proof. First, note that by Lemma 2.2.3

$$\mathbf{R}^1\pi_*^{(p)\text{ét}}F_{X/S*}(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p}) = \mathbf{R}^1\pi_*^{\text{ét}}(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p})$$

Thus, the morphism (2.2.10.2) induces a map

$$\mathbf{R}^1\pi_*^{\text{ét}}(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p}) \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \tag{2.2.11.1}$$

Next, we consider the Grothendieck spectral sequences

$$E_2^{p,q} = \mathbf{R}^p\pi_*^{\text{ét}}(\mathbf{R}^q\varepsilon_{X/S*}\boldsymbol{\mu}_p) \Longrightarrow \mathbf{R}^{p+q}\Theta_*\boldsymbol{\mu}_p$$

$$\overline{E}_2^{p,q} = \mathbf{R}^p\varepsilon_{S*}(\mathbf{R}^q\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p) \Longrightarrow \mathbf{R}^{p+q}\Theta_*\boldsymbol{\mu}_p$$

induced by the square (2.2.9.2). The first spectral sequence gives an exact sequence

$$E_2^{2,0} \to \ker(E_\infty^2 \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}$$

Applying Theorem 2.2.10 to $\mathbf{G}_m$, the Kummer sequence shows that $\mathbf{R}^p\varepsilon_{X/S*}\boldsymbol{\mu}_p = 0$ for $p \geq 2$ Therefore, $E_2^{0,2} = 0$, so we get a map

$$\mathbf{R}^2\Theta_*\boldsymbol{\mu}_p \to \mathbf{R}^1\pi_*^{\text{ét}}\mathbf{R}^1\varepsilon_{X/S*}\boldsymbol{\mu}_p = \mathbf{R}^1\pi_*^{\text{ét}}(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p})$$

The second spectral sequence gives a map $\mathbf{R}^2\Theta_*\boldsymbol{\mu}_p \to \varepsilon_{S*}\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$. Hence, we have a diagram

$$\begin{tikzcd} & \mathbf{R}^2\Theta_*\boldsymbol{\mu}_p \arrow[dl, "f"'] \arrow[dr, "g"] & & \\ \varepsilon_{S*}\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p & & \mathbf{R}^1\pi_*^{\text{ét}}(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p}) \arrow[r] & \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \end{tikzcd}$$

It follows from assumptions (1) and (2) that f is an isomorphism. We therefore obtain the desired map

$$\Upsilon\colon \varepsilon_{S*}\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$$

Suppose that T is a perfect S-scheme. For any étale $U \to X_T^{(p)}$, the map $F_{X/S*}\mathscr{O}_X^\times(U,T) \to \mathscr{O}_{X^{(p)}}^\times(U,T)$ is given by pullback along a base change of $W_{X_T/T}$ followed by pullback along the relative Frobenius $F_{U/X_T^{(p)}}$, both of which are isomorphisms. Therefore, by restricting (2.2.8.1) to the small étale site (which is an exact functor), we get an isomorphism

$$\alpha_{T*}\mathbf{R}^1\pi_{T*}^{\text{ét}}\mathbf{R}^1\varepsilon_{X_T/T*}\boldsymbol{\mu}_p \xrightarrow{\sim} \alpha_{T*}\mathbf{R}^1\pi_{T*}^{(p)\text{ét}}\nu(1)$$

of sheaves on $S_{\text{ét}}$. We have an exact sequence

$$\mathbf{R}^2\pi_{T*}^{\text{ét}}\varepsilon_{X_T/T*}\boldsymbol{\mu}_p \to \varepsilon_{T*}\mathbf{R}^2\pi_{T*}^{\text{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^1\pi_{T*}^{\text{ét}}\mathbf{R}^1\varepsilon_{X_T/T}\boldsymbol{\mu}_p \to \mathbf{R}^3\pi_{T*}^{\text{ét}}\varepsilon_{X_T/T*}\boldsymbol{\mu}_p$$

Because X_T is smooth over a perfect scheme, $\alpha_{X_T*}\varepsilon_{X_T/T*}\boldsymbol{\mu}_p = 0$, and therefore the middle arrow becomes an isomorphism. We have shown that the induced map

$$\alpha_{T*}\varepsilon_{T*}\mathbf{R}^2\pi_{T*}^{\text{fl}}\boldsymbol{\mu}_p \xrightarrow{\sim} \alpha_{T*}\mathbf{R}^1\pi_{T*}^{\text{ét}}\mathbf{R}^1\varepsilon_{X_T/T*}\boldsymbol{\mu}_p \xrightarrow{\sim} \alpha_{T*}\mathbf{R}^1\pi_{T*}^{(p)\text{ét}}\nu(1)$$

of small étale sheaves is an isomorphism. In particular, it is a bijection on global sections, which gives the result. □

In this paper we are interested in the case when $\pi\colon X \to S$ is a relative supersingular K3 surface. If this is so, then the conditions of Proposition 2.2.11 are satisfied. Moreover, by Theorem 2.1.7 the sheaf $\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p$ is representable, and we will see that $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$ is as well in Proposition 3.1.7. The following result therefore applies to show that Υ is a universal homeomorphism.

Proposition 2.2.12. *Let S be a scheme over $\mathbf{F}_p$ and $f\colon X \to Y$ a morphism of S-schemes. If f induces a bijection on T-points for every perfect S-scheme T, then f is a universal homeomorphism.*

Proof. Recall that the *perfection* $X^{\text{pf}} \to X$ of an $\mathbf{F}_p$-scheme X is characterized by the property that any map $T \to X$ from a perfect scheme T factors uniquely through $X^{\text{pf}} \to X$. There is therefore a commutative diagram

$$\begin{array}{ccc} X^{\text{pf}} & \xrightarrow{f^{\text{pf}}} & Y^{\text{pf}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Applying our assumption on f to the S-scheme $Y^{\text{pf}} \to Y \to S$, we get a map $Y^{\text{pf}} \to X$, and hence a map $Y^{\text{pf}} \to X^{\text{pf}}$. By the universal property of the perfection, this map is an inverse to

f^{pf}, so $f^{\mathrm{pf}}\colon X^{\mathrm{pf}} \to Y^{\mathrm{pf}}$ is an isomorphism. By Lemma 3.8 of [6], this is equivalent to f being a universal homeomorphism. $\square$

We next compute the tangent spaces of $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$ and $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$. The former is slightly subtle, as the pushforward under a universal homeomorphism (such as the projection $\pi_X\colon X[\varepsilon] \to X$) is not necessarily exact in the flat topology. We record the following lemma.

Lemma 2.2.13. *If X is a scheme and $\pi_X\colon X[\varepsilon] \to X$ is the natural projection, then $\mathbf{R}^1\pi^{\mathrm{fl}}_{X*}\mathscr{O}^\times_{X[\varepsilon]} = 0$.*

Proof. Consider the diagram

$$\begin{array}{ccc} X[\varepsilon]_{\mathrm{fl}} & \xrightarrow{\pi_X^{\mathrm{fl}}} & X_{\mathrm{fl}} \\ {\scriptstyle \varepsilon_{X[\varepsilon]}}\big\downarrow & \searrow^{\Theta} & \big\downarrow{\scriptstyle \varepsilon_X} \\ X[\varepsilon]_{\text{Ét}} & \xrightarrow[\pi_X^{\text{Ét}}]{} & X_{\text{Ét}} \end{array} \tag{2.2.13.1}$$

and the two induced Grothendieck spectral sequences. On the one hand, we have by Theorem 2.2.10 that $\mathbf{R}^p\varepsilon_{X[\varepsilon]*}\mathscr{O}^\times_{X[\varepsilon]} = 0$ for $p \geq 1$. As in Lemma 2.2.3, the pushforward $\pi^{\text{Ét}}_{X*}$ is exact, and we conclude that $\mathbf{R}^p\Theta_*\mathscr{O}^\times_{X[\varepsilon]} = 0$ for $p \geq 1$. We now consider the other spectral sequence. The exact sequence of low degree terms gives an isomorphism

$$\varepsilon_{X*}\mathbf{R}^1\pi^{\mathrm{fl}}_{X*}\mathscr{O}^\times_{X[\varepsilon]} \xrightarrow{\sim} \mathbf{R}^2\varepsilon_{X*}\pi^{\mathrm{fl}}_{X*}\mathscr{O}^\times_{X[\varepsilon]}$$

We will show that the right hand side vanishes. Consider the standard short exact sequence

$$0 \to \mathscr{O}_X \to \pi^{\mathrm{fl}}_{X*}\mathscr{O}^\times_{X[\varepsilon]} \to \mathscr{O}^\times_X \to 1 \tag{2.2.13.2}$$

of sheaves on the big flat site of X, where the first map is given by $f \mapsto 1 + f\varepsilon$ and the second by $g + f\varepsilon \mapsto g$. Note that this sequence is split. Therefore, $\pi^{\mathrm{fl}}_{X*}\mathscr{O}^\times_{X[\varepsilon]}$ is represented by a smooth group scheme, and hence by Theorem 2.2.10 we have $\mathbf{R}^p\varepsilon_{X*}\pi^{\mathrm{fl}}_{X*}\mathscr{O}^\times_{X[\varepsilon]} = 0$ for all $p \geq 1$. This completes the proof. $\square$

Lemma 2.2.14. *Suppose that $\pi\colon X \to S$ is a smooth morphism of $\mathbf{F}_p$-schemes such that*

1. *The sheaf $\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p$ is representable by an algebraic space,*

2. *The adjunction $\mathscr{O}_S \to \pi_*\mathscr{O}_X$ is universally an isomorphism, and*

3. $\mathbf{R}^1\pi_*\mathscr{O}_X = 0.$

Set $\mathscr{S} = \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$ *and let* $p\colon \mathscr{S} \to S$ *be the forgetful morphism. There is a natural identification*

$$T^1_{\mathscr{S}/S} \xrightarrow{\sim} p^*\mathbf{R}^2\pi_*\mathscr{O}_X$$

Proof. Let $\sigma_e\colon S \to \mathscr{S}$ be the identity section. We will construct an isomorphism

$$\mathbf{R}^2\pi_*\mathscr{O}_X \xrightarrow{\sim} \sigma_e^* T^1_{\mathscr{S}/S}$$

Consider the Cartesian diagram

$$\begin{tikzcd} X[\varepsilon] \arrow[r, "\pi_X"] \arrow[d, "\pi_{[\varepsilon]}"'] & X \arrow[d, "\pi"] \\ S[\varepsilon] \arrow[r, "\pi_S"] & S \end{tikzcd} \tag{2.2.14.1}$$

The group of sections $\sigma_e^* T^1_{\mathscr{S}/S}(U)$ over an open set $U \subset S$ is in natural bijection with the collection of morphisms $t\colon U[\varepsilon] \to \mathscr{S}$ over S such that the diagram

$$\begin{tikzcd} & & \mathscr{S} \arrow[d] \\ U \arrow[r] \arrow[urr, bend left, "\sigma_e"] & U[\varepsilon] \arrow[r] \arrow[ur, "t"] & S \end{tikzcd}$$

commutes. Because $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$ is compatible with base change, this is the the same as the kernel of the natural map

$$\pi_{S*}\mathbf{R}^2\pi^{\mathrm{fl}}_{[\varepsilon]*}\boldsymbol{\mu}_p \to \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p \tag{2.2.14.2}$$

of small Zariski sheaves. Combining (2.2.13.2) with the Kummer sequence, and using the vanishing result of Lemma 2.2.13, we obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
0 \arrow[r] & \mathscr{O}_X \arrow[r] \arrow[d, "\wr"] & \pi^{\mathrm{fl}}_{X*}\boldsymbol{\mu}_p \arrow[r] \arrow[d] & \boldsymbol{\mu}_p \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathscr{O}_X \arrow[r] \arrow[d, "0"] & \pi^{\mathrm{fl}}_{X*}\mathscr{O}^{\times}_{X[\varepsilon]} \arrow[r] \arrow[d, "\cdot p"] & \mathscr{O}^{\times}_X \arrow[r] \arrow[d, "\cdot p"] & 1 \\
0 \arrow[r] & \mathscr{O}_X \arrow[r] \arrow[d, "\wr"] & \pi^{\mathrm{fl}}_{X*}\mathscr{O}^{\times}_{X[\varepsilon]} \arrow[r] \arrow[d] & \mathscr{O}^{\times}_X \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathscr{O}_X \arrow[r, "\sim"] \arrow[d] & \mathbf{R}^1\pi^{\mathrm{fl}}_{X*}\boldsymbol{\mu}_p \arrow[r] \arrow[d] & 1 & \\
 & 0 & 1 & &
\end{tikzcd}$$

of sheaves on the big flat site of X with exact rows and columns. Taking cohomology of the split exact sequence

$$0 \to \mathscr{O}_X \to \pi^{\mathrm{fl}}_{X*}\boldsymbol{\mu}_p \to \boldsymbol{\mu}_p \to 1$$

we get an exact sequence

$$0 \to \mathbf{R}^2\pi^{\mathrm{fl}}_*\mathscr{O}_X \to \mathbf{R}^2\pi^{\mathrm{fl}}_*(\pi^{\mathrm{fl}}_{X*}\boldsymbol{\mu}_p) \to \mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p \to 0$$

of sheaves on the big flat site of S. To compare this to the kernel of the morphism (2.2.14.2), we consider the spectral sequences

$$E_2^{p,q} = \mathbf{R}^p\pi^{\mathrm{fl}}_*(\mathbf{R}^q\pi^{\mathrm{fl}}_{X*}\boldsymbol{\mu}_p) \Longrightarrow \mathbf{R}^{p+q}\Theta^{\mathrm{fl}}_*\boldsymbol{\mu}_p$$

$$\overline{E}_2^{p,q} = \mathbf{R}^p\pi^{\mathrm{fl}}_{S*}(\mathbf{R}^q\pi^{\mathrm{fl}}_{[\varepsilon]*}\boldsymbol{\mu}_p) \Longrightarrow \mathbf{R}^{p+q}\Theta^{\mathrm{fl}}_*\boldsymbol{\mu}_p$$

induced by the commuting square (2.2.14.1), where $\Theta = \pi \circ \pi_X = \pi_S \circ \pi_{[\varepsilon]}$. The first spectral sequence gives an exact sequence

$$0 \to E_2^{1,0} \to E^1_\infty \to E_2^{0,1} \to E_2^{2,0} \to \ker(E^2_\infty \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}$$

By the Kummer sequence and Grothendieck's Theorem, we get that $\mathbf{R}^q\pi^{\mathrm{fl}}_{X*}\boldsymbol{\mu}_p = 0$ for $q \geq 2$, and in particular, $E_2^{0,2} = 0$. The isomorphism

$$\mathscr{O}_X \xrightarrow{\sim} \mathbf{R}^1\pi^{\mathrm{fl}}_{X*}\boldsymbol{\mu}_p \tag{2.2.14.3}$$

and condition (3) imply that $E_2^{1,1} = 0$. Thus, we have an exact sequence

$$0 \to \mathbf{R}^1\pi_*^{\text{fl}}\pi_{X*}^{\text{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^1\Theta_*^{\text{fl}}\boldsymbol{\mu}_p \to \pi_*^{\text{fl}}\mathbf{R}^1\pi_{X*}^{\text{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^2\pi_*^{\text{fl}}\pi_{X*}^{\text{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^2\Theta_*^{\text{fl}}\boldsymbol{\mu}_p \to 0 \tag{2.2.14.4}$$

We next examine the second spectral sequence. We have

$$\begin{aligned}
\overline{E}_2^{p,0} &= \mathbf{R}^p\pi_{S*}^{\text{fl}}(\pi_{[\varepsilon]*}^{\text{fl}}\boldsymbol{\mu}_p) = \mathbf{R}^p\pi_{S*}^{\text{fl}}\boldsymbol{\mu}_p = 0 \text{ for } p \geq 2, \text{ and} \\
\overline{E}_2^{p,1} &= \mathbf{R}^p\pi_{S*}^{\text{fl}}(\mathbf{R}^1\pi_{[\varepsilon]*}^{\text{fl}}\boldsymbol{\mu}_p) = 0 \text{ for } p \geq 0.
\end{aligned}$$

Thus, we have natural isomorphisms

$$\begin{aligned}
\mathbf{R}^1\pi_{S*}^{\text{fl}}(\pi_{[\varepsilon]*}^{\text{fl}}\boldsymbol{\mu}_p) &\xrightarrow{\sim} \mathbf{R}^1\Theta_*^{\text{fl}}\boldsymbol{\mu}_p \\
\mathbf{R}^2\Theta_*^{\text{fl}}\boldsymbol{\mu}_p &\xrightarrow{\sim} \pi_{S*}^{\text{fl}}\mathbf{R}^2\pi_{[\varepsilon]*}^{\text{fl}}\boldsymbol{\mu}_p
\end{aligned}$$

Comparing with the exact sequence (2.2.14.4), we find a surjection

$$\mathbf{R}^2\pi_*^{\text{fl}}\pi_{X*}^{\text{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^2\Theta_*^{\text{fl}}\boldsymbol{\mu}_p \xrightarrow{\sim} \pi_{S*}^{\text{fl}}\mathbf{R}^2\pi_{[\varepsilon]*}^{\text{fl}}\boldsymbol{\mu}_p \tag{2.2.14.5}$$

whose kernel is given by the cokernel of the map

$$\mathbf{R}^1\pi_{S*}^{\text{fl}}(\pi_{[\varepsilon]*}^{\text{fl}}\boldsymbol{\mu}_p) \xrightarrow{\sim} \mathbf{R}^1\Theta_*^{\text{fl}}\boldsymbol{\mu}_p \to \pi_*^{\text{fl}}\mathbf{R}^1\pi_{X*}^{\text{fl}}\boldsymbol{\mu}_p \tag{2.2.14.6}$$

We claim that (2.2.14.6) is an isomorphism. Indeed, using the isomorphism (2.2.14.3) and condition (2), we find a diagram

$$\begin{tikzcd}
\mathbf{R}^1\pi_{S*}^{\text{fl}}(\pi_{[\varepsilon]*}^{\text{fl}}\boldsymbol{\mu}_p) \arrow[r] \arrow[d, "\wr"] & \pi_*^{\text{fl}}\mathbf{R}^1\pi_{X*}^{\text{fl}}\boldsymbol{\mu}_p \arrow[d, "\wr"] \\
\mathscr{O}_S \arrow[r] & \mathscr{O}_S
\end{tikzcd}$$

where the induced map $\mathscr{O}_S \to \mathscr{O}_S$ is the identity. Thus, (2.2.14.6) and hence (2.2.14.5) are isomorphisms, and restricting to the Zariski site we obtain a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \sigma_e^*T^1_{\mathscr{S}/S} \arrow[r] \arrow[d, "\wr"] & \pi_{S*}\mathbf{R}^2\pi_{[\varepsilon]*}^{\text{fl}}\boldsymbol{\mu}_p \arrow[r] \arrow[d, "\wr"] & \mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathbf{R}^2\pi_*^{\text{fl}}\mathscr{O}_X \arrow[r] & \mathbf{R}^2\pi_*^{\text{fl}}(\pi_{X*}^{\text{fl}}\boldsymbol{\mu}_p) \arrow[r] & \mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p \arrow[r] & 0
\end{tikzcd}$$

For any group space $p\colon G \to S$, there is a canonical isomorphism

$$T^1_{G/S} \xrightarrow{\sim} p^*\sigma_e^*T^1_{G/S}$$

The composition

$$T^1_{\mathscr{S}/S} \xrightarrow{\sim} p^*\sigma_e^* T^1_{\mathscr{S}/S} \xrightarrow{\sim} p^*\mathbf{R}^2\pi_*\mathscr{O}_X$$

gives the desired isomorphism. □

Definition 2.2.15. We define a quasicoherent sheaf $B^1_{X/S}$ on $X^{(p)}$ by the short exact sequence

$$0 \to \mathscr{O}_{X^{(p)}} \to F_{X/S*}\mathscr{O}_X \to B^1_{X/S} \to 0$$

Lemma 2.2.16. *Suppose that $\pi\colon X \to S$ is a smooth morphism of $\mathbf{F}_p$-schemes such that the sheaf $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$ is representable by an algebraic space. Set $\mathscr{S}_\nu = \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$ and let $p_\nu\colon \mathscr{S}_\nu \to S$ be the forgetful morphism. There is a natural identification*

$$T^1_{\mathscr{S}_\nu/S} \xrightarrow{\sim} p_\nu^*\mathbf{R}^1\pi_*^{(p)} B^1_{X/S}$$

Proof. Let $\sigma_e\colon S \to \mathscr{S}_\nu$ be the identity section. We will construct an isomorphism

$$\mathbf{R}^1\pi_*^{(p)} B^1_{X/S} \xrightarrow{\sim} \sigma_e^* T^1_{\mathscr{S}_\nu/S}$$

As in Lemma 2.2.14, we identify the sheaf $\sigma_e^* T^1_{\mathscr{S}_\nu/S}$ with the kernel of the natural map

$$\pi_{S*}\mathbf{R}^1\pi_{[\varepsilon]*}^{(p)\text{ét}}\nu(1) \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \tag{2.2.16.1}$$

where $\pi_{[\varepsilon]}^{(p)\text{ét}} = (\pi_{X^{(p)}}, \pi_S)$ is the map of relative étale sites defined as in (2.2.14.1). Restricting (2.2.13.2) to the relative étale site, we get a short exact sequence

$$0 \to \mathscr{O}_X \to \pi^{\text{ét}}_{X*}\mathscr{O}^\times_{X[\varepsilon]} \to \mathscr{O}^\times_X \to 1$$

of sheaves on the relative étale site $(X/S)_{\text{ét}}$. We have a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
0 \arrow[r] & \mathscr{O}_{X^{(p)}} \arrow[r] \arrow[d] & \pi^{\text{ét}}_{X^{(p)}*}\mathscr{O}_{X^{(p)}[\varepsilon]} \arrow[r] \arrow[d] & \mathscr{O}^\times_{X^{(p)}} \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & F_{X/S*}\mathscr{O}_X \arrow[r] \arrow[d] & F_{X/S*}\pi^{\text{ét}}_{X*}\mathscr{O}^\times_{X[\varepsilon]} \arrow[r] \arrow[d] & F_{X/S*}\mathscr{O}^\times_X \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & B^1_{X/S} \arrow[r] \arrow[d] & \pi^{\text{ét}}_{X^{(p)}*}\nu(1) \arrow[r] \arrow[d] & \nu(1) \arrow[r] \arrow[d] & 1 \\
 & 0 & 1 & 1 &
\end{tikzcd}$$

Taking cohomology of the split exact sequence

$$0 \to B^1_{X/S} \to \pi^{\text{ét}}_{X^{(p)}*}\nu(1) \to \nu(1) \to 1$$

we get an exact sequence

$$0 \to \mathbf{R}^1\pi^{(p)\text{ét}}_* B^1_{X/S} \to \mathbf{R}^1\pi^{(p)\text{ét}}_* \pi^{\text{ét}}_{X^{(p)}*}\nu(1) \to \mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1) \to 1$$

By Lemma 2.2.3, the functors $\pi^{\text{ét}}_{X^{(p)}*}$ and $\pi^{\text{ét}}_{S*}$ are exact. Thus, the kernel of the morphism (2.2.16.1) is identified with $\mathbf{R}^1\pi^{(p)}_* B^1_{X/S}$, and we obtain an isomorphism

$$\mathbf{R}^1\pi^{(p)}_* B^1_{X/S} \xrightarrow{\sim} \sigma_e^* T^1_{\mathscr{S}_\nu/S}$$

As in Lemma 2.2.14, this induces an isomorphism

$$T^1_{\mathscr{S}_\nu/S} \xrightarrow{\sim} p_\nu^* \mathbf{R}^1\pi^{(p)}_* B^1_{X/S}$$

□

Lemma 2.2.17. *If $X \to S$ is a morphism of $\mathbf{F}_p$-schemes satisfying the assumptions of Proposition 2.2.11, then the diagram*

$$\begin{array}{ccc} \varepsilon_{S*}\mathbf{R}^2\pi^{\text{fl}}_*\boldsymbol{\mu}_p & \xrightarrow{\Upsilon} & \mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1) \\ \downarrow & & \downarrow \\ \mathbf{R}^2\pi^{\text{ét}}_*\mathscr{O}_X^\times & \longrightarrow & \mathbf{R}^2\pi^{(p)\text{ét}}_*\mathscr{O}^\times_{X^{(p)}} \end{array} \tag{2.2.17.1}$$

commutes, where the left vertical arrow is induced by the inclusion $\boldsymbol{\mu}_p \to \mathscr{O}_X^\times$, the right vertical arrow is the boundary map induced by (2.2.9.1), and the lower horizontal arrow is the canonical map.

Proof. This follows from the construction of the exact sequence of low degree terms of a spectral sequence. □

Remark 2.2.18. Let us interpret the infinitesimal information contained in the diagram (2.2.17.1) in the case when $X \to S = \operatorname{Spec} k$ is a supersingular K3 surface. By Lemma 2.1.10, the left vertical arrow induces an isomorphism on completions at the identity. The right vertical arrow fits into the long exact sequence

$$\ldots \to \mathbf{R}^1\pi^{\text{ét}}_*\mathscr{O}_X^\times \to \mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1) \to \mathbf{R}^2\pi^{(p)\text{ét}}_*\mathscr{O}^\times_{X^{(p)}} \to \mathbf{R}^2\pi^{\text{ét}}_*\mathscr{O}_X^\times \to \ldots$$

The right map induces V upon completion at the identity section. As the formal Brauer group of X has infinite height, multiplication on the formal Brauer group is trivial, and so by Lemma 2.1.8 V is also trivial. Using that the Picard group of X is discrete, it follows as in Lemma 2.1.10 that the right vertical arrow of (2.2.17.1) induces an isomorphism on completions. We therefore have a diagram

$$\begin{tikzcd} \widehat{\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p} \arrow[r, "\widehat{\Upsilon}"] \arrow[d, "\wr"] & \mathbf{R}^1\widehat{\pi^{(p)\text{ét}}_*}\nu(1) \arrow[d, "\wr"] \\ \widehat{\mathrm{Br}(X)} \arrow[r, "0"] & \widehat{\mathrm{Br}(X^{(p)})} \end{tikzcd} \tag{2.2.18.1}$$

In particular, $\widehat{\Upsilon} = 0$. The corresponding diagram on the tangent spaces to the identity element is

$$\begin{tikzcd} \mathrm{H}^2(X, \mathscr{O}_X) \arrow[r, "0"] \arrow[d, "\wr"] & \mathrm{H}^1(X^{(p)}, B^1_{X/S}) \arrow[d, "\wr"] \\ \mathrm{H}^2(X, \mathscr{O}_X) \arrow[r, "0"] & \mathrm{H}^2(X^{(p)}, \mathscr{O}_{X^{(p)}}) \end{tikzcd} \tag{2.2.18.2}$$

2.3 De Rham cohomology on the relative étale site

Let $\pi\colon X \to S$ be a smooth proper morphism of $\mathbf{F}_p$-schemes. We will discuss the Hodge and conjugate filtrations on the de Rham cohomology of π and their relationship under the Cartier operator. We will then relate the de Rham cohomology to the étale cohomology of $\nu(1)$. The material in this section is essentially well known, although we have chosen to work throughout on the relative étale site. This will allow us to cleanly obtain moduli theoretic results later in Section 3.1. For a thorough treatment of the special features of de Rham cohomology in positive characteristic, we refer the reader to [5] and [27]. We remark that the results of this section can be seen as a direct generalization of Section 1 of [40] to the case of a non-perfect base.

Notation 2.3.1. If $\mathscr{E}^\bullet$ is any complex of sheaves of abelian groups on a site, we define

$$Z^i\mathscr{E}^\bullet = \ker(\mathscr{E}^i \xrightarrow{d} \mathscr{E}^{i+1}) \qquad \text{and} \qquad B^i\mathscr{E}^\bullet = \mathrm{im}(\mathscr{E}^{i-1} \xrightarrow{d} \mathscr{E}^i)$$

We consider the algebraic de Rham complex

$$\Omega^\bullet_{X/S} = \left[0 \to \mathscr{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \xrightarrow{d} \dots\right]$$

where $\mathscr{O}_X$ is placed in degree 0. The relation $\frac{d}{dx}(x^p) = 0$ implies that the exterior derivative is $\mathscr{O}_{X^{(p)}}$-linear. This means that $F_{X/S*}\Omega^\bullet_{X/S}$ is a complex of $\mathscr{O}_{X^{(p)}}$-modules, and so the sheaves $F_{X/S*}\Omega^i_{X/S}$, $Z^i(F_{X/S*}\Omega^\bullet_{X/S})$, $B^i(F_{X/S*}\Omega^\bullet_{X/S})$, and $\mathscr{H}^i(F_{X/S*}\Omega^\bullet_{X/S})$ on $X^{(p)}$ have natural $\mathscr{O}_{X^{(p)}}$-module structures.

Lemma 2.3.2. *The sheaves*

$$\Omega^i_{X/S}, F_{X/S*}\Omega^i_{X/S}, Z^i(F_{X/S*}\Omega^\bullet_{X/S}), B^i(F_{X/S*}\Omega^\bullet_{X/S}),\ \textit{and}\ \mathscr{H}^i(F_{X/S*}\Omega^\bullet_{X/S})$$

of modules on the small Zariski sites of X and $X^{(p)}$ are locally free of finite rank.

Proof. Because π is smooth, $\Omega^i_{X/S}$ is locally free of finite rank. For the remainder, see Corollary 3.6 of [5]. ☐

We find a similar story on the relative étale site. We define sheaves of $\mathscr{O}_X$-modules on $(X/S)_{\text{ét}}$ by the formula

$$\Omega^{\text{ét}\,i}_{X/S}(U,T) = \Gamma(U, \Omega^i_{U/T}) \tag{2.3.2.1}$$

Because of the compatibility of the differential with pullback, we obtain a complex

$$\Omega^{\text{ét}\,\bullet}_{X/S} = \left[0 \to \mathscr{O}_X \xrightarrow{d} \Omega^{\text{ét}\,1}_{X/S} \xrightarrow{d} \Omega^{\text{ét}\,2}_{X/S} \xrightarrow{d} \dots\right]$$

of sheaves of abelian groups on $(X/S)_{\text{ét}}$. We will show that the analog of Lemma 2.3.2 holds on the relative étale site.

Proposition 2.3.3. *The sheaves*

$$\Omega^{\text{ét}\,i}_{X/S}, F_{X/S*}\Omega^{\text{ét}\,i}_{X/S}, Z^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S}), B^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S}),\ \textit{and}\ \mathscr{H}^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S})$$

are locally free of finite rank, in the sense of Definition 2.2.4.

Proof. The content of this claim is that the sheaves of Lemma 2.3.2 are of formation compatible with base change by étale morphisms on the source and arbitrary morphisms on the base. Let us first show that these sheaves are quasi-coherent. If (U,T) is an object of $(X/S)_{\text{ét}}$, then there is a natural identification

$$\Omega^{\text{ét}\,i}_{X/S}|_{U_{\text{zar}}} = \Omega^i_{U/T}$$

Thus, condition (1) of Definition 2.2.4 holds. Suppose that $(f, g)\colon (U', T') \to (U, T)$ is a morphism in $(X/S)_{\text{ét}}$. We have a diagram

$$\begin{tikzcd} U' \arrow[r, "a"] \arrow[rr, bend left, "f"] \arrow[dr] & U_{T'} \arrow[r, "b"] \arrow[d] & U \arrow[d] \\ & X_{T'} \arrow[r] \arrow[d] & X_T \arrow[d] \\ & T' \arrow[r, "g"] & T \end{tikzcd} \tag{2.3.3.1}$$

where the squares are Cartesian. Note that a is étale. Thus, the comparison map factors as a composition of isomorphisms

$$\begin{tikzcd} f^*(\Omega^{\text{ét}\, i}_{X/S}|_{U_{\text{zar}}}) \arrow[rr] \arrow[d, equal] & & \Omega^{\text{ét}\, i}_{X/S}|_{U'_{\text{zar}}} \arrow[d, equal] \\ f^*\Omega^i_{U/T} \arrow[r, "\sim"] & a^*\Omega^i_{U_{T'}/T'} \arrow[r, "\sim"] & \Omega^i_{U'/T'} \end{tikzcd}$$

Condition (2) follows, and therefore $\Omega^{\text{ét}\, i}_{X/S}$ is quasi-coherent. Because $F_{X/S}$ is affine, the pushforward of a quasi-coherent sheaf under $F_{X/S}$ is again quasi-coherent, and therefore $F_{X/S*}\Omega^{\text{ét}\, i}_{X/S}$ is quasi-coherent. Restriction to the small Zariski site is a left exact functor, so the natural map

$$Z^i(F_{X/S*}\Omega^{\text{ét}\, \bullet}_{X/S})|_{U_{\text{zar}}} \xrightarrow{\sim} Z^i(F_{X/S*}\Omega^{\text{ét}\, \bullet}_{X/S}|_{U_{\text{zar}}})$$

is an isomorphism. Thus, $Z^i(F_{X/S*}\Omega^{\text{ét}\, \bullet}_{X/S})$ satisfies condition (1). By Lemma 2.3.2, the latter sheaf is locally free of finite rank. Because $F_{X/S*}\Omega^{\text{ét}\, i}_{X/S}$ is quasi-coherent, $Z^i(F_{X/S*}\Omega^{\text{ét}\, \bullet}_{X/S})$ therefore satisfies condition (2), and hence is quasi-coherent. The cokernel of a map of quasi-coherent sheaves is quasi-coherent, so $B^i(F_{X/S*}\Omega^{\text{ét}\, \bullet}_{X/S})$ and $\mathscr{H}^i(F_{X/S*}\Omega^{\text{ét}\, \bullet}_{X/S})$ are quasi-coherent.

By Lemma 2.3.2, and the fact that the restriction to the small Zariski site is acyclic for quasi-coherent sheaves, it follows that these sheaves are locally free of finite rank. □

The *Cartier operator* (as described in Section 3 of [5], and Section 7 of [26]) defines for each i an isomorphism

$$C_{X/S}\colon \mathscr{H}^i(F_{X/S*}\Omega^{\bullet}_{X/S}) \xrightarrow{\sim} \Omega^i_{X^{(p)}/S} \tag{2.3.3.2}$$

of $\mathscr{O}_{X^{(p)}}$-modules.

Lemma 2.3.4. *The Cartier operator extends to an isomorphism*

$$C^{\text{ét}}_{X/S} \colon \mathscr{H}^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S}) \xrightarrow{\sim} \Omega^{\text{ét}\,i}_{X^{(p)}/S}$$

of sheaves of $\mathscr{O}_{X^{(p)}}$*-modules on* $(X^{(p)}/S)_{\text{ét}}$

Proof. To define such a map, it will suffice to give morphisms

$$\mathscr{H}^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S})|_{U_{\text{zar}}} \to \Omega^{\text{ét}\,i}_{X^{(p)}/S}|_{U_{\text{zar}}}$$

for each object (U, T) of $(X^{(p)}/S)_{\text{ét}}$ that behave functorially under pullback. Define U_{X_T} by the Cartesian diagram

$$\begin{tikzcd}
U_{X_T} \arrow[r] \arrow[d] & U \arrow[d] \\
X_T \arrow[r, "F_{X_T/T}"] & (X_T)^{(p/T)}
\end{tikzcd}$$

where we are using the natural isomorphism $(X^{(p/S)})_T \xrightarrow{\sim} (X_T)^{(p/T)}$ of Lemma 2.2.7. As in Lemma 2.2.8, we have a diagram

$$\begin{tikzcd}
U_{X_T} \arrow[r] \arrow[d, "F_{U_{X_T}/T}"', "\wr"] \arrow[dr, "F_{U_{X_T}/T}"] & U \arrow[d, "F_{U/X_T^{(p/T)}}", "\wr"'] & \\
(U_{X_T})^{(p/X_T)} \arrow[r] \arrow[d] & (U_{X_T})^{(p/T)} \arrow[d] & \\
X_T \arrow[r, "F_{X_T/T}"] \arrow[d] & (X_T)^{(p/T)} \arrow[r] \arrow[d] & T \arrow[d] \\
X \arrow[r, "F_{X/S}"] & X^{(p/S)} \arrow[r] & S
\end{tikzcd}$$

with Cartesian squares. Consider the composite arrow $f\colon U \to X^{(p/S)}$. We define a map

$C^{\text{ét}}_{X/S}|_{U_{\text{zar}}}$ by the diagram

$$
\begin{tikzcd}
F^*_{U_{X_T}/T}\mathscr{H}^i(F_{U_{X_T}/T*}\Omega^\bullet_{U_{X_T}/T}) \arrow[r, "C_{U_{X_T}/T}"] \arrow[d, "\wr"] & F^*_{U_{X_T}/T}\Omega^i_{U^{(p/T)}_{X_T}/T} \arrow[d, "\wr"] \\
f^*(\mathscr{H}^i(F_{X/S*}\Omega^\bullet_{X/S})) \arrow[d, "\wr"] & f^*(\Omega^i_{X^{(p)}/S}) \arrow[d, "\wr"] \\
f^*(\mathscr{H}^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S})|_{X^{(p/S)}_{\text{zar}}}) \arrow[d, "\wr"] & f^*(\Omega^{\text{ét}\,i}_{X^{(p)}/S}|_{X^{(p/S)}_{\text{zar}}}) \arrow[d, "\wr"] \\
\mathscr{H}^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S})|_{U_{\text{zar}}} \arrow[r, "C^{\text{ét}}_{X/S}|_{U_{\text{zar}}}"] & \Omega^{\text{ét}\,i}_{X^{(p)}/S}|_{U_{\text{zar}}}
\end{tikzcd}
$$

where the top horizontal arrow is the pullback of $C_{U_{X_T}/T}$ under the isomorphism $F_{U_{X_T}/T}$, and the two lower vertical arrows are the comparison morphisms, which are isomorphisms by Proposition 2.3.3. The explicit description of the Cartier operator in Section 7 of [26] shows that the morphisms $C^{\text{ét}}_{X/S}|_{U_{\text{zar}}}$ are compatible with pullback in the appropriate sense, and so we obtain the desired isomorphism $C_{X/S}$ of sheaves on the relative étale site.

□

We consider the map

$$d\log\colon \mathscr{O}^\times_X \to \Omega^{\text{ét}\,1}_{X/S} \tag{2.3.4.1}$$

of sheaves on $(X/S)_{\text{ét}}$ given by $f \mapsto df/f$. Note that the image of $d\log$ is contained in the subsheaf of closed forms. We also have a map

$$Z^1(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S}) \xrightarrow{W^*} F_{X/S*}W_*\Omega^{\text{ét}\,1}_{X^{(p)}/S} = F_{X^{(p)}*}\Omega^{\text{ét}\,1}_{X^{(p)}/S} \to \Omega^{\text{ét}\,1}_{X^{(p)}/S}$$

where the first map is the pushforward of the map W^* under $F_{X/S}$, and the second is the canonical map.

Lemma 2.3.5. *The sequence*

$$0 \to \mathscr{O}^\times_{X^{(p)}} \to F_{X/S*}\mathscr{O}^\times_X \xrightarrow{d\log} Z^1(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S}) \xrightarrow{C^{\text{ét}}_{X/S}-W^*} \Omega^{\text{ét}\,1}_{X^{(p)}/S} \to 0$$

of sheaves on $(X^{(p)}/S)_{\text{ét}}$ is exact.

Proof. Corollaire 2.1.18 of [25] states that the restriction of this sequence to the small étale site is exact. The result follows by definition of the sheaves involved. □

We next discuss the Hodge and conjugate spectral sequences in the setting of the relative étale site, which converge to the relative de Rham cohomology

$$\mathrm{H}^m_{dR}(X/S)_{\text{ét}} \stackrel{\text{def}}{=} \mathbf{R}^m \pi^{\text{ét}}_* \Omega^{\text{ét}\,\bullet}_{X/S}$$

of $X \to S$. The so-called naïve filtration on $\Omega^{\text{ét}\bullet}_{X/S}$ (see [5], and Chapter 2 of [19]) induces the Hodge spectral sequence

$$E^{p,q}_{H1} = \mathbf{R}^q \pi^{\text{ét}}_* \Omega^{\text{ét}\,p}_{X/S} \Longrightarrow \mathbf{R}^{p+q} \pi^{\text{ét}}_* \Omega^{\text{ét}\,\bullet}_{X/S} \tag{2.3.5.1}$$

On the other hand, we may compute the higher pushforwards of the de Rham complex using the Leray spectral sequence induced by the canonical factorization $\pi = \pi^{(p)} \circ F_{X/S}$. By Lemma 2.2.3 the functor $F_{X/S*}$ is exact on the relative étale site, so this gives a spectral sequence

$$E^{p,q}_{C2} = \mathbf{R}^p \pi^{(p)\text{ét}}_* \mathscr{H}^q(F_{X/S*} \Omega^{\text{ét}\,\bullet}_{X/S}) \Longrightarrow \mathbf{R}^{p+q} \pi^{\text{ét}}_* \Omega^{\text{ét}\,\bullet}_{X/S} \tag{2.3.5.2}$$

called the ***conjugate spectral sequence***. Fix an integer $m \geq 0$. The Hodge and conjugate spectral sequences induce filtrations

$$0 \subset F_H^{m,m} \subset F_H^{m-1,m} \subset \ldots \subset F_H^{i,m} \subset \ldots \subset F_H^{0,m} = \mathrm{H}^m_{dR}(X/S)_{\text{ét}} \tag{2.3.5.3}$$

$$0 \subset F_C^{m,m} \subset F_C^{m-1,m} \subset \ldots \subset F_C^{i,m} \subset \ldots \subset F_C^{0,m} = \mathrm{H}^m_{dR}(X/S)_{\text{ét}} \tag{2.3.5.4}$$

Remark 2.3.6. Working on the relative étale site gives a functorial packaging of the usual Hodge and conjugate spectral sequences on the small Zariski site. If T is a scheme over S, then the restriction of the spectral sequences (2.3.5.1,2.3.5.2) to the small Zariski site of T is naturally isomorphic to the usual Hodge and conjugate spectral sequences induced by the complex $\Omega^\bullet_{X_T/T}$. In particular, there is a natural identification

$$\mathbf{R}^q \pi^{\text{ét}}_* \Omega^{\text{ét}\,p}_{X/S}(T) = \Gamma(T, \mathbf{R}^q \pi_{T*} \Omega^p_{X_T/T})$$

and similarly for $E^{p,q}_{C2}$, $\mathrm{H}^m_{dR}(X/S)_{\text{ét}}$, and the filtrations $F_H^{i,m}$ and $F_C^{i,m}$.

We will make the following assumptions on π.

Definition 2.3.7. We say that the morphism π satisfies $(*)$ if

1. the Hodge spectral sequence degenerates at E_1,
2. the conjugate spectral sequence degenerates at E_2, and
3. the $\mathscr{O}_S$-modules $E_{H1}^{p,q}$, $E_{C2}^{p,q}$, $\mathrm{H}^m_{dR}(X/S)_{\text{ét}}$, $F_H^{i,m}$ and $F_C^{i,m}$ are all locally free of finite rank, in the sense of Definition 2.2.4.

Remark 2.3.8. If $\pi\colon X \to S$ is a relative K3 surface, then π satisfies $(*)$. This follows from the fact that the spectral sequences associated to the usual (small Zariski) De Rham cohomology degenerate, and are of formation compatible with base change.

Under these assumptions, the Hodge and conjugate spectral sequences are related by the Cartier operator.

Lemma 2.3.9. *If $\pi\colon X \to S$ satisfies $(*)$, then there are natural isomorphisms*

$$F_H^{i,m}/F_H^{i+1,m} \xrightarrow{\sim} E_{H1}^{i,m-i} = \mathbf{R}^{m-i}\pi_*^{\text{ét}}(\Omega^{\text{ét}\,i}_{X/S})$$

$$F_C^{i,m}/F_C^{i+1,m} \xrightarrow{\sim} E_{C2}^{i,m-i} = \mathbf{R}^{i}\pi_*^{(p)\text{ét}}(\mathscr{H}^{m-i}(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S}))$$

and the Cartier operator induces isomorphisms

$$F_C^{m-i,m}/F_C^{m-i+1,m} \xrightarrow{\sim} F_S^*(F_H^{i,m}/F_H^{i+1,m})$$

Proof. The first claim follows from the assumed degeneration of the Hodge and conjugate spectral sequences. For the second, we apply $\mathbf{R}\pi_*^{(p)\text{ét}}$ to both sides of (2.3.3.2) to get an isomorphism

$$\mathbf{R}\pi_*^{(p)\text{ét}}\mathscr{H}^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S}) \xrightarrow{\sim} \mathbf{R}\pi_*^{(p)\text{ét}}\Omega^{\text{ét}\,i}_{X^{(p)}/S}$$

Applying the base change isomorphism in the derived category and using our assumption that the cohomology sheaves of $\mathbf{R}\pi_*^{\text{ét}}\Omega^{\text{ét}\,i}_{X/S}$ are locally free, we find isomorphisms

$$\mathbf{R}\pi_*^{(p)\text{ét}}\mathscr{H}^i(F_{X/S*}\Omega^{\text{ét}\,\bullet}_{X/S}) \cong \mathbf{R}\pi_*^{(p)\text{ét}}\Omega^{\text{ét}\,i}_{X^{(p)}/S} \cong \mathbf{R}\pi_*^{(p)\text{ét}}W^*\Omega^{\text{ét}\,i}_{X/S} \cong F_S^*\mathbf{R}\pi_*^{\text{ét}}\Omega^{\text{ét}\,i}_{X/S}$$

Applying the isomorphisms of the previous claim gives the result. $\square$

For the remainder of this paper, we only consider the case $m = 2$, and so we omit m from the notation. We have sheaves F^1_H and F^1_C on the big étale site of S that are locally free of finite rank, and in particular quasi-coherent. By $F^1_H \cap F^1_C$ we will mean the fiber product of sheaves on the big étale site of S. This will not in general be a quasi-coherent sheaf.

Lemma 2.3.10. *If $X \to S$ satisfies $(*)$, then the natural map $\mathbf{R}^1\pi^{\text{ét}}_* Z^1\Omega^\bullet_{X/S} \to \mathbf{R}^2\pi^{\text{ét}}\Omega^\bullet_{X/S}$ induces an identification*

$$\mathbf{R}^1\pi^{\text{ét}}_* Z^1\Omega^\bullet_{X/S} \xrightarrow{\sim} F^1_H \cap F^1_C$$

Proof. We have truncations

$$\Omega^{\text{ét}\,\geq i}_{X/S} = \left[\ldots \to 0 \to \Omega^{\text{ét}\,i}_{X/S} \to \Omega^{\text{ét}\,i+1}_{X/S} \to \ldots\right]$$

where $\Omega^{\text{ét}\,i}_{X/S}$ is placed in degree i. There is an obvious map of complexes $\Omega^{\text{ét}\,\geq i}_{X/S} \to \Omega^{\text{ét}\,\bullet}_{X/S}$, and the image of the induced map $\mathbf{R}^2\pi^{\text{ét}}_*\Omega^{\text{ét}\,\geq i}_{X/S} \to \mathbf{R}^2\pi^{\text{ét}}_*\Omega^{\text{ét}\,\bullet}_{X/S}$ is the i-th level of the Hodge filtration. There is an obvious map $Z^1\Omega^{\text{ét}\,\bullet}_{X/S}[-1] \to \Omega^{\text{ét}\,\geq 1}_{X/S}$, which induces an exact sequence

$$0 \to Z^1\Omega^{\text{ét}\,\bullet}_{X/S}[-1] \to \Omega^{\text{ét}\,\geq 1}_{X/S} \to Q^\bullet \to 0$$

Therefore, we get a triangle

$$\mathbf{R}\pi^{\text{ét}}_* Z^1\Omega^{\text{ét}\,\bullet}_{X/S}[-1] \to \mathbf{R}\pi^{\text{ét}}_*\Omega^{\text{ét}\,\geq 1}_{X/S} \to \mathbf{R}\pi^{\text{ét}}_* Q^\bullet \to$$

Note that $\mathscr{H}^i(Q^\bullet) = 0$ for $i \leq 1$, so by the spectral sequence associated to the canonical filtration on $Q^\bullet$ we get that $\mathbf{R}^1\pi^{\text{ét}}_* Q^\bullet = 0$ and $\mathbf{R}^2\pi^{\text{ét}}_* Q^\bullet \xrightarrow{\sim} \mathbf{R}^0\pi^{\text{ét}}_*\mathscr{H}^2(\Omega^{\text{ét}\,\bullet}_{X/S})$. Therefore, we get an exact sequence

$$0 \to \mathbf{R}^1\pi^{\text{ét}}_* Z^1\Omega^{\text{ét}\,\bullet}_{X/S} \to F^1_H \to F^0_C/F^1_C$$

This gives the result. $\square$

Proposition 2.3.11. *If $X \to S$ satisfies $(*)$ and $\pi^{\text{ét}}_*\Omega^{\text{ét}\,1}_{X/S} = 0$, then there is an exact sequence*

$$0 \to \mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1) \to F^1_H \cap F^1_C \xrightarrow{C\circ\pi_C - F^*_S\circ\pi_H} F^*_S(F^1_H/F^2_H)$$

of sheaves of abelian groups on the big étale site of S.

Proof. Here, the arrow $C \circ \pi_C$ is the projection $F_C^1 \to F_C^1/F_C^2$ followed by the isomorphism $C_{X/S}^{\text{ét}} \colon F_C^1/F_C^2 \xrightarrow{\sim} F_S^*(F_H^1/F_H^2)$, π_H is the projection $F_H^1 \to F_H^1/F_H^2$, and F_S^* is the canonical p-linear map $F_H^1/F_H^2 \to F_S^*(F_H^1/F_H^2)$ given by $s \mapsto s \otimes 1$.

We apply $\mathbf{R}\pi_*^{(p)\text{ét}}$ to the sequence

$$0 \to \nu(1) \to F_{X/S*} Z^1 \Omega_{X/S}^{\text{ét}\,\bullet} \to \Omega_{X^{(p)}/S}^{\text{ét}\,1} \to 0$$

of Lemma 2.3.5 to get

$$0 \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \to \mathbf{R}^1\pi_*^{\text{ét}} Z^1 \Omega_{X/S}^{\text{ét}\,\bullet} \to F_S^*(\mathbf{R}^1\pi_*^{\text{ét}}\Omega_{X/S}^{\text{ét}\,1})$$

Applying Lemma 2.3.10, we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \arrow[r] \arrow[d, equal] & \mathbf{R}^1\pi_*^{\text{ét}} Z^1 \Omega_{X/S}^{\text{ét}\,\bullet} \arrow[r, "C_{X/S}^{\text{ét}} - W^*"] \arrow[d, "\wr"] & F_S^*(\mathbf{R}^1\pi_*^{\text{ét}}\Omega_{X/S}^{\text{ét}\,1}) \arrow[d, "\wr"] \\
0 \arrow[r] & \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \arrow[r] & F_H^1 \cap F_C^1 \arrow[r, "C\circ\pi_C - F_S^*\circ\pi_H"] & F_S^*(F_H^1/F_H^2)
\end{tikzcd}$$

which gives the result. $\square$

Chapter 3

PERIODS

3.0.1 Characteristic subspaces and the crystalline period domain

In this section we consider the crystalline analog of the period domain, Ogus's moduli space of characteristic subspaces. We study certain rational fibrations relating the period spaces of different Artin invariants. We begin by recalling some definitions and results having to do with bilinear forms on vector spaces over $\mathbf{F}_p$. Recall that we are assuming $p \geq 3$.

Definition 3.0.1. Let $U_2 \otimes \mathbf{F}_p$ denote the *hyperbolic plane over* $\mathbf{F}_p$, which is the $\mathbf{F}_p$-vector space generated by the two vectors v, w which satisfy $v^2 = w^2 = 0$ and $v.w = -1$.

This notation is explained by Definition 3.3.4.

Proposition 3.0.2. *[18, Theorem 4.9] Let V be a vector space over $\mathbf{F}_p$ of even dimension $2\sigma_0$, equipped with a non-degenerate $\mathbf{F}_p$-valued bilinear form. If there exists a totally isotropic subspace of V of dimension σ_0, then V is isometric to the orthogonal sum of σ_0 copies of $U_2 \otimes \mathbf{F}_p$. If there does not exist such a subspace, then V is isometric to the orthogonal sum of $\sigma_0 - 1$ copies of $U_2 \otimes \mathbf{F}_p$ and one copy of $\mathbf{F}_{p^2}$.*

Here, we view $\mathbf{F}_{p^2}$ as a two dimensional vector space over $\mathbf{F}_p$ equipped with the quadratic form coming from the trace. In the former case, we say that the form on V is *neutral*, and in the latter we say that it is *non-neutral*.

Theorem 3.0.3. *[18, Theorem 5.2] If V is a vector space over a field of characteristic not equal to 2 equipped with a non-degenerate bilinear form, then any isometry $W \to W'$ between two subspaces of V extends to an isometry $V \to V$.*

In particular, this implies that the group of isometries of V acts transitively on the set of totally isotropic subspaces of any given dimension.

Let V be a vector space over $\mathbf{F}_p$ equipped with a non-degenerate, non-neutral bilinear form. Let S be a scheme over $\mathbf{F}_p$. By viewing V as a coherent sheaf on $\operatorname{Spec}\mathbf{F}_p$, we see that there is a canonical isomorphism

$$F_S^*(V \otimes \mathscr{O}_S) \xrightarrow{\sim} V \otimes \mathscr{O}_S$$

Precomposing with the canonical p-linear map $V \otimes \mathscr{O}_S \to F_S^*(V \otimes \mathscr{O}_S)$, we obtain a map of sheaves

$$F_S^*\colon V \otimes \mathscr{O}_S \to V \otimes \mathscr{O}_S$$

This map is given on sections by $v \otimes s \mapsto v \otimes s^p$. When $S = \operatorname{Spec} k$, we will denote this map by φ.

Definition 3.0.4. [40, Section 4] A *characteristic subspace* of $V \otimes \mathscr{O}_S$ is a submodule $K \subset V \otimes \mathscr{O}_S$ such that

1. both K and $K + F_S^* K$ are locally free and locally direct summands of $V \otimes \mathscr{O}_S$,
2. K is totally isotropic of rank σ_0, and
3. $K + F_S^* K$ has rank $\sigma_0 + 1$.

We define a functor M_V on schemes over $\mathbf{F}_p$ by letting $M_V(S)$ be the set of characteristic subspaces of $V \otimes \mathscr{O}_S$.

The scheme M_V is studied in Section 4 of [40], where it is shown to be smooth and projective over $\mathbf{F}_p$ of dimension $\sigma_0 - 1$. Moreover, M_V is irreducible, and has no $\mathbf{F}_p$-points. The base change $M_V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}_p}$ has two irreducible connected components, which are defined over $\mathbf{F}_{p^2}$ and are interchanged by the action of the Galois group.

Remark 3.0.5. For small values of σ_0, the scheme $M_V \otimes_{\mathbf{F}_p} \mathbf{F}_{p^2}$ admits the following descriptions.

- If $\sigma_0 = 1$, $M_V \otimes_{\mathbf{F}_p} \mathbf{F}_{p^2}$ is a disjoint union of two copies of $\operatorname{Spec}\mathbf{F}_{p^2}$.
- If $\sigma_0 = 2$, $M_V \otimes_{\mathbf{F}_p} \mathbf{F}_{p^2}$ is isomorphic to a disjoint union of two copies of $\mathbf{P}^1_{\mathbf{F}_{p^2}}$.

- If $\sigma_0 = 3$, $M_V \otimes_{\mathbf{F}_p} \mathbf{F}_{p^2}$ is isomorphic to a disjoint union of two copies of the Fermat surface

$$V(x^{p+1} + y^{p+1} + z^{p+1} + w^{p+1}) \subset \mathbf{P}^3_{\mathbf{F}_{p^2}}$$

See Example 4.7 of [40]. The last case is proven in Proposition 5.14 of [15].

By definition, M_V comes with a tautological sub-bundle $K_V \subset V \otimes \mathscr{O}_{M_V}$, and the pairing induces a diagram

$$\begin{tikzcd} 0 \arrow[r] & K_V \arrow[r] \arrow[d, "\wr"] & V \otimes \mathscr{O}_{M_V} \arrow[r] \arrow[d, "\wr"] & Q_V \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & Q_V^* \arrow[r] & V^* \otimes \mathscr{O}_{M_V} \arrow[r] & K_V^* \arrow[r] & 0 \end{tikzcd} \tag{3.0.5.1}$$

Definition 3.0.6. The *Artin invariant* of a characteristic subspace $K \subset V \otimes k$ is

$$\sigma_0(K) = \sigma_0 - \dim_{\mathbf{F}_p}(K \cap V)$$

The subspace K is *strictly characteristic* if $\sigma_0(K) = \sigma_0$.

Definition 3.0.7. Let $W \subset V$ be a subspace. We consider the Cartesian diagram

$$\begin{tikzcd} \underline{K}_V \cap \underline{W} \arrow[r] \arrow[d] & \underline{W} \arrow[d] \\ \underline{K}_V \arrow[r] & \underline{V} \otimes \mathscr{O}_{M_V} \end{tikzcd}$$

of schemes over M_V, where $\underline{K}_V$ is the vector bundle on M_V associated to K_V, and $\underline{W}$ is the associated constant group scheme. The zero section $Z \subset \underline{K}_V \cap \underline{W}$ is open and closed, and so the morphism $\underline{K}_V \cap \underline{W} \setminus Z \to M_V$ is closed. We let $M_V^W \subset M_V$ be the open complement of the image of this morphism. A k-point $[K] \in M_V(k)$ lies in M_V^W exactly when the intersection of K and W inside of $V \otimes k$ is trivial. In particular, we set $U_V = M_V^V$. This is the locus of points with maximal Artin invariant.

We let $M_{V,W} \subset M_V$ denote the closed complement of the union of the $M_V^{\langle w \rangle}$ as w ranges over the non-zero vectors in W. We give $M_{V,W}$ the reduced induced closed subscheme structure. A k-point $[K] \in M_V(k)$ lies in $M_{V,W}$ exactly when $W \subset K$.

Notation 3.0.8. For the remainder of this section, the following notation will be consistently used.

- We let $\widetilde{V}$ be a fixed vector space over $\mathbf{F}_p$ of dimension $2\sigma_0 + 2$ equipped with a non-degenerate non-neutral bilinear form.

- We let $v \in \widetilde{V}$ be a fixed vector that is *isotropic*, meaning that $v \neq 0$ and $v^2 = 0$.

- We set $V = v^\perp / v$. This is a vector space of dimension $2\sigma_0$, and is equipped with a natural bilinear form which is again non-degenerate and non-neutral.

- We fix an orthogonal decomposition $\widetilde{V} = V \oplus (U_2 \otimes \mathbf{F}_p)$, where $U_2 \otimes \mathbf{F}_p$ is defined in Definition 3.0.1 (such a decomposition exists for any v by Theorem 3.0.3).

Lemma 3.0.9. *Let S be a scheme over $\mathbf{F}_p$ and let $\widetilde{K} \subset \widetilde{V} \otimes \mathscr{O}_S$ be a characteristic subspace. If for every geometric point* $\operatorname{Spec} k \to S$ *the fiber $\widetilde{K} \otimes k \subset \widetilde{V} \otimes k$ does not contain* $v(= v \otimes 1)$, *then the image K of $\widetilde{K} \cap (v^\perp \otimes \mathscr{O}_S)$ in $V \otimes \mathscr{O}_S$ is a characteristic subspace, whose formation is compatible with arbitrary base change.*

Proof. Every geometric fiber of K is a maximal totally isotropic subspace of $\widetilde{V} \otimes k$. Therefore by Nakayama's Lemma the map of sheaves $\widetilde{K} \to \mathscr{O}_S$ given by pairing with v is surjective, and we have a short exact sequence

$$0 \to \widetilde{K} \cap (v^\perp \otimes \mathscr{O}_S) \to \widetilde{K} \to \mathscr{O}_S \to 0$$

This shows that $\widetilde{K} \cap (v^\perp \otimes \mathscr{O}_S)$ is a locally free quasi-coherent sheaf of rank σ_0, whose formation is compatible with arbitrary base change. The quotient map $v^\perp \to v^\perp / v = V$ induces an isomorphism

$$\widetilde{K} \cap (v^\perp \otimes \mathscr{O}_S) \xrightarrow{\sim} (\widetilde{K} \cap v^\perp \otimes \mathscr{O}_S)/(v \otimes \mathscr{O}_S) = K$$

Thus, K is locally free of rank σ_0, is locally a direct summand, and is compatible with arbitrary base change. It is clear that K is totally isotropic. The same reasoning applied to $\widetilde{K} + F^*\widetilde{K}$ shows that $(\widetilde{K} + F^*\widetilde{K}) \cap (v^\perp \otimes \mathscr{O}_S)$ and its image under the quotient map are direct summands of rank $\sigma_0 + 1$. $\square$

Note that this condition on the geometric fibers of $\widetilde{K}$ is equivalent to the induced map $S \to M_{\widetilde{V}}$ factoring through $M_{\widetilde{V}}^{\langle v \rangle}$. Because of the compatibility with respect to base change, the association $\widetilde{K} \mapsto K$ defines a morphism

$$\pi_v \colon M_{\widetilde{V}}^{\langle v \rangle} \to M_V \tag{3.0.9.1}$$

This morphism was also studied by Liedtke in [33].

Definition 3.0.10. We define a group scheme $\mathscr{U}_V$ on M_V by the kernel of the map

$$\frac{\underline{V} \otimes \mathscr{O}_{M_V}}{\underline{K}_V} \xrightarrow{1-F_{M_V}^*} \frac{\underline{V} \otimes \mathscr{O}_{M_V}}{\underline{K}_V + F_{M_V}^* \underline{K}_V}$$

of group schemes over M_V.

Proposition 3.0.11. *There is an isomorphism $\mathscr{U}_V \xrightarrow{\sim} M_{\widetilde{V}}^{\langle v \rangle}$ of schemes over M_V.*

Proof. We will first define a morphism $f \colon \mathscr{U}_V \to M_{\widetilde{V}}^{\langle v \rangle}$. That is, we will give a characteristic subspace of $\widetilde{V} \otimes \mathscr{O}_{\mathscr{U}_{\widetilde{V}}}$ whose intersection with $\langle v \rangle \otimes \mathscr{O}_{M_{\widetilde{V}}^{\langle v \rangle}}$ is trivial. By definition, we have an exact sequence

$$0 \to \mathscr{U}_V \to \frac{\underline{V} \otimes \mathscr{O}_{M_V}}{\underline{K}_V} \xrightarrow{1-F_{M_V}^*} \frac{\underline{V} \otimes \mathscr{O}_{M_V}}{\underline{K}_V + F_{M_V}^* \underline{K}_V}$$

of group schemes over M_V. On $\mathscr{U}_V$, there is a tautological section

$$\mathscr{B}_V \in \Gamma\left(\mathscr{U}_V, \left.\frac{V \otimes \mathscr{O}_{M_V}}{K_V}\right|_{\mathscr{U}_V}\right)$$

satisfying $(1 - F^*)\mathscr{B}_V \in K_V + F^* K_V$. Consider the sub-bundle $\widetilde{K} \subset \widetilde{V} \otimes \mathscr{O}_{\mathscr{U}_V}$ defined locally as

$$\widetilde{K} = \left\langle x_1 + (x_1.B)v, \ldots, x_{\sigma_0} + (x_{\sigma_0}.B)v, w + B + \frac{B^2}{2} v \right\rangle \subset \widetilde{V} \otimes \mathscr{O}_{\mathscr{U}_V}$$

where $x_1, \ldots, x_{\sigma_0}$ is a local basis for $K_V|_{\mathscr{U}_V}$, and B denotes a local lift of the tautological section $\mathscr{B}_V$ to $V \otimes \mathscr{O}_{\mathscr{U}_V}$. Note that as K_V is totally isotropic, this does not depend on our choice of B. By construction, $\widetilde{K}$ is locally free of rank $\sigma_0 + 1$, is locally a direct summand, and is totally isotropic. Using the defining property of $\mathscr{B}_V$, and that K_V is characteristic, it follows that $\widetilde{K}$ is characteristic as well. Thus, we get a morphism $f \colon \mathscr{U}_V \to M_{\widetilde{V}}^{\langle v \rangle}$, which is compatible with the respective morphisms to M_V.

Let us construct an inverse. We seek a global section of $\mathscr{U}_V|_{M_{\widetilde{V}}^{\langle v \rangle}}$. Pairing with -$v$ gives a short exact sequence

$$0 \to K_{\widetilde{V}}|_{M_{\widetilde{V}}^{\langle v \rangle}} \cap (v^\perp \otimes \mathscr{O}_{M_{\widetilde{V}}^{\langle v \rangle}}) \to K_{\widetilde{V}}|_{M_{\widetilde{V}}^{\langle v \rangle}} \xrightarrow{\langle \_, -v \rangle} \mathscr{O}_{M_{\widetilde{V}}^{\langle v \rangle}} \to 0$$

and the projection $v^\perp \to V$ induces an isomorphism

$$K_{\widetilde{V}}|_{M_{\widetilde{V}}^{\langle v\rangle}} \cap (v^\perp \otimes \mathscr{O}_{M_{\widetilde{V}}^{\langle v\rangle}}) \xrightarrow{\sim} \pi_v^* K_V$$

Thus, letting $\rho\colon \widetilde{V} \to V$ denote the projection, we have maps

$$\mathscr{O}_{M_{\widetilde{V}}^{\langle v\rangle}} \xrightarrow{\sim} \frac{K_{\widetilde{V}}|_{M_{\widetilde{V}}^{\langle v\rangle}}}{K_{\widetilde{V}}|_{M_{\widetilde{V}}^{\langle v\rangle}} \cap (v^\perp \otimes \mathscr{O}_{M_{\widetilde{V}}^{\langle v\rangle}})} \xrightarrow{\rho} \frac{\rho(K_{\widetilde{V}}|_{M_{\widetilde{V}}^{\langle v\rangle}})}{\pi_v^* K_V} \subset \frac{V \otimes \mathscr{O}_{M_{\widetilde{V}}^{\langle v\rangle}}}{\pi_v^* K_V}$$

The image of $1 \in \Gamma(M_{\widetilde{V}}^{\langle v\rangle}, \mathscr{O}_{M_{\widetilde{V}}^{\langle v\rangle}})$ under the left isomorphism can be lifted locally to a section of $K_{\widetilde{V}}$ of the form $w + b + \frac{b^2}{2}v$ where b is a section of $V \otimes \mathscr{O}_V$, and the above composition sends 1 to b. Notice that

$$(1 - F^*)\left(w + b + \frac{b^2}{2}v\right) \in (K_{\widetilde{V}} + F^* K_{\widetilde{V}}) \cap (v^\perp \otimes \mathscr{O}_{M_{\widetilde{V}}})$$

Because v is fixed by the Frobenius, the projection $v^\perp \to V$ also induces an isomorphism

$$(K_{\widetilde{V}} + F^* K_{\widetilde{V}}) \cap (v^\perp \otimes \mathscr{O}_{M_{\widetilde{V}}}) \xrightarrow{\sim} \pi_v^*(K_V + F^* K_V)$$

Thus, the image of our global section under the map

$$\frac{V \otimes \mathscr{O}_{M_{\widetilde{V}}^{\langle v\rangle}}}{\pi_v^* K_V} \xrightarrow{1-F^*} \frac{V \otimes \mathscr{O}_{M_{\widetilde{V}}^{\langle v\rangle}}}{\pi_v^*(K_V + F^* K_V)}$$

is zero. Therefore, b gives a global section of $\mathscr{U}_V|_{M_{\widetilde{V}}^{\langle v\rangle}}$. We have constructed a morphism $g\colon M_{\widetilde{V}}^{\langle v\rangle} \to \mathscr{U}_V$ over M_V. One checks that this map is an inverse to f, and this gives the result. $\square$

Remark 3.0.12. Via crystalline cohomology, the period domain $M_{\widetilde{V}}$ classifies supersingular K3 surfaces (in a sense that is made precise by Ogus's crystalline Torelli theorem). In the following sections, we will interpret $\mathscr{U}_V$ as the connected component of the relative second cohomology of $\boldsymbol{\mu}_p$ on the universal K3 surface. That is, $\mathscr{U}_V$ classifies (certain) twisted supersingular K3 surfaces. The isomorphism of Proposition 3.0.11 therefore suggests that there should be a relationship between the collection of supersingular K3 surfaces of Artin invariant $\sigma_0 + 1$ and the collection of twisted supersingular K3 surfaces whose coarse space has Artin invariant σ_0. A central goal of this paper is to find a geometric interpretation of this isomorphism.

Lemma 3.0.13. *The morphisms $\mathscr{U}_V \to M_V$ and $\pi_v \colon M_{\tilde{V}}^{\langle v \rangle} \to M_V$ are smooth.*

Proof. We will show that $\mathscr{U}_V \to M_V$ is smooth. Because the differential of the Frobenius vanishes, the morphism

$$1 - F_{M_V}^* : \frac{\underline{V} \otimes \mathscr{O}_{M_V}}{\underline{K}_V} \to \frac{\underline{V} \otimes \mathscr{O}_{M_V}}{\underline{K}_V + F_{M_V}^* \underline{K}_V}$$

is smooth. But $\mathscr{U}_V \to M_V$ is the pullback of this morphism along the zero section. By Proposition 3.0.11, π_v is smooth as well. $\square$

By definition, M_V comes equipped with two natural line bundles $K_V / K_V \cap F^* K_V$ and $F^* K_V / K_V \cap F^* K_V$. The pairing on V induces an isomorphism

$$\frac{K_V}{K_V \cap F^* K_V} \otimes \frac{F^* K_V}{K_V \cap F^* K_V} \xrightarrow{\sim} \mathscr{O}_{M_V}$$

and so these line bundles are naturally dual. Consider the open subset $U_V \subset M_V$ parameterizing strictly characteristic subspaces. Write $U = U_V$, $K_U = K_V|_U$, and $\mathscr{U}_U = \mathscr{U}_V|_U$. As discussed in Section 4 of [40], the subsheaf

$$\mathscr{L}_U = K_U \cap F_U^* K_U \cap \ldots \cap F_U^{*\sigma_0 - 1} K_U$$

is locally free of rank 1, and we have

$$\mathscr{L}_U + F^* \mathscr{L}_U + \cdots + F^{*\sigma_0 - 1} \mathscr{L}_U = F^{*\sigma_0 - 1} K_U \tag{3.0.13.1}$$

$$\mathscr{L}_U + F^* \mathscr{L}_U + \cdots + F^{*2\sigma_0 - 1} \mathscr{L}_U = V \otimes \mathscr{O}_M \tag{3.0.13.2}$$

Furthermore, the natural map

$$\mathscr{L}_U \to F^{*\sigma_0 - 1} \left(\left. \frac{K_V}{K_V \cap F^* K_V} \right|_U \right)$$

is an isomorphism, and the pairing on V induces an isomorphism

$$\mathscr{L}_U \otimes F^{*\sigma_0} \mathscr{L}_U \xrightarrow{\sim} \mathscr{O}_U$$

Consider the composition

$$F^{*\sigma_0 - 1} \mathscr{U}_U \subset \frac{\underline{V} \otimes \mathscr{O}_U}{F^{*\sigma_0 - 1} \underline{K}_U} \xrightarrow{\sim} (F^{*\sigma_0 - 1} \underline{K}_U)^\vee \to \underline{\mathscr{L}}_U^\vee \tag{3.0.13.3}$$

Lemma 3.0.14. *The map $F^{*\sigma_0 - 1} \mathscr{U}_U \to \underline{\mathscr{L}}_U^\vee$ (3.0.13.3) is an isomorphism of group schemes.*

Proof. We claim that it suffices to show that (3.0.13.3) is injective (as a map of group schemes). Indeed, we may check isomorphy on geometric fibers. By Lemma 3.0.13, $F^{*\sigma_0-1}\mathscr{U}_U$ is a smooth, p-torsion group scheme of relative dimension 1. Hence, its geometric fibers are abstractly isomorphic to a disjoint union of copies of $\mathbf{A}^1$. Of course, the geometric fibers of $\underline{\mathscr{L}}_U^\vee$ are also isomorphic to $\mathbf{A}^1$. The claim follows from the fact that any monomorphism $\mathbf{A}^1 \to \mathbf{A}^1$ is an isomorphism.

Suppose that S is a scheme over M_V, and consider a section in $F^{*\sigma_0-1}\mathscr{U}_U(S)$ whose image is equal to 0. Passing to an open cover of S, we may assume that $\mathscr{L}_U$ is trivial, generated by some section $e \in \Gamma(S, \mathscr{L}_U|_S)$, and that our section lifts to some element $b \in \Gamma(S, V \otimes \mathscr{O}_S)$. We have that $b.e = 0$. By assumption,

$$b - F^*(b) \in \Gamma(S, F^{*\sigma_0-1}K + F^{*\sigma_0}K)$$

and therefore $b - F^*(b)$ is perpendicular to $F^{*\sigma_0-1}K \cap F^{*\sigma_0}K$. It follows that

$$b.F^*(e) = (b - F^*(b)).F^*(e) + F^*(b).F^*(e) = F^*(b).F^*(e) = F^*(e.b) = 0$$

Similarly, we find $b.F^{*i}(e) = 0$ for $0 \leq i \leq \sigma_0 - 1$. By (3.0.13.1), b is orthogonal to $F^{\sigma_0-1}K$. As $F^{*\sigma_0-1}K$ is a maximal totally isotropic sub-bundle, it follows that $b \in \Gamma(S, F^{*\sigma_0-1}K)$. This means that our original section was equal to 0. □

Lemma 3.0.15. *If $K \subset V \otimes k$ is a characteristic subspace of Artin invariant σ, then the fiber of $\mathscr{U}_V$ over $[K]$ is isomorphic to $\mathbf{A}^1 \times (\mathbf{Z}/p\mathbf{Z})^{\oplus \sigma_0 - \sigma}$. In particular, the group scheme $\mathscr{U}_U$ is an $\mathbf{A}^1$-bundle (that is, a smooth morphism whose geometric fibers are isomorphic to $\mathbf{A}^1$).*

Proof. The last claim is implied by Lemma 3.0.14. For the first claim, consider $\cap_i \varphi^i(K) \subset V \otimes k$. This is fixed by φ, hence is equal to $W \otimes k$ for some totally isotropic subspace $W \subset V$, and has dimension $\sigma_0 - \sigma$. Using Theorem 3.0.3, we find a subspace $V_0 \subset V$ such that the form on V_0 is neutral and nondegenerate and $W \subset V_0$ is a maximal totally isotropic subspace. There is a direct sum decomposition $V = V_0 \oplus V/V_0$ and an induced isomorphism

$$\frac{V \otimes k}{K} \cong \frac{V_0 \otimes k}{K \cap (V_0 \otimes k)} \oplus \frac{V \otimes k / V_0 \otimes k}{(K + V_0 \otimes k)/V_0 \otimes k}$$

Because $W \otimes k$ is a maximal totally isotropic subspace of $V_0 \otimes k$, and K is totally isotropic, $W \otimes k = K \cap (V_0 \otimes k)$. On the other hand, $K \cap V = K \cap V_0 = W$. It follows that that

$K \cap (V_0 \otimes k) = (K \cap V_0) \otimes k$, so the first term is isomorphic to $(V_0/W) \otimes k$. The kernel of $1 - F$ on this term is therefore $(\mathbf{Z}/p\mathbf{Z})^{\oplus \sigma_0 - \sigma}$. In the second term, note that the subspace $(K + V_0 \otimes k)/V_0 \otimes k \subset (V/V_0) \otimes k$ has trivial intersection with the $\mathbf{F}_p$-subspace V/V_0 and therefore is strictly characteristic. $\square$

Remark 3.0.16. By Lemma 3.0.15, $\mathscr{U}_U \to U$ is an $\mathbf{A}^1$-bundle. Furthermore, its Frobenius pullback $F^{*\sigma_0 - 1}\mathscr{U}_U$ is isomorphic to a line bundle, and hence locally trivial in the Zariski topology. We do not expect this to be the case for $\mathscr{U}_U$ as long as $\sigma_0 \geq 2$. We can verify this for $\sigma_0 = 2$ using the explicit descriptions in Remark 3.0.5. Indeed, the $\sigma_0 = 2$ period space is geometrically a disjoint union of two copies of $\mathbf{P}^1$, and the group scheme $\mathscr{U}_U$ over the $\sigma_0 = 2$ period space is isomorphic to an open subset of the $\sigma_0 = 3$ period space. Hence, if $\mathscr{U}_U$ were Zariski locally trivial, then the $\sigma_0 = 3$ period space would be (geometrically) rational. But, by Remark 3.0.5, the $\sigma_0 = 3$ period space is not geometrically rational for any $p \geq 3$.

We have the following consequence of Lemma 3.0.14.

Proposition 3.0.17. *Each connected component of $M_V \otimes_{\mathbf{F}_p} \mathbf{F}_{p^2}$ is purely inseparably unirational over $\mathbf{F}_{p^2}$.*

Proof. We will induct on σ_0. The cases $\sigma_0 = 1$ and $\sigma_0 = 2$ are covered by Remark 3.0.5. Suppose that the result is true for some σ_0. By Proposition 3.0.11, $\mathscr{U}_V$ is isomorphic to an open subset of $M_{\widetilde{V}}$. Choosing an open subset of M_V where $\mathscr{L}_U$ is trivial, we find by Lemma 3.0.14 a birational map

$$M_{\widetilde{V}}^{(p^{\sigma_0 - 1}/M)} \dashrightarrow M_V \times \mathbf{A}^1 \tag{3.0.17.1}$$

where $M_{\widetilde{V}}^{(p^{\sigma_0 - 1}/M)}$ is defined by the Cartesian diagram

$$\begin{tikzcd} M_{\widetilde{V}}^{(p^{\sigma_0-1}/M)} \arrow[r, "W^{\sigma_0-1}"] \arrow[d] & M_V^{\langle v \rangle} \arrow[d, "\pi_v"] \\ M_V \arrow[r, "F_{M_V}^{\sigma_0-1}"] & M_V \end{tikzcd}$$

Composing the inverse of (3.0.17.1) with $W^{\sigma_0 - 1}$, we find a dominant rational map

$$M_V \times \mathbf{A}^1 \dashrightarrow M_{\widetilde{V}}$$

over M_V which is generically finite and purely inseparable on fibers. The result follows by induction. $\square$

Remark 3.0.18. In Proposition 10.3 of [43] it is shown that M_V is unirational. We have not checked whether their unirational parameterization is related to ours.

Remark 3.0.19. Here is a strange observation. If $p = 3$, then by Remark 3.0.5 the $\sigma_0 = 3$ period space is geometrically a disjoint union of two copies of the Fermat quartic, which is known to be isomorphic to the unique supersingular K3 surface X_1 of Artin invariant 1. The above gives an explicit description of a number of quasi-elliptic fibrations $X_1 \to \mathbf{P}^1$.

3.0.2 Twistor lines

We will fix for the remainder of this paper an algebraically closed field k of characteristic $p > 0$. We will continue to assume that $p \geq 3$, unless explicitly noted otherwise.

Notation 3.0.1. We write

$$\overline{M}_V = M_V \times_{\operatorname{Spec} \mathbf{F}_p} \operatorname{Spec} k$$

for the base change of $\overline{M}_V$ to $\operatorname{Spec} k$. Equivalently, this is the functor on the category of schemes over k whose value $\overline{M}_V(S)$ on a k-scheme S is the set of characteristic subspaces of $V \otimes \mathscr{O}_S$. We let $\overline{M}_V^W$ and $\overline{M}_{V,W}$ be the base changes of the subschemes defined in Definition 3.0.7, and we will denote the base change of the universal bundle K_V and the morphism π_v by the same symbols.

It is shown in [40] that $\overline{M}_V$ has two connected components, each of which is defined over $\mathbf{F}_{p^2}$. We maintain Notation 3.0.8, so that $\widetilde{V}$ is a vector space over $\mathbf{F}_p$ of dimension $2\sigma_0 + 2$ equipped with a non-degenerate, non-neutral bilinear form.

Definition 3.0.2. A *twistor line* in $\overline{M}_{\widetilde{V}}$ is a subvariety $L \subset \overline{M}_{\widetilde{V}}$ that is a connected component of a fiber of π_v over a k-point $[K] \in \overline{M}_{v^\perp/v}(k)$ for some isotropic $v \in \widetilde{V}$.

By Lemma 3.0.15, any twistor line is isomorphic to $\mathbf{A}^1$. We will eventually construct certain families of twisted K3 surfaces whose periods correspond to the twistor lines. In this section, we will record some facts about the geometry of twistor lines in $\overline{M}_{\widetilde{V}}$.

Remark 3.0.3. Fix a characteristic subspace $K \subset V \otimes k$. Let $U(K)$ be the group of k-points of the fiber of $\mathscr{U}_V$ over $[K] \in \overline{M}_V(k)$, so that

$$U(K) = \{B \in V \otimes k | B - \varphi(B) \in K + \varphi(K)\} / K$$

Under the isomorphism $U(K) \xrightarrow{\sim} \pi_v^{-1}([K])(k)$ of Proposition 3.0.11, we obtain an explicit description of the fiber of π_v over $[K]$, which is useful for making computations. Let $\{x_1, \ldots, x_{\sigma_0}\}$ be a basis for K and $B \in V \otimes k$ an element such that $B - \varphi(B) \in K + \varphi(K)$. Set

$$K(B) = \left\langle x_1 + (x_1.B)v, \ldots, x_{\sigma_0} + (x_{\sigma_0}.B)v, w + B + \frac{B^2}{2}w \right\rangle \subset \widetilde{V} \otimes k$$

Then $K(B)$ determines (and is determined by) B modulo K, and $K(B)$ is the characteristic subspace corresponding to (the image of) B in $U(K)$. If K is strictly characteristic, then a further description of the fiber $\pi_v^{-1}([K])$ is implied by Lemma 3.0.14. Specifically, if e is a generator for the line

$$l_K = K \cap \varphi(K) \cap \cdots \cap \varphi^{\sigma_0 - 1}(K)$$

then the map

$$B \mapsto B.\varphi^{-\sigma_0+1}(e)$$

gives a bijection $\mathscr{U}(K) \xrightarrow{\sim} k$.

Let $L \subset \overline{M}_{\widetilde{V}}$ be a twistor line that is a connected component of a fiber of π_v over a k-point $[K] \in \overline{M}_V(k)$.

Lemma 3.0.4. *If* $[\widetilde{K}] \in L(k)$ *is a* k*-point, then*

$$\sigma_0(\widetilde{K}) = \begin{cases} \sigma_0(K) + 1 & \textit{if } \widetilde{K} \cap \widetilde{V} \subset v^{\perp} \\ \sigma_0(K) & \textit{otherwise} \end{cases}$$

In particular, the twistor line L *has generic Artin invariant* $\sigma_0(K) + 1$.

Proof. By assumption, $K = (\widetilde{K} \cap v^{\perp})/v \subset V \otimes k$. As in the proof of Lemma 3.0.9, the quotient map $v^{\perp} \to v^{\perp}/v = V$ induces an isomorphism

$$(\widetilde{K} \cap \widetilde{V} \cap v^{\perp})/v \xrightarrow{\sim} K \cap V$$

It follows that $\sigma_0(\widetilde{K}) = \sigma_0(K) + 1$ if $\widetilde{K} \cap \widetilde{V} \subset v^{\perp}$, and $\sigma_0(\widetilde{K}) = \sigma_0(K)$ otherwise. ∎

Lemma 3.0.5. *If $v' \in \widetilde{V}$ is an isotropic vector such that $v.v' \neq 0$, then L intersects $\overline{M}_{\widetilde{V},v'}$ in at most one point.*

Proof. Recall that in Notation 3.0.8 we have fixed an orthogonal decomposition $\widetilde{V} = V \oplus (U_2 \otimes \mathbf{F}_p)$, where $U_2 \otimes \mathbf{F}_p$ is generated by v and w, which satisfy $v^2 = w^2 = 0$ and $v.w = -1$. This decomposition gives rise to a diagram

$$\begin{tikzcd} \overline{M}_{\widetilde{V}}^{\langle v\rangle} \arrow[d, "\pi_v"] \\ \overline{M}_V \arrow[u, bend left, "\sigma_w"] \end{tikzcd}$$

where σ_w is the section of π_v defined by $K \mapsto \langle K, w\rangle$. Note that the image of σ_w is $\overline{M}_{\widetilde{V},w} \subset \overline{M}_{\widetilde{V}}$. From this, it is clear that L intersects the locus $\overline{M}_{\widetilde{V},w}$ in at most one point. We will transfer this result to v', essentially by using the group structure given by the isomorphism of Proposition 3.0.11. In terms of our orthogonal decomposition, we have $v' = w + b + (b^2/2)v$ for some $b \in V$. Consider the linear transformation $\exp(b)\colon \widetilde{V} \to \widetilde{V}$ given by

$$\begin{aligned} &v \mapsto v, \\ &w \mapsto w + b + (b^2/2)v, \quad \text{and} \\ &x \mapsto x + (x.b)v \quad \text{for } x \in V. \end{aligned}$$

One checks that $\exp(b)$ is in fact an isometry. We have a commutative diagram

$$\begin{tikzcd} \overline{M}_{\widetilde{V}}^{\langle v\rangle} \arrow[rr, "\exp(b)", "\sim"'] \arrow[dr, "\pi_v"'] & & \overline{M}_{\widetilde{V}}^{\langle v\rangle} \arrow[dl, "\pi_v"] \\ & \overline{M}_V & \end{tikzcd}$$

Furthermore, the image of $\exp(b) \circ \sigma_w$ is exactly $\overline{M}_{\widetilde{V},v'}$. This gives the result. $\square$

Lemma 3.0.6. *If L has generic Artin invariant $\sigma \geq 2$, then L intersects the locus of points with Artin invariant $\sigma - 1$ in $p^{2\sigma-2}$ distinct points.*

Proof. Under the isomorphism of Proposition 3.0.11 (see also Remark 3.0.3), the points in the fiber $\pi_v^{-1}([K])$ with Artin invariant $\sigma - 1$ correspond to the subgroup

$$\frac{V}{K \cap V} \subset \frac{V \otimes k}{K}$$

The dimension of the $\mathbf{F}_p$-vector space $V/(K\cap V)$ is $\sigma_0+\sigma-2$. By Lemma 3.0.15, the fiber $\pi_v^{-1}([K])$ has $p^{\sigma_0-\sigma}$ connected components. It follows that L contains $p^{2\sigma-2}$ points of Artin invariant $\sigma-1$. $\square$

We next study the tangent spaces of twistor lines.

Lemma 3.0.7. *The canonical connection on $F^*_{M_{\widetilde{V}}}K_{\widetilde{V}}$ induces an isomorphism*

$$T^1_{M_{\widetilde{V}}}\xrightarrow{\sim}\mathscr{H}om_{M_{\widetilde{V}}}\left(K_{\widetilde{V}}\cap F^*_{M_{\widetilde{V}}}K_{\widetilde{V}},\frac{F^*_{M_{\widetilde{V}}}K_{\widetilde{V}}}{K_{\widetilde{V}}\cap F^*_{M_{\widetilde{V}}}K_{\widetilde{V}}}\right)$$

Proof. See [40], Proposition 4.6. $\square$

Lemma 3.0.8. *Write $\widetilde{K}=K_{\widetilde{V}}|_{M^{\langle v\rangle}_{\widetilde{V}}}$ and $K=K_V$. Under the isomorphism of Lemma 3.0.7, the sub-bundle*

$$T^1_{M^{\langle v\rangle}_{\widetilde{V}}/M_V}\subset T^1_{M^{\langle v\rangle}_{\widetilde{V}}}$$

is identified with

$$\mathscr{H}om_{M^{\langle v\rangle}_{\widetilde{V}}}\left(\frac{\widetilde{K}\cap F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}\cap v^\perp},\frac{F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}}\right)\cong\frac{F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}}$$

Proof. There are isomorphisms

$$\pi_v^*T^1_{M_V}\cong\pi_v^*\mathscr{H}om_{M_V}\left(K\cap F^*K,\frac{F^*K}{K\cap F^*K}\right)\cong\mathscr{H}om_{M^{\langle v\rangle}_{\widetilde{V}}}\left(\widetilde{K}\cap F^*\widetilde{K}\cap v^\perp,\frac{F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}}\right)$$

The morphism π_v induces a commutative diagram

$$\begin{array}{ccc}
T^1_{M^{\langle v\rangle}_{\widetilde{V}}} & \xrightarrow{\sim} & \mathscr{H}om_{M^{\langle v\rangle}_{\widetilde{V}}}\left(\widetilde{K}\cap F^*\widetilde{K},\dfrac{F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}}\right)\\
\downarrow & & \downarrow\\
\pi_v^*T^1_{M_V} & \xrightarrow{\sim} & \mathscr{H}om_{M^{\langle v\rangle}_{\widetilde{V}}}\left(\widetilde{K}\cap F^*\widetilde{K}\cap v^\perp,\dfrac{F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}}\right)
\end{array}$$

and the right vertical map sends a homomorphism $f\colon\widetilde{K}\cap F^*\widetilde{K}\to F^*\widetilde{K}/\widetilde{K}\cap F^*\widetilde{K}$ to the composition

$$\widetilde{K}\cap F^*\widetilde{K}\cap v^\perp\hookrightarrow\widetilde{K}\cap F^*\widetilde{K}\xrightarrow{f}\frac{F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}}$$

Thus, its kernel is

$$T^1_{M^{\langle v\rangle}_{\widetilde{V}}/M_V} = \mathscr{H}om_{M^{\langle v\rangle}_{\widetilde{V}}}\left(\frac{\widetilde{K}\cap F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}\cap v^\perp}, \frac{F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}}\right)$$

Finally, note that pairing with v gives a trivialization

$$\frac{\widetilde{K}\cap F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}\cap v^\perp} \xrightarrow{\sim} \mathscr{O}_{M^{\langle v\rangle}_{\widetilde{V}}}$$

□

Consider the open subset $U_{\widetilde{V}} \subset M_{\widetilde{V}}$ consisting of those characteristic subspaces with maximal Artin invariant. This is the intersection of the domains of definition of the rational maps π_v as v ranges over all isotropic vectors in $\widetilde{V}$. Thus, for any isotropic $v \in V$, we get a morphism $U_{\widetilde{V}} \to U_{v^\perp/v}$, whose fibers over k-points are open subsets of twistor lines in $M_{\widetilde{V}}$.

Lemma 3.0.9. *Let $v_0, \ldots, v_n$ be an enumeration of the isotropic vectors in $\widetilde{V}$. The morphism*

$$U_{\widetilde{V}} \to U_{v_0^\perp/v_0} \times \cdots \times U_{v_n^\perp/v_n}$$

induced by the fibrations π_{v_i} is unramified. In other words, for any $x \in U_{\widetilde{V}}(k)$, the tangent vectors to the twistor lines through x span $T^1_{U_{\widetilde{V}},x}$

Proof. Take a point $x \in U_{\widetilde{V}}(k)$ corresponding to a characteristic subspace $\widetilde{K} \subset V \otimes k$. Fix an isomorphism $(F^*\widetilde{K}/(\widetilde{K} + F^*\widetilde{K}))|_x \xrightarrow{\sim} k$. If $v_i \in \widetilde{V}$ is isotropic, then the tangent space to the fiber of π_{v_i} containing x is

$$\mathrm{Hom}\left(\frac{\widetilde{K}\cap F^*\widetilde{K}}{\widetilde{K}\cap F^*\widetilde{K}\cap v_i^\perp}, k\right) \subset \mathrm{Hom}(\widetilde{K}\cap F^*\widetilde{K}, k)$$

Thus, it will suffice to show that the functions $\langle \_, v_i\rangle$ span $\mathrm{Hom}(\widetilde{K}\cap F^*\widetilde{K}, k)$. Let $e \in \widetilde{K}$ be a vector that spans the line $l_{\widetilde{K}} = \widetilde{K}\cap\varphi(\widetilde{K})\cap\ldots\varphi^{\sigma_0}(\widetilde{K})$, so that $\left\{\varphi^{-\sigma_0+1}(e), \ldots, e\right\}$ is a basis for $\widetilde{K}\cap\varphi(\widetilde{K})$. Note that for any vector v defined over $\mathbf{F}_p$ we have

$$(v.e)^{p^j} = \sigma^j(v.e) = v.\varphi^j(e)$$

for all j. In particular, each v_i is uniquely determined by $\lambda_i = v_i.\varphi^{-\sigma_0+1}(e)$, and the λ_i are distinct and non-zero. We will be finished if we can show that the matrix

$$A = \begin{bmatrix} \lambda_0 & \lambda_0^p & \dots & \lambda_0^{p^{\sigma_0-1}} \\ \lambda_1 & \lambda_1^p & \dots & \lambda_1^{p^{\sigma_0-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n & \lambda_n^p & \dots & \lambda_n^{p^{\sigma_0-1}} \end{bmatrix}$$

has full rank. Take a $\sigma_0 \times \sigma_0$ minor of A, which we may assume to be $(\lambda_i^{p^j})_{0\leq i,j\leq\sigma_0-1}$ without loss of generality. Consider its determinant $D(\lambda_0, \dots, \lambda_{\sigma_0-1})$ as a polynomial in the λ_i. We claim that

$$D(\lambda_0, \dots, \lambda_{\sigma_0-1}) = c \prod_{\alpha\in\mathbf{P}^{\sigma_0-1}(\mathbf{F}_p)} \lambda_\alpha$$

where $c \in \mathbf{F}^\times$ is a non-zero constant, and for $\alpha = [\alpha_0 : \cdots : \alpha_{\sigma_0-1}] \in \mathbf{P}^{\sigma_0-1}(\mathbf{F}_p)$ we set $\lambda_\alpha = \sum_{k=0}^{\sigma_0-1} \alpha_k\lambda_k$, which is well defined up to elements of $\mathbf{F}_p^\times$. To see this, note that if one of the λ_α vanishes, then $D(\lambda_0, \dots, \lambda_{\sigma_0-1})$ vanishes as well. Thus, by the nullstellensatz, the product of the λ_α divides $D(\lambda_0, \dots, \lambda_{\sigma_0-1})$. By comparing degrees we get the result. So, to show that A has full rank, we need to show that we can find σ_0 vectors among the v_i such that the corresponding λ_i are linearly independent over $\mathbf{F}_p$. Because the λ_i are distinct and non-zero, we can do this if $n \geq p^{\sigma_0-1}$. By Lemma 4.12 of [40], the number of isotropic vectors in $\widetilde{V}$ is $p^{2\sigma_0+1} - p^{\sigma_0+1} + p^{\sigma_0} - 1$. □

3.1 The relative Brauer group of a family of supersingular K3 surfaces

Fix a relative supersingular K3 surface $\pi\colon X \to S$. In this section we will study the relationship between the cohomology sheaves $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$ and $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$, and relate them to the Hodge and conjugate filtrations on the de Rham cohomology of $X \to S$. In order to allow for the variation of the Artin invariant of the fibers, we will find it convenient to introduce an appropriate marking of the relative Picard group.

Definition 3.1.1. A *supersingular K3 lattice* is a free abelian group Λ of rank 22 equipped with an even symmetric bilinear form such that

1. $\mathrm{disc}(\Lambda \otimes \mathbf{Q}) = -1$ in $\mathbf{Q}^\times/\mathbf{Q}^{\times 2}$,

2. the signature of $\Lambda \otimes \mathbf{R}$ is $(1, 21)$, and

3. the discriminant group Λ^*/Λ is p-torsion.

For the definitions of these terms and other lattice theoretic background we refer to Chapter 14 of [21]. It is shown in [41] that the supersingular K3 lattices are precisely those lattices that occur as the Picard group of a supersingular K3 surface. If Λ is a supersingular K3 lattice, then there are inclusions

$$p\Lambda \subset p\Lambda^* \subset \Lambda \subset \Lambda^*$$

We set

$$\Lambda_0 = \frac{p\Lambda^*}{p\Lambda} \qquad \Lambda_1 = \frac{\Lambda}{p\Lambda^*}$$

By condition (3), Λ_0 and Λ_1 are vector spaces over $\mathbf{F}_p$. The dimension of Λ_0 over $\mathbf{F}_p$ is equal to $2\sigma_0$ for some integer $1 \leq \sigma_0 \leq 10$, called the ***Artin invariant*** of Λ, and the dimension of Λ_1 is $22 - 2\sigma_0$. The Artin invariant determines Λ uniquely up to isometry (see Theorem 7.4 of [40]). There is a natural symmetric bilinear form $\Lambda_0 \otimes \Lambda_0 \to \mathbf{F}_p$ given by

$$\overline{v}.\overline{w} = p^{-1}\langle v, w\rangle_\Lambda \mod p$$

for $v, w \in p\Lambda^* \subset \Lambda$. This form is non-degenerate and non-neutral, so that $V = \Lambda_0$ satisfies the assumptions of section 3.0.1. We fix a supersingular K3 lattice Λ with Artin invariant σ_0.

Definition 3.1.2. If S is a k-scheme, we define a ***family of Λ-marked supersingular K3 surfaces over S*** to be an algebraic space X equipped with a smooth, proper morphism $X \to S$ whose geometric fibers are supersingular K3 surfaces, along with a morphism $m\colon \underline{\Lambda}_S \to \mathrm{Pic}_{X/S}$ of sheaves of groups that is compatible with the intersection forms. We let S_Λ denote the functor whose S-points are isomorphism classes of families of Λ-marked supersingular K3 surfaces over S.

Using Artin's representability theorems, Ogus proved the following.

Theorem 3.1.3. *[41, Theorem 2.7] The functor S_Λ is representable by an algebraic space over k that is locally of finite presentation, locally separated, and smooth of dimension $\sigma_0 - 1$.*

We recall the definition of Ogus's crystalline period morphism.

Definition 3.1.4. Let $\pi\colon X \to S$ be a relative supersingular K3 surface equipped with a marking $m\colon \underline{\Lambda}_S \to \mathrm{Pic}_{X/S}$. Composing with the Chern class map $\mathrm{Pic}_{X/S} \to \mathrm{H}^2_{dR}(X/S)$, we obtain a map $\underline{\Lambda}_S \to \mathrm{H}^2_{dR}(X/S)$, which induces a map $\underline{\Lambda}_S \otimes \mathscr{O}_S \to \mathrm{H}^2_{dR}(X/S)$. Because $p\mathscr{O}_S = 0$, there is a natural inclusion $\underline{\Lambda}_{0S} \otimes \mathscr{O}_S \subset \underline{\Lambda}_S \otimes \mathscr{O}_S$. Suppose that S is smooth over k. As described in Section 3 of [41], the kernel of the map $\underline{\Lambda}_S \otimes \mathscr{O}_S \to \mathrm{H}^2_{dR}(X/S)$ is a characteristic subspace $K' \subset \underline{\Lambda}_{0S} \otimes \mathscr{O}_S$, and that the Gauss-Manin connection induces a descent datum on K' with respect to the Frobenius F_S. Thus we find a characteristic subspace $K \subset \underline{\Lambda}_{0S} \otimes \mathscr{O}_S$ equipped with an isomorphism $F_S^* K \cong K'$. In particular, the moduli space S_Λ is smooth, so we may apply the construction of K to produce (via étale descent) a characteristic subspace $K_\Lambda \subset \underline{\Lambda}_{0S} \otimes \mathscr{O}_{S_\Lambda}$, which in turn gives a morphism

$$\rho\colon S_\Lambda \to \overline{M}_{\Lambda_0} \tag{3.1.4.1}$$

Remark 3.1.5. The morphism ρ can be interpreted in terms of crystalline cohomology (see Remark 3.2.5). We will take this viewpoint in Section 3.3 when we define a period morphism for twisted supersingular K3 surfaces.

Remark 3.1.6. Note that this definition of the period morphism is not purely moduli-theoretic, in the sense of giving an "intrinsic" procedure that associates to any marked family over an arbitrary k-scheme S a characteristic subspace. The difficulty lies in the descent through the Frobenius. If S is a smooth k-scheme, the Gauss Manin connection induces a descent of the kernel of the map $\Lambda \otimes \mathscr{O}_S \to \mathrm{H}^2_{dR}(X/S)$ through the Frobenius, and the resulting characteristic subspace is of formation compatible with base change along morphisms $T \to S$, where T is a smooth k-scheme. If, however, S is not smooth, then it is not clear which descent of K' to choose. By defining the period morphism in terms of the universal characteristic subspace, we have in effect specified a choice of Frobenius descent of the kernel for every marked family over an arbitrary base.

Proposition 3.1.7. *If $\pi\colon X \to S$ is a relative supersingular K3 surface, then the sheaf $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$ is representable by a group algebraic space of finite presentation over S.*

Proof. It will suffice to prove the result after taking an étale cover of S. Thus, we may assume that there exists a marking $\underline{\Lambda}_S \to \mathrm{Pic}_{X/S}$ (see page 1522 of [42]). Let $\rho_S\colon S \to \overline{M}_{\Lambda_0}$ be the corresponding morphism. The sheaf K_{Λ_0} on $\overline{M}_{\Lambda_0}$ pulls back to a sheaf K on S, whose associated vector bundle we will denote by $\underline{K}$. By results of [40], we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & F_S^*\underline{K} \arrow[r] & \underline{\Lambda}_S \otimes \mathscr{O}_S \arrow[r] & F_H^0 \arrow[r] & F_S^*\underline{K} \arrow[r] & 0 \\
0 \arrow[r] & F_S^*\underline{K} \arrow[r] \arrow[u, equal] & \underline{\Lambda}_S \otimes \mathscr{O}_S \arrow[r] \arrow[u, equal] & F_H^1 \arrow[r] \arrow[u, hook] & \underline{K} \cap F_S^*\underline{K} \arrow[r] \arrow[u, hook] & 0 \\
0 \arrow[r] & F_S^*\underline{K} \arrow[r] \arrow[u, equal] & \underline{K} + F_S^*\underline{K} \arrow[r] \arrow[u, hook] & F_H^2 \arrow[r] \arrow[u, hook] & 0 \arrow[u, hook] &
\end{tikzcd} \tag{3.1.7.1}$$

of big étale sheaves on S with exact rows, where

$$0 \subset F_H^2 \subset F_H^1 \subset F_H^0 = \mathrm{H}^2_{dR}(X/S)_{\text{ét}}$$

is the Hodge filtration (2.3.5.3) on the second de Rham cohomology $\mathrm{H}^2_{dR}(X/S)_{\text{ét}}$. Note that the entries of this diagram are representable by groups schemes on S. The first Chern class map $\mathrm{Pic}_{X/S} \to \mathbf{R}^2\pi_*^{\text{ét}}\Omega^{\text{ét}\bullet}_{X/S}$ factors through $\mathbf{R}^1\pi_*^{\text{ét}}Z^1\Omega^{\text{ét}\bullet}_{X/S}$. By Lemma 2.3.10, there is an identification $\mathbf{R}^1\pi_*^{\text{ét}}Z^1\Omega^{\text{ét}\bullet}_{X/S} \xrightarrow{\sim} F_H^1 \cap F_C^1$, and the inclusion

$$F_H^1 \cap F_C^1 \hookrightarrow \mathrm{H}^2_{dR}(X/S)_{\text{ét}}$$

together with (3.1.7.1) gives a diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\Lambda}_S \otimes \mathscr{O}_S/F_S^*\underline{K} \arrow[r] \arrow[d] & F_H^1 \cap F_C^1 \arrow[r] \arrow[d] & \underline{K} \cap F_S^*\underline{K} \cap F_S^{*2}\underline{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\Lambda}_S \otimes \mathscr{O}_S/F_S^*\underline{K} \arrow[r] & \mathrm{H}^2_{dR}(X/S)_{\text{ét}} \arrow[r] & F_S^*\underline{K} \arrow[r] & 0
\end{tikzcd} \tag{3.1.7.2}$$

with exact rows. We emphasize that by $F_H^1 \cap F_C^1$ and $\underline{K} \cap F_S^*\underline{K} \cap F_S^{*2}\underline{K}$ we mean the fiber products of sheaves (or modules) on the big étale site of S. The rank of these sheaves jumps on the superspecial locus, and they will not be quasi-coherent in general. Furthermore, by (3.1.7.1) there is a short exact sequence

$$0 \to \frac{\underline{\Lambda}_S \otimes \mathscr{O}_S}{F_S^*\underline{K} + F_S^{*2}\underline{K}} \to F_S^*(F_H^1/F_H^2) \to F_S^*\underline{K} \cap F_S^{*2}\underline{K} \to 0$$

We are led to the following diagram of big étale sheaves on S, with exact rows and columns.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathscr{V} \arrow[r] \arrow[d] & \dfrac{\underline{\Lambda}_S \otimes \mathscr{O}_S}{F_S^*\underline{K}} \arrow[r, "1-F_S^*"] \arrow[d] & \dfrac{\underline{\Lambda}_S \otimes \mathscr{O}_S}{F_S^*\underline{K} + F_S^{*2}\underline{K}} \arrow[d] \\
0 \arrow[r] & \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \arrow[r] \arrow[d] & F_H^1 \cap F_C^1 \arrow[r, "C\circ\pi_C - F_S^*\circ\pi_H"] \arrow[d] & F_S^*(F_H^1/F_H^2) \arrow[d] \\
0 \arrow[r] & \mathscr{W} \arrow[r] \arrow[d] & \underline{K} \cap F_S^*\underline{K} \cap F_S^{*2}\underline{K} \arrow[r, "1-F_S^*"] \arrow[d] & F_S^*\underline{K} \cap F_S^{*2}\underline{K} \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}
\tag{3.1.7.3}
$$

The sheaf $\mathscr{V}$ is the kernel of a map of group schemes, and so is representable. We have a map of exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{W} \arrow[r] \arrow[d, equal] & \underline{K} \cap F_S^*\underline{K} \cap F_S^{*2}\underline{K} \arrow[r, "1-F_S^*"] \arrow[d, hook] & F_S^*\underline{K} \cap F_S^{*2}\underline{K} \arrow[d, hook] \\
0 \arrow[r] & \mathscr{W} \arrow[r] & F_S^*\underline{K} \arrow[r, "1-F_S^*"] & F_S^*\underline{K} + F_S^{*2}\underline{K}
\end{tikzcd}
$$

Thus $\mathscr{W}$ is also the kernel of a map of group schemes and hence representable. By Lemma 3.1.8, it follows that $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$ is representable by an algebraic space. $\square$

Lemma 3.1.8. *If X is a scheme and $0 \to A \to B \to C \to 0$ is a short exact sequence of sheaves of abelian groups on $X_{\text{Ét}}$, then if A and C are representable by algebraic spaces, so is B.*

Proof. Let us think of B as a sheaf of groups on C via the map $B \to C$. The map $A \to B$ gives an action of A on B over C. Because $B \to C$ is a surjection of big étale sheaves, it admits a section étale locally. Thus, B is an A-torsor, and étale locally on C there is an isomorphism $B \xrightarrow{\sim} A \times C$. We conclude that B is an algebraic space. $\square$

If $\pi\colon X \to S$ is a family of supersingular K3 surfaces, then the conditions of Proposition 2.2.11 are satisfied, and so we obtain a morphism

$$\Upsilon\colon \mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \tag{3.1.8.1}$$

By Theorem 2.1.7 and Proposition 3.1.7, these sheaves are represented by group algebraic spaces over S.

Remark 3.1.9. It follows from Proposition 2.2.12 that Υ is a universal homeomorphism. By Remark 2.2.18, it is totally ramified relative to S. We will deduce in Proposition 3.3.10 that Υ is essentially the relative Frobenius over S.

Suppose that the family $X \to S$ is equipped with a marking $\underline{\Lambda}_S \to \mathrm{Pic}_{X/S}$. We will identify some particular subgroups of the sheaves $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$ and $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$. We will use the notation of the proof of Proposition 3.1.7, and in particular diagram (3.1.7.3). Let $\mathscr{U}_S$ be the pullback of $\mathscr{U}_{\Lambda_0}$ under the induced map $\rho_S \colon S \to M_{\Lambda_0}$. Consider the exact sequence

$$0 \to F_S^{-1}\mathscr{U}_S \to \frac{\underline{\Lambda}_{0S} \otimes \mathscr{O}_S}{F_S^*\underline{K}} \xrightarrow{1-F_S^*} \frac{\underline{\Lambda}_{0S} \otimes \mathscr{O}_S}{F_S^*\underline{K} + F_S^{*2}\underline{K}}$$

As $F_S^*\underline{K} \subset \underline{\Lambda}_{0S} \otimes \mathscr{O}_S$, we get a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & F_S^{-1}\mathscr{U}_S \arrow[r] \arrow[d, hook] & \frac{\underline{\Lambda}_{0S} \otimes \mathscr{O}_S}{F_S^*\underline{K}} \arrow[r, "1-F_S^*"] \arrow[d] & \frac{\underline{\Lambda}_{0S} \otimes \mathscr{O}_S}{F_S^*\underline{K} + F_S^{*2}\underline{K}} \arrow[d] \\
0 \arrow[r] & \mathscr{V} \arrow[r] & \frac{\underline{\Lambda}_S \otimes \mathscr{O}_S}{F_S^*\underline{K}} \arrow[r, "1-F_S^*"] & \frac{\underline{\Lambda}_S \otimes \mathscr{O}_S}{F_S^*\underline{K} + F_S^{*2}\underline{K}}
\end{tikzcd}$$

Definition 3.1.10. If $\pi \colon X \to S$ is a relative supersingular K3 surface and $\underline{\Lambda} \to \mathrm{Pic}_{X/S}$ is a marking, we define the subsheaf

$$\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o \subset \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$$

to be the image of $F_S^{-1}\mathscr{U}_S$ under the inclusions

$$F_S^{-1}\mathscr{U}_S \subset \mathscr{V} \subset \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$$

We define

$$(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o \subset \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$$

to be the preimage of $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o$ under the morphism (3.1.8.1).

Note that these subsheaves depend on the marking, although it is suppressed from the notation.

Lemma 3.1.11. *The subgroups* $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o$ *and* $(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o$ *are open and closed and are represented by group algebraic spaces. There is a short exact sequence*

$$0 \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \to \mathscr{D} \to 0 \tag{3.1.11.1}$$

where $\mathscr{D}$ *is a group scheme that fits into a short exact sequence*

$$0 \to \underline{\Lambda}_{1S} \to \mathscr{D} \to \mathscr{W} \to 0$$

Proof. Because $F_S^{-1}\mathscr{U}_S$ is representable, so are $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o$ and $(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o$. We have a short exact sequence

$$0 \to \mathscr{V} \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \to \mathscr{W} \to 0$$

The cokernel of the map $F_S^{-1}\mathscr{U}_S \to \mathscr{V}$ is the fixed points of F_S^* acting on $\underline{\Lambda}_S \otimes \mathscr{O}_S/\underline{\Lambda}_{0S} \otimes \mathscr{O}_S$, which is just $\underline{\Lambda}_{1S}$. We obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_S^{-1}\mathscr{U}_S \arrow[r] \arrow[d] & \mathscr{V} \arrow[r] \arrow[d] & \underline{\Lambda}_{1S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o \arrow[r] \arrow[d] & \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \arrow[r] \arrow[d] & \mathscr{D} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathscr{W} \arrow[r] \arrow[d] & \mathscr{W} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{3.1.11.2}$$

with exact columns and rows. Because $\underline{\Lambda}_{1S}$ is separated over S, the morphism $\mathscr{D} \to \mathscr{W}$ is separated. The sheaf $\mathscr{W}$ is equal to the intersection $\underline{K} \cap \underline{\Lambda}_{0S}$, and in particular is separated over S. We conclude that $\mathscr{D}$ is separated over S, and therefore the zero section of $\mathscr{D}$ is closed. The immersion $\mathscr{W} \subset \underline{\Lambda}_{0S}$ shows that the zero section of $\mathscr{W}$ is open, and hence $\underline{\Lambda}_{1S} \to \mathscr{D}$ is open. It follows that the zero section of $\mathscr{D}$ is also open. Thus, the subgroups $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o$ and $(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o$ are open and closed. $\square$

Lemma 3.1.12. *The group schemes* $(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o$ *and* $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o$ *are smooth over* S *of relative dimension 1. Every geometric fiber of either has connected component isomorphic to* $\mathbf{G}_a$.

Proof. By Lemma 3.0.13, $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o \to S$ is smooth, and by Lemma 3.0.15, every geometric fiber has connected component isomorphic to $\mathbf{G}_a$. Lemma 3.1.11 gives that the morphism $(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o \subset \mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p$ is open and closed, so by Proposition 2.1.11 each geometric fiber of $(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o$ is regular of dimension 1, and has connected component isomorphic to $\mathbf{G}_a$. It remains to show that $(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o \to S$ is flat. The morphism (3.1.8.1) gives a diagram

$$\begin{tikzcd}
(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o \arrow[r, "\Upsilon^o"] \arrow[dr] & \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o \arrow[d] \\
& S
\end{tikzcd}$$

We have already seen that the vertical arrow is smooth. By Proposition 2.2.12 the horizontal arrow is a universal homeomorphism. It follows that $(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)^o \to S$ is universally open. Moreover, its geometric fibers are reduced. It will suffice to prove the result in the universal case when $S = S_\Lambda$, so we may in addition assume that S is reduced. By 15.2.3 of [14], these conditions imply flatness. $\square$

Let us now suppose that $S = \operatorname{Spec} k$, and that $\pi\colon X \to \operatorname{Spec} k$ is a supersingular K3 surface equipped with a marking $m\colon \Lambda \to \operatorname{Pic}(X)$. Evaluating the morphism (3.1.8.1) on $\operatorname{Spec} k$, we obtain a diagram

$$\begin{tikzcd}
(\mathbf{R}^2\pi_*^{\text{fl}}\boldsymbol{\mu}_p)(k) \arrow[r, "\Upsilon", "\sim"'] \arrow[d, equal] & (\mathbf{R}^1\pi_*^{\text{ét}(p)}\nu(1))(k) \arrow[d, equal] \\
\mathrm{H}^2(X_{\text{fl}}, \boldsymbol{\mu}_p) \arrow[r, "\sim"] & \mathrm{H}^1(X_{\text{ét}}, \nu(1))
\end{tikzcd} \tag{3.1.12.1}$$

where the vertical arrows are the canonical identifications induced by the respective Leray spectral sequences.

Definition 3.1.13. If X is a K3 surface, we say that a class $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ is *transcendental* if its image in $\mathrm{H}^2_{dR}(X/k)$ is orthogonal to the image of the first Chern class map

$$\operatorname{Pic}(X) \to \mathrm{H}^2_{dR}(X/k)$$

If X is supersingular and $m\colon \Lambda \to \operatorname{Pic}(X)$ is a marking, then we say that α is *transcendental with respect to* m if its image in $\mathrm{H}^2_{dR}(X/k)$ is orthogonal to the image of

$$\Lambda \to \operatorname{Pic}(X) \to \mathrm{H}^2_{dR}(X/k)$$

Lemma 3.1.14. *A class $\alpha \in \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k)$ is transcendental with respect to a marking $\Lambda \to \mathrm{Pic}(X)$ if and only if it lies in the subgroup $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o(k) \subset \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k)$.*

Proof. The subspace $\Lambda \otimes k/F_k^*K \subset \mathrm{H}^2_{dR}(X/k)$ has rank $22 - \sigma_0$, where σ_0 is the Artin invariant of Λ. The subgroup $p\Lambda^* \subset \Lambda$ consists of those elements $x \in \Lambda$ such that $\langle x, y\rangle \equiv 0 \mod p$ for every $y \in \Lambda$. The span of the image of $p\Lambda^*$ in $\mathrm{H}^2_{dR}(X/k)$ is therefore in the orthogonal complement of $\Lambda \otimes k/F_k^*K$. But the subspace $\Lambda_0 \otimes k/F_k^*K \subset \mathrm{H}^2_{dR}(X/k)$ has rank σ_0, and the pairing on $\mathrm{H}^2_{dR}(X/k)$ is perfect. Therefore

$$\left(\frac{\Lambda \otimes k}{F_k^*K}\right)^{\perp} = \left(\frac{\Lambda_0 \otimes k}{F_k^*K}\right)$$

as subspaces of $\mathrm{H}^2_{dR}(X/k)$. In particular, the subgroup of transcendental elements of $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k)$ is equal to the intersection of $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k)$ with $\Lambda_0 \otimes k/F_k^*K$ inside of $\mathrm{H}^2_{dR}(X/k)$. By definition, this is equal to $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o(k)$. $\square$

Finally, let us further specialize to the case when $S = \operatorname{Spec} k$, and the marking $\Lambda \to \mathrm{Pic}(X)$ is an isomorphism.

Lemma 3.1.15. *If the marking m is an isomorphism, then $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o$ is the connected component of the identity.*

Proof. Using the notation of the proof of Proposition 3.1.7, we have a short exact sequence

$$0 \to \mathscr{V} \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \to \mathscr{W} \to 0$$

Note that in this case $\mathscr{W}$ is the trivial group. Thus, the identity component of $\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$ is identified with the image of the identity component of $F_S^{-1}(\mathscr{U}_S)$. But by Lemma 3.0.15, this group is isomorphic to $\mathbf{A}^1$, and in particular connected. By Proposition 2.2.12, the morphism $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p \to \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)$ is a homeomorphism, so $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o$ is the connected component of the identity. $\square$

Remark 3.1.16. We will eventually show that if $\pi \colon X \to S_\Lambda$ is the universal marked supersingular K3 surface, then the subgroup $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o \subset \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$ is the connected component of the identity. More generally, the same is true for any marked family such that the marking is generically an isomorphism.

Remark 3.1.17. Let us consider the k-points of the diagram (3.1.11.2). Evaluating the short exact sequence (3.1.11.1) on k, and applying the isomorphism $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k) \xrightarrow{\sim} \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)(k)$ of diagram 3.1.12.1, we get a short exact sequence

$$0 \to (\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o(k) \to \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k) \to \mathscr{D}(k) \to 0$$

Recall that under the identification $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k) = \mathrm{H}^2(X, \boldsymbol{\mu}_p)$, the subgroup $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o(k)$ corresponds to the group $\mathrm{U}^2(X, \boldsymbol{\mu}_p)$ of (2.1.12.2), and $\mathscr{D}(k)$ corresponds to $\mathrm{D}^2(X, \boldsymbol{\mu}_p)$. Finally, note that as the marking is an isomorphism, the sheaf $\mathscr{W}$ in (3.1.11.2) vanishes. Thus, the diagram (3.1.11.2) on k-points recovers (part of) diagram (2.1.13.1). We will find a strengthened form of this observation in Remark 3.3.12.

3.2 Twisted K3 crystals

Let $W = W(k)$ be the ring of Witt vectors of k, $K = W[\frac{1}{p}]$ its field of fractions, and $F_W : W \to W$ the homomorphism induced by the Frobenius on k. We begin this section by recalling from [40] the definition of a K3 crystal over W, and their connection to characteristic subspaces. We then introduce a crystalline analog of the Hodge-theoretic B-fields studied in [23]. Unlike in the complex case, crystalline B-fields satisfy a non-trivial relation, given in Lemma 3.2.11. Accordingly, they seem to posses a somewhat richer structure than those over the complex numbers. In particular, in the supersingular case they have nontrivial moduli. Using these, we show how to associate a K3 crystal to a pair (X, α), where X is a K3 surface and $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ (we will prefer to express the data (X, α) as a $\boldsymbol{\mu}_p$-gerbe $\mathscr{X} \to X$, although in this section this is just language). This construction is the crystalline analog of the twisted Hodge structures defined in [23]. We then discuss these constructions in the relative setting.

Suppose that $\pi : X \to \operatorname{Spec} k$ is a K3 surface over k. The second crystalline cohomology group $\mathrm{H}^2(X/W)$ of X is a free W-module of rank 22, which is equipped with a F_W-linear endomorphism $\Phi : \mathrm{H}^2(X/W) \to \mathrm{H}^2(X/W)$ induced by the absolute Frobenius on X. The pair $(\mathrm{H}^2(X/W), \Phi)$ is an F-crystal, in the sense of Definition 1.1.3. It also is equipped with a perfect pairing

$$\mathrm{H}^2(X/W) \otimes_W \mathrm{H}^2(X/W) \to W$$

which satisfies a certain compatibility with Φ. Following Ogus [40], we abstract these properties in the following definition.

Definition 3.2.1. A *K3 crystal of rank* n is a W-module H of rank n equipped with a F_W-linear endomorphism $\Phi\colon H \to H$ and a symmetric bilinear form $H \otimes_W H \to W$ such that

1. $p^2 H \subset \Phi(H)$,

2. $\Phi \otimes k$ has rank 1,

3. the pairing $\langle \_, \_ \rangle$ is perfect, and

4. $\langle \Phi(x), \Phi(y) \rangle = p^2 F_W \langle x, y \rangle$ for all $x, y \in H$.

A K3 crystal H is ***supersingular*** if H is isogenous to a crystal such that Φ acts as multiplication by p. In this case, we use the notation $\varphi = p^{-1}\Phi$.

Remark 3.2.2. Condition (1) is equivalent to the existence of a map $V\colon H \to H$ such that $\Phi \circ V = V \circ \Phi = p^2$ (see [39, Proposition 1.6.4]). If H is supersingular, then Corollary 3.8 of [40] shows that $(2) \implies (1)$.

Definition 3.2.3. The *Tate module* of a K3 crystal H is

$$T_H = \{h \in H | \Phi(h) = ph\}$$

This has a natural structure of $\mathbf{Z}_p$-module, and is equipped with the restriction of the bilinear form on H. If H is supersingular, then by Proposition 3.13 of [40] the p-adic ordinal of the discriminant of T_H is equal to $2\sigma_0$ for some integer $\sigma_0 \geq 1$, called the ***Artin invariant*** of H.

Supersingular K3 crystals give rise to characteristic subspaces via the following procedure. If H is a supersingular K3 crystal with Tate module T, then we have a chain of inclusions

$$T \otimes W \subset H \subset T^* \otimes W$$

Proposition 3.2.4. *[40, Proposition 3.2] If H is a supersingular K3 crystal, then T^*/T is a vector space over $\mathbf{F}_p$ of dimension $2\sigma_0$ whose induced bilinear form is non-degenerate and non-neutral, and the image $\overline{H}$ of H in $T^* \otimes W/T \otimes W = (T^*/T) \otimes k$ is a strictly characteristic subspace, as is $K_H = \varphi^{-1}(\overline{H})$.*

In fact, Ogus shows that this procedure is reversible, so the above correspondence gives an equivalence between certain appropriately defined categories of supersingular K3 crystals and strictly characteristic subspaces.

Remark 3.2.5. If $H = \mathrm{H}^2(X/W)$, where X is a supersingular K3 surface, then the characteristic subspace $\varphi(K_H)$ produced by Proposition 3.2.4 is essentially the same as the characteristic subspace appearing in the definition of the period morphism. Indeed, if H is any supersingular K3 crystal, then the image of H under the map

$$H \to \frac{T^* \otimes W}{T \otimes W} \xrightarrow{\cdot p} \frac{pT^* \otimes W}{pT \otimes W} \subset \frac{T \otimes W}{pT \otimes W} = T \otimes k \tag{3.2.5.1}$$

is equal to the kernel of the map $T \otimes k \to H \otimes k$. Because $\mathrm{H}^3(X/W)$ is torsion free there is a canonical identification

$$\mathrm{H}^2(X/W) \otimes k \xrightarrow{\sim} \mathrm{H}^2_{dR}(X/k) \tag{3.2.5.2}$$

of the reduction modulo p of the second crystalline cohomology group of X and the second de Rham cohomology group of X (see Summary 7.26 in [4]). By the Tate conjecture, $\mathrm{Pic}(X) \otimes \mathbf{Z}_p \xrightarrow{\sim} T$. Thus, the image of H in $(T^*/T) \otimes k$ corresponds under (3.2.5.1) to the kernel of the Chern class map

$$\mathrm{Pic}(X) \otimes k \to \mathrm{H}^2_{dR}(X/k)$$

Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe on the K3 surface X. We will show how to associate to $\mathscr{X}$ a K3 crystal of rank 24. This construction is essentially the isomorphism of Proposition 3.0.11 translated into supersingular K3 crystals. We begin by recalling the Mukai crystal associated to a K3 surface X, as introduced in [32]. Let $K(1)$ denote the F-isocrystal with underlying vector space K and Frobenius action given by multiplication by $1/p$. For any F-isocrystal M and integer n, we set $M(n) = M \otimes K(1)^{\otimes n}$.

Definition 3.2.6. The ***Mukai crystal*** of X is the W-module

$$\widetilde{\mathrm{H}}(X/W) = \mathrm{H}^0(X/W)(\text{-}1) \oplus \mathrm{H}^2(X/W) \oplus \mathrm{H}^4(X/W)(1)$$

equipped with the twisted Frobenius $\widetilde{\Phi} : \widetilde{\mathrm{H}}(X/W) \to \widetilde{\mathrm{H}}(X/W)$. We define the ***Mukai pairing*** on $\widetilde{\mathrm{H}}(X/W)$ by

$$(a, b, c).(a', b', c') = -ac' + b.b' - a'c \in \mathrm{H}^4(X/W) = W$$

Notation 3.2.7. Given two classes (a,b,c) and (a',b',c') in $\widetilde{\mathrm{H}}(X/W)$, we now have two possible operations: the cup product and the Mukai pairing. We will reserve the notation $(a,b,c).(a',b',c')$ for the Mukai pairing and will use the juxtaposition $(a,b,c)(a',b',c')$ for the cup product. For example, we will often translate the lattice $\widetilde{\mathrm{H}}(X/W)$ inside the rational cohomology $\widetilde{\mathrm{H}}(X/K)$ by taking the cup product with a class of the form $e^B = (1, B, B^2/2)$, and we will write this as $e^B\,\widetilde{\mathrm{H}}(X/W) \subset \widetilde{\mathrm{H}}(X/K)$.

Both $\mathrm{H}^0(X/W)$ and $\mathrm{H}^4(X/W)$ are canonically isomorphic (as W-modules) to W. Under these identifications, the twisted Frobenius action is given by

$$\widetilde{\Phi}(a,b,c) = (pF_W(a), \Phi(b), pF_W(c))$$

It follows from the definitions that $\widetilde{\mathrm{H}}(X/W)$ is a K3 crystal of rank 24. Because $\mathrm{H}^0(\text{-}1)$ and $\mathrm{H}^4(1)$ have slope 1, $\widetilde{\mathrm{H}}(X/W)$ is supersingular if and only if X is supersingular.

Recall the identifications of diagram (3.1.12.1). By Proposition 2.3.11, there is an exact sequence

$$0 \to \mathrm{H}^2(X, \boldsymbol{\mu}_p) \xrightarrow{d\log} F_H^1 \cap F_C^1(k) \xrightarrow{C\circ\pi_C - \pi_H} F_H^1/F_H^2(k) \tag{3.2.7.1}$$

of abelian groups. In particular, we get an injective homomorphism

$$\mathrm{H}^2(X, \boldsymbol{\mu}_p) \hookrightarrow F_H^1 \cap F_C^1(k) \hookrightarrow \mathrm{H}^2_{dR}(X/k)$$

and a diagram

$$\begin{array}{ccc} & & \mathrm{H}^2(X/W) \\ & & \Big\downarrow \scriptstyle{\mathrm{mod}\ p} \\ \mathrm{H}^2(X, \boldsymbol{\mu}_p) & \overset{d\log}{\hookrightarrow} & \mathrm{H}^2_{dR}(X/k) \end{array}$$

where the vertical arrow is induced by the canonical identification (3.2.5.2) of the reduction modulo p of the crystalline cohomology with the de Rham cohomology.

The following definition is the crystalline analog of the Hodge theoretic B-fields defined in [23].

Definition 3.2.8. An element $B = \frac{a}{p} \in \mathrm{H}^2(X/K)$ is a ***B-field*** if $a \in \mathrm{H}^2(X/W)$ and the image of a in $\mathrm{H}^2_{dR}(X/k)$ lies in the image of $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$. We write α_B for the unique element of $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$ such that $d\log(\alpha_B) \equiv a \mod p$, and we say that B is a ***B-field lift*** of α_B.

Remark 3.2.9. In this work we only discuss B-fields associated to p-torsion Brauer classes. Using the de Rham-Witt theory, one can make a similar definition that works for p^n-torsion classes as well. As the Brauer group of a supersingular K3 surface is p-torsion, the mod p theory presented here suffices for applications to supersingular K3 surfaces.

We will give an alternative characterization of B-fields that uses only the crystal structure on $\mathrm{H}^2(X/W)$. The Frobenius $\Phi\colon\ \mathrm{H}^m(X/W) \to \mathrm{H}^m(X/W)$ induces filtrations

$$\begin{aligned} M^i\,\mathrm{H}^m(X/W) &= (p^{-i}\Phi)^{-1}(\mathrm{H}^m(X/W)) \\ N^i\,\mathrm{H}^m(X/W) &= p^{-(m-i)}\Phi(M^{m-i}\,\mathrm{H}^m(X/W)) \end{aligned}$$

Note that there is an isomorphism $p^{-i}\Phi\colon\ M^i\,\mathrm{H}^m \to N^{m-i}\,\mathrm{H}^m$. Suppose that $\mathrm{H}^{m+1}(X/W)$ is torsion free, and consider the natural map

$$\rho\colon\ \mathrm{H}^m(X/W) \to \mathrm{H}^m_{dR}(X/k)$$

given by reduction modulo p. The following theorem of Mazur relates the image of these filtrations under ρ to the Hodge and conjugate filtrations on de Rham cohomology.

Theorem 3.2.10 ([4], Theorem 8.26 and Lemma 8.30). *Suppose that X is a smooth proper variety satisfying $(*)$. If $\mathrm{H}^{m+1}(X/W)$ is torsion free, then*

1. *the image of $M^i\,\mathrm{H}^m(X/W)$ under ρ is $F^i_H\,\mathrm{H}^m_{dR}(X/k)$,*
2. *the image of $N^i\,\mathrm{H}^m(X/W)$ under ρ is $F^i_C\,\mathrm{H}^m_{dR}(X/k)$, and*
3. *the following diagram commutes*

$$\begin{array}{ccccc} M^i\,\mathrm{H}^m(X/W) & \xrightarrow{\rho} & F^i_H\,\mathrm{H}^m_{dR}(X/k) & \xrightarrow{\pi_H} & \mathrm{gr}^i_{F_H}\,\mathrm{H}^m_{dR}(X/k) \\ {\scriptstyle p^{-i}\Phi}\downarrow & & & & \downarrow{\scriptstyle C^{-1}} \\ N^{m-i}\,\mathrm{H}^m(X/W) & \xrightarrow{\rho} & F^{m-i}_C\,\mathrm{H}^m_{dR}(X/k) & \xrightarrow{\pi_C} & \mathrm{gr}^{m-i}_{F_C}\,\mathrm{H}^m_{dR}(X/k) \end{array}$$

Lemma 3.2.11. *An element $B \in p^{-1}\,\mathrm{H}^2(X/W)$ is a B-field if and only if*

$$B - \varphi(B) \in \mathrm{H}^2(X/W) + \varphi(\mathrm{H}^2(X/W))$$

Proof. Suppose that $B = \frac{a}{p}$ is a B-field. By Lemma 2.3.10, $\rho(a) \in F_H^1 \cap F_C^1$, which implies that $a \in M^1 \cap N^1$. By the exact sequence (3.2.7.1), we know that $C \circ \pi_C(\alpha) = \pi_H(\alpha)$. Part (3) of Theorem 3.2.10 then shows that $\pi_C(\rho(a - \varphi(a))) = 0$, which implies that $a - \varphi(a) \in N^2 + pH$, so B satisfies the claimed relation.

Conversely, suppose that $B = \frac{a}{p} \in p^{-1}\,\mathrm{H}^2(X/W)$ is an element satisfying the relation. This implies that $\varphi(a) \in \mathrm{H}^2(X/W)$, so $a \in M^1$. Using that $p^2\,\mathrm{H}^2 \subset \Phi(\mathrm{H}^2)$, we also get that $a \in \varphi(\mathrm{H}^2)$, so $a \in N^1$. Thus, $\rho(a)$ is contained in $F_H^1 \cap F_C^1$. Part (3) of Theorem 3.2.10 then implies that $\pi_C(\rho(a) - \rho(\varphi(a))) = 0$, so $\rho(a)$ is in the image of $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$, and hence B is a B-field. $\square$

Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe with cohomology class α, and let B be a B-field lift of α. Cupping with $e^B = (1, B, B^2/2)$ defines an isometry $\widetilde{\mathrm{H}}(X/K) \to \widetilde{\mathrm{H}}(X, K)$, given explicitly by

$$e^B(a, b, c) = \left(a, b + aB, c + b.B + a\frac{B^2}{2}\right)$$

Definition 3.2.12. The *twisted Mukai crystal* associated to $\mathscr{X}$ is the W-module

$$\widetilde{\mathrm{H}}(\mathscr{X}/W) = e^B\,\widetilde{\mathrm{H}}(X/W) \subset \widetilde{\mathrm{H}}(X/K)$$

Note that if $h \in \mathrm{H}^2(X/W)$ then $e^h \in \mathrm{H}^*(X/W)$, and therefore the submodule $\widetilde{\mathrm{H}}(\mathscr{X}/W)$ is independent of the choice of B-field. We will show that $\widetilde{\mathrm{H}}(\mathscr{X}/W)$ has a natural K3 crystal structure.

Lemma 3.2.13. *The submodule $\widetilde{\mathrm{H}}(\mathscr{X}/W)$ is preserved by the action of the Frobenius $\widetilde{\Phi}$ on $\widetilde{\mathrm{H}}(X/K)$.*

Proof. We must show that $\widetilde{\Phi}(\widetilde{\mathrm{H}}(\mathscr{X}/W)) \subset \widetilde{\mathrm{H}}(\mathscr{X}/W)$. Consider an element $e^B(a, b, c) \in \widetilde{\mathrm{H}}(\mathscr{X}/W)$. A consequence of the Mukai twist is the useful relation

$$\widetilde{\Phi}(e^B(a, b, c)) = e^{\varphi(B)}\widetilde{\Phi}(a, b, c)$$

Thus, the lemma is equivalent to the statement that

$$e^{\varphi(B)-B}\widetilde{\Phi}(a, b, c) \in \widetilde{\mathrm{H}}(X/W)$$

for all $(a, b, c) \in \widetilde{\mathrm{H}}(X/W)$. Write $B' = \varphi(B) - B$. We have

$$e^{B'}\widetilde{\Phi}(a, b, c) = \left(pF_W(a), \Phi(b) + pF_W(a)B', pF_W(c) + \Phi(b).B' + pF_W(a)\frac{B'^2}{2}\right)$$

By assumption, $B' \in \mathrm{H}^2(X/W) + \varphi(\mathrm{H}^2(X/W))$. This implies that $pB' \in \mathrm{H}^2(X/W)$, $\Phi(b).B' \in W$, and $p\frac{B'^2}{2} \in W$. □

Proposition 3.2.14. *The W-module $\widetilde{\mathrm{H}}(\mathscr{X}/W)$ equipped with the endomorphism $\widetilde{\Phi}$ and the Mukai pairing is a K3 crystal of rank 24, which is supersingular if and only if X is supersingular.*

Proof. Because cupping with e^B is an isometry with respect to the Mukai pairing, conditions (3) and (4) are immediate. For condition (1), we must show that for all $(a, b, c) \in \widetilde{\mathrm{H}}(\mathscr{X}/W)$

$$e^{B-\varphi(B)}(p^2a, p^2b, p^2c) \in \widetilde{\Phi}(\widetilde{\mathrm{H}}(\mathscr{X}/W))$$

Because $\mathrm{H}^2(X/W)$ is a K3 crystal,

$$p^2(B - \varphi(B)) \in p^2\,\mathrm{H}^2(X/W) + p\varphi(\mathrm{H}^2(X/W)) \subset \Phi(\mathrm{H}^2(X/W))$$

and $p^2b \in \Phi(\mathrm{H}^2(X/W))$. To check condition (2), we must compute the image of $\widetilde{\Phi}\colon\ \widetilde{\mathrm{H}}(\mathscr{X}/W) \to \widetilde{\mathrm{H}}(\mathscr{X}/W)$ modulo $p\,\widetilde{\mathrm{H}}(\mathscr{X}/W)$. This is isomorphic to the image of $e^{-B} \circ \widetilde{\Phi}$ modulo $p\,\widetilde{\mathrm{H}}(X/W)$. By assumption, $B' = \varphi(B) - B = h + \varphi(h')$ for some $h, h' \in \mathrm{H}^2(X/W)$. We compute

$$\begin{aligned} e^{-B}\widetilde{\Phi}(e^B(a,b,c)) &= e^{B'}.\widetilde{\Phi}(a,b,c) \\ &\equiv e^{B'}(pF_W(a), 0, 0) + e^{B'}(0, \Phi(b), 0) \\ &\equiv (0, \Phi(ah'), \Phi(ah').h) + (0, \Phi(b), \Phi(b).h) \\ &\equiv (0, \Phi(b + ah'), \Phi(b + ah').h) \end{aligned}$$

Because $\mathrm{H}^2(X/W)$ is a K3 crystal, the result follows. □

Definition 3.2.15. If $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$ gerbe on a supersingular K3 surface, we define the *Artin invariant* $\sigma_0(\mathscr{X})$ of $\mathscr{X}$ to be the Artin invariant of the supersingular K3 crystal $\widetilde{\mathrm{H}}(\mathscr{X}/W)$.

Remark 3.2.16. Definition 3.2.12 makes sense in the abstract setting where H is a K3 crystal and B satisfies the conditions of Lemma 3.2.11. Much of the rest of this section is valid in this generality as well. For the sake of exposition we have chosen to phrase our results in the geometric context.

We next introduce a twisted versions of the Néron-Severi lattice.

Definition 3.2.17. If $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$-gerbe on a K3 surface, we define the *extended Néron-Severi group* of X and $\mathscr{X}$ by

$$\widetilde{N}(X) = \langle (1,0,0)\rangle \oplus N(X) \oplus \langle (0,0,1)\rangle$$

and

$$\widetilde{N}(\mathscr{X}) = (\widetilde{N}(X) \otimes \mathbf{Q}) \cap \widetilde{\mathrm{H}}(\mathscr{X}/W) \subset \widetilde{\mathrm{H}}(X/K).$$

Note that $\widetilde{N}(\mathscr{X})$ only depends upon the cohomology class $[\mathscr{X}] \in \mathrm{H}^2(X, \boldsymbol{\mu}_p)$.

Notation 3.2.18. For an integer n, we will write $\widetilde{N}^{(n)}(\mathscr{X})$ for $\widetilde{N}(\mathscr{X}')$, where $\mathscr{X}'$ is a $\boldsymbol{\mu}_p$ gerbe on X whose cohomology class is $n[\mathscr{X}]$. This is independent of the choice of $\mathscr{X}'$.

In the supersingular case, we can give a very explicit presentation of $\widetilde{N}(\mathscr{X})$.

Lemma 3.2.19. *If X is a supersingular K3 surface, then any $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ can be written as $\alpha = \alpha' + \beta$, where α' is transcendental (see Definition 3.1.13) and β is in the image of the boundary map* $\mathrm{H}^1(X, \mathbf{G}_m) \to \mathrm{H}^2(X, \boldsymbol{\mu}_p)$.

Proof. Equip X with the tautological marking by $\Lambda = \mathrm{Pic}(X)$. Consider the short exact sequence

$$0 \to (\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o(k) \to \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k) \to \mathscr{D}(k) \to 0$$

described in Remark 3.1.17. By Lemma 3.1.14, the subgroup $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o(k)$ consists of exactly the transcendental classes. Because the marking is an isomorphism, $\Lambda_1 \cong \mathscr{D}(k)$, and so every element of $\mathscr{D}(k)$ lifts to an element of $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p(k) = \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ that is in the image of the boundary map $\mathrm{H}^1(X, \mathbf{G}_m) \to \mathrm{H}^2(X, \boldsymbol{\mu}_p)$. This gives the result. $\square$

Proposition 3.2.20. *Suppose that X is supersingular, let $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ be the cohomology class of $\mathscr{X} \to X$, and let α_0 be the image of α in $\mathrm{Br}(X)$. If α is transcendental and $\alpha_0 \neq 0$, then*

$$\widetilde{N}(\mathscr{X}) = \langle (p,0,0)\rangle \oplus N(X) \oplus \langle (0,0,1)\rangle$$

Let α be arbitrary, and suppose that $\alpha = \alpha' + \beta$, where α' is transcendental and β has trivial Brauer class. If β is the image of a line bundle $\mathscr{L}$ under the boundary map, and $t = c_1(\mathscr{L})$, then

$$\widetilde{N}(\mathscr{X}) = \begin{cases} \langle (1, \frac{t}{p}, \frac{(t/p)^2}{2})\rangle \oplus \langle (0, D, D.\frac{t}{p})\rangle \oplus \langle (0,0,1)\rangle & \textit{if } 0 = \alpha_0 \in \mathrm{Br}(X), \\ \langle (p, t, \frac{t^2}{2p})\rangle \oplus \langle (0, D, D.\frac{t}{p})\rangle \oplus \langle (0,0,1)\rangle & \textit{if } 0 \neq \alpha_0 \in \mathrm{Br}(X) \end{cases}$$

Proof. Suppose that α is transcendental, and that the class of α in $\mathrm{Br}(X)$ is non-zero. Let B be a B-field lift of α. An element $(a,b,c)\in\widetilde{N}(X)\otimes\mathbf{Q}$ lies in $\widetilde{N}(\mathscr{X})$ if and only if

$$e^{-B}(a,b,c)=(a,b-aB,c-b.B+a\frac{B^2}{2})$$

is in $\widetilde{\mathrm{H}}(X/W)$. This implies $a\in\mathbf{Z}$. Note that $b=\frac{l}{p}$ for some $l\in N(X)$. If $b-aB=h$ for some $h\in\mathrm{H}^2(X/W)$, then $aB=\frac{l}{p}-h$. Because the class of α in $\mathrm{Br}(X)$ is non-zero, this implies that a is divisible by p. Hence, $b=aB+h\in N(X)$. Finally, as α is transcendental, $b.B\in W$ and $p\frac{B^2}{2}\in W$. Therefore, $c\in\mathbf{Z}$. We have shown that in this case

$$\widetilde{N}(\mathscr{X})=\langle(p,0,0)\rangle\oplus N(X)\oplus\langle(0,0,1)\rangle$$

Next, let $\alpha\in\mathrm{H}^2(X,\boldsymbol{\mu}_p)$ be arbitrary. By Lemma 3.2.19, we can write $\alpha=\alpha'+\beta$, where α' is transcendental and β is the image of a line bundle $\mathscr{L}$ under the boundary map $\mathrm{H}^1(X,\mathbf{G}_m)\to\mathrm{H}^2(X,\boldsymbol{\mu}_p)$. Let B' be a B-field lift of α' and $t=c_1(\mathscr{L})$. Then $B=B'+\frac{t}{p}$ is a B-field lift of α, and we have an isomorphism

$$e^{\frac{t}{p}}:e^{B'}\,\widetilde{\mathrm{H}}(X/W)\xrightarrow{\sim}e^{B}\,\widetilde{\mathrm{H}}(X/W)$$

of K3 crystals. This induces an isomorphism on Tate modules, and one checks that it also induces an isomorphism on extended Néron-Severi groups. The result therefore follows by the previous case. $\square$

Write $T(X),\widetilde{T}(X)$, and $\widetilde{T}(\mathscr{X})$ for the Tate modules of $\mathrm{H}^2(X/W),\widetilde{\mathrm{H}}(X/W)$, and $\widetilde{\mathrm{H}}(\mathscr{X}/W)$.

Proposition 3.2.21. *The inclusion $\widetilde{N}(\mathscr{X})\hookrightarrow\widetilde{\mathrm{H}}(\mathscr{X}/W)$ factors through $\widetilde{T}(\mathscr{X})$. If X is supersingular, the induced map*

$$\widetilde{N}(\mathscr{X})\otimes\mathbf{Z}_p\to\widetilde{T}(\mathscr{X})$$

is an isomorphism.

Proof. The first claim follows from the Tate twists in the definition of $\widetilde{\mathrm{H}}(X/W)$. Suppose that X is supersingular. It follows from the definitions that

$$\widetilde{T}(\mathscr{X})=(\widetilde{T}(X)\otimes\mathbf{Q})\cap\widetilde{\mathrm{H}}(\mathscr{X}/W)$$

By the Tate conjecture (or by assumption), the map $N(X) \otimes \mathbf{Z}_p \xrightarrow{\sim} T(X)$ is an isomorphism. The calculations of Proposition 3.2.21 then apply as written to show that the $\mathbf{Z}_p$-span of the given vectors is equal to $\widetilde{T}(\mathscr{X})$, which gives the result. $\square$

Corollary 3.2.22. *If $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$-gerbe over a supersingular K3 surface, then $\sigma_0(\mathscr{X}) = \sigma_0(X) + 1$ if the Brauer class of $\mathscr{X}$ is non-zero, and $\sigma_0(\mathscr{X}) = \sigma_0(X)$ otherwise.*

We will next discuss these constructions in the relative setting. Given a relative K3 surface $\pi\colon X \to S$ and a $\boldsymbol{\mu}_p$-gerbe $\mathscr{X} \to X$, we would like to define a relative twisted Mukai crystal $\widetilde{\mathrm{H}}(\mathscr{X}/S)$. The correct notion of a crystal over a non-perfect base carries significant technicalities. We will restrict our attention to the following situation.

Situation 3.2.23. Suppose that $S = \operatorname{Spec} A$ is affine, where A is a smooth k-algebra. Fix a smooth, p-adically complete lift $S' = \operatorname{Spec} A'$ of S over W, together with a lift $F_{S'}$ of the absolute Frobenius of S (that is, a morphism $F_{S'}\colon S' \to S'$ that reduces to F_S modulo p).

If S is perfect, then there is a unique such choice of A', given by the Witt vectors $A' = W(A)$. In general, there will be many choices of S'. Note however that our assumption that S is smooth ensures that they are at least locally isomorphic.

Definition 3.2.24. Suppose that we are in Situation 3.2.23. A *crystal on S/W (with respect to S')* is a pair (M, ∇), such that

1. M is a finitely generated p-adically complete projective A'-module, and
2. $\nabla\colon M \to M \mathbin{\widehat{\otimes}} \widehat{\Omega}^1_{S'/W}$ is a connection such that
3. ∇ is integrable and topologically quasi-nilpotent.

An *F-crystal on S/W (with respect to $(S', F_{S'})$)* is a triple (M, ∇, Φ) where (M, ∇) is a crystal on S/W and

$$\Phi\colon F_{S'}^*(M, \nabla) \to (M, \nabla)$$

is a horizontal[1] map of A'-modules that becomes an isomorphism after inverting p.

[1]Recall that if $(\mathscr{E}, \nabla)$ and $(\mathscr{F}, \nabla')$ are modules with connection, a map $f\colon \mathscr{E} \to \mathscr{F}$ is *horizontal* if $\nabla' \circ f = (f \otimes \mathrm{id}) \circ \nabla$.

For the definitions of these terms, we refer the reader to [47, 07GI].

Remark 3.2.25. This is essentially the same as Definition 1.1.3 of [39], although we have chosen to work with complete objects, while Ogus works with formal objects on formal schemes. In addition, Ogus omits the nilpotence condition, as he works only with F-crystals, where it is automatic by [39, Corollary 1.7].

Remark 3.2.26. As explained in Remark 1.8 of [39], the definition of an F-crystal on S/W is in a certain sense independent of the choice of S' and $F_{S'}$. Suppose that (M, ∇) is a crystal on S/W with respect to the lifting S'. If T' is another smooth, complete lift of S over W, then locally S' and T' are isomorphic over S. The connection ∇ induces an isomorphism between the pullbacks of M along any two such local isomorphisms, and the integrability assumption implies that these isomorphisms satisfy the cocycle condition. Thus, (M, ∇) induces in a canonical way a crystal on S/W with respect to T'.

If (M, ∇, Φ) is an F-crystal on S/W with respect to $(S', F_{S'})$, and $G_{S'}$ is another choice of lifting of the Frobenius, then the connection induces a horizontal isomorphism $\varepsilon \colon G_{S'}^*(M, \nabla) \to F_{S'}^*(M, \nabla)$. The triple $(M, \nabla, \Phi \circ \varepsilon)$ is then an F-crystal on S/W with respect to $(S', G_{S'})$.

This independence is explained by the site-theoretic approach to crystals and crystalline cohomology, as developed in generality in [3]. See [4, Proposition 6.8] and [47, 07JH] for the equivalence of these approaches.

Definition 3.2.27. [40, Section 5] A ***K3 crystal on*** S/W (with respect to $(S', F_{S'})$) of rank n is an F-crystal (H, ∇, Φ) on S/W (with respect to $(S', F_{S'})$) where H is an A'-module of rank n, endowed with a horizontal symmetric pairing $H \otimes H \to \mathscr{O}_{S'}$ such that

1. there exists a horizontal map $V \colon (H, \nabla) \to F_{S'}^*(H, \nabla)$ satisfying $\Phi \circ V = V \circ \Phi = p^2$,
2. $\mathrm{gr}_F^\bullet H$ is a locally free $\mathscr{O}_S$-module of rank one,[2]
3. the pairing is perfect, and
4. $\langle \Phi(x), \Phi(y) \rangle = p^2 F_{S'}^* \langle x, y \rangle$ for any two sections x, y of $F_{S'}^* H$.

[2]For the definition of the Hodge filtration on an abstract F-crystal we refer to [40].

We say that H is ***supersingular*** if for all geometric points $s \to S$ the restricted crystal $H(s)$ is a supersingular K3 crystal in the sense of Definition 3.2.1.

Definition 3.2.28. Suppose that we are in Situation 3.2.23, and that $X \to S$ is a relative K3 surface. We let

$$(\mathrm{H}^2(X/S'), \nabla_{S'}, \Phi_{S'})$$

be the F-crystal on S/W relative to $(S', F_{S'})$ corresponding to the second crystalline cohomology of $X \to S$ with its canonical F-structure induced by the Frobenius. When equipped with the pairing induced by the cup product, this is a K3 crystal of rank 22, which is supersingular if and only if $X \to S$ is supersingular. We let

$$(\widetilde{\mathrm{H}}(X/S'), \widetilde{\nabla}_{S'}, \widetilde{\Phi}_{S'})$$

be the F-crystal on S/W relative to $(S', F_{S'})$ corresponding to the total crystalline cohomology of $X \to S$ with the twisted F-structure (as in Definition 3.2.6). When equipped with the Muaki pairing, this is a K3 crystal of rank 24, which is supersingular if and only if $X \to S$ is supersingular.

Remark 3.2.29. As explained in [40], the relative analogs of Proposition 3.2.4 and Remark 3.2.5 hold as well.

Set $\mathrm{H}^2(X/S'_K) = \mathrm{H}^2(X/S') \otimes_W K$ and $\widetilde{\mathrm{H}}(X/S'_K) = \widetilde{\mathrm{H}}(X/S') \otimes_W K$. We will extend Definition 3.2.12 to the relative setting. We have maps

$$\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p \xrightarrow{\Upsilon} \mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1) \xrightarrow{d\log} \mathbf{R}^2\pi^{\text{ét}}_*\Omega^\bullet_{X/S}$$

There is a canonical isomorphism

$$\mathrm{H}^2(X/S') \otimes_{A'} A \xrightarrow{\sim} \mathrm{H}^2_{dR}(X/S)$$

As in the case when $S = \operatorname{Spec} k$, we find a diagram

$$\begin{array}{ccc} & & \Gamma(S', \mathrm{H}^2(X/S')) \\ & & \downarrow \\ \Gamma(S, \mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p) & \longrightarrow & \Gamma(S, \mathrm{H}^2_{dR}(X/S)) \end{array} \tag{3.2.29.1}$$

although the horizontal map is no longer necessarily injective.

Definition 3.2.30. Let $X \to S$ be a relative K3 surface and $\mathscr{X} \to X$ a $\boldsymbol{\mu}_p$-gerbe with cohomology class $\alpha \in \Gamma(S, \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)$. Suppose that we are in Situation 3.2.23. Let $\mathscr{B} \in \Gamma(S', \mathrm{H}^2(X/S'_K))$ be a section such that $p\mathscr{B} \in \Gamma(S', \mathrm{H}^2(X/S'))$ and the image of $p\mathscr{B}$ in $\Gamma(S, \mathrm{H}^2_{dR}(X/S))$ is equal to the image of α. Consider the composition

$$\widetilde{\mathrm{H}}(X/S') \xrightarrow{h \mapsto e^{\mathscr{B}} \otimes h} \widetilde{\mathrm{H}}(X/S'_K) \otimes \widetilde{\mathrm{H}}(X/S') \to \widetilde{\mathrm{H}}(X/S'_K)$$

where the second map is given by the cup product. We define

$$\widetilde{\mathrm{H}}(\mathscr{X}/S') = e^{\mathscr{B}}\, \widetilde{\mathrm{H}}(X/S') \subset \mathrm{H}^*(X/S'_K)$$

to be its image. As before, note that this does not depend on our choice of $\mathscr{B}$. This definition makes sense more generally for a cohomology class $\alpha \in \Gamma(S, \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)$ that is not in the image of $\mathrm{H}^2(X, \boldsymbol{\mu}_p)$, and hence may only be represented by a gerbe flat locally on S.

Proposition 3.2.31. *The A'-module $\widetilde{\mathrm{H}}(\mathscr{X}/S')$ is of formation compatible with base change, in the following sense. Suppose B is a smooth k-algebra and B' is a p-adically complete lift of B to W. Write $T = \operatorname{Spec} B$ and $T' = \operatorname{Spec} B'$. Given a commutative diagram*

$$\begin{array}{ccc} T' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

of W-schemes, the natural map

$$\widetilde{\mathrm{H}}(\mathscr{X}_T/T') \to \widetilde{\mathrm{H}}(\mathscr{X}/S') \otimes_{A'} B'$$

is an isomorphism of B'-modules.

Proof. The crystalline cohomology of a relative K3 surface is of formation compatible with base change, in the sense that the natural map

$$\widetilde{\mathrm{H}}(X/S') \otimes_{A'} B' \to \widetilde{\mathrm{H}}(X_T/T')$$

is an isomorphism. Let $\mathscr{B}_T = \mathscr{B} \otimes 1 \in \widetilde{\mathrm{H}}(X/S') \otimes_{A'} B'$ be the pullback of $\mathscr{B}$. The cup product is compatible with base change as well, so

$$\widetilde{\mathrm{H}}(\mathscr{X}/S') \otimes_{A'} B' = e^{\mathscr{B}}\, \widetilde{\mathrm{H}}(X/S') \otimes_{A'} B' \xrightarrow{\sim} e^{\mathscr{B}_T}\, \widetilde{\mathrm{H}}(X_T/T')$$

Let α_T be the cohomology class of the $\boldsymbol{\mu}_p$-gerbe $\mathscr{X}_T \to X_T$. It follows that $\mathscr{B}_T$ is a B-field lift of α_T, and therefore that $e^{\mathscr{B}_T}\,\widetilde{\mathrm{H}}(X_T/T') = \widetilde{\mathrm{H}}(\mathscr{X}_T/T')$. $\square$

To endow $\widetilde{\mathrm{H}}(\mathscr{X}/S')$ with the structure of a K3 crystal, we need to give it a connection. Under our assumption that S is smooth, the second de Rham cohomology of $X \to S$ is equipped with the Gauss-Manin connection

$$\nabla_0\colon\ \mathrm{H}^2_{dR}(X/S) \to \mathrm{H}^2_{dR}(X/S) \otimes \Omega^1_{S/k}$$

For any $D \in \Gamma(S, T^1_{S/k})$, composing ∇_0 with D gives a map $\nabla_0(D)\colon\ \mathrm{H}^2_{dR}(X/S) \to \mathrm{H}^2_{dR}(X/S)$. Morevoer, via the isomorphism

$$\mathrm{H}^2(X/S') \otimes_W k \xrightarrow{\sim} \mathrm{H}^2_{dR}(X/S),$$

the connection $\nabla_{S'}$ reduces to ∇_0.

Lemma 3.2.32. *If $\alpha \in \Gamma(S, \mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p)$ is a cohomology class, then $\nabla_0(\beta) = 0$, where $\beta = d\log\circ\Upsilon(\alpha)$ is the image of α in $\Gamma(S, \mathrm{H}^2_{dR}(X/S))$ under the horizontal map of (3.2.29.1).*

Proof. We recall from Proposition 2.2.11 that Υ is defined to be the composite of morphisms

$$\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p \to \mathbf{R}^1\pi^{\text{ét}}_*(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p}) \to \mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1)$$

We consider the induced maps on global sections

$$\Gamma(S, \mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p) \to \Gamma(S, \mathbf{R}^1\pi^{\text{ét}}_*(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p})) \to \Gamma(S, \mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1))$$

We will show that the image of any global section of $\mathbf{R}^1\pi^{\text{ét}}_*(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p})$ is horizontal (in fact, this is equivalent to the result, because under the assumption that S is smooth the first map is an isomorphism on global sections). We will do this using the explicit description of the Gauss-Manin connection in terms of cocycles due to Katz. We will follow the presentation in Section 3 of [26], and also refer to the slightly different formulation in [28].

Let r be the relative dimension of $X \to S$ (in our case $r = 2$). Choose a finite covering $\{U_\alpha\}$ of X by open affines such that each U_α is étale over $\mathbf{A}^r_S$, and such that on each U_α the sheaf $\Omega^1_{X/S}$ is a free $\mathscr{O}_X$-module with basis $\{dx_1^\alpha, \ldots, dx_r^\alpha\}$. We consider the double complex $\mathscr{C}^{\bullet,\bullet}$ with terms

$$\mathscr{C}^{p,q} = \mathscr{C}^p(\{U_\alpha\}, \Omega^q_{X/S})$$

where $\mathscr{C}^p(\{U_\alpha\}, \Omega^q_{X/S})$ is the set of alternating Čech cochains on the covering $\{U_\alpha\}$. This is equipped with a vertical differential $d\colon \mathscr{C}^{p,q} \to \mathscr{C}^{p,q+1}$ and a horizontal differential $\delta\colon \mathscr{C}^{p,q} \to \mathscr{C}^{p+1,q}$, defined in [28]. We consider the associated total complex $\mathscr{K}^\bullet$ with terms

$$\mathscr{K}^r = \sum_{p+q=n} \mathscr{C}^{p,q}$$

and differential $d+\delta$. As explained in [28], the hypercohomology of $\mathscr{K}^\bullet$ computes the de Rham cohomology of $X \to S$.

Let $D \in \mathrm{Der}_k(\mathscr{O}_S, \mathscr{O}_S)$ be a k-derivation of $\mathscr{O}_S$. For each index α, let $D_\alpha \in \mathrm{Der}_k(U_\alpha, U_\alpha)$ be the unique extension of D which kills $dx_1^\alpha, \ldots, dx_r^\alpha$. In [26], Katz defines maps $\widetilde{D}\colon \mathscr{C}^{p,q} \to \mathscr{C}^{p,q}$ and $\lambda(D)\colon \mathscr{C}^{p,q} \to \mathscr{C}^{p+1,q-1}$, and shows that the sum

$$\widetilde{D} + \lambda(D)\colon \mathscr{K}^r \to \mathscr{K}^r$$

upon passage to cohomology computes the map $\nabla_0(D)$.

We now make our computation. Our cohomology class in $\Gamma(S, \mathbf{R}^1\pi_*^{\text{ét}}(\mathscr{O}_X^\times/\mathscr{O}_X^{\times p})$ may be represented by a cocycle $f_{i,j} \in \mathscr{C}^1(\{U_\alpha\}, \mathscr{O}_X^\times/\mathscr{O}_X^{\times p})$. After possibly shrinking our cover, we may find lifts $\zeta_{i,j} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^\times)$ of each of the $f_{i,j}$. Because we are in characteristic p, the quotients $d\log(\zeta_{i,j})$ are independent of our choice of lifts, and hence give rise to a cocycle

$$d\log(\zeta_{i,j}) \in \mathscr{C}^{1,1}$$

that represents the image of our cohomology class in the second de Rham cohomology. Fix a k-derivation D of $\mathscr{O}_S$. We wish to show that the cocycle

$$(\widetilde{D} + \lambda(D))(d\log(\zeta_{i,j})) \in \mathscr{C}^{1,1} \oplus \mathscr{C}^{2,0}$$

is in the image of $d+\delta$. Consider the cocycle

$$D_i \log(\zeta_{i,j}) = \frac{D_i(\zeta_{i,j})}{\zeta_{i,j}} \in \mathscr{C}^{1,0}$$

Note that, because D_i is a derivation, this is also independent of the choice of $\zeta_{i,j}$. One computes that

$$(d+\delta)(D_i \log(\zeta_{i,j})) = (\widetilde{D} + \lambda(D))(d\log(\zeta_{i,j}))$$

which gives the result. □

Proposition 3.2.33. *The submodule* $\widetilde{\mathrm{H}}(\mathscr{X}/S') \subset \widetilde{\mathrm{H}}(X/S'_K)$ *is horizontal with respect to the connection* $\widetilde{\nabla}_{S'} \otimes K$.

Proof. The assertion is equivalent to the statement that for all $(a,b,c) \in \widetilde{\mathrm{H}}(X/S')$,

$$(e^{\mathscr{B}} \otimes \mathrm{id}) \circ \widetilde{\nabla}_{S'}(e^{-\mathscr{B}}(a,b,c)) \in \widetilde{\mathrm{H}}(X/S') \mathbin{\widehat{\otimes}} \widehat{\Omega}^1_{S'/W}$$

By the compatibility of the connection with the cup product, we have

$$\begin{aligned}(e^{\mathscr{B}} \otimes \mathrm{id}) \circ \widetilde{\nabla}_{S'}(e^{-\mathscr{B}}(a,b,c)) &= \widetilde{\nabla}_{S'}(a,b,c) + (e^{\mathscr{B}} \otimes \mathrm{id})((a,b,c)\widetilde{\nabla}_{S'}(e^{-\mathscr{B}})) \\ &= \widetilde{\nabla}_{S'}(a,b,c) + (e^{\mathscr{B}} \otimes \mathrm{id})((a,b,c)(0, \nabla_{S'}(-\mathscr{B}), \mathscr{B}.\nabla_{S'}(\mathscr{B}))) \\ &= \widetilde{\nabla}_{S'}(a,b,c) + (e^{\mathscr{B}} \otimes \mathrm{id})(0, a\nabla_{S'}(-\mathscr{B}), a\mathscr{B}.\nabla_{S'}(\mathscr{B}) + b.\nabla_{S'}(-\mathscr{B})) \\ &= \widetilde{\nabla}_{S'}(a,b,c) + (0, a\nabla_{S'}(-\mathscr{B}), b.\nabla_{S'}(-\mathscr{B}))\end{aligned}$$

By Lemma 3.2.32, we have $\nabla_{S'}(\mathscr{B}) \in \mathrm{H}^2(X/S') \mathbin{\widehat{\otimes}} \widehat{\Omega}^1_{S'/W}$, and the result follows. □

Thus, we obtain a connection

$$\widetilde{\nabla}_{S'} \colon \widetilde{\mathrm{H}}(\mathscr{X}/S') \to \widetilde{\mathrm{H}}(\mathscr{X}/S') \mathbin{\widehat{\otimes}} \widehat{\Omega}^1_{S'/W}$$

on $\widetilde{\mathrm{H}}(\mathscr{X}/S')$. It is immediate that it satisfies condition (3) of Definition 3.2.24. The remaining properties can be shown by the same methods as in the punctual case, using the relative version of Theorem 3.2.10. We will instead deduce them by reduction to the punctual case. Recall that given a closed point $s \in S$, there is a unique map $s' \colon (\operatorname{Spec} W, F_W) \to (S', F_{S'})$ called the *Teichmüller lifting* of s. If s' is a Teichmüller lifting of a closed point s, then the restricted F-crystal

$$(s'^* \, \widetilde{\mathrm{H}}(X/S'), s'^* \widetilde{\nabla}_{S'}, s'^* \widetilde{\Phi}_{S'})$$

is identified with $(\widetilde{\mathrm{H}}(X_s/W), \widetilde{\Phi})$.

Lemma 3.2.34. *The submodule* $\widetilde{\mathrm{H}}(\mathscr{X}/S')$ *is preserved by the action of the Frobenius* $\widetilde{\Phi}_{S'}$ *on* $\widetilde{\mathrm{H}}(X/S'_K)$.

Proof. We will show that the quotient

$$M = (\widetilde{\Phi}_{S'}(F^*_{S'} \, \widetilde{\mathrm{H}}(\mathscr{X}/S')) + \widetilde{\mathrm{H}}(\mathscr{X}/S')) / \, \widetilde{\mathrm{H}}(\mathscr{X}/S')$$

of A'-modules is zero. Consider a closed point $s \in S$, with Teichmüller lift

$$s' \colon (\operatorname{Spec} W(s), F_{W(s)}) \to (S', F_{S'})$$

The restriction of M to such a point vanishes by Lemma 3.2.13. It follows that $M \otimes W/pW = 0$. This means that multiplication by p on M is an isomorphism. Because M is p-adically complete, we conclude that $M = 0$. □

Proposition 3.2.35. *The triple* $(\widetilde{\mathrm{H}}(\mathscr{X}/S'), \widetilde{\nabla}_{S'}, \widetilde{\Phi}_{S'})$ *is an* F*-crystal. Equipped with the Mukai pairing, it is a K3 crystal on* S/W *with respect to* $(S', F_{S'})$ *of rank 24, which is supersingular if and only if* $X \to S$ *is supersingular.*

Proof. That $(\widetilde{\mathrm{H}}(\mathscr{X}/S'), \widetilde{\nabla}_{S'}, \widetilde{\Phi}_{S'})$ is an F-crystal follows immediately from the fact that $(\widetilde{\mathrm{H}}(X/S'), \widetilde{\nabla}_{S'}, \widetilde{\Phi}_{S'})$ is an F-crystal. Next, we will check the conditions of Definition 3.2.27. Because $\widetilde{\mathrm{H}}(X/S')$ is a K3 crystal, it admits a map $\widetilde{V}_{S'} \colon \widetilde{\mathrm{H}}(X/S') \to F_{S'}^* \widetilde{\mathrm{H}}(X/S')$ satisfying condition (1). We will show that $\widetilde{V}_{S'}$ restricts to such a map on $\widetilde{\mathrm{H}}(\mathscr{X}/S')$. Indeed, by Proposition 3.2.14, this is true for the restriction of $\widetilde{\mathrm{H}}(\mathscr{X}/S')$ to any closed point of S. By the same argument as Lemma 3.2.34, this implies the result. Because $\mathrm{gr}_F^\bullet H$ is of formation compatible with base change, condition (2) also follows from Proposition 3.2.14. Conditions (3) and (4) hold because they are true for $\widetilde{\mathrm{H}}(X/S')$, and cupping with $e^{\mathscr{B}}$ is an isometry with respect to the Mukai pairing.

Finally, supersingularity is by definition a condition on the fibers, so the final claim follows from Proposition 3.2.14. □

Remark 3.2.36. Note that we have only defined the crystal $\widetilde{\mathrm{H}}(\mathscr{X}/S')$ in Situation 3.2.23. If S is smooth but not necessarily affine, the compatibility of our construction with respect to base change shows that our locally defined crystals glue to a crystal on S/W, in the sense of [47, 07IS].

3.3 Moduli of twisted supersingular K3 surfaces and the twisted period morphism

In this section we discuss the moduli space of marked twisted supersingular K3 surfaces. We then use the twisted K3 crystals of the previous section to define a twisted crystalline period morphism. We compute its differential, and show that it is étale.

Let Λ be a supersingular K3 lattice and let $\pi \colon X \to S_\Lambda$ denote the universal marked supersingular K3 surface. For psychological reasons, we adopt the following notation.

Definition 3.3.1.

$$\mathscr{S}_\Lambda = \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$$
$$\mathscr{S}_\Lambda^o = (\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o$$

Recall from Definition 3.1.10 that $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o \subset \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$ is the subgroup whose fiber over a marked K3 surface is the group of transcendental classes.

Theorem 3.3.2. *The functor $\mathscr{S}_\Lambda^o$ is representable by an algebraic space that is locally of finite presentation, locally separated, and smooth over* $\operatorname{Spec} k$. *The morphism $\mathscr{S}_\Lambda^o \to S_\Lambda$ is a smooth group scheme of relative dimension 1, and the connected component of any of its geometric fibers is isomorphic to $\mathbf{G}_a$.*

Proof. This follows from Theorem 3.1.3 and Lemma 3.1.12. □

The k-points of the moduli space $\mathscr{S}_\Lambda^o$ are given by pairs $(\mathscr{X}, m)$ where $\mathscr{X}$ is a $\boldsymbol{\mu}_p$-gerbe over a supersingular K3 surface X, and m is a marking of X such that $\mathscr{X}$ is transcendental with respect to m. Using the assumption that $\mathscr{X}$ is transcendental, we can give an alternate description of such a pair as a $\boldsymbol{\mu}_p$-gerbe $\mathscr{X}$ equipped with a certain marking of the extended Néron-Severi group of $\mathscr{X}$.

Definition 3.3.3. An *extended supersingular K3 lattice* is a free abelian group $\widetilde{\Lambda}$ of rank 24 equipped with an even symmetric bilinear form such that

1. $\operatorname{disc}(\Lambda \otimes \mathbf{Q}) = 1$ in $\mathbf{Q}^\times/\mathbf{Q}^{\times 2}$,

2. the signature of $\Lambda \otimes \mathbf{R}$ is $(2, 22)$, and

3. the discriminant group Λ^*/Λ is p-torsion.

These lattices behave similarly to supersingular K3 lattices. In particular, if $\widetilde{\Lambda}$ is an extended supersingular K3 lattice, then $\widetilde{\Lambda}_0 = p\widetilde{\Lambda}^*/p\widetilde{\Lambda}$ is a vector space over $\mathbf{F}_p$ of dimension $2\sigma_0$ for some integer $1 \leq \sigma_0 \leq 11$, called the *Artin invariant* of $\widetilde{\Lambda}$. This vector space has a natural bilinear form, which is non-degenerate and non-neutral.

Notation 3.3.4.

- We let U_2 denote the rank 2 lattice which is generated by two elements e, f satisfying $e^2 = f^2 = 0$ and $e.f = -1$.

- Given a lattice L and an integer n, we write $L(n)$ for the lattice with underlying abelian group L but with the form multiplied by n. Thus, $U_2(p)$ denotes the lattice generated by two elements e, f satisfying $e^2 = f^2 = 0$ and $e.f = -p$.

- We set $\widetilde{\Lambda} = \Lambda \oplus U_2(p)$, where Λ is our fixed supersingular K3 lattice.

Lemma 3.3.5. *If Λ is a supersingular K3 lattice of Artin invariant σ_0, then $\Lambda \oplus U_2$ and $\Lambda \oplus U_2(p)$ are extended supersingular K3 lattices of Artin invariants σ_0 and $\sigma_0 + 1$. Moreover, every extended supersingular K3 lattice is of this form for some Λ.*

Proof. In [40] it is shown that the Artin invariant determines a supersingular K3 lattice up to isometry. The proof implies the same for extended supersingular K3 lattices. It is then easy to compute that $\Lambda \oplus U_2$ and $\Lambda \oplus U_2(p)$ have the required properties. □

By the calculations of Proposition 3.2.20, we see that extended supersingular K3 lattices are exactly those lattices that occur as the extended Néron-Severi group of a twisted supersingular K3 surface.

Lemma 3.3.6. *Let X be a supersingular K3 surface with a marking $m\colon \Lambda \to \mathrm{Pic}(X)$. If $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$-gerbe that is transcendental with respect to m, then the map*

$$\widetilde{\Lambda} = \Lambda \oplus U_2(p) \to \widetilde{\mathrm{H}}(\mathscr{X}/W)$$

given by $e \mapsto (0,0,1)$ and $f \mapsto (p,0,0)$ factors through $\widetilde{N}(\mathscr{X})$.

Proof. We must check that if $l \in \Lambda$ and $s \in \mathbf{Z}$ then $(p,l,s) \in \widetilde{N}(\mathscr{X})$. If B is a B-field lift of α, we compute

$$e^{-B}(p,l,s) = (p, l - pB, s - l.B + p\frac{B^2}{2})$$

As α is transcendental relative to the marking m, $l.B \in W$ and $p\frac{B^2}{2} \in W$, which gives the result. □

We will next define our twisted period morphism. We first briefly recall some material from [40]. Let T be a $\mathbf{Z}_p$-lattice that is isometric to the Tate module of a supersingular K3 crystal of rank n (in [40] these are called "K3 lattices"). For instance, we may take $\Lambda \otimes \mathbf{Z}_p$ where Λ is a supersingular K3 lattice (and $n = 22$), or $\widetilde{\Lambda} \otimes \mathbf{Z}_p$, where $\widetilde{\Lambda}$ is an extended supersingular K3 lattice (and $n = 24$). Suppose that we are in Situation 3.2.23, and that $H_{S'}$ is a supersingular K3 crystal of rank n on S/W with Tate module T_H. A T*-structure* on $H_{S'}$ is an isometry $T \otimes \mathscr{O}_{S'} \to T_H$. The image of H in $T^* \otimes \mathscr{O}_{S'}/T \otimes \mathscr{O}_{S'} \cong T_0 \otimes \mathscr{O}_S$ descends uniquely through the absolute Frobenius of S, giving rise to a characteristic subspace $K_H \subset T_0 \otimes \mathscr{O}_S$.

Definition 3.3.7. Let $\widetilde{\pi}\colon \widetilde{X} \to \mathscr{S}^o_\Lambda$ be the pullback of $\pi\colon X \to S_\Lambda$ to $\mathscr{S}^o_\Lambda$, and let $\alpha \in (\mathbf{R}^2\widetilde{\pi}^{\mathrm{fl}}_*\boldsymbol{\mu}_p)(\mathscr{S}^o_\Lambda)$ be the restriction of the universal cohomology class. Given an affine étale open S of $\mathscr{S}^o_\Lambda$ and data as in Situation 3.2.23, we consider the twisted K3 crystal $\widetilde{\mathrm{H}}(\mathscr{X}/S')$ corresponding to $(\widetilde{X}, \alpha)|_S$. Let $T = \widetilde{\Lambda} \otimes \mathbf{Z}_p$. By Lemma 3.3.6 the induced map $\underline{\widetilde{\Lambda}} \otimes \mathscr{O}_{S'} \to \widetilde{\mathrm{H}}(\mathscr{X}/S')$ factors through the Tate module, and thus gives rise to a T-structure. As described in [40], the image of $\widetilde{\mathrm{H}}(\mathscr{X}/S')$ in

$$\frac{T^* \otimes \mathscr{O}_{S'}}{T \otimes \mathscr{O}_{S'}} = \frac{\widetilde{\Lambda}^*}{\widetilde{\Lambda}} \otimes \mathscr{O}_S \cong \underline{\widetilde{\Lambda}}_0 \otimes \mathscr{O}_S$$

descends through the Frobenius, and we obtain a characteristic subspace $K \subset \underline{\widetilde{\Lambda}}_0 \otimes \mathscr{O}_S$. By Proposition 3.2.31, this is independent of our choices of lifts, and in particular glues to a global characteristic subspace

$$\widetilde{K}_\Lambda \subset \underline{\widetilde{\Lambda}}_0 \otimes \mathscr{O}_{\mathscr{S}^o_\Lambda}$$

Note that for every geometric point $s \in \mathscr{S}^o_\Lambda$ we have $e \notin (\widetilde{K}_\Lambda)_s$. We therefore obtain a morphism

$$\widetilde{\rho}\colon \mathscr{S}^o_\Lambda \to \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \tag{3.3.7.1}$$

which we call the *twisted period morphism.*

We refer to Remark 3.1.6 for some of the subtleties of this definition. We remark that the non-twisted period morphism (3.1.4.1) has a similar interpretation in terms of crystalline cohomology (see Remark 3.2.5). The twisted period morphism and the usual one are related by a commutative diagram

$$\begin{tikzcd} \mathscr{S}^o_\Lambda \arrow[r, "p"] \arrow[d, "\widetilde{\rho}"'] & S_\Lambda \arrow[d, "\rho"] \\ \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \arrow[r, "\pi_e"] & \overline{M}_{\Lambda_0} \end{tikzcd} \tag{3.3.7.2}$$

Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe over a supersingular K3 surface and let $m\colon \Lambda \to \mathrm{Pic}(X)$ be a marking. By Lemma 3.3.6, we obtain a map $\widetilde{\Lambda} \to \widetilde{\mathrm{H}}(\mathscr{X}/W)$, and hence a $T = \widetilde{\Lambda} \otimes \mathbf{Z}_p$-structure $\widetilde{\Lambda} \otimes \mathbf{Z}_p \to \widetilde{\mathrm{H}}(\mathscr{X}/W)$ on the supersingular K3 crystal $\widetilde{\mathrm{H}}(\mathscr{X}/W)$. We will let $K(\mathscr{X}) \subset \widetilde{\Lambda}_0 \otimes k$ be the corresponding characteristic subspace. In other words, this gives the image of the point corresponding to $\mathscr{X}$ under $\widetilde{\rho}$.

Proposition 3.3.8. *The twisted period morphism $\widetilde{\rho}$ is étale.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathscr{S}^o_\Lambda \arrow[r, "(\widetilde{\rho},p)"] \arrow[rr, bend left, "p"] \arrow[dr, "\widetilde{\rho}"'] & \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \times_{\overline{M}_{\Lambda_0}} S_\Lambda \arrow[r] \arrow[d] & S_\Lambda \arrow[d, "\rho"] \\ & \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \arrow[r, "\pi_e"] & \overline{M}_{\Lambda_0} \end{tikzcd} \tag{3.3.8.1}$$

with Cartesian square induced by the commuting square (3.3.7.2). Ogus has shown in [40] that ρ is étale. Thus, it will suffice to show that $(\widetilde{\rho}, p)$ is étale. The base change of $(\widetilde{\rho}, p)$ by the absolute Frobenius $F_{S_\Lambda}\colon S_\Lambda \to S_\Lambda$ gives a map

$$(\widetilde{\rho}, p)^{(p/S_\Lambda)}\colon (\mathscr{S}^o_\Lambda)^{(p/S_\Lambda)} \to \left(\overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \times_{\overline{M}_{\Lambda_0}} S_\Lambda\right)^{(p/S_\Lambda)}$$

which is a étale if and only if $(\widetilde{\rho}, p)$ is étale. Let $\mathscr{U}_{S_\Lambda}$ be the pullback of $\mathscr{U}_{\Lambda_0}$ to S_Λ. By Proposition 3.0.11, we have an isomorphism

$$\mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1)^o = F_{S_\Lambda}^{-1}\mathscr{U}_{S_\Lambda} \xrightarrow{\sim} \left(\overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \times_{\overline{M}_{\Lambda_0}} S_\Lambda\right)^{(p)}$$

We will identify the two for the remainder of this proof. By the definition of the twisted period morphism, we have a commuting diagram

$$\begin{tikzcd} \mathscr{S}^o_\Lambda \arrow[r, "F_{\mathscr{S}^o_\Lambda/S_\Lambda}"] \arrow[rr, bend right, "\Upsilon^o"'] & (\mathscr{S}^o_\Lambda)^{(p)} \arrow[r, "F_{S_\Lambda}^{-1}(\widetilde{\rho},p)"] & (\overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \times_{\overline{M}_{\Lambda_0}} S_\Lambda)^{(p)} \end{tikzcd} \tag{3.3.8.2}$$

Because its source and target are smooth, $(\widetilde{\rho}, p)$ is étale if and only if it is étale on the fiber over every geometric point s of S_Λ. Thus, we may replace S_Λ with the spectrum $\operatorname{Spec} k'$ of an algebraically closed field, and the universal K3 surface $\pi\colon X \to S_\Lambda$ with its fiber $\pi_s\colon X_s \to \operatorname{Spec} k'$. As in Lemma 2.2.17, we consider the diagram

$$\begin{tikzcd}
(\mathbf{R}^2\pi^{\mathrm{fl}}_{s*}\boldsymbol{\mu}_p)^o \arrow[r, "F_{\mathscr{S}^o_s/k'}"] \arrow[d] & (\mathbf{R}^2\pi^{(p)\,\mathrm{fl}}_{s*}\boldsymbol{\mu}_p)^o \arrow[r, "F^{-1}_{k'}(\widetilde{\rho},p)"] \arrow[d, "\iota"] & \mathbf{R}^1\pi^{(p)\mathrm{\acute{e}t}}_{s*}\nu(1)^o \arrow[dl, "\delta"] \\
\mathbf{R}^2\pi^{\mathrm{\acute{e}t}}_{s*}\mathscr{O}^\times_{X_s} \arrow[r] & \mathbf{R}^2\pi^{(p)\mathrm{\acute{e}t}}_{s*}\mathscr{O}^\times_{X^{(p)}_s} &
\end{tikzcd} \tag{3.3.8.3}$$

We do not know yet that this diagram commutes, merely that the outer compositions agree (by Lemma 2.2.17) and that the square commutes. This implies that

$$\delta \circ F^{-1}_{k'}(\widetilde{\rho}, p) \circ F_{\mathscr{S}^o_s/k'} = \iota \circ F_{\mathscr{S}^o_s/k'}$$

By Theorem 2.2.10 and the Leray spectral sequence, there is an isomorphism

$$\mathbf{R}^2\pi^{\mathrm{\acute{e}t}}_{s*}\mathscr{O}^\times_{X_s} \cong \varepsilon_{s*}\mathbf{R}^2\pi^{\mathrm{fl}}\mathscr{O}^\times_{X_s}$$

Thus, the functors $\mathbf{R}^2\pi^{\mathrm{\acute{e}t}}_{s*}\mathscr{O}^\times_{X_s}$ and $\mathbf{R}^2\pi^{(p)\mathrm{\acute{e}t}}_{s*}\mathscr{O}^\times_{X^{(p)}_s}$ on the category of schemes over $\operatorname{Spec} k'$ are sheaves in the flat topology. Being representable, the functors in the top row of diagram (3.3.8.3) are also sheaves in the flat topology. The relative Frobenius $F_{\mathscr{S}^o_s/k'}$ is faithfully flat, and hence an epimorphism in the category of sheaves on the big flat site. Thus, we have

$$\delta \circ F^{-1}_{k'}(\widetilde{\rho}, p) = \iota$$

and so in fact the diagram commutes. After passing to completions at the identity section, both ι and δ become isomorphisms (see Remark 2.2.18). Therefore, $F^{-1}_{k'}(\widetilde{\rho}, p)$ induces an isomorphism on completions at the identity section, and so is étale. □

Proposition 3.3.9. *The map*

$$(\widetilde{\rho}, p)\colon \mathscr{S}^o_\Lambda \to \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \times_{\overline{M}_{\Lambda_0}} S_\Lambda$$

induced by the commuting square (3.3.7.2) is an isomorphism. In other words, the square is Cartesian.

Proof. For readability, let us we write $M^+ = \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}$ and $M = \overline{M}_{\Lambda_0}$. With the same identifications as in the proof of Proposition 3.3.8, we then have a commutative diagram

$$
\begin{tikzcd}
\mathscr{S}^o_\Lambda \arrow[r, "(\widetilde{\rho}{,}p)"'] \arrow[rr, bend left, "\Upsilon^o"] \arrow[rrr, bend left=40, "p"] \arrow[dr, "\widetilde{\rho}"'] & M^+\times_M S_\Lambda \arrow[r] \arrow[d] & M^+\times_M S_\Lambda \arrow[r] \arrow[d] & S_\Lambda \arrow[d, "\rho"] \\
 & M^+ \arrow[r, "F_{M^+/M}"] \arrow[dr, "F_{M^+}"'] & M^{+(p/M)} \arrow[r, "\pi_e^{(p/M)}"] \arrow[d, "W_{M^+/M}"] & M \arrow[d, "F_M"] \\
 & & M^+ \arrow[r, "\pi_e"] & M
\end{tikzcd}
\tag{3.3.9.1}
$$

where the three squares are Cartesian. Because $F_{M^+/M}$ and Υ^o are universal homeomorphisms, it follows that $(\widetilde{\rho}, p)$ is a universal homeomorphism as well. By Proposition 3.3.8, $(\widetilde{\rho}, p)$ is étale. By Zariski's Main Theorem (for algebraic spaces, see [47, 082K]), an étale universal homeomorphism is an isomorphism. $\square$

Proposition 3.3.9 could be viewed as a relative Torelli theorem for the twisted period morphism over the non-twisted period morphism. Let us describe a few consequences for an individual supersingular K3 surface $\pi\colon X \to \operatorname{Spec} k$. We define groups schemes $\mathscr{U}$ and $\mathscr{H}$ over k by the exact sequences

$$0 \to \mathscr{U} \to \frac{\operatorname{Pic}(X)_0 \otimes k}{K(X)} \xrightarrow{1-F^*} \frac{\operatorname{Pic}(X)_0 \otimes k}{K(X) + F^*K(X)}$$

$$0 \to \mathscr{H} \to \frac{\operatorname{Pic}(X) \otimes k}{K(X)} \xrightarrow{1-F^*} \frac{\operatorname{Pic}(X) \otimes k}{K(X) + F^*K(X)}$$

where $K(X)$ is the characteristic subspace corresponding to X (equipped with the trivial marking $\mathrm{id} : \operatorname{Pic}(X) \to \operatorname{Pic}(X)$).

Proposition 3.3.10. *There are natural isomorphisms* $\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p \xrightarrow{\sim} \mathscr{H}$ *and* $\mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1) \xrightarrow{\sim} \mathscr{H}^{(p/\operatorname{Spec} k)}$ *of group schemes fitting into a commutative diagram*

$$
\begin{tikzcd}
\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p \arrow[r, "\Upsilon"] \arrow[d, "\wr"] & \mathbf{R}^1\pi^{(p)\text{ét}}_*\nu(1) \arrow[d, "\wr"] \\
\mathscr{H} \arrow[r, "F_{\mathscr{H}/k}"] & \mathscr{H}^{(p/\operatorname{Spec} k)}
\end{tikzcd}
$$

Proof. Let $\Lambda = \mathrm{Pic}(X)$, and equip X with the trivial marking $\mathrm{id} : \Lambda \to \mathrm{Pic}(X)$. We obtain the right hand vertical isomorphism from diagram (3.1.11.2) (as the marking is an isomorphism, the sheaf $\mathscr{W}$ is trivial). By Proposition 3.3.9 (and the isomorphism of Proposition 3.0.11) the twisted period morphism induces an isomorphism

$$(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o \xrightarrow{\sim} \mathscr{U}$$

which by the definition of the twisted period morphism fits into a commuting diagram

$$\begin{tikzcd}
(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o \arrow[r, hook] \arrow[d, "\wr"] & \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p \arrow[r, "\Upsilon"] \arrow[d, dashed] & \mathbf{R}^1\pi_*^{(p)\text{ét}}\nu(1) \arrow[d, "\wr"] \\
\mathscr{U} \arrow[r, hook] & \mathscr{H} \arrow[r, "F_{\mathscr{H}/k}"] & \mathscr{H}^{(p/\operatorname{Spec} k)}
\end{tikzcd}$$

As Υ and $F_{\mathscr{H}/k}$ are universal homeomorphisms, they induce isomorphisms on the respective groups of connected components. We may therefore produce by translation an isomorphism filling in the dashed arrow. $\square$

Remark 3.3.11. The exact sequence (of group schemes!)

$$0 \to \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p \to \frac{\mathrm{Pic}(X)\otimes k}{K(X)} \xrightarrow{1-F^*} \frac{\mathrm{Pic}(X)\otimes k}{K(X)+F^*K(X)}$$

produced by Proposition 3.3.10 is a strengthened form of the equality mentioned in Remark 3.26 of [40] (we warn that the referenced formula appears to contain some typos).

Remark 3.3.12. The natural inclusion $\mathscr{U} \subset \mathscr{H}$ gives rise to a short exact sequence

$$0 \to \mathscr{U} \to \mathscr{H} \to \mathscr{D} \to 0$$

where $\mathscr{D}$ is the quotient (this is the same as the sheaf $\mathscr{D}$ in the diagram (3.1.11.2)). Write $\Lambda = \mathrm{Pic}(X)$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{p\Lambda^*}{p\Lambda} \arrow[r] \arrow[d] & \dfrac{\Lambda}{p\Lambda} \arrow[r] \arrow[d] & \dfrac{\Lambda}{p\Lambda^*} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathscr{U} \arrow[r] & \mathscr{H} \arrow[r] & \mathscr{D} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Evaluating on k-points and applying Proposition 3.3.10, we recover part of diagram (2.1.13.1) (see also Remark 3.1.17).

A basic fact about the Gauss-Manin connection is that it may be computed in terms of the Kodaira-Spencer map (see [27]). A similar result is true in our twisted setting. As explained in [40], the connections on the crystals $\widetilde{\mathrm{H}}(\mathscr{X}/S')$ defined on an affine cover of $\mathscr{S}^o_\Lambda$ induce a map

$$T^1_{\mathscr{S}^o_\Lambda/k} \to \mathscr{H}om_{\mathscr{S}^o_\Lambda}\left(\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda, \frac{F^*\widetilde{K}_\Lambda}{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda}\right) \tag{3.3.12.1}$$

which, under the identification in Lemma 3.0.7 of the tangent space to the period domain, is exactly the differential

$$d\widetilde{\rho}\colon T^1_{\mathscr{S}^o_\Lambda/k} \to \widetilde{\rho}^* T^1_{M^{\langle e\rangle}_{\widetilde{\Lambda}_0}/k}$$

of $\widetilde{\rho}$. The diagram (3.3.8.1) gives rise to a diagram

$$\begin{tikzcd}
0 \arrow[r] & T^1_{\mathscr{S}^o_\Lambda/S_\Lambda} \arrow[r] \arrow[d, "d(\widetilde{\rho},p)"] & T^1_{\mathscr{S}^o_\Lambda/k} \arrow[r] \arrow[d, "d\widetilde{\rho}"] & p^*T^1_{S_\Lambda/k} \arrow[r] \arrow[d, "p^*(d\rho)"] & 0 \\
0 \arrow[r] & \widetilde{\rho}^*T^1_{\overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}/\overline{M}_{\Lambda_0}} \arrow[r] & \widetilde{\rho}^*T^1_{\overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}/k} \arrow[r] & \widetilde{\rho}^*\pi_e^*T^1_{\overline{M}_{\Lambda_0}/k} \arrow[r] & 0
\end{tikzcd} \tag{3.3.12.2}$$

of tangent sheaves with exact rows. Pulling back the identification of Lemma 3.0.8, we find identifications

$$\begin{tikzcd}
T^1_{\mathscr{S}^o_\Lambda/S_\Lambda} \arrow[r] \arrow[d, hook] & \mathscr{H}om_{\mathscr{S}^o_\Lambda}\left(\dfrac{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda}{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda \cap e^\perp}, \dfrac{F^*\widetilde{K}_\Lambda}{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda}\right) \arrow[d, hook] \\
T^1_{\mathscr{S}^o_\Lambda/k} \arrow[r] & \mathscr{H}om_{\mathscr{S}^o_\Lambda}\left(\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda, \dfrac{F^*\widetilde{K}_\Lambda}{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda}\right)
\end{tikzcd}$$

As described in the proof of Lemma 3.0.8, pairing with e gives rise to isomorphisms

$$\mathscr{H}om_{\mathscr{S}^o_\Lambda}\left(\frac{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda}{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda \cap e^\perp}, \frac{F^*\widetilde{K}_\Lambda}{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda}\right) \cong \frac{F^*\widetilde{K}_\Lambda}{\widetilde{K}_\Lambda \cap F^*\widetilde{K}_\Lambda} \cong p^*\frac{F^*K_\Lambda}{K_\Lambda \cap F^*K_\Lambda} \cong p^*\mathbf{R}^2\pi_*\mathscr{O}_X$$

Thus, $d(\widetilde{\rho}, p)$ induces a map

$$T^1_{\mathscr{S}^o_\Lambda/S_\Lambda} \to p^*\mathbf{R}^2\pi_*\mathscr{O}_X \tag{3.3.12.3}$$

which we have proved to be an isomorphism.

Proposition 3.3.13. *The isomorphism (3.3.12.3) agrees with the "Kodaira-Spencer" isomorphism constructed in Lemma 2.2.14.*

Proof. Omitted. □

Chapter 4

SUPERSINGULAR TWISTOR SPACE

4.1 Moduli of twisted sheaves

Let X be a K3 surface over our fixed algebraically closed field k of characteristic $p \geq 3$, and let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe. In this section we will study the derived category of twisted sheaves on $\mathscr{X}$ with the aim of extending various results that are well known over the complex numbers to our setting. In particular, we define Chern characters for twisted sheaves on $\mathscr{X}$, consider the action of derived equivalences on the twisted Mukai crystals, and establish the existence of moduli spaces of twisted sheaves.

In establishing these results, we will need to reduce certain statements for twisted K3 surfaces to the non-twisted case. A considerable simplification occurs here if we assume that X is supersingular. This is because any $\boldsymbol{\mu}_p$-gerbe over a supersingular K3 surface sits in a canonical flat family with a $\boldsymbol{\mu}_p$-gerbe over the same surface that has trivial Brauer class. Indeed, consider the connected component $\mathbf{A}^1 \subset \mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$ of the group of $\boldsymbol{\mu}_p$-gerbes on X such that $\mathscr{X}$ is a fiber of the tautological family $\widetilde{\mathscr{X}} \to \mathbf{A}^1$ (see Section 2.1, where these are discussed as the basic examples of twistor families). We say that $\mathscr{X}$ *deforms the trivial gerbe* if this component is the identity component. In any case, the map $\mathbf{A}^1(k) \to \mathrm{Br}(X)$ induced by the Kummer sequence is surjective (for instance, by diagram (3.1.11.2)) so some fiber of the family $\widetilde{\mathscr{X}} \to \mathbf{A}^1$ has trivial Brauer class.

The corresponding reductions in the finite height case are significantly more involved, and require lifting to the complex numbers and comparison with the Hodge-theoretic constructions of [23]. Thus, although the main results of this section are true for with no assumptions on the height of X, we will for the most part restrict our attention to the supersingular case.

Let $p\colon \mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe on a smooth projective variety over k. Suppose that $\mathscr{X}$ has the resolution property. We have in mind the case when $\mathscr{X}$ is a $\boldsymbol{\mu}_p$-gerbe over a K3 surface, or the exterior sum of two $\boldsymbol{\mu}_p$-gerbes on the product of two K3 surfaces.

Definition 4.1.1. If $\mathscr{E}$ is a locally free sheaf of positive rank on $\mathscr{X}$, the *twisted Chern character* of $\mathscr{E}$ is

$$\mathrm{ch}_{\mathscr{X}}(\mathscr{E}) = \sqrt[p]{\mathrm{ch}(p_*(\mathscr{E}^{\otimes p}))}$$

where by convention we choose the p-th root so that $\mathrm{rk}(\mathrm{ch}_{\mathscr{X}}(\mathscr{E})) = \mathrm{rk}(\mathscr{E})$. We define the twisted Mukai vector of $\mathscr{E}$ by

$$v_{\mathscr{X}}(\mathscr{E}) = \mathrm{ch}_{\mathscr{X}}(\mathscr{E}).\sqrt{\mathrm{Td}(X)}$$

If $\mathscr{E}$ is 0-twisted, then its twisted Chern character is the same as the usual Chern character of its pushforward to X:

$$\mathrm{ch}_{\mathscr{X}}(\mathscr{E}) = \mathrm{ch}(p_*\mathscr{E})$$

In most cases we will consider, $\mathscr{E}$ will be a twisted sheaf. The twisted Chern character determines an additive map

$$\mathrm{ch}_{\mathscr{X}} \colon K^{(1)}(\mathscr{X}) \to A^*(X)_{\mathbf{Q}}$$

where $K^{(1)}(\mathscr{X})$ is the Grothendieck group of locally free twisted sheaves on $\mathscr{X}$ and $A^*(X)_{\mathbf{Q}}$ is the numerical Chow theory of X tensored with $\mathbf{Q}$. We record a few straightforward lemmas extending properties of the usual Chern characters to our twisted Chern characters.

Lemma 4.1.2. *If $\mathscr{E}$ and $\mathscr{F}$ are locally free twisted sheaves on $\mathscr{X}$, then*

$$\chi(\mathscr{E}, \mathscr{F}) = \deg(\mathrm{ch}_{\mathscr{X}}(\mathscr{E}^\vee \otimes \mathscr{F}).\,\mathrm{Td}(X))$$

Proof. The sheaves $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ and $\mathscr{E}^\vee \otimes \mathscr{F}$ are 0-twisted. Thus,

$$\mathrm{ch}_{\mathscr{X}}(\mathscr{E}^\vee \otimes \mathscr{F}) = \mathrm{ch}(p_*(\mathscr{E}^\vee \otimes \mathscr{F}))$$

and the result follows from the Grothendieck-Riemann-Roch theorem as in the nontwisted case. □

If $f \colon \mathscr{X} \to \mathscr{Y}$ is a map of $\boldsymbol{\mu}_p$-gerbes, we obtain a homomorphism

$$f^* \colon K^{(1)}(\mathscr{Y}) \to K^{(1)}(\mathscr{X})$$

determined by $[\mathscr{E}] \mapsto [\mathbf{L}f^*\mathscr{E}]$.

Lemma 4.1.3. *Let $\pi\colon T \to X$ be a morphism of smooth projective k-schemes, and consider the Cartesian diagram*

$$\begin{array}{ccc} \mathscr{X}_T & \xrightarrow{\pi_{\mathscr{X}}} & \mathscr{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\pi} & X \end{array}$$

If $\alpha \in K^{(1)}(\mathscr{X})$, then $\pi^* \operatorname{ch}_{\mathscr{X}}(\alpha) = \operatorname{ch}_{\mathscr{X}_T}(\pi^*_{\mathscr{X}}\alpha)$.

Proof. If $\mathscr{E}$ is a locally free twisted sheaf on $\mathscr{X}$, then

$$\pi^* \operatorname{ch}_{\mathscr{X}}(\mathscr{E}) = \pi^* \sqrt[p]{\operatorname{ch}(p_*(\mathscr{E}^{\otimes p}))} = \sqrt[p]{\operatorname{ch}(\mathbf{L}\pi^* p_*(\mathscr{E}^{\otimes p}))} = \sqrt[p]{\operatorname{ch}(p_{T*}\mathbf{L}\pi^*_{\mathscr{X}}(\mathscr{E}^{\otimes p}))} = \operatorname{ch}_{\mathscr{X}_T}(\mathbf{L}\pi^*_{\mathscr{X}}\mathscr{E})$$

This implies the result. $\square$

Let $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ be $\boldsymbol{\mu}_p$-gerbes over smooth projective varieties satisfying the resolution property. In the following lemma we formulate a version of the Grothendieck-Riemann-Roch formula for the projection $\mathscr{X} \times \mathscr{Y} \to \mathscr{X}$.

Lemma 4.1.4. *For any $\alpha \in K^{(1,0)}(\mathscr{X} \times \mathscr{Y})$ we have*

$$\operatorname{ch}_{\mathscr{X}}(\pi_{\mathscr{X}*}\alpha) = \pi_{X*}(\operatorname{ch}_{\mathscr{X}\times\mathscr{Y}}(\alpha).\operatorname{Td}(\pi_X))$$

where $\pi_{\mathscr{X}}\colon \mathscr{X} \times \mathscr{Y} \to \mathscr{X}$ is the projection and $\pi_X\colon X \times Y \to X$ is the induced map on coarse spaces.

Proof. We observe that by the usual Grothendieck-Riemann-Roch formula, the result holds when $\mathscr{X}$ is trivial. In the general case, we choose a finite flat cover $f_X\colon U \to X$ by a smooth projective k-scheme U such that the gerbe $\mathscr{X}_U = \mathscr{X} \times_X U$ is trivial (for instance, we may take f_X to be the absolute Frobenius). Consider the Cartesian diagrams

$$\begin{array}{ccc} \mathscr{X}_U \times \mathscr{Y} & \xrightarrow{f_{\mathscr{X}\times\mathscr{Y}}} & \mathscr{X} \times \mathscr{Y} \\ {\scriptstyle \pi_{\mathscr{X}_U}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathscr{X}}} \\ \mathscr{X}_U & \xrightarrow{f_{\mathscr{X}}} & \mathscr{X} \end{array} \qquad \begin{array}{ccc} X_U \times Y & \xrightarrow{f_{X\times Y}} & X \times Y \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ U & \xrightarrow{f_X} & X \end{array}$$

Using Lemma 4.1.3, we compute

$$\begin{aligned} f_X^*(\pi_{X*}(\mathrm{ch}_{\mathscr{X}\times\mathscr{Y}}(\alpha).\,\mathrm{Td}(\pi_X))) &= \pi_{U*} f_{X\times Y}^*(\mathrm{ch}_{\mathscr{X}\times\mathscr{Y}}(\alpha).\,\mathrm{Td}(\pi_X)) \\ &= \pi_{U*}(\mathrm{ch}_{\mathscr{X}_U\times\mathscr{Y}}(f_{\mathscr{X}\times\mathscr{Y}}^*\alpha).\,\mathrm{Td}(\pi_U)) \\ &= \mathrm{ch}_{\mathscr{X}_U}(\pi_{\mathscr{X}_U*}(f_{\mathscr{X}\times\mathscr{Y}}^*\alpha)) \\ &= \mathrm{ch}_{\mathscr{X}_U}(f_{\mathscr{X}}^*\pi_{\mathscr{X}*}\alpha) \\ &= f_X^*\,\mathrm{ch}_{\mathscr{X}}(\pi_{\mathscr{X}*}\alpha) \end{aligned}$$

The map $f_{X*}f_X^*\colon A^*(X)_{\mathbf{Q}} \to A^*(X)_{\mathbf{Q}}$ is given by multiplication by the degree of f_X, so this computation implies the result. $\square$

Remark 4.1.5. The proof gives the same formula in various other situations. We have not attempted to give a general formulation. However, we note that one must be somewhat careful in applying the formula of Lemma 4.1.4, as it is easily seen to fail, for instance, for a $(1,1)$ twisted sheaf on $\mathscr{X}\times\mathscr{Y}$, or for a twisted sheaf on $\mathscr{X}$ and the coarse space morphism $\mathscr{X}\to X$.

We next discuss the relationship between the derived category and cohomological equivalences. Generally speaking, we find the same behaviors as in the untwisted case (see Chapter 5 of [19]) and in the twisted case over the complex numbers (see [23]). We denote by $D^{(n)}(\mathscr{X})$ the bounded derived category associated to $\mathbf{Coh}^{(n)}(\mathscr{X})$. We consider a perfect complex $\mathscr{P}^\bullet \in D^{(1,1)}(\mathscr{X}\times\mathscr{Y})$ of twisted sheaves. Using $\mathscr{P}^\bullet$ as a kernel, we get a Fourier-Mukai transform

$$\Phi_{\mathscr{P}^\bullet}\colon D^{(\text{-}1)}(\mathscr{X}) \to D^{(1)}(\mathscr{Y})$$

The twisted Mukai vector $v_{\mathscr{X}\times\mathscr{Y}}(\mathscr{P}^\bullet) \in A^*(X\times Y)\otimes\mathbf{Q}$ also gives a cohomological transform

$$\Phi^{cris}_{v_{\mathscr{X}\times\mathscr{Y}}(\mathscr{P}^\bullet)}\colon \mathrm{H}^*(X/K) \to \mathrm{H}^*(Y/K)$$

Lemma 4.1.6. *If $\mathscr{P}^\bullet \in D^{(1,1)}(\mathscr{X}\times\mathscr{Y})$ is a perfect complex of twisted sheaves, then the diagram*

$$\begin{array}{ccc} K^{(\text{-}1)}(\mathscr{X}) & \xrightarrow{\Phi_{\mathscr{P}^\bullet}} & K^{(1)}(\mathscr{Y}) \\ {\scriptstyle v_{\mathscr{X}}}\downarrow & & \downarrow{\scriptstyle v_{\mathscr{Y}}} \\ \mathrm{H}^*(X/K) & \xrightarrow{\Phi^{cris}_{v_{\mathscr{X}\times\mathscr{Y}}(\mathscr{P}^\bullet)}} & \mathrm{H}^*(Y/K) \end{array}$$

commutes.

Proof. In the untwisted case, this follows by applying the Grothedieck-Riemann-Roch formula to the projections $X \times Y \to X$ and $X \times Y \to Y$ (see Corollary 5.29 of [19]). Using the twisted Grothendieck-Riemann-Roch formula of Lemma 4.1.4, the same proof immediately gives the result in the twisted case as well. $\square$

Proposition 4.1.7. *Suppose that $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ are $\boldsymbol{\mu}_p$-gerbes over K3 surfaces, and $\mathscr{P}^\bullet \in D^{(1,1)}(\mathscr{X} \times \mathscr{Y})$ is a perfect complex of twisted sheaves inducing an equivalence of categories $\Phi_{\mathscr{P}^\bullet}: D^{(\text{-}1)}(\mathscr{X}) \to D^{(1)}(\mathscr{Y})$. The cohomological transform*

$$\Phi^{cris}_{v_{\mathscr{X}\times\mathscr{Y}}(\mathscr{P}^\bullet)} \colon \widetilde{\mathrm{H}}(X/K) \to \widetilde{\mathrm{H}}(Y/K)$$

is an isomorphism of K-vector spaces, an isometry with respect to the Mukai pairing, and commutes with the respective Frobenius operators.

Proof. Using Lemma 4.1.6 and 4.1.2, one shows that the cohomological transform is an isomorphism of vector spaces and an isometry exactly as in the untwisted case (see Proposition 5.33 and Proposition 5.44 of [19]). To see that it is compatible with the crystal structure, recall that the i-th component $\mathrm{ch}(\mathscr{E})_i$ of the Chern character of a sheaf $\mathscr{E}$ on a smooth projective variety satisfies

$$\Phi(\mathrm{ch}(\mathscr{E})_i) = p^i\,\mathrm{ch}(\mathscr{E})_i$$

The result follows upon expanding $v_{\mathscr{X}\times\mathscr{Y}}(\mathscr{P}^\bullet)$ under the Kunneth decomposition. $\square$

Let us record a first example of an equivalence of derived categories and its action on cohomology. Suppose that $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$-gerbe over a K3 surface, and let $\mathscr{L} \in \mathrm{Pic}(X)$ be a line bundle. In Definition 2.1.4, we defined a $\boldsymbol{\mu}_p$-gerbe $\mathscr{X}\{\mathscr{L}^{1/p}\} \to X$ over X, which is equipped with a universal line bundle $\mathscr{M} \in \mathrm{Pic}^{(\text{-}1,1)}(\mathscr{X} \times_X \mathscr{X}\{\mathscr{L}^{1/p}\})$. Consider the natural map

$$\iota\colon \mathscr{X} \times_X \mathscr{X}\{\mathscr{L}^{1/p}\} \to \mathscr{X} \times \mathscr{X}\{\mathscr{L}^{1/p}\}.$$

Proposition 4.1.8. *The Fourier-Mukai transform induced by the sheaf $\iota_*\mathscr{M} \in \mathbf{Coh}^{(\text{-}1,1)}(\mathscr{X} \times \mathscr{X}\{\mathscr{L}^{1/p}\})$ is an equivalence of categories*

$$\Phi_{\iota_*\mathscr{M}}\colon D^{(1)}(\mathscr{X}) \to D^{(1)}(\mathscr{X}\{\mathscr{L}^{1/p}\})$$

The induced map on cohomology is given by

$$e^{l/p}\colon \widetilde{\mathrm{H}}(\mathscr{X}/W) \to \widetilde{\mathrm{H}}(\mathscr{X}\{\mathscr{L}^{1/p}\}/W)$$

where $l \in \mathrm{Pic}(X)$ *is the first Chern class of* $\mathscr{L}$*.*

Proof. To see that $\Phi_{\iota_*\mathscr{M}}$ is an equivalence, note that it is inverse to $\Phi_{\iota_*\mathscr{M}^\vee}$. To determine the action on cohomology, we use that the induced map on cohomology

$$\Phi^{cris}_{\iota_*\mathscr{M}}\colon \widetilde{\mathrm{H}}(X/K) \to \widetilde{\mathrm{H}}(X/K)$$

satisfies

$$\left(\Phi^{cris}_{\iota_*\mathscr{M}}\right)^p = \Phi^{cris}_{\iota_*\mathscr{L}} = e^l$$

The result follows upon noting that $\Phi^{cris}_{\iota_*\mathscr{M}}$ sends sends sheaves of positive rank to sheaves of positive rank. □

In fact, in this example $\Phi_{\iota_*\mathscr{M}}$ induces also an equivalence of abelian categories. For the remainder of this section, we will further specialize to the case when X and Y are supersingular. As noted at the beginning of this section, many of these results hold without this assumption, but the proofs are more involved.

Proposition 4.1.9. *If* $p\colon \mathscr{X} \to X$ *is a* $\boldsymbol{\mu}_p$*-gerbe on a supersingular K3 surface over* k*, then for any twisted sheaf* $\mathscr{E} \in \mathbf{Coh}^{(1)}(\mathscr{X})$*, the twisted Chern character* $\mathrm{ch}_{\mathscr{X}}(\mathscr{E})$ *lies in the twisted Néron-Severi lattice* $\widetilde{N}(\mathscr{X})$*.*

Proof. Let $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ be the class of $\mathscr{X}$. We assume first that $\mathscr{X}$ is essentially trivial, so that there exists a line bundle $\mathscr{L}$ such that $B = \frac{c_1(\mathscr{L})}{p}$ is a B-field lift of α. The boundary map $\mathrm{H}^1(X, \mathbf{G}_m) \to \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ takes a line bundle to its gerbe of p-th roots, so on $\mathscr{X}$ there is a universal line bundle $\mathscr{M}$ equipped with an isomorphism $\mathscr{M}^{\otimes p} \xrightarrow{\sim} p^*\mathscr{L}$. Using $\mathscr{M}$, we compute

$$\mathscr{L}^\vee \otimes p_*(\mathscr{E}^{\otimes p}) \cong p_*(\mathscr{M}^\vee \otimes \mathscr{E})^{\otimes p}$$

which implies that

$$\mathrm{ch}_{\mathscr{X}}(\mathscr{E}) = e^{\frac{c_1(\mathscr{L})}{p}}\,\mathrm{ch}(p_*(\mathscr{M}^\vee \otimes \mathscr{E}))$$

which gives the result.

Next, suppose that α is not essentially trivial, so that $[\alpha] \in \mathrm{Br}(X)$ has order p. We make some reductions. It will suffice to show the result for $\mathscr{E}$ a torsion free twisted sheaf of positive rank. Let η denote the generic point of X. There exists a $\mathscr{X}_\eta$-twisted sheaf $\mathscr{F}$ of rank p and a surjection $\mathscr{E}|_\eta \to \mathscr{F}$. Thus we get a map $\mathscr{E} \to \eta_*\mathscr{F}$. We find a surjection $\mathscr{E} \to \mathscr{F}'$ for some coherent $\mathscr{X}$-twisted sub-sheaf $\mathscr{F}'$ of rank p, and by modding out by torsion we may assume $\mathscr{F}'$ is torsion free. As $\mathscr{E}$ is torsion free, so is the kernel of this map. So, by induction we may reduce to the case when $\mathscr{E}$ is a torsion free twisted sheaf of rank p. We claim that such an $\mathscr{E}$ is simple, that is, that $\mathrm{Hom}(\mathscr{E},\mathscr{E}) \cong k$. Indeed, as $\mathscr{E}$ is torsion free, the natural map $\mathrm{Hom}(\mathscr{E},\mathscr{E}) \to \mathrm{Hom}(\mathscr{E}^{\vee\vee},\mathscr{E}^{\vee\vee})$ is an injection, and because $[\alpha] \in \mathrm{Br}(X)$ has order p, any locally free sheaf of rank p is simple. There is a deformation theory for torsion free twisted sheaves with unobstructed determinant with obstruction space

$$\ker(\mathrm{Ext}^2_X(\mathscr{E},\mathscr{E}) \xrightarrow{\mathrm{Tr}} \mathrm{H}^2(X,\mathscr{O}_X))$$

Under Serre duality, Tr is dual to the natural map $\mathrm{H}^0(X,\mathscr{O}_X) \to \mathrm{Hom}(\mathscr{E},\mathscr{E})$, which is an isomorphism because $\mathscr{E}$ is simple. Consider the connected component $\mathbf{A}^1$ of the group of $\boldsymbol{\mu}_p$-gerbes on X that contains $\mathscr{X}$ as the fiber over some $t \in \mathbf{A}^1(k)$. The coarse space of the corresponding twistor family $\widetilde{\mathscr{X}} \to \mathbf{A}^1$ is the trivial family $X \times \mathbf{A}^1 \to \mathbf{A}^1$, so in particular the determinant of $\mathscr{E}$ is unobstructed. Hence, $\mathscr{E}$ is unobstructed. By the Grothendieck existence theorem for twisted sheaves, we find a twisted sheaf $\mathscr{E}'$ on $\widetilde{\mathscr{X}}_{k[[x-t]]}$ that is flat over $\mathrm{Spec}(k\,[[x-t]])$ and whose restriction to the closed fiber is isomorphic to $\mathscr{E}$. By Proposition 2.3.1.1 of [30], the stack of coherent twisted sheaves on the morphism $\widetilde{\mathscr{X}} \to \mathbf{A}^1$ is in particular limit preserving. Thus we may apply Artin approximation (see [47, 07XB]) to produce an étale morphism

$$f : (U,u) \to (\mathbf{A}^1, t)$$

and a coherent $\widetilde{\mathscr{X}} \times_{\mathbf{A}^1} U$-twisted sheaf $\mathscr{E}''$ that is U-flat such that the restrictions of $\mathscr{E}''$ and $f^*\mathscr{E}$ to the closed fiber $(\widetilde{\mathscr{X}} \times_{\mathbf{A}^1} U) \times_U u \cong \widetilde{X}_t$ are isomorphic. Taking the normalization of $\mathbf{A}^1$ in U, we find a factorization $U \to C \to \mathbf{A}^1$ such that $U \to C$ is an open immersion and $C \to \mathbf{A}^1$ is finite and flat. Thus, every connected component of C maps surjectively onto $\mathbf{A}^1$. Consider a flat extension of $\mathscr{E}''$ to $\mathscr{X}_C$. The twisted Chern class is constant in a flat family, and the origin $0 \in \mathbf{A}^1$ corresponds to an essentially trivial $\boldsymbol{\mu}_p$-gerbe. We therefore obtain by the previous case that

$$\mathrm{ch}_{\mathscr{X}}(\mathscr{E}) \in \widetilde{N}(\widetilde{\mathscr{X}_0}) \subset \widetilde{N}(X) \otimes \mathbf{Q}$$

But $\mathscr{E}$ has rank p, so by the explicit presentations in Proposition 3.2.20 we see that $\mathrm{ch}_{\mathscr{X}}(\mathscr{E}) \in \widetilde{N}(\mathscr{X})$. This completes the proof.

□

Proposition 4.1.10. *If $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$-gerbe on a supersingular K3 surface, then the map*

$$\mathrm{ch}_{\mathscr{X}} : K^{(1)}(\mathscr{X}) \to \widetilde{N}(\mathscr{X})$$

is surjective.

Proof. If the Brauer class of $\mathscr{X}$ is trivial, then $\mathscr{X}$ is the gerbe of p-th roots of some line bundle $\mathscr{L}$ on X. This means that there is an invertible $\mathscr{X}$-twisted sheaf $\mathscr{M}$ equipped with an isomorphism $\mathscr{M}^{\otimes p} \xrightarrow{\sim} p^*\mathscr{L}$. We have a commutative diagram

$$\begin{tikzcd} \mathbf{Coh}(X) \arrow[r, "p^*(\_)\otimes\mathscr{M}"] \arrow[d, "\mathrm{ch}"'] & \mathbf{Coh}^{(1)}(\mathscr{X}) \arrow[d, "\mathrm{ch}_{\mathscr{X}}"] \\ \widetilde{N}(X) \arrow[r, "\cdot\frac{c_1(\mathscr{L})}{p}"] & \widetilde{N}(\mathscr{X}) \end{tikzcd}$$

where the top horizontal arrow is an equivalence of categories, and the lower horizontal arrow is an isometry. Because the left vertical arrow is surjective, so is the right vertical arrow.

Next, suppose that the Brauer class of $\mathscr{X}$ is non-trivial. We refer to Proposition 3.2.20 for an explicit description of $\widetilde{N}(\mathscr{X})$. The structure sheaf of a closed point gives a $\mathscr{X}$-twisted sheaf with twisted Chern class $(0, 0, 1)$. If $D \subset X$ is a closed integral subscheme of dimension 1, then by Tsen's Theorem the Brauer class of the gerbe $\mathscr{X}_D = \mathscr{X} \times_X D \to D$ is trivial, and hence there is an invertible $\mathscr{X}_D$-twisted sheaf. The pushforward of such a sheaf under the map $\mathscr{X}_D \hookrightarrow \mathscr{X}$ gives a $\mathscr{X}$-twisted sheaf whose twisted Chern character is of the form $(0, D, s)$. Finally, by a theorem of Grothendieck, there exists a locally free $\mathscr{X}$-twisted sheaf $\mathscr{E}$ of rank p. □

Remark 4.1.11. Combining Proposition 4.1.10 with Proposition 3.2.21 shows that the natural map

$$\mathrm{im}\left(\mathrm{ch}_{\mathscr{X}} : K^{(1)}(\mathscr{X}) \to \widetilde{N}(\mathscr{X})\right) \otimes \mathbf{Z}_p \to \widetilde{T}(\mathscr{X})$$

is an isomorphism, where $\widetilde{T}(\mathscr{X})$ is the Tate module of $\widetilde{\mathrm{H}}(\mathscr{X}/W)$. Thus, the analog of the Tate conjecture holds for twisted supersingular K3 surfaces.

We next discuss stability conditions.

Definition 4.1.12. A *polarization* of X is an element of the cone $\mathscr{C} \subset N(X) \otimes \mathbf{R}$ generated by ample divisors.

Fix a polarization H of X. Recall the following definition (see Definition 2.2.7.2 of [30]).

Definition 4.1.13. If $\mathscr{E}$ is a $\mathscr{X}$-twisted sheaf, then the *geometric Hilbert polynomial* of $\mathscr{E}$ is the function

$$P_{\mathscr{E}}(m) = \deg(\mathrm{ch}_{\mathscr{X}}(\mathscr{E}(m)).\,\mathrm{Td}(X))$$

where $\mathscr{E}(m) = \mathscr{E} \otimes p^* \mathrm{H}^{\otimes m}$ (if H is only a linear combination of divisors, then we compute $\mathrm{ch}_{\mathscr{X}}(\mathscr{E}(m))$ formally). Let α_d be the leading coefficient of $P_{\mathscr{E}}$, and write

$$p_{\mathscr{E}}(m) = \frac{1}{\alpha_d} P_{\mathscr{E}}(m)$$

As explained in Section 2.2.7 of [30], $P_{\mathscr{E}}$ is a numerical polynomial with the usual properties. In particular, we can use it to define stability and semistable of sheaves.

Definition 4.1.14. A $\mathscr{X}$-twisted sheaf $\mathscr{E}$ is *stable* (resp. *semistable*) if it is pure and for all proper non-trivial subsheaves $\mathscr{F} \subset \mathscr{E}$

$$p_{\mathscr{F}}(m) < p_{\mathscr{E}}(m)$$

(resp. $\leq$) for m sufficiently large.

We have the following result.

Proposition 4.1.15. *If $v = (r, l, s) \in \widetilde{N}(\mathscr{X})$ is a primitive Mukai vector, then there exists a locally finite collection of hyperplanes $W \subset N(X) \otimes \mathbf{R}$ such that if H does not lie on any of these hyperplanes, then any H-semistable $\mathscr{X}$-twisted sheaf $\mathscr{E}$ with $v_{\mathscr{X}}(\mathscr{E}) = v$ is H-stable.*

Proof. We need some form of the Bogomolov inequality for $\boldsymbol{\mu}_H$-semistable twisted sheaves on $\boldsymbol{\mu}_p$-gerbes. For our purposes, little more than the existence of such a lower bound will suffice. By Lemma 3.2.3.13 of [30], if $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$-gerbe on a smooth proper surface, then there exists a constant $C \geq 0$ such that for any polarization H of X and any H-semistable $\mathscr{X}$-twisted sheaf $\mathscr{E}$ of rank r,

$$\Delta(\mathscr{E}) \geq -Cr^4 \tag{4.1.15.1}$$

The discriminant $\Delta(\mathscr{E})$ of a twisted sheaf is defined in by Definition 3.2.1.1 of [30]. For the remainder of the proof we will closely follow the proof of Theorem 4.C.3 of [22]. We say that a class $\xi \in N(X)$ is of type v if

$$-r^2(\Delta + 2Cr^4) \leq \xi^2 < 0$$

The wall determined by ξ is the set

$$W_\xi = \{H \in \mathscr{C} | \xi.H = 0\} \subset \mathscr{C}$$

We say that W_ξ is of type v if ξ is of type v. The proof of Theorem 4.C.2 of [22] shows that the set of walls of type v is locally finite in the ample cone $\mathscr{C}_X$. Suppose that $\mathscr{E}$ is a $\mathscr{X}$-twisted H-semistable sheaf. If $\mathscr{E}$ fails to be H-stable, then there exists a subsheaf $\mathscr{E}' \subset \mathscr{E}$ with Mukai vector $v' = (r', l', s')$ such that $r' < r$ and $p_{\mathscr{E}'}(m) \equiv p_{\mathscr{E}}(m)$. Suppose that $r > 0$. We have

$$p_{\mathscr{E}}(m) = \frac{m^2}{2} + m\frac{l.H}{r\,\mathrm{H}^2} + \frac{s}{r\,\mathrm{H}^2} + \frac{1}{\mathrm{H}^2}$$

so this condition is equivalent to

$$\xi_{\mathscr{E}',\mathscr{E}}.H = 0 \qquad \text{and} \qquad r's = rs'$$

where

$$\xi = \xi_{\mathscr{E}',\mathscr{E}} = r'l - rl'$$

We may assume that $\mathscr{E}'$ is saturated, so that $\mathscr{E}'' = \mathscr{E}/\mathscr{E}'$ is torsion-free of rank $r'' = r' - r$. By the Hodge index theorem, either $\xi = 0$ or $\xi^2 < 0$. If $\xi = 0$, then we compute

$$\frac{r}{r'}v' = v$$

But $r' < r$, so this contradicts our assumption that v was primitive. Therefore, $\xi^2 < 0$. We have the identity

$$\Delta(\mathscr{E}) - \frac{r}{r'}\Delta(\mathscr{E}') - \frac{r}{r''}\Delta(\mathscr{E}'') = -\frac{\xi^2}{r'r''}$$

Applying (4.1.15.1) to $\mathscr{E}'$ and $\mathscr{E}''$, we get

$$-\Delta(\mathscr{E})r^2 - 2Cr^6 \leq \xi^2 < 0$$

We conclude that if such an $\mathscr{E}'$ exists, then H lies on a wall of type v. Therefore, if H is not on any wall of type v, then $\mathscr{E}$ is stable.

The $r = 0$ case is proved exactly as in Theorem 10.2.5 of [21]. $\square$

Definition 4.1.16. A polarization H is *v-generic* if any H-semistable twisted sheaf with Mukai vector v is H-stable.

For future use, we record the following observation.

Lemma 4.1.17. *Suppose that X is supersingular, and let $\widetilde{\mathscr{X}} \to \mathbf{A}^1$ be a universal family of $\boldsymbol{\mu}_p$-gerbes over a connected component of $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$. If $v = (r, l, s)$ is a primitive isotropic Mukai vector, then there exists a polarization H of X such that in each fiber H is v-generic.*

Proof. The extended Néron-Severi groups of the fibers are given by Proposition 3.2.20. In particular, we see that, among the fibers of the universal family over each connected component of $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$, there are only finitely many possibilities for the twisted Néron-Severi group (viewed as a subgroup of $\widetilde{N}(X) \otimes \mathbf{Q}$). For each, we find by Proposition 4.1.15 a locally finite union of hyperplanes in $N(X) \otimes \mathbf{R}$. But the union of all of these is again a locally finite union of hyperplanes, and we may therefore find an H with the desired property. $\square$

We make the following definition.

Definition 4.1.18. Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe over a K3 surface, $v = (r, l, s) \in \widetilde{N}(\mathscr{X})$ a Mukai vector with $r > 0$, and H a polarization on X. The *moduli space of $\mathscr{X}$-twisted stable sheaves with Mukai vector* v is the stack $\mathscr{M}_{\mathscr{X}}(v)$ on $\operatorname{Spec} k$ whose objects over a k-scheme T are T-flat $\mathscr{X}_T$-twisted sheaves $\mathscr{E}$ locally of finite presentation such that for each geometric point $t \in T$ the fiber $\mathscr{E}_t$ is H-stable and has Mukai vector v.

Theorem 4.1.19. *Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe on a supersingular K3 surface and $v = (r, l, s) \in \widetilde{N}(\mathscr{X})$ a primitive Mukai vector with $v^2 = 0$. If H is a sufficiently generic polarization, then the moduli space $\mathscr{M}_{\mathscr{X}}(v)$ of H-stable twisted sheaves on $\mathscr{X}$ with twisted Mukai vector v is either empty or satisfies*

1. *$\mathscr{M}_{\mathscr{X}}(v)$ is a $\mathbf{G}_m$-gerbe over a supersingular K3 surface $M_{\mathscr{X}}(v)$,*

2. *the universal sheaf $\mathscr{P}$ on $\mathscr{M}_{\mathscr{X}}(v) \times \mathscr{X}$ induces a Fourier-Mukai equivalence*

$$\Phi_{\mathscr{P}} \colon D^{(-1)}(\mathscr{M}_{\mathscr{X}}(v)) \to D^{(1)}(\mathscr{X})$$

and

3. *the Brauer class of the gerbe $\mathscr{M}_{\mathscr{X}}(v) \to M_{\mathscr{X}}(v)$ is trivial if and only if there exists a vector $w \in \widetilde{N}(\mathscr{X})$ such that $v.w$ is coprime to p.*

Proof. Our strategy to prove (1) is to lift to characteristic 0. We note that if $\mathscr{X}$ deforms the trivial gerbe, then (1) can be easily reduced to the case when $\mathscr{X}$ is trivial, where the result follows by results of Mukai [37] (see also Corollary 10.2.4 and Proposition 10.2.5 of [21]), which hold independent of characteristic. After establishing (1), we will deduce (2) and (3) by characteristic independent means. Thus, in this special case, this theorem can be proved without lifting.

Let H be a v-generic polarization. Suppose that $\mathscr{M}_{\mathscr{X}}(v)$ is non-empty. By tensoring with an appropriate line bundle, we may assume without loss of generality that l is ample and not divisible by p in the Picard group of X. Using results of Deligne [13], we find a smooth pointed W-scheme (M, m), a relative projective K3 surface $X_M \to M$, a $\boldsymbol{\mu}_p$-gerbe $\mathscr{X}_M \to X_M$, a Mukai vector $v_M \in A^*(X_M) \otimes \mathbf{Q}$, and a class $H_M \in \operatorname{Pic}(X_M) \otimes \mathbf{R}$, together with an isomorphism between the fiber of the triple $(\mathscr{X}_M, v_M, H_M)$ over m and $(\mathscr{X}, v, H)$. Choose a Hensalian DVR R with algebraically closed residue field and a dominant morphism $\operatorname{Spec} R \to M$ with center m. Write $(\mathscr{X}_R, v_R, H_R)$ for the restriction of the triple on M to $\operatorname{Spec} R$. Note that in each geometric fiber v_R is primitive and isotropic.

Consider the relative moduli space $\mathscr{M}_{\mathscr{X}_R}(v_R) \to \operatorname{Spec} R$ of $\mathscr{X}_R$-twisted H_R-stable sheaves with Mukai vector v. By the usual results on stability (summarized in Section 3.2.1 of [30]), each geometric fiber of $\mathscr{M}_{\mathscr{X}_R}(v_R)$ is either empty or a $\mathbf{G}_m$-gerbe over an algebraic space $M_{\mathscr{X}_R}(v_R)$. A twisted sheaf with unobstructed determinant is unobstructed, so $M_{\mathscr{X}_R}(v_R)$ is smooth over $\operatorname{Spec} R$. By Langton's theorem (Lemma 2.3.3.2 of [30]), any semistable sheaf on the generic fiber sits in a flat family with a semistable sheaf on the special fiber, after possibly taking a finite extension of R. But by our choice of H, every H-semistable twisted sheaf on the special fiber is stable. As stability is open in flat families, it follows that H_R is v_R-generic in every geometric fiber. Thus, by Langton's theorem, $M_{\mathscr{X}_R}(v_R)$ is proper over $\operatorname{Spec} R$. As the closed fiber is non-empty, the geometric generic fiber is non-empty as well. By Theorem 3.16 of [49], it follows that the geometric generic fiber of $M_{\mathscr{X}_R}(v_R)$ is a K3 surface. This implies that the special fiber is as well.

We have shown that $\mathscr{M}_{\mathscr{X}}(v)$ is a $\mathbf{G}_m$-gerbe over a K3 surface. Condition (2) then follows from a criterion of Bridgeland (Theorem 2.3 and Theorem 3.3 of [9]). By Theorem 4.1.19 and Proposition 4.1.7 we know therefore that the rational cohomological transform is an isomorphism of K-vector spaces that is compatible with the Mukai pairing and the Frobenius operators. This shows that $M_{\mathscr{X}}(v)$ is supersingular, completing the proof of (1).

Finally, we show (3). The universal sheaf

$$\mathscr{P} \in \mathbf{Coh}^{(1,1)}(\mathscr{M}_{\mathscr{X}}(v) \times \mathscr{X})$$

induces an isometry

$$g\colon \widetilde{N}(\mathscr{M}_{\mathscr{X}}(v)) \to \widetilde{N}(\mathscr{X})$$

sending $(0,0,1)$ to v. Suppose that the Brauer class of the gerbe $\mathscr{M}_{\mathscr{X}}(v) \to M_{\mathscr{X}}(v)$ is trivial. This is equivalent to the existence of an invertible twisted sheaf, say $\mathscr{L}$, on $\mathscr{M}_{\mathscr{X}}(v)$. We have $\mathrm{ch}_{\mathscr{X}}(\mathscr{L}).(0,0,1) = 1$, so $g(\mathrm{ch}_{\mathscr{X}}(\mathscr{L})).v = 1$. Conversely, suppose that there exists a $w \in \widetilde{N}(\mathscr{X})$ such that $(v.w, p) = 1$. By Proposition 4.1.10, there exists a $\mathscr{X}$-twisted sheaf $\mathscr{E}$ with $v_{\mathscr{X}}(\mathscr{E}) = w$. Consider the perfect complex

$$\mathbf{R}q_*(p^*(\mathscr{E}) \overset{\mathbf{L}}{\otimes} \mathscr{P}^\vee) \in D^{(-1)}(\mathscr{M}_{\mathscr{X}}(v))$$

The rank of this complex over a geometric point $x \in M_{\mathscr{X}}(v)$ is

$$\chi(\mathscr{E}, \mathscr{P}_x) = -v.w$$

The existence of a such a complex implies that

$$[\mathscr{M}_{\mathscr{X}}(v)] \in \mathrm{Br}(M_{\mathscr{X}}(v))[v.w]$$

The result follows. □

Proposition 4.1.20. *Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe on a supersingular K3 surface. If $v = (r, l, s)$ is a primitive Mukai vector with $v^2 = 0$, then the moduli space $\mathscr{M}_{\mathscr{X}}(v)$ with respect to a v-generic polarization is non-empty if one of the following holds:*

1. *$r > 0$ and the order of the Brauer class $[\mathscr{X}]$ of $\mathscr{X}$ divides r.*

2. *$r = 0$ and E is a smooth fiber of an elliptic fibration $X \to \mathbf{P}^1$.*

3. $r = l = 0$ ***and*** $s > 0$*.*

Proof. Suppose that we are in case (1). Consider a lift to characteristic 0 as in the proof of Theorem 4.1.19, and form the moduli space $\mathscr{M}_{\mathscr{X}_R}(v_R)$. By Theorem 3.16 of [49], the geometric generic fiber is non-empty. By Langton's theorem, this gives the result.

Suppose that we are in case (2). By Tsen's theorem, the gerbe $\mathscr{X} \times_X E$ has trivial Brauer class, and hence admits an invertible twisted sheaf, say $\mathscr{L}$. Tensoring with $\mathscr{L}^\vee$ and pushing forward to E reduces us to finding an appropriate stable vector bundle on E, which can be done by results of Atiyah.

Finally, note that if we are in case (3), we must in fact have $s = 1$. □

Proposition 4.1.21. *Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe on a supersingular K3 surface and $v \in \widetilde{N}(\mathscr{X})$ a primitive Mukai vector with $v^2 = 0$. Let H be a sufficiently generic polarization. Suppose that one of the conditions of Proposition 4.1.20 holds, and consider the moduli space $\mathscr{M}_{\mathscr{X}}(v)$. Let*

$$\mathscr{P} \in \mathbf{Coh}^{(1,1)}(\mathscr{M}_{\mathscr{X}}(v) \times \mathscr{X})$$

be the universal twisted sheaf. The cohomological correspondence

$$\Phi^{cris}_{v_{\mathscr{M}\times\mathscr{X}}(\mathscr{P})} \colon \widetilde{\mathrm{H}}^{(\text{-}1)}(\mathscr{M}_{\mathscr{X}}(v)/W) \to \widetilde{\mathrm{H}}(\mathscr{X}/W)$$

is an isomorphism of K3 crystals that is compatible with the inclusions of the respective twisted Néron-Severi lattices.

Proof. Form the moduli space $\mathscr{M}_{\mathscr{X}}(v)$ of sheaves stable with respect to H. Define the sublattice

$$\widetilde{N}(\mathscr{M}_{\mathscr{X}}(v) \times \mathscr{X}) \overset{\text{def}}{=} \widetilde{N}(\mathscr{M}_{\mathscr{X}}(v)) \boxplus \widetilde{N}(\mathscr{X}) \subset A^*(M_{\mathscr{X}}(v) \times X) \otimes \mathbf{Q}$$

By the results of Theorem 4.1.19 and Proposition 4.1.7, it remains only to show that

$$v_{\mathscr{M}\times\mathscr{X}}(\mathscr{P}) \in \widetilde{N}(\mathscr{M}_{\mathscr{X}}(v) \times \mathscr{X})$$

Let us first suppose that the Brauer classes of the gerbes $\mathscr{X}$ and $\mathscr{M}_{\mathscr{X}}(v)$ are both trivial. By tensoring with twisted invertible sheaves, we may reduce to the case that the gerbes $\mathscr{X}$ and $\mathscr{M}_{\mathscr{X}}(v)$ are themselves trivial. The result then follows by applying the Grothendieck-Riemann-Roch theorem to the respective projections as in the untwisted case (see Lemma 10.6 of [19]).

We now prove the result in general. Expand v in the basis considered in Proposition 3.2.20, and let $s \in \mathbf{Z}$ be the coefficient of $(0,0,1)$. By tensoring with a line bundle of non-zero degree, we may assume without loss of generality that s is not divisible by p. Consider the connected component $\widetilde{\mathscr{X}} \to \mathbf{A}^1$ of the universal family of $\boldsymbol{\mu}_p$-gerbes on X containing $\mathscr{X}$ as a fiber. By Lemma 4.1.17, we may choose H to be v-generic in each fiber. Consider the relative moduli space

$$\mathscr{M}_{\widetilde{\mathscr{X}}}(v) \to \mathbf{A}^1$$

and the relative universal twisted sheaf

$$\widetilde{\mathscr{P}} \in \mathbf{Coh}^{(1,1)}(\mathscr{M}_{\widetilde{\mathscr{X}}}(v) \times_{\mathbf{A}^1} \widetilde{\mathscr{X}})$$

Consider a point $t \in \mathbf{A}^1$ such that the gerbe $\widetilde{\mathscr{X}_t}$ has trivial Brauer class. There exists then a vector $w \in \widetilde{N}(\widetilde{\mathscr{X}_t})$ with rank 1. As p does not divide s, part (3) of Theorem 4.1.19 gives that the fiber $(\mathscr{M}_{\widetilde{\mathscr{X}}}(v))_t$ of the moduli space also has trivial Brauer class. Thus, by the previous case the result holds for $\widetilde{\mathscr{P}_t}$. By the constancy of the twisted Chern character in flat families, the result also holds for $\mathscr{P}$. $\square$

Finally, we record the following lemma, which appears to be well known.

Lemma 4.1.22. *If $v = (p,l,s)$ is primitive and $v^2 = 0$ and H is v-generic, then any object $\mathscr{E} \in \mathscr{M}_{\mathscr{X}}(v)(T)$ is locally free.*

Proof. By the local criterion for flatness, it will suffice to prove this when $T = \operatorname{Spec} k$. If $\mathscr{E}$ is stable, then $\mathscr{E}^{\vee\vee}$ is also stable. The Mukai vector of $\mathscr{E}^{\vee\vee}$ is (r,l,s') for some $s' \leq s$, with equality if and only if $\mathscr{E}$ is locally free. The sheaf $\mathscr{E}^{\vee\vee}$ gives a point in the moduli space of stable sheaves on $\mathscr{X}$ with Mukai vector (r,l,s'). These moduli spaces are always smooth, and are either empty or of the expected dimension $l^2 - 2rs'$. If $s' < s$, this number is negative, a contradiction. $\square$

Remark 4.1.23. In this section we have appealed to results of Yoshioka on the existence of semi-stable sheaves with prescribed invariants and on analytic deformations trivializing Brauer classes. These represent the unique points in this paper where we (implicitly) use analytic techniques. It would be interesting to try to remove this dependence.

4.2 Twistor families of positive rank

We maintain Notation 3.3.4, so that Λ is a fixed supersingular K3 lattice and $\widetilde{\Lambda} = \Lambda \oplus U_2(p)$. In Section 3.0.2, we defined a twistor line $f_v \colon \mathbf{A}^1 \to \overline{M}_{\widetilde{\Lambda}_0}$ to be a connected component of a fiber of π_v for some isotropic vector $v \in \widetilde{\Lambda}_0$, where π_v fits into the diagram

$$\begin{tikzcd} \overline{M}_{\widetilde{\Lambda}_0} \arrow[dr, dashed] & & \overline{M}^{\langle v \rangle}_{\widetilde{\Lambda}_0} \arrow[ll, hook'] \arrow[dl, "\pi_v"] \\ & \overline{M}_{v^\perp/v} & \end{tikzcd}$$

We will make the following (somewhat preliminary) definition.

Definition 4.2.1. A family of twisted supersingular K3 surfaces over an open subset $U \subset \mathbf{A}^1$ is a *twistor family* if it admits a marking such that the induced map from U to the period domain is an isomorphism onto an open subset of a twistor line.

In Theorem 5.1.17, we will give another characterization of twistor families as relative moduli spaces of twisted sheaves on universal families of $\boldsymbol{\mu}_p$-gerbes over an open subset of a connected component of the group of $\boldsymbol{\mu}_p$-gerbes on a supersingular K3 surface. For reasons that will become clear shortly, we will make the following distinction.

Definition 4.2.2. A twistor line (or twistor family) is *of positive rank* if $v.e \neq 0$, and is *Artin-Tate* if $v.e = 0$.

We will restrict our attention in this section to the positive rank case, and postpone our study of the Artin-Tate case to Section 4.3. The main result of this section is Theorem 4.2.8, which says that, under certain restrictions, twistor lines of positive rank lift to families of twisted K3 surfaces, in a strong sense. We will apply Theorem 4.2.8 in Section 5.1 to deduce consequences for moduli spaces of marked twisted supersingular K3 surfaces.

We have already constructed moduli spaces of twisted sheaves in Theorem 4.1.19, which for a generic choice of polarization are $\mathbf{G}_m$-gerbes over supersingular K3 surfaces. In order to examine these stacks in terms of our period morphism, we will identify an associated $\boldsymbol{\mu}_p$-gerbe.

Definition 4.2.3. Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe over a supersingular K3 surface, $v = (r, l, s) \in \widetilde{N}(\mathscr{X})$ a Mukai vector with $r > 0$, and H a polarization on X. Let $\mathscr{L}$ be a line bundle on $\mathscr{X}$

with first Chern class l. We define a stack $\mathscr{M}^{\det}_{\mathscr{X}}(v)$ on $\operatorname{Spec} k$ whose objects over a k-scheme T are pairs $(\mathscr{E}, \phi)$ where $\mathscr{E}$ is a T-flat $\mathscr{X}_T$-twisted sheaf that is locally of finite presentation such that for each geometric point $t \in T$ the fiber $\mathscr{E}_t$ is H-stable and has Mukai vector v, and $\phi\colon \det \mathscr{E} \xrightarrow{\sim} \mathscr{L}_T$ is an isomorphism of invertible sheaves on $\mathscr{X}_T$.

Note that this definition is restricted to the case of positive rank. If $X \to S$ is a supersingular K3 surface, we define the *relative extended Néron-Severi lattice*

$$\underline{\widetilde{N}}_{X/S} = \underline{\mathbf{Z}}_S \oplus \operatorname{Pic}_{X/S} \oplus \underline{\mathbf{Z}}_S$$

which we equip with the Mukai pairing. We make the following observation regarding markings of tautological families of $\boldsymbol{\mu}_p$-gerbes.

Lemma 4.2.4. *Let $\pi\colon X \to \operatorname{Spec} k$ be a supersingular K3 surface and $\mathscr{X}_0 \to X$ a $\boldsymbol{\mu}_p$ gerbe. Let $\mathbf{A}^1 \subset \mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p$ be the connected component containing $\mathscr{X}_0$ as a fiber, and $\mathscr{X} \to \mathbf{A}^1$ the corresponding tautological family. If the Brauer class of $\mathscr{X}_0$ is non-trivial, then any marking $\widetilde{\Lambda} \to \widetilde{N}(\mathscr{X}_0)$ extends to a map $\underline{\widetilde{\Lambda}}_{\mathbf{A}^1} \to \underline{\widetilde{N}}_{X\times\mathbf{A}^1/\mathbf{A}^1} \otimes \mathbf{Q}$ which on each geometric fiber $t \in \mathbf{A}^1$ restricts to a map $\widetilde{\Lambda} \to \widetilde{N}(\mathscr{X}_t)$.*

Proof. This follows immediately from the calculations of Proposition 3.2.20. □

We record some purely lattice-theoretic facts. Consider an isometric inclusion $\widetilde{\Lambda} \subset \widetilde{N}$ of extended supersingular K3 lattices (or of supersingular K3 lattices). We have a chain of inclusions

$$\widetilde{\Lambda} \subset \widetilde{N} \subset \widetilde{N}^* \subset \widetilde{\Lambda}^*$$

Lemma 4.2.5. *The $\mathbf{F}_p$-vector spaces $\widetilde{N}/\widetilde{\Lambda}$ and $\widetilde{\Lambda}^*/\widetilde{N}^*$ are naturally dual. In particular, they have the same dimension.*

Proof. Applying $\operatorname{Hom}_{\mathbf{Z}}(\_, \mathbf{Z})$ to the short exact sequence

$$0 \to \widetilde{\Lambda} \to \widetilde{N} \to \widetilde{N}/\widetilde{\Lambda} \to 0$$

we get a short exact sequence

$$0 \to \widetilde{N}^* \to \widetilde{\Lambda}^* \to \operatorname{Ext}^1_{\mathbf{Z}}(\widetilde{N}/\widetilde{\Lambda}, \mathbf{Z}) \to 0$$

Applying $\mathrm{Hom}_{\mathbf{Z}}(\widetilde{N}/\widetilde{\Lambda}, \_)$ to the short exact sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$, we find a natural isomorphism

$$\mathrm{Hom}_{\mathbf{F}_p}(\widetilde{N}/\widetilde{\Lambda}, \mathbf{F}_p) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathbf{Z}}(\widetilde{N}/\widetilde{\Lambda}, \mathbf{Z})$$

This gives the result. □

Consider the containments

$$\frac{p\widetilde{N}}{p\widetilde{\Lambda}} \subset \frac{p\widetilde{N}^*}{p\widetilde{\Lambda}} \subset \frac{p\widetilde{\Lambda}^*}{p\widetilde{\Lambda}} = \widetilde{\Lambda}_0$$

of $\mathbf{F}_p$-vector spaces.

Lemma 4.2.6. *As subspaces of $\widetilde{\Lambda}_0$, we have*

$$\left(\frac{p\widetilde{N}}{p\widetilde{\Lambda}}\right)^{\perp} = \frac{p\widetilde{N}^*}{p\widetilde{\Lambda}}$$

Proof. It is immediate that the right hand side is contained in the left hand side. We will show that they have the same dimensions. Let σ_0 be the Artin invariant of $\widetilde{\Lambda}$ and σ the Artin invariant of $\widetilde{N}$, so that $p\widetilde{\Lambda}^*/p\widetilde{\Lambda}$ has dimension $2\sigma_0$, and $p\widetilde{N}^*/p\widetilde{N}$ has dimension 2σ. Consider the short exact sequences

$$0 \to \frac{p\widetilde{N}^*}{p\widetilde{\Lambda}} \to \frac{p\widetilde{\Lambda}^*}{p\widetilde{\Lambda}} \to \frac{p\widetilde{\Lambda}^*}{p\widetilde{N}^*} \to 0 \qquad\qquad 0 \to \frac{p\widetilde{N}}{p\widetilde{\Lambda}} \to \frac{p\widetilde{N}^*}{p\widetilde{\Lambda}} \to \frac{p\widetilde{N}^*}{p\widetilde{N}} \to 0$$

Combined with Lemma 4.2.5, we find that

$$\dim_{\mathbf{F}_p}\left(\frac{p\widetilde{N}}{p\widetilde{\Lambda}}\right) = \dim_{\mathbf{F}_p}\left(\frac{p\widetilde{\Lambda}^*}{p\widetilde{N}^*}\right) = \sigma_0 - \sigma, \text{ and } \quad \dim_{\mathbf{F}_p}\left(\frac{p\widetilde{N}^*}{p\widetilde{\Lambda}}\right) = \sigma_0 + \sigma$$

This gives the result. □

Lemma 4.2.7. *If N is a supersingular K3 lattice or an extended supersingular K3 lattice, and $v \in N_0$ is an isotropic vector, then there exists a primitive isotropic vector $l \in pN^* \subset N$ whose image in N_0 is equal to v.*

Proof. The existence of an isotropic vector in N_0 implies that $\sigma_0(N) \geq 2$. By the explicit presentation of supersingular K3 lattices in [42] and Lemma 3.3.5, we see that there is an orthogonal decomposition $N = N' \oplus U_2(p)$. Let $\{e, f\}$ be the standard generators for $U_2(p)$, and let w be the image of e in N_0. By Witt's Lemma, there exists an isometry $g \in O(N_0)$ taking w to v. By a result of Nikulin (see Theorem 14.2.4 of [21]), the map

$$O(N) \to O(N_0)$$

is surjective, so we find an orthogonal transformation $h \in O(N)$ inducing g. The vector $h(e) \in N$ is primitive and isotropic, and its image in N_0 is equal to V, as desired. $\square$

We are now ready to prove the main result of this section. This should be viewed as a (partial) supersingular analog of Proposition 3.9, Chapter 4 of [21], which describes those curves in the Hodge-theoretic period domain that lift to twistor spaces of complex analytic K3 surfaces.

Theorem 4.2.8. *Let $x \in \mathscr{S}^o_\Lambda(k)$ be a k-point. Let $L \subset \overline{M}_{\widetilde{\Lambda}_0}$ be a twistor line corresponding to an isotropic vector $v \in \widetilde{\Lambda}_0$, and let $f_v \colon L \to \overline{M}_{\widetilde{\Lambda}_0}$ be the inclusion. Suppose that $\widetilde{\rho}(x) \in L$ and that the Artin invariant of $\widetilde{\rho}(x)$ is equal to the generic Artin invariant of L. Set $U = L \cap \overline{M}^{\langle e \rangle}_{\widetilde{\Lambda}_0}$. If $v.e \neq 0$, then there exists a lift $f \colon U \to \mathscr{S}^o_\Lambda$ such that the diagram*

$$\begin{tikzcd} & \mathscr{S}^o_\Lambda \arrow[d, "\widetilde{\rho}"] \\ U \arrow[ur, "f"] \arrow[r, "f_v|_U"'] & \overline{M}^{\langle e \rangle}_{\widetilde{\Lambda}_0} \end{tikzcd}$$

commutes and $x \in f(U)$.

Remark 4.2.9. Let us explain the idea behind the proof. Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_p$-gerbe corresponding to x. Using v, we form an appropriate moduli space of sheaves, say $\mathscr{Y} \to Y$, on $\mathscr{X}$ (in fact, we may need to first replace $\mathscr{X}$ by another $\boldsymbol{\mu}_p$-gerbe with the same Brauer class). Let $\widetilde{\mathscr{Y}} \to \mathbf{A}^1$ be the corresponding universal twistor family containing $\mathscr{Y}$ as a fiber. By our relative twisted Torelli theorem (Proposition 3.3.9), the induced map from the base $\mathbf{A}^1$ to the period domain identifies $\mathbf{A}^1$ with a twistor line. We show that $\mathscr{X}$ is also a moduli space of twisted sheaves on $\mathscr{Y}$, an instance of Mukai duality. By taking an appropriate relative moduli space of sheaves on the family $\widetilde{\mathscr{Y}}$, we obtain a family of twisted surfaces containing the original surface $\mathscr{X}$ as a fiber. Our assumptions on v will allow us to ensure that moduli space $\mathscr{Y}$ parametrizes sheaves of rank p, and that the gerbe $\mathscr{Y}$ has Brauer class of order p. Together, these conditions enable us to ignore stability conditions at a key moment. It is possible that this could be eliminated with enough knowledge of the stability of "wrong-way" slices of universal sheaves. After proving our twisted crystalline Torelli theorem (a consequence of this result!) we will explain in Theorem 5.1.17 how to remove certain of our assumptions on v.

Proof. The k-point x of $\mathscr{S}^o_\Lambda$ corresponds to a $\boldsymbol{\mu}_p$-gerbe $\mathscr{X} \to X$ over a supersingular K3 surface X along with a marking $m\colon \Lambda \to \mathrm{Pic}(X)$. Let us identify Λ with its image in $\mathrm{Pic}(X)$, so that m is just the canonical inclusion. By Lemma 3.3.6, we obtain an induced inclusion $\widetilde{\Lambda} \subset \widetilde{N}(\mathscr{X})$ identifying e with $(0,0,1)$ and f with $(p,0,0)$.

Claim 4.2.10. There exists an element $x = (p, l, l^2/2p) \in p\widetilde{\Lambda}^*$ such that the image of x in $\widetilde{\Lambda}_0$ is a non-zero scalar multiple of v.

By Lemma 4.2.7, we may find a primitive isotropic vector $(r, l, s) \in p\widetilde{\Lambda}^* \subset \widetilde{\Lambda}$ whose image in $\widetilde{\Lambda}_0$ is v. Note that r is necessarily of the form pa for some integer a, and a must be invertible modulo p by our assumption that $v.e \neq 0$. Let b be an integer such that $ab \equiv 1 \mod p$. Consider the vector $x = (p, bl, ab^2 s) \in \widetilde{\Lambda}$. It is immediate that this vector is isotropic. As (r, l, s) is primitive, so is $(p, bl, ab^2 s)$. Finally, note that $ab^2 - b$ is divisible by p, and that

$$x = (p, bl, ab^2 s) = b(pa, l, s) + p\left(1 - ab, 0, s\frac{ab^2 - b}{p}\right)$$

Because $1 - ab$ is divisible by p, $(1 - ab, 0, s(ab^2 - b)/p) \in \widetilde{\Lambda}$. It follows that $x \in p\widetilde{\Lambda}^*$ and that the image of x in $\widetilde{\Lambda}_0$ is a non-zero scalar multiple of v. This completes the proof of the claim.

We will henceforth let $x = (p, l, l^2/2p) \in p\widetilde{\Lambda}^* \subset \widetilde{\Lambda}$ denote a fixed vector with these properties.

Claim 4.2.11. The image of x in $\widetilde{N}(\mathscr{X})$ is primitive and isotropic, and is contained in $p\widetilde{N}(\mathscr{X})^*$.

Let K be the characteristic subspace corresponding to $\widetilde{\rho}(x)$. The twistor line L is by definition contained in the locus of characteristic subspaces not containing v, so $v \notin K$. But as subspaces of $\widetilde{\Lambda}_0$ we have

$$K \cap \widetilde{\Lambda}_0 = \frac{p\widetilde{N}(\mathscr{X})}{p\widetilde{\Lambda}}$$

so the image of x in $\widetilde{N}(\mathscr{X})$ remains primitive. By Lemma 4.2.5, we have

$$\left(\frac{p\widetilde{N}(\mathscr{X})}{p\widetilde{\Lambda}}\right)^\perp = \frac{p\widetilde{N}(\mathscr{X})^*}{p\widetilde{\Lambda}}$$

By Lemma 3.0.4, our assumption on the Artin invariant of x implies that $v \in (K \cap \widetilde{\Lambda}_0)^{\perp}$, and therefore the image of x in $\widetilde{N}(\mathscr{X})$ is contained in $p\widetilde{N}(\mathscr{X})^{*}$. This completes the proof of the claim.

Using this lift, we will first construct a particular family of twisted supersingular K3 surfaces over an open subset of $\mathbf{A}^1$ that contains $\mathscr{X}$ as a fiber. This family will come with a natural marking by $\widetilde{\Lambda}$, and we will show that it satisfies the conclusions of the theorem.

Fix a line bundle $\mathscr{L}$ on X with first Chern class l, and consider the stack $\mathscr{X}' = \mathscr{X}\{\mathscr{L}^{\vee 1/p}\}$ (see Definition 2.1.4). By Proposition 4.1.8, the universal bundle induces a derived equivalence, and the corresponding map on cohomology is given by

$$e^{-l/p} : \widetilde{\mathrm{H}}(\mathscr{X}/W) \to \widetilde{\mathrm{H}}(\mathscr{X}'/W)$$

In particular, note that $e^{-l/p}(p, l, l^2/2p) = (p, 0, 0)$. By Theorem 4.1.19, the moduli space $\mathscr{Y} = \mathscr{M}^{\det}_{\mathscr{X}'}(p, 0, 0)$ of stable twisted sheaves on $\mathscr{X}'$ with respect to a generic polarization is a $\boldsymbol{\mu}_p$-gerbe over a supersingular K3 surface Y, and there is a universal object

$$\mathscr{P} \in \mathbf{Coh}^{(1,1)}(\mathscr{Y} \times \mathscr{X}')$$

equipped with an isomorphism

$$\det \mathscr{P} \xrightarrow{\sim} \mathscr{O}_{\mathscr{Y} \times \mathscr{X}'}$$

The isometry

$$g\colon \widetilde{N}^{(-1)}(\mathscr{Y}) \to \widetilde{N}(\mathscr{X}')$$

(see Notation 3.2.18) induced by the kernel $v(\mathscr{P})$ satisfies $(0, 0, 1) \mapsto (p, 0, 0)$ and $(p, 0, 0) \mapsto (0, 0, 1)$. We have $(p, 0, 0) \in p\widetilde{N}(\mathscr{X}')^{*}$, so by Theorem 4.1.19 the Brauer class of the gerbe $\mathscr{Y} \to Y$ is non-trivial. Let $\pi\colon Y \to \operatorname{Spec} k$ be the structure map, and consider the connected component $\mathbf{A}^1 \subset \mathbf{R}^2\pi^{\mathrm{fl}}_{*}\boldsymbol{\mu}_p$ that contains $\mathscr{Y}$ as a fiber. Let $\widetilde{\mathscr{Y}} \to \mathbf{A}^1$ be the universal family. Let $V \subset \mathbf{A}^1$ be the locus of points $t \in \mathbf{A}^1(k)$ where the image of the class $(p, 0, 0) = g^{-1}(0, 0, 1)$ in $\widetilde{N}^{(-1)}(\mathscr{Y}_t)$ remains primitive. Note that if $\mathscr{Y}$ deforms the trivial gerbe then V is the complement of the origin, and otherwise $V = \mathbf{A}^1$. We form the relative moduli space

$$\widetilde{\mathscr{X}'} = \mathscr{M}^{\det}_{\widetilde{\mathscr{Y}}|_V/V}(p, 0, 0) \to V$$

of twisted sheaves on the fibers of $\widetilde{\mathscr{Y}}|_V \to V$ that are stable with respect to a sufficiently generic polarization, which comes with a universal object

$$\widetilde{\mathscr{Q}} \in \mathbf{Coh}^{(1,1)}(\widetilde{\mathscr{Z}'} \times_V \widetilde{\mathscr{Y}}|_V)$$

Let $\mathscr{Z}'$ be the fiber corresponding to $\mathscr{Y}$. We will construct an isomorphism $\mathscr{X}' \xrightarrow{\sim} \mathscr{Z}'$. By Lemma 4.1.22, the universal twisted sheaf $\mathscr{P}$ is locally free, and in particular flat over $\mathscr{Y}$. Because the Brauer class of $\mathscr{Y} \to Y$ is non-trivial, stability conditions for rank p twisted sheaves on $\mathscr{Y}$ are vacuous. Thus, $\mathscr{P}$ gives (by descent) an object of $\mathscr{M}^{\det}_{\mathscr{Y}}(p,0,0)(\mathscr{X}')$, and hence a morphism

$$\Theta = [\mathscr{P}] \colon \mathscr{X}' \to \mathscr{Z}'$$

We claim that this is an isomorphism. The restriction $\mathscr{Q}$ of $\widetilde{\mathscr{Q}}$ to $\mathscr{Z}' \times \mathscr{Y}$ induces a Fourier-Mukai equivalence $D^{(\text{-}1)}(\mathscr{Z}') \to D^{(1)}(\mathscr{Y})$. The sheaf $\mathscr{Q}^\vee \in \mathbf{Coh}^{(\text{-}1,\text{-}1)}(\mathscr{Z}' \times \mathscr{Y})$ also induces a Fourier-Mukai equivalence $D^{(1)}(\mathscr{Z}') \to D^{(\text{-}1)}(\mathscr{Y})$ (see for instance Theorem 1.6.15 of [38]). Consider the "wrong-way" Fourier-Mukai transforms

$$\Phi^o_{\mathscr{Q}} \colon D^{(\text{-}1)}(\mathscr{Y}) \to D^{(1)}(\mathscr{Z}') \qquad\qquad \Phi^o_{\mathscr{Q}^\vee} \colon D^{(1)}(\mathscr{Z}') \to D^{(\text{-}1)}(\mathscr{Y})$$

One shows as in Theorem 1.6.15 of [38] that these are both equivalences, and that there is a natural isomorphism of functors $\Phi^o_{\mathscr{Q}^\vee} \circ \Phi^o_{\mathscr{Q}} \cong [-2]$ (see Remark 5.8 of Chapter 5 in [19]). By construction, the map

$$\Theta \times \mathrm{id} \colon \mathscr{X}' \times \mathscr{Y} \to \mathscr{Z}' \times \mathscr{Y}$$

satisfies $(\Theta \times \mathrm{id})^* \mathscr{Q} \cong \mathscr{P}$. Using this and the projection formula, we obtain a natural isomorphism

$$\mathbf{L}\Theta^* \circ \Phi^o_{\mathscr{Q}} \cong \Phi_{\mathscr{P}}$$

where

$$\Phi_{\mathscr{P}} \colon D^{(\text{-}1)}(\mathscr{Y}) \to D^{(1)}(\mathscr{X}')$$

is the natural Fourier-Mukai transform associated to $\mathscr{P}$. We therefore find that

$$\mathbf{L}\Theta^* \cong \Phi_{\mathscr{P}} \circ \Phi^o_{\mathscr{Q}^\vee}[2]$$

In particular, we conclude that $\mathbf{L}\Theta^*$ is an equivalence of categories. By looking at structure sheaves of closed points, this shows that the map Θ induces an isomorphism $X \xrightarrow{\sim} Z$. As $\mathscr{P}$ is a $\mathscr{X}'$-twisted sheaf, its $\mathscr{O}_{\mathscr{X}'}^{\times}$-action is identified with its canonical action by the inertia group of $\mathscr{X}'$. It follows that Θ is in fact a morphism of $\boldsymbol{\mu}_p$-gerbes, and hence is an isomorphism.

We have realized $\mathscr{X}'$ as a fiber of a family of twisted K3 surfaces. We wish to do the same for $\mathscr{X}$. Consider the isometries

$$\widetilde{N}(\mathscr{Z}') \xrightarrow{h} \widetilde{N}^{(\text{-}1)}(\mathscr{Y}) \xrightarrow{g} \widetilde{N}(\mathscr{X}')$$

induced by $\Phi_{\mathscr{Q}}$ and $\Phi_{\mathscr{P}}$. Let $\widetilde{Z}'$ be the coarse space of $\widetilde{\mathscr{Z}'}$ and Z' the coarse space of $\mathscr{Z}'$. Let $\mathscr{L}' = \Theta_*(\mathscr{L}) \in \operatorname{Pic}(Z')$. We claim that $\mathscr{L}'$ is the restriction of a line bundle on $\widetilde{Z}'$. Indeed, consider the class $(p, -l, l^2/2p) \in \widetilde{N}(\mathscr{X}')$. We have that

$$h^{-1} \circ g^{-1}(p, -l, l^2/2p) = (p, -l', l'^2/2p)$$

where l' is the first Chern class of $\mathscr{L}'$. Any class in $\widetilde{N}^{(\text{-}1)}(\mathscr{Y})$ extends to a section of $\underline{\widetilde{N}}^{(\text{-}1)}_{Y_{\mathbf{A}^1}/\mathbf{A}^1} \otimes \mathbf{Q}$ which restricts over each geometric point $t \in \mathbf{A}^1$ to a class in $\widetilde{N}^{(\text{-}1)}(\widetilde{\mathscr{Y}_t})$. Thus, $(p, -l', l'^2/2p)$ extends to a section of $\underline{\widetilde{N}}_{\widetilde{Z}'/V} \otimes \mathbf{Q}$ which restricts over each geometric point $t \in V$ to a class in $\widetilde{N}(\widetilde{\mathscr{Z}'_t})$. It follows that l' extends to a class in the relative Picard group of $\widetilde{Z}'$, and so $\mathscr{L}'$ extends to a line bundle, say $\widetilde{\mathscr{L}'}$, on $\widetilde{Z}'$. Define

$$\widetilde{\mathscr{Z}} = \widetilde{\mathscr{Z}'}\{\widetilde{\mathscr{L}'}^{1/p}\}$$

This is a $\boldsymbol{\mu}_p$-gerbe over $\widetilde{Z}'$, and is equipped with a morphism $\widetilde{\mathscr{Z}} \to V$ (see Definition 2.1.4). If $\mathscr{Z}$ is the fiber corresponding to $\mathscr{Z}'$, then Θ induces an isomorphism

$$\mathscr{X} = \mathscr{X}'\{\mathscr{L}^{1/p}\} \xrightarrow{\sim} \mathscr{Z}'\{\mathscr{L}'^{1/p}\} = \mathscr{Z}$$

Thus, $\mathscr{X}$ is isomorphic to a fiber of the family $\widetilde{\mathscr{Z}} \to V$.

We will now complete the proof. The derived equivalences we have constructed induce isometries

$$\begin{tikzcd} \widetilde{N}(\mathscr{X}) \arrow[r, "e^{-l/p}"] & \widetilde{N}(\mathscr{X}') \arrow[r, "g^{-1}"] \arrow[rr, bend right, "\Theta_*"'] & \widetilde{N}^{(\text{-}1)}(\mathscr{Y}) \arrow[r, "h^{-1}"] & \widetilde{N}(\mathscr{Z}') \arrow[r, "e^{l'/p}"] & \widetilde{N}(\mathscr{Z}) \end{tikzcd} \tag{4.2.11.1}$$

By our construction, the family $\widetilde{\mathscr{Z}}$ carries a natural marking by Λ. Indeed, the marking $\widetilde{\Lambda} = \Lambda \oplus U_2(p) \to \widetilde{N}(\mathscr{X})$ of the extended Néron-Severi group of $\mathscr{X}$ induces a marking $\widetilde{\Lambda} \to \widetilde{N}(\mathscr{X}')$, and hence a map $\widetilde{\Lambda} \to \widetilde{N}^{(\text{-}1)}(\mathscr{Y})$, which by Lemma 4.2.4 extends to a map $\underline{\widetilde{\Lambda}}_{\mathbf{A}^1} \to \underline{\widetilde{N}}^{(\text{-}1)}_{Y_{\mathbf{A}^1}/\mathbf{A}^1} \otimes \mathbf{Q}$ restricting over each geometric point $t \in \mathbf{A}^1$ to a map $\widetilde{\Lambda} \to \widetilde{N}^{(\text{-}1)}(\widetilde{\mathscr{Y}_t})$. We thus find the same for the families $\widetilde{\mathscr{Z}'}$ and $\widetilde{\mathscr{Z}}$. The composition $\widetilde{N}(\mathscr{X}) \to \widetilde{N}(\mathscr{Z})$ satisfies $(0,0,1) \mapsto (0,0,1)$ and $(p,0,0) \mapsto (p,0,0)$, so there is an induced marking $\Lambda \to \mathrm{Pic}_{\widetilde{\mathscr{Z}}/V}$. The resulting marked family gives a morphism

$$f_0 \colon V \to \mathscr{S}^o_{\Lambda}$$

The isomorphism $\mathscr{X} \xrightarrow{\sim} \mathscr{Z}$ is compatible with the respective markings by Λ, so the image of f_0 contains x. Consider the composition

$$\widetilde{\rho} \circ f_0 \colon V \to \overline{M}^{\langle e \rangle}_{\widetilde{\Lambda}_0}$$

It follows directly from our constructions that the image of V under $\widetilde{\rho} \circ f_0$ is U, and moreover that the map $V \to U$ is a bijection on closed points. To obtain the stronger result that we have claimed, we need to show that in fact $\widetilde{\rho} \circ f_0$ maps V isomorphically onto U. To see this, consider the induced isometry $\widetilde{\Lambda} \to \widetilde{N}(\mathscr{Y})$, which satisfies $(p, l, l^2/2p) \mapsto (0,0,1)$ and $e = (0,0,1) \mapsto (p,0,0)$. Set $\Lambda' = (p,l,s)^{\perp}/(p,l,s)$ (a subquotient of $\widetilde{\Lambda}$). Set $\widetilde{\Lambda}' = \Lambda' \oplus U_2(p)$, and for clarity let us denote the standard basis of this copy of $U_2(p)$ by e', f'. The family $\widetilde{\mathscr{Y}} \to \mathbf{A}^1$ carries a natural marking by Λ', and by Proposition 3.3.9 this family induces a diagram

$$\begin{tikzcd}
 & \mathscr{S}^o_{\Lambda'} \arrow[d, "\widetilde{\rho}"] \\
\mathbf{A}^1 \arrow[ur, "f_1"] \arrow[r, hook] & \overline{M}^{\langle e' \rangle}_{\widetilde{\Lambda}'_0}
\end{tikzcd}$$

where the lower horizontal arrow is the inclusion of the twistor line (up to our identification of this line with $\mathbf{A}^1$). Let $K(\widetilde{\mathscr{Y}})$ be the sub-bundle of $\widetilde{\Lambda}_0 \otimes \mathscr{O}_{\mathbf{A}^1}$ corresponding to the relative Mukai crystal of $\widetilde{\mathscr{Y}}$ with its $T = \widetilde{\Lambda} \otimes \mathbf{Z}_p$-structure, and let $K(\widetilde{\mathscr{Z}})$ be the sub-bundle of $\widetilde{\Lambda}_0 \otimes \mathscr{O}_V$ corresponding to the relative Mukai crystal of $\widetilde{\mathscr{Z}}$ with its $\widetilde{\Lambda} \otimes \mathbf{Z}_p$-structure. Applying Proposition 4.1.7 (and Proposition 4.1.21), we see that $K(\widetilde{\mathscr{Y}})|_V$ and $K(\widetilde{\mathscr{Z}})$ are equal as sub-bundles of $\widetilde{\Lambda}_0 \otimes \mathscr{O}_V$. Furthermore, the map $\widetilde{\Lambda}' \to \widetilde{\Lambda}$ induced by the derived equivalence between $\mathscr{Y}$ and $\mathscr{Z}$ and satisfies $e' \mapsto (p, l, l^2/2p)$ and $f' \mapsto (0,0,1)$. It follows that the map $f_0 \colon V \to \mathscr{S}^o_{\Lambda}$ induced by $\widetilde{\mathscr{Z}} \to V$ maps V isomorphically onto U. Composing with the inverse of this isomorphism, we find a lift f of $f_v|_U$ and a diagram

$$\begin{tikzcd} & \mathscr{S}^o_\Lambda \arrow[d, "\widetilde{\rho}"] \\ U \arrow[ur, "f"] \arrow[r, "f_v|_U"] & \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \end{tikzcd}$$

As previously noted, the image of f contains x, so this completes the proof.

□

To apply this theorem, we will use the following technical lemma. Let us adopt Notation 3.0.8, except that we shall write e, f for the generators v, w of U_2. Let $\widetilde{K} \subset \widetilde{V} \otimes k$ be a characteristic subspace such that $e \notin \widetilde{K}$, and let $K = \pi_e(K)$. Let σ be the Artin invariant of K.

Lemma 4.2.12. *If $\sigma \geq 2$, then there exists an isotropic vector $v \in \widetilde{V}$ such that $v.e \neq 0$, $v \notin \widetilde{K}$, and $v \in (\widetilde{K} \cap \widetilde{V})^\perp$.*

Proof. As discussed in Remark 3.0.3, there exists an element $B \in V \otimes k$, uniquely determined up to elements of K, such that

$$\widetilde{K} = \left\langle x_1 + (x_1.B)e, \ldots, x_{\sigma_0} + (x_{\sigma_0}.B)e, f + B + \frac{B^2}{2}e \right\rangle$$

Consider the vector space $(K \cap V)^\perp/(K \cap V)$, which has dimension 2σ. The natural form on this space is non-degenerate, and as $\sigma \geq 2$, it has dimension greater than or equal to 4. Thus, it contains a nonzero isotropic vector (by, for instance, Proposition 3.0.2). We may therefore find a nonzero isotropic vector $x \in (K \cap V)^\perp$ such that $x \notin K \cap V$.

Let $y_1, \ldots, y_{\sigma_0 - \sigma}$ be a basis for $K \cap V$. We consider first the case that the image of B in $V \otimes k/K$ is not contained in the subgroup

$$\frac{V}{K \cap V} = \frac{V + K}{K} \subset \frac{V \otimes k}{K}$$

We have that

$$\widetilde{K} \cap \widetilde{V} = \langle y_1 + (y_1.B)e, \ldots, y_{\sigma_0 - \sigma} + (y_{\sigma_0 - \sigma}.B)e \rangle$$

It is immediate that the vector $v = f + x$ has the desired properties.

Next, suppose that the image of B is contained in $V/K \cap V$. Without loss of generality, we may then assume that $B \in V$, and therefore

$$\widetilde{K} \cap \widetilde{V} = \left\langle y_1 + (y_1.B)e, \ldots, y_{\sigma_0 - \sigma} + (y_{\sigma_0 - \sigma}.B)e, f + B + \frac{B^2}{2}e \right\rangle$$

One checks that the vector $v = f + x + B + (x.B + B^2/2)e$ has the desired properties.

$\square$

4.3 Artin-Tate twistor families

In this section we discuss Artin-Tate twistor families. We begin by interpreting moduli spaces of rank 0 twisted sheaves in terms of elliptic fibrations. We then interpret the fibers of Artin-Tate families in terms of torsors, giving a geometric manifestation of the isomorphism $\mathrm{Br} \cong Ш$. We will apply this in Section 5.2 to study unirationality of supersingular K3 surfaces.

Definition 4.3.1. An *elliptic fibration* on a surface X is a proper flat morphism $f\colon X \to \mathbf{P}^1$ whose generic fiber is a smooth genus 1 curve. An elliptic fibration is ***Jacobian*** if it admits a section $\mathbf{P}^1 \to X$, and is ***non-Jacobian*** otherwise. A ***multisection of degree*** n is an integral subscheme $\Sigma \subset X$ such that $f|_\Sigma \colon \Sigma \to \mathbf{P}^1$ is finite flat of degree n.

Let $f\colon X \to \mathbf{P}^1$ be an elliptic fibration on a supersingular K3 surface, and $p\colon \mathscr{X} \to X$ a $\boldsymbol{\mu}_p$-gerbe. Let $E \subset X$ be a smooth fiber of f. Fix an integer s, and consider the Mukai vector

$$v = (0, E, s)$$

Note that v is primitive and $v^2 = 0$. Fix a generic polarization H on X. By Theorem 4.1.19 the moduli stack $\mathscr{M}_{\mathscr{X}}(v)$ is a $\mathbf{G}_m$-gerbe over a supersingular K3 surface. We wish to identify a natural choice of a corresponding $\boldsymbol{\mu}_p$-gerbe, as in Definition 4.2.3 in the positive rank case. Fix a multisection $\Sigma \subset X$ whose degree is equal to the index of the fibration $X \to \mathbf{P}^1$. In our case, X is supersingular, so the index is equal to 1 or p. Let $\widetilde{\Sigma} = \mathscr{X} \times_X \Sigma$ be the restriction of $\mathscr{X}$ to $\Sigma \subset X$. By Tsen's theorem, the Brauer class of the gerbe $\widetilde{\Sigma}$ is trivial. We fix also an invertible twisted sheaf $\mathscr{L}$ on $\widetilde{\Sigma}$. We define the following relative stacks over $\mathbf{P}^1$.

Definition 4.3.2.

1. Let $\mathscr{R}^{(\Sigma,\mathscr{L})}_{\mathscr{X}}(s) \to \mathbf{P}^1$ be the stack whose objects over $T \to \mathbf{P}^1$ are pairs $(\mathscr{E}, \phi)$, where $\mathscr{E}$ is a T-flat $\mathscr{X} \times_{\mathbf{P}^1} T$-twisted quasi-coherent sheaf of finite presentation such that for each geometric point $t \in T$, the pushforward of the fiber $\mathscr{E}_t$ along the natural closed immersion

$$\mathscr{X} \times_{\mathbf{P}^1} t \to \mathscr{X} \times t$$

is H-stable with twisted Mukai vector $v = (0, E, s)$, and ϕ is an isomorphism

$$\phi\colon \det(q_{\widetilde{\Sigma}_T *}(\mathscr{E}|_{\widetilde{\Sigma}_T} \otimes \mathscr{L}^\vee)) \xrightarrow{\sim} \det(q_{\Sigma_T *}\mathscr{O}_\Sigma)$$

where $\widetilde{\Sigma}_T = \widetilde{\Sigma} \times_{\mathbf{P}^1} T$, $\Sigma_T = \Sigma \times_{\mathbf{P}^1} T$, and $q_{\widetilde{\Sigma}_T *}\colon \widetilde{\Sigma}_T \to T$ and $q_{\Sigma_T}\colon \Sigma_T \to T$ are the projections.

2. Let $\mathscr{R}_{\mathscr{X}}(s) \to \mathbf{P}^1$ be the same but omitting the isomorphisms ϕ.

Example 4.3.3. For our applications in Section 5.2, it will suffice to consider the case when $\mathscr{X} \to X$ is the trivial gerbe $X \times \mathbf{B}\boldsymbol{\mu}_p$. There is then an invertible $\mathscr{X}$-twisted sheaf $\mathscr{L}'$ such that $\mathscr{L}'^{\otimes n} \cong \mathscr{O}_{\mathscr{X}}$. Tensoring with $\mathscr{L}'^\vee$ and pushing forward to X defines an isomorphism between $\mathscr{M}_{\mathscr{X}}(v)$ and the stack of coherent pure 1-dimensional sheaves on X with determinant $\mathscr{O}(E)$ and second Chern class s. As shown in Section 4 of [8], this stack is isomorphic to the relative moduli stack of stable sheaves on the fibers of $f\colon X \to \mathbf{P}^1$ of rank 1 and degree s.

Lemma 4.3.4. *Pushforward defines an isomorphism $\mathscr{R}_{\mathscr{X}}(s) \to \mathscr{M}_{\mathscr{X}}(v)$ of k-stacks.*

Proof. This is Lemma 5.13 of [31]. □

Corollary 4.3.5. *If $X \to \mathbf{P}^1$ has index p, then the stack*

$$\mathscr{R}_{\mathscr{X}}^{(\Sigma,\mathscr{L})}(s) \to R_{\mathscr{X}}(s)$$

is a $\boldsymbol{\mu}_p$-gerbe over a supersingular K3 surface, with associated $\mathbf{G}_m$-gerbe $\mathscr{R}_{\mathscr{X}}(s) \to R_{\mathscr{X}}(s)$.

Proof. By Theorem 4.1.19, $\mathscr{M}_{\mathscr{X}}(v) \to M_{\mathscr{X}}(v)$ is a $\mathbf{G}_m$-gerbe over a supersingular K3 surface. By Lemma 4.3.4, $\mathscr{R}_{\mathscr{X}}(s) \to R_{\mathscr{X}}(s)$ is as well. There is a natural forgetful map

$$\mathscr{R}_{\mathscr{X}}^{(\Sigma,\mathscr{L})}(s) \to \mathscr{R}_{\mathscr{X}}(s)$$

Consider a morphism $T \to \mathbf{P}^1$ and an object $(\mathscr{E}, \phi)$ over T. The sheaf $\mathscr{E}|_{\widetilde{\Sigma}_T} \otimes \mathscr{L}^\vee$ on $\widetilde{\Sigma}_T$ has rank 1 and is 0-twisted. Because $\Sigma \to \mathbf{P}^1$ is finite flat of degree p, the pushforward

$$q_{\widetilde{\Sigma}_T *}(\mathscr{E}|_{\widetilde{\Sigma}_T} \otimes \mathscr{L}^\vee)$$

is of rank p. An automorphism $f \in \mathrm{Aut}(\mathscr{E}) \xrightarrow{\sim} \mathbf{G}_m$ acts on its determinant via the p-th power map. It follows that $\mathscr{R}_{\mathscr{X}}^{(\Sigma,\mathscr{L})}(s)$ is a $\boldsymbol{\mu}_p$-gerbe over its sheafification, and that $\mathscr{R}_{\mathscr{X}}(s)$ is its associated $\mathbf{G}_m$-gerbe. □

We consider the universal twistor family $\widetilde{\mathscr{X}} \to \mathbf{A}^1$ of $\boldsymbol{\mu}_p$-gerbes over the connected component of $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$. Form the relative moduli space of sheaves

$$\mathscr{M}_{\widetilde{\mathscr{X}}}(v) \to \mathbf{A}^1$$

By the above isomorphism, each fiber of this family is equipped with an elliptic fibration. These fibrations turn out to be closely related, and in fact are all étale forms of each other.

Proposition 4.3.6. *If $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$-gerbe that deforms the trivial gerbe, then for any s the morphism $\mathscr{R}_{\mathscr{X}}(s) \to \mathbf{P}^1$ is an étale form of the morphism $\mathscr{R}_X(s) \to \mathbf{P}^1$.*

Proof. This is included in Proposition 5.17 of [31]. □

Recall that if E is an elliptic curve over a field L, then $\text{Ш}(L, E)$ is the group of E-torsors over L. In particular, we may consider $\text{Ш}(k(t), \mathrm{Jac}(X_{k(t)}))$. Each fiber of the Artin-Tate twistor family $M_{\widetilde{\mathscr{X}}}(v) \cong R_{\widetilde{\mathscr{X}}}(1)$ is equipped with an elliptic structure, which is an étale form of the fiber $R_{\widetilde{\mathscr{X}}}(1)|_0 \cong \mathrm{Jac}(X_{k(t)})$. Recall that the subgroup $\mathrm{U}^2(X, \boldsymbol{\mu}_p) \subset \mathrm{H}^2(X, \boldsymbol{\mu}_p)$ is the k-points of the connected component of the identity of $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$, and therefore classifies those $\boldsymbol{\mu}_p$-gerbes on X that deform the trivial gerbe. The association $\mathscr{X} \mapsto [R_{\mathscr{X}}(1)]$ gives a map

$$\mathrm{U}^2(X, \boldsymbol{\mu}_p) \to \text{Ш}(k(t), \mathrm{Jac}(X_{k(t)}))$$

We will interpret this map as a geometric manifestation of the Artin-Tate isomorphism. The following result is Theorem 3.1 of [46].

Theorem 4.3.7 (Artin-Tate). *Let $f\colon X \to \mathbf{P}^1$ be an elliptic fibration on a K3 surface. The edge map in the E^2 term of the Leray spectral sequence for $\mathbf{G}_m$ on f yields an isomorphism*

$$\mathrm{Br}(X) \to \text{Ш}(k(t), \mathrm{Pic}_{X_{k(t)}/k(t)})$$

resulting in a natural surjection

$$\text{Ш}(k(t), \mathrm{Jac}(X_{k(t)})) \twoheadrightarrow \mathrm{Br}(X)$$

with kernel isomorphic to $\mathbf{Z}/i\mathbf{Z}$, where i is the index of the generic fiber $X_{k(t)}$ over $k(t)$. In particular, if $X \to \mathbf{P}^1$ has a section, the latter arrow is an isomorphism.

Proof. This is Proposition 4.5 (and "cas particulier (4.6)") of [17]. □

It follows by descent theory that any element of $Ш(k(t), \mathrm{Jac}(X_{k(t)}))$ corresponds to an étale form X' of X, and X' is also a K3 surface.

Proposition 4.3.8. *Let $X \to \mathbf{P}^1$ be an elliptic supersingular K3 surface with Jacobian $J(X) \to \mathbf{P}^1$. If $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_p$-gerbe that deforms the trivial gerbe, then for any s the morphism $R_{\mathscr{X}}(s) \to \mathbf{P}^1$ is an étale form of the morphism $J(X) \to \mathbf{P}^1$. The resulting map $U^2(X, \boldsymbol{\mu}_p) \to Ш(k(t), \mathrm{Jac}(X_{k(t)}))$ defined by*

$$\mathscr{X} \mapsto [R_{\mathscr{X}}(1)]$$

fits into a commutative diagram

$$\begin{tikzcd} \mathrm{U}^2(X, \boldsymbol{\mu}_p) \arrow[d] \arrow[dr, two heads] & \\ Ш(k(t), \mathrm{Jac}(X_{k(t)})) \arrow[r, two heads] & \mathrm{Br}(X) \end{tikzcd}$$

where the horizontal arrow is the Artin-Tate map of Proposition 4.3.7.

Proof. By Proposition 4.3.6, $\mathscr{R}_{\mathscr{X}}(s) \to \mathbf{P}^1$ is an étale form of $\mathscr{R}_X(s) \to \mathbf{P}^1$. The same is therefore true for $R_{\mathscr{X}}(s) \to \mathbf{P}^1$ and $R_X(s) \to \mathbf{P}^1$. The latter is isomorphic as an elliptic surface to the Jacobian $J^s(X) \to \mathbf{P}^1$, which is an étale form of $J(X) \to \mathbf{P}^1$ (and also of $X \to \mathbf{P}^1$) (see Chapter 11, Remark 4.4 of [21]). This gives the first claim.

For the second claim, we consider the corresponding genus 1 curves X_η and $R_\eta = (R_{\mathscr{X}}(1))_\eta$ over the generic point $\eta = \operatorname{Spec} k(t)$ of $\mathbf{P}^1$. The Leray spectral sequence and Tsen's theorem show that the edge map gives an isomorphism

$$\mathrm{Br}(X_\eta) \xrightarrow{\sim} \mathrm{H}^1(\eta, \mathrm{Jac}(X_\eta))$$

which we can describe concretely as follows. Over $\overline{k(t)}$ the gerbe $\mathscr{X}_\eta \to X_\eta$ has trivial Brauer class, hence carries an invertible twisted sheaf $\mathscr{L}$ such that $\mathscr{L}^{\otimes p}$ has degree $pa(\mathscr{X})$, where

$$a(\mathscr{X}) \in \frac{1}{p}\mathbf{Z}/\mathbf{Z}$$

is the unique element that corresponds to the cohomology class of the restriction of $\mathscr{X}$ to any smooth fiber $E \subset X$ of f under the natural isomorphism

$$\frac{1}{p}\mathbf{Z}/\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}.$$

By our assumption that $\mathscr{X}$ deforms the trivial gerbe, $a(\mathscr{X}) = 0$. Given an element σ of the Galois group of $\overline{k(t)}$ over $k(t)$, there is an invertible sheaf $L_\sigma \in \mathrm{Pic}(X_{\overline{k(t)}})$ such that $\sigma^*\mathscr{L} \otimes \mathscr{L}^\vee \cong L_\sigma|_{\mathscr{X}_\eta}$. This defines a 1-cocycle in the sheaf $\mathrm{Pic}_{X_\eta/\eta}$, and its cohomology class is the image of a unique class in $\mathrm{H}^1(\eta, \mathrm{Jac}(X_\eta))$, as desired.

On the other hand, tensoring with $\mathscr{L}^\vee$ gives an isomorphism between the stack of invertible $\mathscr{X}_\eta$-twisted sheaves of degree 1 and the stack of invertible sheaves on X_η of degree 1. The latter stack is a gerbe over X_η, and the Galois group induces the cocycle given by the translation action of $\mathrm{Jac}(X_\eta)$ on X_η. But this gives the edge map in the Leray spectral sequence.

Finally, the surjectivity of the diagonal arrow is implied by Theorem 2.1.13. $\square$

Chapter 5

APPLICATIONS

5.1 A crystalline Torelli theorem for twisted supersingular K3 surfaces

In this section we use twistor lines to prove a crystalline Torelli theorem for twisted supersingular K3 surfaces over an algebraically closed field k of characteristic $p \geq 3$. Our approach is inspired by Verbitsky's proof of a Torelli theorem for hyperkähler manifolds over the complex numbers using classical twistor space theory.[1] As a corollary, we obtain an alternative proof of Ogus's crystalline Torelli theorem (Theorem III of [40]).

Let us briefly discuss the differences between our proof and that of Ogus. A key input in [41] is the result of Rudakov, Zink, and Shafarevich that supersingular K3 surfaces in characteristic $p \geq 5$ do not degenerate (see Theorem 3 of [44] and Theorem 5.1.11). This is applied to show that the moduli space S_Λ is almost proper over k (meaning it satisfies the surjectivity part of the valuative criterion with respect to DVRs). Using that the period domain is proper, it is then deduced that the period morphism is almost proper, and eventually (after adding ample cones) an isomorphism. Because of this, the main result of [41] is restricted to characteristic $p \geq 5$, although the surrounding theory works in characteristic $p \geq 3$. In our proof, we will show using twistor lines that the period morphism is an isomorphism without using the almost properness of the moduli space. Thus, we do not need to restrict to characteristic $p \geq 5$, and we are able to extend Ogus's Torelli theorem to characteristic 3 (characteristic 2 carries its own difficulties, which we will not discuss here). In fact, we can reverse the flow of information in Ogus's proof to obtain in particular a proof of the non-degeneration result of [44] in characteristic $p \geq 3$. This result was previously known only for $p \geq 5$, and $p = 2$ via different methods (see [43]). We end the section by applying our twisted Torelli theorem to prove that every twistor family admits a modular interpretation, and to deduce some consequences for the structure of the moduli space

[1] See [48] for the proof, and [20] for a skillful exposition of these ideas. Also see [21], where the same strategy is applied in a simplified form to deduce the Torelli theorem for complex K3 surfaces.

of supersingular K3 surfaces which do not have any analog over the complex numbers.

The period domain M_{Λ_0} is a projective variety, and in particular separated. However, the moduli space S_Λ is not separated. To correct this, we will follow Ogus [41] and equip characteristic subspaces with ample cones. If Λ is a supersingular K3 lattice, we set

$$\Delta_\Lambda = \left\{\delta \in \Lambda | \delta^2 = -2\right\}$$

Associated to an element $\delta \in \Delta_\Lambda$ is the reflection

$$s_\delta(w) = w + (\delta.w)\delta$$

The *Weyl group* of Λ is the subgroup $W_\Lambda \subset O(\Lambda)$ generated by the s_δ. We set

$$V_\Lambda = \left\{x \in \Lambda \otimes \mathbf{R} | x^2 > 0 \text{ and } x.\delta \neq 0 \text{ for all } \delta \in \Delta_\Lambda\right\},$$

and let C_Λ be the set of connected components of V_Λ.

Proposition 5.1.1 ([41], Proposition 1.10). *The Weyl group W_Λ acts simply transitively on C_Λ.*

If $\Lambda = \mathrm{Pic}(X)$ for some K3 surface X, then it is shown in [41] that the ample cone of X corresponds to a connected component of V_Λ, and hence to a certain element of C_Λ. Let $K \in M_{\Lambda_0}(S)$ be a characteristic subspace. For each (possibly non-closed) point $s \in S$, we set

$$\Lambda(s) = \{x \in \Lambda \otimes \mathbf{Q} | px \in \Lambda \text{ and } \overline{px} \in K(s)\}$$

We caution that this notation is not completely compatible with that of [41].

Definition 5.1.2. If $K \in M_{\Lambda_0}(S)$ is a characteristic subspace, then an *ample cone* for K is a choice of elements $\alpha_s \in C_{\Lambda(s)}$ for each $s \in S$, such that if s specializes to s_0 then $\alpha_s \subset \alpha_{s_0}$. We let P_Λ denote the functor on schemes over $\overline{M}_{\Lambda_0}$ whose value on a scheme is the set of ample cones of the corresponding characteristic subspace.

Proposition 5.1.3 ([41], Proposition 1.16). *The functor P_Λ is representable and locally of finite type over k, and the natural map $P_\Lambda \to \overline{M}_{\Lambda_0}$ is étale and surjective.*

Ogus also shows that $\rho\colon S_\Lambda \to \overline{M}_{\Lambda_0}$ factors through P_Λ, resulting in a map

$$\rho^a\colon S_\Lambda \to P_\Lambda$$

This map is shown to be étale and separated. Ogus's crystalline Torelli theorem asserts that this map is an isomorphism. We seek an extension of these ideas to the twisted setting. We define a functor $\mathscr{P}_{\widetilde{\Lambda}}$ by the Cartesian diagram

$$\begin{tikzcd} \mathscr{P}_{\widetilde{\Lambda}} \arrow[r, "\pi_e^a"] \arrow[d] & P_\Lambda \arrow[d] \\ \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \arrow[r, "\pi_e"] & \overline{M}_{\Lambda_0} \end{tikzcd}$$

It is immediate from Proposition 5.1.3 that the functor $\mathscr{P}_{\widetilde{\Lambda}}$ is representable and locally of finite type over k, and that the natural map $\mathscr{P}_{\widetilde{\Lambda}} \to \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}$ is étale and surjective.

Proposition 5.1.4. *The morphism $\widetilde{\rho}\colon \mathscr{S}^o_{\widetilde{\Lambda}} \to \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}$ factors through $\mathscr{P}_{\widetilde{\Lambda}}$. The resulting diagram*

$$\begin{tikzcd} \mathscr{S}^o_{\widetilde{\Lambda}} \arrow[r, "p"] \arrow[d, "\widetilde{\rho}^a"'] & S_\Lambda \arrow[d, "\rho^a"] \\ \mathscr{P}_{\widetilde{\Lambda}} \arrow[r, "\pi_e^a"] & P_\Lambda \end{tikzcd} \tag{5.1.4.1}$$

is Cartesian, and the map $\widetilde{\rho}^a\colon \mathscr{S}^o_{\widetilde{\Lambda}} \to \mathscr{P}_{\widetilde{\Lambda}}$ is separated and étale.

Proof. By the universal property of the fiber product and Proposition 3.3.9, we get a diagram

$$\begin{tikzcd} \mathscr{S}^o_{\widetilde{\Lambda}} \arrow[r, "p"] \arrow[d, "\widetilde{\rho}^a"] \arrow[dd, bend right, "\widetilde{\rho}"'] & S_\Lambda \arrow[d, "\rho^a"'] \arrow[dd, bend left, "\rho"] \\ \mathscr{P}_{\widetilde{\Lambda}} \arrow[r, "\pi_e^a"] \arrow[d] & P_\Lambda \arrow[d] \\ \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \arrow[r, "\pi_e"] & \overline{M}_{\Lambda_0} \end{tikzcd}$$

where the lower and outer squares are Cartesian. It follows that the upper square is Cartesian. Because ρ^a is separated and étale, the same is true for $\widetilde{\rho}^a$. $\square$

Our crystalline Torelli theorem for twisted supersingular K3 surfaces asserts that the twisted period morphism $\widetilde{\rho}^a$ is an isomorphism. We prove this in several steps.

Let us define the Artin invariant of a k-point of $\mathscr{P}_{\widetilde{\Lambda}}$ (respectively, P_Λ) to be the Artin invariant of its image in $\overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}$ (respectively $\overline{M}_{\Lambda_0}$). Using Kummer surfaces, Ogus showed in [40] that the crystalline Torelli theorem is true at all points of Artin invariant $\sigma_0 \leq 2$ (this part of his argument needs only that $p \geq 3$). Using Proposition 5.1.4, we will extend this to the twisted setting.

Proposition 5.1.5. *The fiber of $\widetilde{\rho}^a$ over any k-point of Artin invariant $\sigma_0 \leq 2$ is a singleton. Each connected component of $\mathscr{P}_{\widetilde{\Lambda}}$ contains a point of Artin invariant ≤ 2.*

Proof. It is shown in [41] that the fiber of ρ^a over any k-point of Artin invariant $\sigma_0 \leq 2$ is a singleton (see Step 4 of the proof of Theorem III'). Consider a point $x \in \mathscr{P}_{\widetilde{\Lambda}}$ of Artin invariant ≤ 2. The image of x in P_Λ also has Artin invariant ≤ 2, so by Proposition 5.1.4 the fiber $(\widetilde{\rho}^a)^{-1}(x)$ is a singleton.

Following Ogus [41], let us say that a morphism of schemes is *almost proper* if it satisfies the surjectivity part of the valuative criterion with respect to DVR's. In the proof of Proposition 1.16 of [41], Ogus shows that $P_\Lambda \to \overline{M}_{\Lambda_0}$ is almost proper. This property is clearly preserved under base change, so the morphism

$$\mathscr{P}_{\widetilde{\Lambda}} \to \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}$$

is also almost proper. Let $\mathscr{P} \subset \mathscr{P}_{\widetilde{\Lambda}}$ be a connected component. By Proposition 5.1.4, its image in $\overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}$ is open. The scheme $\overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}$ has two connected components, each of which are irreducible and contain a (unique) point of Artin invariant 1 (see the discussion following Definition 3.0.4). We may therefore find a DVR R with residue field k and fraction field K and a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathscr{P} \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0} \end{tikzcd}$$

such that the closed point of $\operatorname{Spec} R$ is mapped to a point of Artin invariant ≤ 2. As the right vertical arrow is almost proper, we may find the dashed arrow, and this gives the result. $\square$

We next show that $\widetilde{\rho}^a$ is surjective.

Proposition 5.1.6. *The morphism $\widetilde{\rho}^a \colon \mathscr{S}^o_\Lambda \to \mathscr{P}_{\widetilde{\Lambda}}$ is surjective.*

Proof. Clearly, $\widetilde{\rho}^a$ is surjective if and only if ρ^a is surjective. By Proposition 5.1.1, the group of reflections about (-2)-classes in Λ acts transitively on the set of ample cones of Λ. Moreover, this group preserves any characteristic subspace. Therefore, ρ^a is surjective if and only if ρ is surjective (see [41], Step 4 of proof of Theorem III' for a similar argument). Because ρ is surjective if and only if $\widetilde{\rho}$ is surjective, we are reduced to showing that

$$\widetilde{\rho}\colon \mathscr{S}^o_\Lambda \to \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0}$$

is surjective. We will induct on the Artin invariant. For each $1 \leq \sigma_0 \leq 10$ fix a supersingular K3 lattice Λ^{σ_0}, and define $\widetilde{\Lambda}^{\sigma_0+1} = \Lambda^{\sigma_0} \oplus U_2(p)$. Let

$$\rho_{\sigma_0}\colon S_{\Lambda^{\sigma_0}} \to \overline{M}_{\Lambda_0^{\sigma_0}} \qquad \text{and} \qquad \widetilde{\rho}_{\sigma_0+1}\colon \mathscr{S}_{\Lambda^{\sigma_0}} \to \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0^{\sigma_0+1}}$$

be the period morphisms. As described in Step 4 of the proof of Theorem III′ of [41], ρ_1 is surjective. We will show that

1. if ρ_{σ_0} is surjective, then $\widetilde{\rho}_{\sigma_0+1}$ is surjective, and

2. if $\widetilde{\rho}_{\sigma_0}$ is surjective, then ρ_{σ_0} is surjective.

The first claim follows immediately from Proposition 3.3.9. To show the second, suppose that $\widetilde{\rho}_{\sigma_0}$ is surjective, and take a point $p \in \overline{M}_{\Lambda^{\sigma_0}}(k)$ corresponding to a characteristic subspace $K \subset \Lambda_0^{\sigma_0} \otimes k$. The orthogonal sum $\Lambda^{\sigma_0} \oplus U_2$ where $U_2 = \langle a, b\rangle$ and $a^2 = b^2 = 0$, $a.b = -1$, is an extended supersingular K3 lattice of Artin invariant σ_0. Therefore, there exists an isometry

$$g\colon \Lambda^{\sigma_0} \oplus U_2 \xrightarrow{\sim} \widetilde{\Lambda}^{\sigma_0}$$

Let $v = g(a)$. We may assume that $v.e \neq 0$. There is an induced isometry $g_0\colon \Lambda_0^{\sigma_0} \xrightarrow{\sim} \widetilde{\Lambda}_0^{\sigma_0}$ and an induced isomorphism

$$g_0^*\colon \overline{M}_{\widetilde{\Lambda}_0^{\sigma_0}} \xrightarrow{\sim} \overline{M}_{\Lambda_0^{\sigma_0}}$$

Suppose $g_0^*(q) = p$. It follows that $q \in \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0^{\sigma_0}}$. By assumption we may find a point $y \in \mathscr{S}^o_{\Lambda^{\sigma_0-1}}(k)$ corresponding to a marked twisted supersingular K3 surface $\mathscr{Y} \to Y$ such that $\widetilde{\rho}_{\sigma_0}(y) = q$. Note that the vector $w = g(b)$ satisfies $v.w = -1$. It follows that the image of v under the marking $\widetilde{\Lambda}^{\sigma_0} \to \widetilde{N}(\mathscr{Y})$ is primitive and isotropic. By Theorem 4.1.19, the moduli space $\mathscr{X} = \mathscr{M}^{\det}_{\mathscr{Y}}(v)$ is an essentially trivial $\boldsymbol{\mu}_p$-gerbe over a supersingular K3 surface X, and we may find a commutative diagram

$$\begin{tikzcd} \widetilde{N}(X) \arrow[r] \arrow[d, "h"', "\wr"] & \widetilde{\mathrm{H}}(X/W) \arrow[d, "\wr"] \\ \widetilde{N}(\mathscr{Y}) \arrow[r] & \widetilde{\mathrm{H}}(\mathscr{Y}/W) \end{tikzcd}$$

where h is an isometry such that $(0,0,1) \mapsto v$. The K3 surface X with the induced marking

$$\Lambda^{\sigma_0} = a^{\perp}/a \xrightarrow{g} v^{\perp}/v \xrightarrow{h^{-1}} N(X)$$

gives a point in $S_{\Lambda^{\sigma_0}}$ whose image under ρ is p. $\square$

We now know that $\widetilde{\rho}^a \colon \mathscr{S}^o_\Lambda \to \mathscr{P}_{\widetilde{\Lambda}}$ is separated, étale, and surjective, and that there exists a point in each connected component of $\mathscr{P}_{\widetilde{\Lambda}}$ whose fiber is a singleton. Combining this with our construction of twistor families in Theorem 4.2.8, we will show that $\widetilde{\rho}^a$ is an isomorphism.

Theorem 5.1.7. *If $p \geq 3$, then the twisted period morphism $\widetilde{\rho}^a \colon \mathscr{S}^o_\Lambda \to \mathscr{P}_{\widetilde{\Lambda}}$ is an isomorphism.*

Proof. Let $\mathscr{P}$ be a connected component of $\mathscr{P}_{\widetilde{\Lambda}}$, $\mathscr{S}$ its preimage in $\mathscr{S}^o_\Lambda$, and $\widetilde{\rho}' \colon \mathscr{S} \to P$ the restriction of $\widetilde{\rho}^a$. Consider the diagonal

$$\Delta_{\widetilde{\rho}'} \colon \mathscr{S} \to \mathscr{S} \times_{\mathscr{P}} \mathscr{S}$$

Because $\widetilde{\rho}'$ is separated, $\Delta_{\widetilde{\rho}'}$ is a closed immersion, and because $\widetilde{\rho}'$ is étale, it is also an open immersion. Thus, $\Delta_{\widetilde{\rho}'}$ is an isomorphism onto its image $\Delta = \Delta_{\widetilde{\rho}'}(\mathscr{P})$, which is a connected component. We will show that $\Delta = \mathscr{S} \times_{\mathscr{P}} \mathscr{S}$. Take a point $(x_0, x_1) \in \mathscr{S} \times_P \mathscr{S}$. We will construct a connected subvariety $C \subset \mathscr{S} \times_{\mathscr{P}} \mathscr{S}$ that contains (x_0, x_1) and a point whose image in $\mathscr{P}$ has Artin invariant 1.

Let σ be the Artin invariant of x_0 and x_1. Suppose that $\sigma > 2$. By induction, it will suffice to construct a connected subvariety containing (x_0, x_1) and a point of Artin invariant $\sigma - 1$. Let $\widetilde{K}$ be the characteristic subspace corresponding to $\widetilde{\rho}(x_0) = \widetilde{\rho}(x_1)$. By Lemma 4.2.12, we may find an isotropic vector v such that $v.e \neq 0$, $v \notin \widetilde{K}$, and $v \in (\widetilde{K} \cap \widetilde{\Lambda}_0)^\perp$. That is, v satisfies the assumptions of Theorem 4.2.8 (see Lemma 3.0.4). Applying Theorem 4.2.8 to both x_0 and x_1, we find an open subset $U \subset L$ and lifts

$$\varphi_0, \varphi_1 \colon U \to \mathscr{S}$$

of f, which agree upon composing with $\widetilde{\rho}$. Consider the diagram

$$\begin{tikzcd}
U \arrow[dr, "\varphi_i"] \arrow[drr, bend left, "\varphi_i'"] \arrow[ddr, bend right] & & \\
& \mathscr{S}^o_\Lambda \arrow[r, "p"] \arrow[d, "\widetilde{\rho}^a"'] & S_\Lambda \arrow[d, "\rho"] \\
& \mathscr{P}_{\widetilde{\Lambda}} \arrow[r] & P_\Lambda
\end{tikzcd}$$

where $i = 0, 1$. We will show that by performing an appropriate sequence of elementary modifications, we can modify the φ_i so that they agree upon composing with $\widetilde{\rho}^a$. By definition of $\mathscr{P}_{\widetilde{\Lambda}}$, this is the same as $\rho \circ \varphi_0 = \rho \circ \varphi_1$.

By Proposition 5.1.1, the Weyl group of Λ acts transitively on the set of ample cones C_Λ. By precomposing the marking of one of our families by an appropriate reflection, we may therefore ensure that $\rho \circ \varphi_0' = \rho \circ \varphi_1'$ at the generic point η of U. The same is then true also on the open locus of points in U with Artin invariant σ. Suppose that $s_0 \in U$ is one of the finitely many closed points with Artin invariant $\sigma - 1$. There is an inclusion $\Lambda \subset \Lambda(s)$, which induces an inclusion $V_\Lambda \supset V_{\Lambda(s)}$. The morphisms $\rho \circ \varphi_i'$ applied to η give an ample cone $\alpha \subset V_\Lambda$, and applied to s_0 give two possibly different ample cones $\alpha_0, \alpha_1 \subset V_{\Lambda(s)}$, which have the property that $\alpha_0, \alpha_1 \subset \alpha$. By Proposition 5.1.1, there exists $\delta \in \Lambda(s)$ whose associated reflection sends α_0 to α_1. We may assume that δ is not in the image of the Picard group of the generic fiber. Suppose that δ is an irreducible effective curve. By Proposition 2.8 of [41], taking the elementary modification of the family $X \to U$ corresponding to φ_1' with respect to δ produces a new family $Y \to U$, along with isomorphisms

$$\Theta_{U-s_0} \colon Y_{U-s_0} \xrightarrow{\sim} X_{U-s_0} \qquad \text{and} \qquad \Theta_{s_0} \colon Y_{s_0} \xrightarrow{\sim} X_{s_0}$$

Moreover, these isomorphisms act on the Picard groups in such a way so that $Y \to U$ inherits a marking by Λ, and the induced morphism $U \to S_\Lambda \to P_\Lambda$ now agrees with φ_0 at s_0. By Remark 8.2.4 of [21], the Weyl group is generated by irreducible effective curves, so we achieve the same result for a general δ as well. Replacing φ_1 with the induced $U \to \mathscr{S}_\Lambda^o$, we have modified our lift φ_1 so that $\widetilde{\rho} \circ \varphi_0 = \widetilde{\rho} \circ \varphi_1$ at every point of Artin invariant σ, and at s_0. Furthermore, note that x_1 is still in the image of φ_1. Repeating this procedure at each of the finitely many points of U with Artin invariant $\sigma - 1$, we find the desired lifts φ_0, φ_1 satisfying $\widetilde{\rho}^a \circ \varphi_0 = \widetilde{\rho}^a \circ \varphi_1$.

We have constructed a morphism

$$(\varphi_0, \varphi_1) \colon U \to \mathscr{S} \times_{\mathscr{P}} \mathscr{S}$$

whose image contains (x_0, x_1). By Lemma 3.0.5, U is equal to $L \cong \mathbf{A}^1$ minus at most one point. Therefore, by Lemma 3.0.6, the image of U contains a point, say (x_0', x_1'), where x_0' and x_1' have Artin invariant $\sigma - 1$. Continuing in this manner, we find a chain of (open subsets of) twistor lines C and a map $C \to \mathscr{S} \times_{\mathscr{P}} \mathscr{S}$ whose image contains (x_0, x_1), such that the image of the composition $C \to \mathscr{S} \times_{\mathscr{P}} \mathscr{S} \to \mathscr{P}$ contains a point of Artin invariant ≤ 2.

By Proposition 5.1.5, the preimage of this point under $\widetilde{\rho}'$ is a singleton. It follows that C intersects Δ, and hence C is contained in Δ. As (x_0, x_1) was arbitrary, this shows that $\Delta = \mathscr{S} \times_{\mathscr{P}} \mathscr{S}$. The diagonal morphism $\Delta_{\widetilde{\rho}'}$ is therefore an isomorphism, so $\widetilde{\rho}'$ is a monomorphism. Because $\widetilde{\rho}'$ is étale, it is an open immersion. By Proposition 5.1.6, $\widetilde{\rho}'$ is surjective. It follows that $\widetilde{\rho}'$ is an isomorphism. $\square$

Note that the pullback of the section $\sigma_f \colon \overline{M}_{\Lambda_0} \subset \overline{M}^{\langle e \rangle}_{\widetilde{\Lambda}_0}$ along ρ gives the locus $S_\Lambda \subset \mathscr{S}^o_\Lambda$ of marked surfaces with trivial $\boldsymbol{\mu}_p$-gerbe. Thus, Theorem 5.1.7 implies Ogus's crystalline Torelli theorem (Theorem III′ of [41], as well as its consequences Theorems I,II,II′,II″, and III), and extends these results to characteristic $p \geq 3$.

Corollary 5.1.8. *If $p \geq 3$, then the period morphism $\rho^a \colon S_\Lambda \to P_\Lambda$ is an isomorphism.*

Of course, because the diagram (5.1.4.1) is Cartesian, the opposite implication holds as well. Let us translate our result from characteristic subspaces to K3 crystals. Under the inclusion $\widetilde{\mathrm{H}}(\mathscr{X}/W) \subset \widetilde{\mathrm{H}}(X/K)$, the codimension filtration on $\widetilde{\mathrm{H}}(X/K)$ induces a filtration on $\widetilde{\mathrm{H}}(\mathscr{X}/W)$. Theorem 5.1.7 implies the following pointwise statement (compare to Theorem II of [41]).

Theorem 5.1.9. *Let k be an algebraically closed field of characteristic $p \geq 3$ and $\mathscr{X}$ and $\mathscr{Y}$ be $\boldsymbol{\mu}_p$-gerbes on supersingular K3 surfaces over k. If $\Theta \colon \widetilde{\mathrm{H}}(\mathscr{X}/W) \to \widetilde{\mathrm{H}}(\mathscr{Y}/W)$ is an isomorphism of W-modules such that*

1. *Θ is compatible with the bilinear forms and commutes with the respective Frobenius operators,*

2. *Θ preserves the extended Néron-Severi groups, so that there is a commutative diagram*

$$\begin{tikzcd} \widetilde{N}(\mathscr{X}) \arrow[r] \arrow[d, hook] & \widetilde{N}(\mathscr{Y}) \arrow[d, hook] \\ \widetilde{\mathrm{H}}(\mathscr{X}/W) \arrow[r, "\Theta"] & \widetilde{\mathrm{H}}(\mathscr{Y}/W) \end{tikzcd}$$

3. *Θ preserves the codimension filtration, and*

4. *Θ maps an ample class to an ample class,*

then Θ is induced by a unique isomorphism $\mathscr{X} \xrightarrow{\sim} \mathscr{Y}$. Conversely, any isomorphism induces a map Θ satisfying these conditions.

Using the same methods as in [40], one can deduce the following (compare to Theorem I of [41]).

Theorem 5.1.10. *If $p \geq 3$ and $\mathscr{X}$ and $\mathscr{Y}$ are $\boldsymbol{\mu}_p$-gerbes on supersingular K3 surfaces, then $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic if and only if there exists a filtered isomorphism*

$$\widetilde{\mathrm{H}}(\mathscr{X}/W) \xrightarrow{\sim} \widetilde{\mathrm{H}}(\mathscr{Y}/W)$$

of W-modules that is compatible with the bilinear forms and with the Frobenius operators.

We obtain an alternate proof of the following theorem of Rudakov, Zink, and Shafarevich (Theorem 3 of [44]). As discussed in the introduction, this result is new in characteristic 3.

Theorem 5.1.11. *Let k be an algebraically closed field of characteristic $p > 0$, and let $R = k[[t]]$ and $K = k((t))$. If X is a K3 surface over K whose geometric fiber has Picard rank 22, then there exists a finite extension R'/R and a smooth surface X' over R' such that $X'_{K'} \cong X_{K'}$.*

Proof. The result holds in characteristic $p = 2$ by results of [43]. Suppose that $p \geq 3$. After taking a finite extension R' of R, we may arrange so that the family $X \to \operatorname{Spec} K$ admits a marking by some supersingular K3 lattice Λ (see page 1522 of [42]), and hence gives a morphism $K' \to S_\Lambda$. As explained in Proposition 1.16 of [41], the forgetful map $P_{\Lambda_0} \to \overline{M}_{\Lambda_0}$ satisfies the existence part of the valuative criterion with respect to DVR's. Because the period domain $\overline{M}_{\Lambda_0}$ is proper over $\operatorname{Spec} k$, we conclude by Theorem 5.1.7 that $S_\Lambda \to \operatorname{Spec} k$ satisfies the existence part of the valuative criterion with respect to DVR's. This gives the result. $\square$

Our results give an interesting interpretation of the moduli space of supersingular K3 surfaces in terms of the sheaf $\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p$. Let $1 \leq \sigma_0 \leq 8$ be an integer, fix a supersingular K3 lattice Λ^{σ_0+1} of Artin invariant $\sigma_0 + 1$ and set $\widetilde{\Lambda}^{\sigma_0+1} = \Lambda^{\sigma_0+1} \oplus U_2$. Pick a primitive isotropic vector $v = (p, l, s) \in p\widetilde{\Lambda}^{\sigma_0+1}$, let $\Lambda^{\sigma_0} = v^\perp/v$, and set $\widetilde{\Lambda}^{\sigma_0+1} = \Lambda^{\sigma_0} \oplus U_2(p)$. To a Λ^{σ_0+1}-marked supersingular K3 surface X we may associate the moduli space $\mathscr{M}_X(v)$, whose extended Néron-Severi group has an induced marking by $\widetilde{\Lambda}^{\sigma_0+1}$. This is reflected at the level of period domains by a diagram

$$\begin{tikzcd} \overline{M}^{\langle v\rangle}_{\Lambda_0^{\sigma_0+1}} \arrow[rr, "\sim"] \arrow[dr, "\pi_v"'] & & \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0^{\sigma_0+1}} \arrow[dl, "\pi_e"] \\ & \overline{M}_{\Lambda_0^{\sigma_0}} & \end{tikzcd} \tag{5.1.11.1}$$

This shows that the crystalline period domain for marked supersingular K3 surfaces of Artin invariant $\leq \sigma_0+1$ is covered by open subsets that are isomorphic to the sheaf of groups $(\mathbf{R}^2\pi_*^{\mathrm{fl}}\boldsymbol{\mu}_p)^o$ over the crystalline period domain for supersingular K3 surfaces of Artin invariant $\leq \sigma_0$. With a little extra work, we may obtain a statement at the level of moduli spaces.

Definition 5.1.12. If $W \subset V$ is a subspace, we let $U_V^W \subset \overline{M}_V$ be the locus of characteristic subspaces $K \subset V \otimes k$ such that $K \cap V \subset W$. In particular, $U_V^0 = U_V$ is the open subset of strictly characteristic subspaces.

Lemma 5.1.13. *For any totally isotropic subspace $W \subset \Lambda_0$, the morphism $P_\Lambda \to \overline{M}_{\Lambda_0}$ admits a section over the open subset $U_{\Lambda_0}^W \subset \overline{M}_{\Lambda_0}$.*

Proof. A section over an open subset $U \subset \overline{M}_{\Lambda_0}$ corresponds to a choice of ample cone for the restriction of the universal characteristic subspace K_{Λ_0} to U. Let $W \subset \Lambda_0$ be a totally isotropic subspace. Define

$$\Lambda_W = \{x \in \Lambda \otimes \mathbf{Q} | px \in \Lambda \text{ and } \overline{px} \in W\}$$

This is a supersingular K3 lattice, and we have $\Lambda \subset \Lambda_W$. For any (possibly non-closed) point $s \in U_{\Lambda_0}^W$, we have that $K(s) \cap \Lambda_0 \subset W$, so there are inclusions

$$\Lambda \subset \Lambda(s) \subset \Lambda_W$$

It follows that

$$V_{\Lambda_W} \subset V_{\Lambda(s)} \subset V_\Lambda$$

Pick an ample cone $\alpha \in C_{\Lambda_W}$. For each s, let $\alpha_s \in C_{\Lambda(s)}$ be the unique connected component containing the image of α under the above inclusion. These ample cones are compatible, and hence we get an ample cone for the restriction of K_{Λ_0} to $U^W_{\Lambda_0}$. In other words, we obtain for each $\alpha \in C_{\Lambda_W}$ a diagram

$$\begin{tikzcd} & P_\Lambda \arrow[d] \\ U^W_{\Lambda_0} \arrow[ur, "{[\alpha]}"] \arrow[r, hook] & \overline{M}_{\Lambda_0} \end{tikzcd}$$

which in particular implies the result. $\square$

Combining this with our observation about the period domains, we obtain the following statement.

Proposition 5.1.14. *The moduli spaces $S_{\Lambda^{\sigma_0+1}}$ and $\mathscr{S}_{\Lambda^{\sigma_0}}$ are birational.*

Proof. By Lemma 5.1.13, the morphisms $\mathscr{P}_{\widetilde{\Lambda}^{\sigma_0+1}} \to \overline{M}^{\langle e\rangle}_{\widetilde{\Lambda}_0^{\sigma_0+1}}$ and $P_{\Lambda^{\sigma_0+1}} \to \overline{M}_{\Lambda^{\sigma_0+1}}$ admit sections defined on open subsets of the target. A section of an étale morphism is an open immersion, so the isomorphism $\overline{M}_{\Lambda_0^{\sigma_0+1}} \xrightarrow{\sim} \overline{M}_{\widetilde{\Lambda}_0^{\sigma_0+1}}$ (5.1.11.1) induces the desired birational correspondence. $\square$

This iterated bundle structure has the following pointwise consequence.

Proposition 5.1.15. *Every supersingular K3 surface can be obtained from the unique supersingular K3 surface X of Artin invariant $\sigma_0 = 1$ by iteratively taking moduli spaces of twisted sheaves. That is, for any supersingular K3 surface Y, there exists a sequence $v_1, \dots, v_n$ of Mukai vectors and $\alpha_1, \dots, \alpha_n$ of cohomology classes such that $v_i \in \widetilde{N}(X_i)$ and $\alpha_i \in \mathrm{H}^2(X_i, \boldsymbol{\mu}_p)$, where $X_1 = X$ and*

$$X_{i+1} = M_{(X_i, \alpha_i)}(v_i)$$

such that $X_n = Y$ (in fact, we may take $n = \sigma_0(Y) - 1$).

Finally, we will show that every twistor family is a moduli space of twisted sheaves on a universal twistor family. In particular, this will remove the assumptions on v in Theorem 4.2.8. To treat the Artin-Tate case, we will use the following lemma.

Lemma 5.1.16. *Suppose that $p \geq 5$ and that $v \in \widetilde{N}(\mathscr{X})_0$ is an isotropic vector with $v.e = 0$, where $e = (0, 0, 1)$. If v is not a multiple of e, then there exists an elliptic fibration $X \to \mathbf{P}^1$ with smooth fiber E and an integer s such that the image of the vector $(0, E, s) \in p\widetilde{N}(\mathscr{X})^* \subset \widetilde{N}(\mathscr{X})$ in $\widetilde{N}(\mathscr{X})_0$ is a scalar multiple of v.*

Proof. Consider any lift of v to an isotropic vector $(0, l, s) \in p\widetilde{N}(\mathscr{X})^* \subset \widetilde{N}(\mathscr{X})$. We will modify this vector while ensuring that its image in $\widetilde{N}(\mathscr{X})_0$ remains a non-zero scalar multiple of v. We claim that we may assume that l is primitive. Suppose that $l = nl'$ for some primitive l' and some n, which is necessarily invertible modulo p. We pick an integer s' such that $ns' \equiv s$ modulo p. Replacing $(0, l, s)$ with $(0, l', s')$ achieves our goal. Next, after possibly multiplying by -1, we may arrange so that l is in the closure of the positive cone, and hence is effective. By Proposition 3.3.10 of [21], we find a sequence of (-2)-curves C_i such that

$$s_{C_n} \circ \ldots \circ s_{C_1}(l)$$

is linearly equivalent to a smooth fiber $E \subset X$ of an elliptic fibration $f\colon X \to \mathbf{P}^1$. Because the reflections s_{C_i} act trivially on $\widetilde{N}(X)_0$, the Mukai vector $(0, E, s)$ also lifts v. $\square$

Theorem 5.1.17. *If $p \geq 5$, then every twistor family is a moduli space of twisted sheaves on (an open subset of) a connected component of a universal family of $\boldsymbol{\mu}_p$-gerbes on some supersingular K3 surface. If $p \geq 3$, the same is true if $v.e \neq 0$ (with the notation of Theorem 4.2.8).*

Proof. It is likely possible to give a direct proof for this result along the lines of Theorem 4.2.8. Allowing ourselves to use the twisted crystalline Torelli theorem, the proof is greatly simplified. Choose a lift $x \in \widetilde{\Lambda}$ as in Theorem 4.2.8 (in the positive rank case), or with the properties of Lemma 5.1.16 (in the Artin-Tate case). The remainder of the proof follows Theorem 4.2.8, except that we deduce the existence of the desired isomorphism from the Torelli theorem. $\square$

5.2 Unirationality of supersingular K3 surfaces

In this section we give some applications of the Artin-Tate twistor families studied in Section 4.3. Our first result is that every elliptic fibration on a supersingular K3 surface admits a purely inseparable multisection of degree p. We apply this result to show that unirationality can be transferred along Artin-Tate twistor families associated to elliptic fibrations. By induction on the Artin invariant, this implies that all supersingular K3 surfaces in characteristic $p \geq 5$ are unirational. This gives another proof of the main result of [33]. Our method extends to show that supersingular K3 surfaces in characteristic 3 with Artin invariant $\sigma_0 \geq 7$ are unirational, which was previously unknown. The remaining cases of $p = 2$ or $p = 3$ and $\sigma_0 \leq 6$ have long been known by results of Rudakov and Shafarevich on quasi-elliptic fibrations (see [43], and Theorem 5.2.7 and Proposition 5.2.8 below). It follows that supersingular K3 surfaces in any characteristic are unirational, verifying a conjecture of Artin, Rudakov, Shafarevich, and Shioda. For a history of this conjecture, we refer to the introduction of [33].

Remark 5.2.1. In [33], Liedtke uses deformations of the trivial torsor over the formal Brauer group to produce purely inseparable multisections, and eventually purely inseparable isogenies relating supersingular K3 surfaces of different Artin invariants. Unirationality of all supersingular K3 surfaces is then shown to follow from the unirationality of a single supersingular K3 surface. Where our method differs from Liedtke's is that we construct families of torsors over the group scheme $(\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p)^o \cong \mathbf{A}^1$, instead of over the formal Brauer group $\widehat{\mathrm{Br}(X)} \cong \operatorname{Spec} k[[x]]$. Our families recover Liedtke's by restricting to the complete local ring at the identity and applying the natural isomorphism

$$(\widehat{\mathbf{R}^2\pi^{\mathrm{fl}}_*\boldsymbol{\mu}_p})^o \xrightarrow{\sim} \widehat{\mathrm{Br}(X)}$$

of Lemma 2.1.10. This difference allows us to considerably simplify portions of the strategy of [33]. Notably, we do not need to use any strong results on the global structure of the moduli space of supersingular K3 surfaces, such as the crystalline Torelli theorem or the potential good reduction of supersingular K3 surfaces (Theorem 5.1.11). In particular, the results of this section are independent of those of Section 5.1.

We first consider families of torsors under an elliptic curve in a somewhat general situation. The key result that we will need is that a nontrivial deformation of the trivial torsor admits a purely inseparable multisection. To state this precisely, we fix the following notation.

1. k is an algebraically closed field of characteristic p,

2. L/k is a finitely generated regular field extension of k,

3. E is an elliptic curve over L with identity section $\mathbf{0}$,

4. R is a DVR over k with residue field k, field of fractions K, and uniformizer x,

5. $S(R) = (L \otimes_k R)_{(x)}$. Note that $S(R)$ is in fact a dvr with residue field L, which follows from the fact that $L \otimes_k R$ is a domain (L is regular over k), and the fact that R has residue field k, so that $S(R)/xS(R) \cong L$, making x generate a maximal ideal.

6. η is the generic point of $\operatorname{Spec} S(R)$, and

7. η_∞ is the generic point of $\operatorname{Spec} S(R) \otimes_R \overline{K} = \operatorname{Spec} L \otimes_k \overline{K}$.

In our application, we will take $R = k[[x]]$ and $L = k(t)$.

Definition 5.2.2. If A is a k-algebra, we say that a *family of E-torsors over L parametrized by A* is an $E \otimes_k A$-torsor over $L \otimes_k A$.

The following result is due to Max Lieblich, and we refer to Section 5.2 of [7] for the proof.

Proposition 5.2.3. *If $C \to \operatorname{Spec}(R \otimes_k L)$ is a family of E-torsors over L parametrized by R such that*

1. the curve C_{η_∞} has index p over $\kappa(\eta_\infty)$ (and thus has order p in $\mathrm{H}^1(\eta_\infty, E_{\eta_\infty})$) and

2. there is an isomorphism of E-torsors

$$\tau\colon E \xrightarrow{\sim} C_0,$$

where $C_0 = C \otimes_R R/tR$,

then there is a point P on C_{η_∞} that is purely inseparable of degree p over η_∞.

We now apply Proposition 5.2.3 to study elliptic fibrations on supersingular K3 surfaces.

Theorem 5.2.4. *If X is a supersingular K3 surface, then any elliptic fibration $X \to \mathbf{P}^1$ admits a purely inseparable multisection of degree p.*

The strategy is as follows. Suppose that $X \to \mathbf{P}^1$ is non-Jacobian. Using Proposition 5.2.3, we construct a certain twistor family $Y \to \mathbf{A}^1$ whose fibers are equipped with compatible elliptic structures, and show that there are closed points $x_0, x_1 \in \mathbf{A}^1$ such that Y_{x_0} is isomorphic to X, and Y_{x_1} is isomorphic to the Jacobian of X. In both cases, the isomorphism respects the elliptic structure. In other words, we construct a family of torsors under the Jacobian of X which interpolates between X and the trivial torsor. Because this family contains the trivial torsor, Proposition 5.2.3 shows that its generic fiber admits a purely inseparable multisection of degree p. We then specialize this multisection to $Y_{x_0} \cong X$ to get the result.

Proof of Theorem 5.2.4. Note that if $X \to \mathbf{P}^1$ admits a section, say $\sigma\colon \mathbf{P}^1 \to X$, then $F_X \circ \sigma$ is a purely inseparable multisection of degree p. Suppose that X is non-Jacobian, and let $J \to \mathbf{P}^1$ be its Jacobian fibration. Consider the universal family $\mathscr{J} \to \mathbf{A}^1$ of $\boldsymbol{\mu}_p$-gerbes on J that contains the trivial gerbe, and form the relative moduli space $\mathscr{R}_{\mathscr{J}}(1) \to \mathbf{A}^1$. Note that the class of the gerbe $\mathscr{R}_{\mathscr{J}}(1) \to R_{\mathscr{J}}(1)$ is trivial. Its coarse space gives a family

$$R_{\mathscr{J}}(1) \to \mathbf{A}^1$$

of K3 surfaces. The fibers of this family come equipped with compatible elliptic structures, giving a morphism $\mathbf{R}_{\mathscr{J}}(1) \to \mathbf{A}^1 \times \mathbf{P}^1$ over $\mathbf{A}^1$. Restricting to the generic point η of $\mathbf{P}^1$, we find a family

$$(R_{\mathscr{J}}(1))_\eta \to \mathbf{A}^1 \times \eta \tag{5.2.4.1}$$

of genus 1 curves over $\eta = \operatorname{Spec} k(t)$ parametrized by $\mathbf{A}^1$. To avoid confusion, we will set $\mathbf{A}^1 = \operatorname{Spec} k[x]$.

We make some observations about this family. First, the fibers admit compatible actions by J_η. Applying Proposition 4.3.8 to the elliptic surface $J \to \mathbf{P}^1$, we see that each fiber is an étale form of J_η, so $(R_{\mathscr{J}}(1))_\eta$ has the structure of a family of J_η-torsors over $k[x]$, in the sense of Definition 5.2.2. Furthermore, the fiber over the origin $0 \in \mathbf{A}^1(k)$ is the trivial torsor, and the generic fiber is non-trivial (this follows from Proposition 4.3.8 and the fact that the Brauer class of $\mathscr{J} \to \mathbf{A}^1$ is generically non-trivial). Finally, we claim that this family contains the torsor X_η as a fiber. To see this, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{U}^2(J, \boldsymbol{\mu}_p) \arrow[d, two heads] \arrow[dr, two heads] & \\
Ш(k(t), \mathrm{Jac}(J_{k(t)})) \arrow[r, "\sim"'] & \mathrm{Br}(J)
\end{tikzcd}$$

defined in Proposition 4.3.8 (we recall that the diagonal arrow is surjective by Theorem 2.1.13). The elliptic surface $X \to \mathbf{P}^1$ corresponds to an element of $Ш(k(t), J_{k(t)})$. Because $J \to \mathbf{P}^1$ is

Jacobian, the horizontal arrow is an isomorphism, and therefore the vertical arrow is surjective. It follows that there is a $x_0 \in \mathbf{A}^1(k)$ such that $(R_{\mathscr{J}}(1)_\eta)_{x_0}$ is isomorphic to X_η as a J_η-torsor.

Let $R = k[[x]]$ be the completion of the local ring of $\mathbf{A}^1 = \operatorname{Spec} k[x]$ at the origin, and let $L = k(t)$. The restriction of the family (5.2.4.1) to $\operatorname{Spec} R \times \eta = \operatorname{Spec}(R \otimes_k L)$ gives a morphism

$$R_{\mathscr{J}}(1)_{\eta \times \operatorname{Spec} R} \to \operatorname{Spec}(R \otimes_k L)$$

We check that this family satisfies the conditions of Proposition 5.2.3. Fix an algebraic closure $K \subset \overline{K}$ of $K = k(x)$. As we have observed, the elliptic surface $R_{\mathscr{J}}(1)_{\overline{K}} \to \mathbf{P}^1 \times \overline{K}$ is non-Jacobian, and the corresponding torsor is necessarily of index p. We also know that the fiber over the closed point $0 \in \operatorname{Spec} R$ is the trivial torsor. Therefore, by Proposition 5.2.3, $R_{\mathscr{J}}(1)_{\eta_\infty} \to \eta_\infty$ admits a purely inseparable point P of degree p, where η_∞ is the generic point of $\operatorname{Spec} L \otimes_k \overline{K}$. We may realize P over some finite extension K' of K. Let A be the normalization of R in K'. Taking the closure of P in the base change of our family along the map $\operatorname{Spec} A \to \mathbf{A}^1$ gives a multisection Σ of the family $\mathbf{R}_{\mathscr{J}}(1)_\eta \to \eta \times \operatorname{Spec} A$ of torsors parametrized by A.

Consider the point $x_0 \in \mathbf{A}^1(k)$ such that $(R_{\mathscr{J}}(1)_\eta)_{x_0}$ is isomorphic to X_η as a J_η-torsor. This lifts to a closed point of $\operatorname{Spec} A$, which we will also denote by x_0. Restricting Σ, we find a multisection

$$\Sigma_{x_0} \subset R_{\mathscr{J}}(1)_{x_0}$$

If Σ_{x_0} were non-integral (e.g., a p-th power in the Picard group) then Σ would be non-integral because the specialization map

$$\operatorname{Pic}(R_{\mathscr{J}}(1)_{\eta_A}) \to \operatorname{Pic}(R_{\mathscr{J}}(1)_{x_0})$$

is an isomorphism, where η_A is the generic point of $\operatorname{Spec} A$. Thus, Σ_{x_0} is integral, giving a point of $R_{\mathscr{J}}(1)_{\eta\, x_0}$ that is purely inseparable of degree p over η. Taking its closure gives a purely inseparable multisection of the elliptic surface $R_{\mathscr{J}}(1)_{x_0} \to \mathbf{P}^1$. As $R_{\mathscr{J}}(1)_{x_0}$ and X were isomorphic as elliptic surfaces, this completes the proof. $\square$

The existence of a purely inseparable multisection has the following consequence.

Lemma 5.2.5. *If $X \to \mathbf{P}^1$ is an elliptic fibration on a K3 surface that admits a purely inseparable multisection $\Sigma \subset X$ of degree p, then there exists a purely inseparable rational map*

$$J(X) \dashrightarrow X$$

of degree p^2. In particular, if $J(X)$ is unirational then so is X.

Proof. An inseparable multisection of degree p gives a diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \mathbf{P}^1 \arrow[ur, "\Sigma"] \arrow[r, "F_{\mathbf{P}^1}"] & \mathbf{P}^1 \end{tikzcd}$$

where the horizontal morphism is the absolute Frobenius. In particular, $X^{(p/\mathbf{P}^1)} \to \mathbf{P}^1$ admits a section. Let η be the generic point of $\mathbf{P}^1$. The generic fiber $X_\eta^{(p/\mathbf{P}^1)} \to \eta$ is a smooth genus 1 curve with a section, and hence is isomorphic to its own Jacobian. We therefore find isomorphisms

$$(J(X)^{(p/\mathbf{P}^1)})_\eta \cong J(X_\eta^{(p/\mathbf{P}^1)}) \cong X_\eta^{(p/\mathbf{P}^1)}$$

and hence a birational correspondence $J(X)^{(p/\mathbf{P}^1)} \dashrightarrow X^{(p/\mathbf{P}^1)}$. The composition

$$J(X) \xrightarrow{F_{J(X)/\mathbf{P}^1}} J(X)^{(p/\mathbf{P}^1)} \dashrightarrow X^{(p/\mathbf{P}^1)} \xrightarrow{W_{X/\mathbf{P}^1}} X$$

is purely inseparable of degree p^2. □

Proposition 5.2.6. *Let k be an algebraically closed field of characteristic $p > 0$. Suppose that either*

1. *$p \geq 5$ and $2 \leq \sigma_0 \leq 10$, or*
2. *$p = 3$ and $7 \leq \sigma_0 \leq 10$.*

If X is a supersingular K3 surface over k of Artin invariant σ_0, then X admits a non-Jacobian elliptic fibration.

Proof. Fix such a supersingular K3 surface X. Because $\sigma_0 \geq 2$, by [42] there exists an orthogonal decomposition $N(X) = N' \oplus U_2(p)$. Let $l \in U_2(p)$ be a primitive nonzero isotropic vector. After possibly multiplying by -1, we may ensure l is effective. We find a sequence of (-2)-curves C_i such that

$$s_{C_n} \circ \ldots \circ s_{C_1}(l)$$

is linearly equivalent to an integral curve that is a fiber of some fibration $X \to \mathbf{P}^1$ (see Proposition 2.3.10 of [19]). It is shown in [43] that if we are in either case (1) or case (2), then X does not admit any quasi-elliptic fibration. Thus, $X \to \mathbf{P}^1$ is necessarily an elliptic fibration. Because $l \in pN(X)^*$, $X \to \mathbf{P}^1$ is non-Jacobian. □

In order to treat the remaining cases, we recall some results on quasi-eliptic fibrations from [43].

Theorem 5.2.7. *Let k be an algebraically closed field of characteristic $p > 0$. Suppose that either*

1. *$p = 2$, or*

2. *$p = 3$ and $1 \leq \sigma_0 \leq 6$.*

If X is a supersingular K3 surface over k of Artin invariant σ_0, then X admits a quasi-elliptic fibration.

Proof. This is proved in Section 5 of [43]. See the theorem on page 1496 and Remark 3 on page 1497. $\square$

By a result of Rudakov and Shafarevich (see Section 2, Proposition 2 of [43] on page 1485), a surface possessing a quasi-elliptic pencil is unirational. The proof is short, so we shall give it.

Proposition 5.2.8. *If X is a smooth proper surface over k that admits a quasi-elliptic pencil $X \to \mathbf{P}^1$, then there exists a purely inseparable rational map*

$$\mathbf{P}^2 \dashrightarrow X$$

of degree p. In particular, X is unirational.

Proof. Let us recall some facts on quasi-elliptic pencils from [43]. There is a curve $\Sigma \subset X$ such that the restriction $\Sigma \to \mathbf{P}^1$ is purely inseparable of degree p, and $X - \Sigma \to \mathbf{P}^1$ has smooth generic fiber. Moreover, if η is the generic point of $\mathbf{P}^1$, then $(X - \Sigma)_\eta$ is a quasi-elliptic curve over $\eta = \operatorname{Spec} k(t)$. If such a curve has a rational point, then on the open complement of its unique singular point it is a form of $\mathbf{G}_a$ over $k(t)$. Finally, just as for elliptic curves, a quasi-elliptic curve without a rational point is a torsor under one with a rational point. We have an induced diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \Sigma \arrow[ur] \arrow[r] & \mathbf{P}^1 \end{tikzcd}$$

where the horizontal arrow is purely inseparable of degree p. The base change $X' = X \times_\Sigma \mathbf{P}^1$ admits a pencil $X' \to \Sigma$ with a section. The generic fiber is therefore a rational curve. Because Σ is rational, X' is rational, and the result follows. $\square$

Theorem 5.2.9. *Every supersingular K3 surface over an algebraically closed field of characteristic* $p > 0$ *is unirational.*

Proof. Assume $p \geq 5$. We will induct on the Artin invariant. By [45], the unique supersingular K3 surface of Artin invariant $\sigma_0 = 1$ is unirational. Let X be a supersingular K3 surface with Artin invariant $\sigma_0 \geq 2$. By Proposition 5.2.6, X admits a non-Jacobian elliptic fibration $X \to \mathbf{P}^1$. By Theorem 5.2.4, there is a purely inseparable multi-section $\Sigma \subset X$. The Artin invariant of the Jacobian $J(X)$ is strictly lower than that of X. Thus, by Lemma 5.2.5 and the induction hypothesis we conclude that X is unirational.

By Proposition 5.2.8 and Theorem 5.2.7, the result is proved for $p = 2$, and for $p = 3$ and $1 \leq \sigma_0(X) \leq 6$. We obtain the result for $p = 3$ and $7 \leq \sigma_0(X) \leq 10$ by the same induction as in the previous case. $\square$

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