Topics in Chiral Symmetry on the Lattice

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Abstract

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There currently does not exist a good regulator for chiral gauge theories on the lattice. A number of approaches have been attempted, including domain wall fermions and overlap fermions. It has been shown that these fermions both satisfy the Ginsparg-Wilson (GW) relation, a relation constraining chiral symmetry on the lattice. In this thesis, we will discuss various generalizations of GW fermions, as well as developing a novel (continuum) theory of domain wall fermions which may evade some of the shortcomings of ordinary domain wall fermions in describing chiral gauge theories.

In Chapter 2, we give a general derivation of Ginsparg-Wilson relations for both Dirac and Majorana fermions in any dimension. These relations encode continuous and discrete chiral, parity, and time-reversal anomalies and will apply to the various classes of free-fermion topological insulators and superconductors (in the framework of a relativistic quantum field theory in Euclidean spacetime). We show how to formulate the exact symmetries of the lattice action and the relevant index theorems for the anomalies.

In Chapter 3, we derive the Hamiltonian for a fermion satisfying the GW equation. We work with a solution to the GW equation which is fractional linear in time derivatives. The resulting Hamiltonian is non-local and has ghosts, but is free of doublers and has the correct continuum limit. This construction works in general odd spatial dimensions, and we provide an explicit expression for the Hamiltonian in 1 spatial dimension.

In Chapter 4, the theory of fermions in odd dimensional bulk with radial domain wall mass profile is discussed. Edge states localized near the boundary describe the theory of double-valued Weyl fermions on even dimensional spheres, and may be able to evade the doubling problem present on the lattice for ordinary domain wall fermions.

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DEDICATION

to Grandpa Horii, in loving memory

Chapter 1
INTRODUCTION

The standard model (SM) is made up of Weyl fermions with interactions mediated by gauge bosons. Weyl fermions can come in two chiralities: left- and right- handed. In the electroweak interaction, the $W^{\pm}$ bosons couple only to left-handed particles. Gauge theories in which the left- and right- handed particles transform differently under the gauge group are said to be chiral. Chiral gauge theories (and therefore the SM) are susceptile to gauge anomalies, and there still does not exist an anomaly free non-perturbative regulator for them in general. The lattice is one example of a nonperturbative regulator on which a satisfactory definition of a chiral gauge theory remains elusive. The primary obstruction is the Nielsen-Ninomiya (NN) no-go theorem [1], which states that chiral symmetry is in general impossible to realize on the lattice without sacrificing some desirable qualities, even for vectorlike (i.e. non-chiral) theories. There are various workarounds to this theorem, including domain wall fermions [2] and overlap fermions [3]. Both of these models satisfy the so-called Ginsparg-Wilson (GW) relation [4], which constrains lattice Dirac operators that aim to recover chiral symmetry in the continuum limit, and derive their forms as the gapless edge theories of five dimensional theories which are gapped in bulk.[1]

Recent work in the theory of topological insulators and superconductors has clarified the relationship between anomalous symmetries of massless fermions and extra dimensions[7, 8, 9]. This has motivated the work of Chapter 2, which explores Ginsparg-Wilson relations and overlap operators for theories with anomalous symmetry other than chiral symmetry of Dirac fermions in four dimensions. The work of Chapter 3 regards the construction of a Hamiltonian

[1] Domain wall fermions require an extra dimension to simulate, and satisfaction of the GW relation occurs at the level of an effective edge operator [5]. The overlap operator is technically defined in four-dimensions, but its simulation can still be viewed as introducing an extra dimension [6].

for a theory whose Lagrangian satisfies the Ginsparg-Wilson relation, as generalized by Chapter 2. Finally, Chapter 4 explores the possibility of domain wall fermions for a spherical domain wall, whose latticization may evade the so-called doubling problem, which remains an issue for ordinary domain wall fermions [10, 11]. The goals of this thesis are two-fold: progress toward realization of chiral gauge theories on the lattice, and generally toward a more unified understanding of anomalous symmetries, overlap operators, and edge states of topological insulators and superconductors.

Subsequent chapters have their own self-contained introductions; the remainder of this chapter is dedicated to a high-level overview of chiral symmetry on the lattice relevant to the material in this thesis. There already exist a number of excellent reviews of the subject matter, including Refs. [12, 13, 5]. The author wishes to summarize these reviews so as to contextualize and motivate the work of this thesis.

1.1 Chiral symmetry in the continuum

Recall the continuum Euclidean Dirac operator in d dimensions

$$D = \gamma^\mu(\partial_\mu - iA_\mu), \quad \mu = 1, \cdots, d, \tag{1.1}$$

where A_μ is a gauge field, and the matrices γ^μ satisy the Clifford algebra relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu}. \tag{1.2}$$

For $d = 2k$ even, the Lorentz generators

$$\Sigma_{\mu\nu} = \frac{1}{4i}[\gamma^\mu, \gamma^\nu] \tag{1.3}$$

form a representation of the Euclidean Lorentz (rotation) group $SO(2k)$. With these one may define

$$\gamma_\chi \sim \gamma^1 \cdots \gamma^{2k}, \tag{1.4}$$

which anti-commutes with all the previous γ matrices and squares to unity, so that $\gamma_\chi \sim \gamma^{2k+1}$. Were we to include this matrix in eq. (1.3), we would end up with a representation for

$SO(2k+1)$, with $2k$ more generators $\Sigma_{\mu,2k+1}$, $\mu = 1, \cdots, 2k$. However, we proceed working in $d = 2k$. Let us define $\psi, \overline{\psi}$ as spinors[2] transforming in the representation of $SO(2k)$

$$\psi \to e^{\frac{i}{2}\omega_{\mu\nu}\Sigma_{\mu\nu}}\psi, \quad \overline{\psi} \to \overline{\psi}e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma_{\mu\nu}}, \tag{1.5}$$

we can write down the $SO(2k)$ invariant action

$$S = \int d^d x \overline{\psi} D \psi. \tag{1.6}$$

The spurned γ_χ refuses to stay hidden, however: the action eq. (1.6) has the "axial" or "chiral"[3] $U(1)$ symmetry

$$\psi \to e^{i\alpha\gamma_\chi}\psi, \quad \overline{\psi} \to \overline{\psi}e^{i\alpha\gamma_\chi}, \tag{1.7}$$

or infinitesimally,[4]

$$\delta\psi = i\alpha\gamma_\chi\psi, \quad \delta\overline{\psi} = i\alpha\overline{\psi}\gamma_\chi. \tag{1.8}$$

This follows from the fact that

$$\{\gamma_\chi, D\} = 0, \tag{1.9}$$

which would not be the case in $2k+1$ dimensions, as D would contain a $\gamma_\chi\partial_{2k+1}$. Note, as well, that

$$[\gamma_\chi, \Sigma_{\mu\nu}] = 0, \quad \mu, \nu = 1, ..., d. \tag{1.10}$$

As a result, the spinor representation of the even-dimensional Euclidean Lorentz group SO(2k) is reducible: the spinors decompose into eigenstates of the chirality matrix $\gamma_\chi = \pm 1$. We denote the chiral projection operators

$$P_\pm = \frac{1 \pm \gamma_\chi}{2}, \tag{1.11}$$

[2]Eventually we will have to analytically continue to Minkowski spacetime, and if we take the mostly plus metric, this constrains $\psi^\dagger o \gamma^0 = \overline{\psi}$.

[3]These words will be used interchangeably in this context

[4]In later chapters, sometimes the α is removed from $\delta\psi$, e.g. transformations are denoted $\psi + \alpha\delta\psi$, with $\delta\psi = i\gamma_\chi\psi$.

about which one finds

$$P_+P_- = 0, \quad P_\pm^2 = P_\pm = P_\pm^\dagger, \quad P_+ + P_- = 1, \quad \gamma_\chi P_\pm \psi = \pm P_\pm \psi. \tag{1.12}$$

Therefore the spinor representation decomposes into left- and right-handed eigenspaces, which we denote

$$\psi_- = P_- \psi, \; \psi_+ = P_+ \psi, \quad \overline{\psi}_- = \overline{\psi} P_+, \; \overline{\psi}_+ = \overline{\psi} P_-, \tag{1.13}$$

with $-$ denoting left-handed fermions and $+$ denoting right-handed fermions. The $\pm$ switch for $\overline{\psi}$ in eq. (1.13) is that D acts from the right on $\overline{\psi}$, and $P_+ D = D P_-$. We can write the chiral transformation eq. (1.7) as

$$\psi_+ \to e^{i\alpha} \psi_+, \quad \psi_- \to e^{-i\alpha} \psi_-, \tag{1.14}$$

$$\overline{\psi}_+ \to \overline{\psi}_+ e^{-i\alpha}, \quad \overline{\psi}_- \to \overline{\psi}_- e^{i\alpha}. \tag{1.15}$$

Associated to chiral symmetry is the axial current

$$J_A^\mu = \overline{\psi} \gamma^\mu \gamma_\chi \psi, \tag{1.16}$$

which, as derived from Noether's theorem [14], is classically conserved:

$$\partial_\mu J_A^\mu = 0. \tag{1.17}$$

Note from eq. (1.14) that any term of the form

$$\overline{\psi}_- \psi_+, \quad \overline{\psi}_+ \psi_- \tag{1.18}$$

is not an invariant under chiral symmetry, and terms like $\overline{\psi}_- \psi_-$ are not Lorentz invariant. In general under the infinitesimal transformation eq. (1.8) we find

$$\delta(\overline{\psi} \psi) = 2i \overline{\psi} \gamma_\chi \psi, \tag{1.19}$$

so that masses violate chiral symmetry. In the quantum theory with massless action in eq. (1.6), chiral symmetry is violated by quantum effects, and the current eq. (1.17) is not conserved. This symmetry breaking is known as the chiral anomaly. This is connected to eq. (1.19), and has a number of important implications for the physics of chiral fermions. The violation of chiral symmetry at loop order in Feynman diagrams (i.e. quantum effects) is responsible for neutral pion decay, while the conservation of chiral symmetry at the classical level is responsible for protecting fermions from additive mass renormalization. Therefore, any satisfactory theory of chiral symmetry on the lattice ought to have the same properties. As it turns out, Ginsparg-Wilson fermions do [15], as we shall explore shortly.

1.2 Chiral anomaly in the continuum

One may wish to write down the quantum theory of massless fermions. We begin with a formal expression for the path integral:

$$Z = \int d\overline{\psi} d\psi e^{-S}. \tag{1.20}$$

Under the infinitesimal chiral transformation in eq. (1.8), the measure varies as [16]:

$$d\psi d\overline{\psi} \to d\psi d\overline{\psi} \left[1 - i\alpha \int d^4x \sum_k \psi_k(x) \gamma_5 \psi_k(x) \right], \tag{1.21}$$

where ψ_k is any basis of wave-functions, usually taken to be the eigenbasis of the Dirac operator D. Note that this term can be written $\mathrm{Tr}\,\gamma_5$, but without the introduction of a cutoff scale in the measure $d\psi d\overline{\psi}$ this trace cannot be evaluated. After regularization (e.g. heat-kernel regularization [16]), one finds in the presence of background gauge fields A_μ that

$$\mathrm{Tr}\,\gamma_5 = -\int d^4x \frac{1}{16\pi^2} \mathrm{Tr}\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{1.22}$$

By noting that $\{\gamma_5, D\} = 0$, any non-zero modes of D come paired and cannot contribute to the trace. Therefore the left-hand side of eq. (1.22) is $n_+ - n_-$, where $n_\pm$ denotes the number of zero-modes of D with chirality ± 1. The right hand side is a topological winding number ν, known as the instanton number of the gauge field configuration A_μ. Thus eq. (1.22) may be written:

$$n_+ - n_- = \nu. \tag{1.23}$$

This is an example of the celebrated Atiyah-Singer index theorem [17].

Separate from (but related to) the chiral anomaly are chiral gauge anomalies. Since left- and right- handed fermions transform differently under the gauge group in chiral gauge theories, the symmetry breaking in the measure is not even gauge invariant, unless there is some cancellation of the anomalies. See Ref. [5] for details; in the remainder of this thesis, we will only be concerned with details of the chiral anomaly and not the anomalies of chiral gauge theories.

1.3 Chiral symmetry on the lattice

Lattice theories are regulated from conception. The continuum variable x is replaced by discrete points $x = a\hat{n}$, where $\hat{n}$ denotes a vector of integers, and a denotes an arbitrarily chosen length scale called the lattice spacing. Lattice theories at finite lattice spacing are thus incapable of describing physics at energy scales higher than the inverse lattice spacing $1/a$. The continuum limit corresponds to taking $a \to 0$, and at finite lattice spacing one expects physics at mass scales much lower than $1/a$ to resemble the continuum. Theories can then be simulated on lattice for finite (but small) lattice spacings via Monte-Carlo methods [18].

In writing down a theory on the lattice which one hopes has the right continuum limit, one is quite free to define any lattice Dirac operator (matrix) D_{nm} they like, as long as it has the properties they desire in the limit $a \to 0$. In space, one would have

$$S = \sum_{n,m} \overline{\psi}_n D_{nm} \psi_m. \tag{1.24}$$

In momentum space, the effect of the lattice spacing is to restrict the allowed momentum values to the Brillouin zone $[-\pi/a, \pi/a]$:

$$\psi(x) = \int_{-\pi/a}^{\pi/a} e^{ipx} \tilde{\psi}(p). \tag{1.25}$$

The lattice has both forward and backward derivatives, denoted, respectively:[5]

$$\partial_\mu \psi(x) = \frac{\psi(x + a\hat{\mu}) - \psi(x)}{a}, \quad \partial^*_\mu \psi(x) = \frac{\psi(x) - \psi(x - a\hat{\mu})}{a}, \tag{1.26}$$

where $\hat{\mu}$ denotes a unit vector with $\mathbf{1}$ in the μth slot and $\mathbf{0}$ otherwise.

It is important to note that $\partial_\mu \psi(x)$ and $\partial^*_\mu \psi(x)$ are completely independent. In order to get a Dirac operator which is Hermitian (up to order a), we must at least take their linear combination. One could then imagine defining the free Dirac operator via the ansatz

$$D_0 = \frac{1}{2}\left(\partial_\mu + \partial^*_\mu\right). \tag{1.27}$$

Under the transformation eq. (1.8), the action eq. (1.24) is invariant for $D = D_0$. However, it is easy to see in momentum space and $d = 1$ that $D_0(p) \sim \sin ap$. It thus has the correct dispersion relation at small values of the momentum, but introduces a pole in the propagator $D_0^{-1}(p)$ at $p = \pi/a$. Poles in the propagator correspond to the introduction of massless particles in Green's functions $\langle\overline{\psi}(x)\psi(y)\rangle$, so the number of particles is doubled in $d = 1$. In general d dimensions, one finds doublers at all corners of the Brillouin zone, and thus finds 2^d flavors of fermion where they started with one. Therefore, a common choice is to augment D_0 with a so-called Wilson term[19], defining the Wilson-Dirac operator:

$$D_W = \frac{1}{2}\left(\partial_\mu + \partial^*_\mu - a\partial^*_\mu \partial_\mu\right). \tag{1.28}$$

The term quadratic in momentum serves to remove the poles in the corners of the Brillouin zone, but at a price: the Wilson term $a\partial^*_\mu \partial_\mu$ violates chiral symmetry explicitly. Thus, the Wilson-Dirac operator finds widespread use, but not in the cases where chiral symmetry is important.

Another consequence of fermion doubling is the following: if one attempts to study a theory of Weyl fermions with the operator in eq. (1.28), e.g. if one writes down an action

[5] wherever these operators appear, one may replace them with the gauge covariant derivative via insertion of appropriate gauge link variables, e.g. $\psi(x) \to U(x, x + a\hat{\mu})\psi(x)$ in ∂_μ, c.f. Ref. [5] or any text on lattice field theory [18]. In most cases throughout this thesis, the correct chiral symmetry in the free theory is emphasized.

with $D = P_R D_W P_L$ for fermions satisfying $P_R\psi = \psi$, one finds [5]

$$D = \frac{1}{2} P_+ \gamma^\mu \left(\partial^*_\mu + \partial_\mu \right). \tag{1.29}$$

Once again, this suffers the same fate at extremal values of the momenta that eq. (1.27) disk, and doubles the Weyl fermions.

1.4 Domain wall fermions

An early suggestion [2] for simulating chiral gauge theories is to start in $2k+1$ dimensions, with coordinates (x^μ, s), with $\mu = 1, ..., 2k$, and denoting $\gamma^{2k+1} = \gamma^s$. We then add a spatially varying mass with a domain wall profile:

$$m(s) = \begin{cases} -m & s < 0, \\ m & s > 0. \end{cases} \tag{1.30}$$

Starting in the continuum, we may write down the Dirac operator

$$D = \gamma^\mu \partial_\mu + \gamma^s \partial_s + m(r). \tag{1.31}$$

In the limit $m \to \infty$, one finds Weyl modes with chirality $\gamma_s = +1$ bound to the domain wall at $s = 0$ with profile $e^{-m|s|}$, which can be described by the four-dimensional action

$$D = P_- \gamma^\mu \partial_\mu P_+. \tag{1.32}$$

When one moves to a system of finite size in the s direction with antiperiodic boundary conditions, one finds a wall with localized fermions of one chirality, and an antiwall with fermions of the opposite chirality, preventing the definition of a chiral gauge theory [20, 10]. A recent paper suggests avoiding the issue entirely by working on a disc [21], simply connecting the fermion on one domain wall with its antiwall. In Chapter 4, we generalize this method to spheres, where the bulk Lorentz symmetry may be more manifest.

1.5 Ginsparg-Wilson fermions and the overlap solution

The fermion doubling problem and issues with chiral symmetry on the lattice can be summarized via the Nielsen-Ninomiya theorem [22, 15], which states that there is no local doubler-free Dirac operator on the lattice in four dimensions with the correct continuum limit, and which satisfies

$$\{\gamma_5, D\} = 0. \tag{1.33}$$

Ginsparg and Wilson [4] noted that instead of demanding eq. (1.33), one finds by block-spin averaging a continuum theory with chiral symmetry, the resulting lattice theory is described by an operator D constrained by the relation

$$\{\gamma_5, D\} = aD\gamma_5 D. \tag{1.34}$$

This is known as the Ginsparg-Wilson relation; it states that chiral symmetry is violated at the order of the lattice spacing. In the following chapters, we discuss a generalization of this relation to different symmetries, and a Hamiltonian for a theory satisfying the GW relation.

The first solution to eq. (1.34) was found by Neuberger and Naranayan[23], which begins with a five dimensional "time"-dependent Hamiltonian

$$H(s) = \gamma_5(D_W + m(s)), \quad m(s) = \begin{cases} -m & s < 0, \\ m_2 & s > 0. \end{cases} \tag{1.35}$$

where D_W denotes the Wilson-Dirac operator in eq. (1.28), s labels the fifth coordinate, and $m, m_2 > 0$. They derive a path integral, $\det D$, as the overlap of the ground state in the far past ($s \to -\infty$) with the ground state in the far future ($s \to \infty$), with D the "overlap" operator:[6]

$$D = \frac{1}{2}(1 + V), \tag{1.36}$$

[6] for simplicity taking $a = 1$

where

$$V = \frac{D_W - m}{\sqrt{(D_W - m)^\dagger (D_W - m)}}. \tag{1.37}$$

It is immediately obvious that V is unitary and

$$\gamma_5 V \gamma_5 = V^\dagger. \tag{1.38}$$

Remarkably (and trivially), one may see that any operator of the form in eq. (1.36) which sastifies eq. (1.38) *also* satisfies the GW relation. A crucial observation about the operator described by eq. (1.36) with V in eq. (1.37) is that it reproduces the anomaly correctly; i.e. one obtains the lattice version of the index theorem in eq. (1.23). The relationship between the anomaly, overlap operator, and GW relation was clarified by Lüscher in Refs. [15, 5]: the action

$$S = \overline{\psi} D \psi \tag{1.39}$$

with D the aforementioned overlap has the symmetry[7]

$$\delta\psi = -\gamma_5 V \psi, \quad \delta\overline{\psi} = \gamma_5 \psi, \tag{1.40}$$

which tends toward chiral symmetry in the continuum limit. The existence of this symmetry follows from eq. (1.38), and we refer to eq. (1.40) as a Lüscher symmetry. Let us now consider the effect of this tranformation on quantum expectation values. The variation of the measure[8] $d\psi d\overline{\psi}$ under the transformation in eq. (1.40) is

$$d\psi d\overline{\psi} \to -\left(\mathrm{Tr}\,\gamma_5 D\right) d\psi d\overline{\psi}. \tag{1.41}$$

Note the similarities with eq. (1.22). The variation in eq. (1.41) can be evaluated [15] to find

$$-\mathrm{Tr}\,\gamma_5 D = \mathbf{2} \times \mathrm{index}(D), \tag{1.42}$$

[7] this symmetry is not unique, as we explore in later chapters.

[8] it should be noted ψ here is used to a lattice variable in this section and a continuous variable in the previous section

i.e., the lattice version of what is in the continuum $\mathrm{Tr}\,\gamma_5$. This allows one to define the instanton number in a lattice gauge field configuration. This relies only on the spectral properties of the operator of the form in eq. (1.36) satisfying eq. (1.38) and not the specific form in eq. (1.37). Note what has happened here: chiral symmetry is classicaly exact in the continuum with chiral anomaly given by eq. (1.23), and Lüscher symmetry is classically exact on the lattice but violated by the variation in the (well-defined) path integral measure. It is therefore appealing to refer to Lüscher symmetry as the lattice version of chiral symmetry [15]. Furthermore, the properties of Lüscher symmetry produce the same desirable features of the chiral anomaly, such as protection of masses from additive renormalization and chiral symmetry violation in quantum effects.

Since GW fermions (at least in the form of eq. (1.36)) have these symmetries, they are very desirable candidates for the realization of chiral symmetry (or potentially any anomalous symmetry) on the lattice.

Chapter 2
GENERALIZED GINSPARG-WILSON RELATIONS

2.1 Introduction

The GW relations govern how massless lattice fermions without doublers can optimally realize anomalous continuum symmetries [4, 3, 25, 15]. They were originally derived for describing massless Dirac fermions with chiral symmetries in even spacetime dimensions, while analogous relations were posited for a massless Dirac fermion in three dimensions with a parity anomaly [26]. Lattice operators which satisfy these relations realize anomalous symmetries in the "best" possible way: the fermion propagator respects the symmetry at any nonzero spacetime separation, and as in the continuum, the lattice action possesses an exact, nearly local form of the symmetry [15], which is therefore respected by the Feynman rules in perturbative calculations. On the other hand, the lattice integration measure is not invariant under this "Lüscher symmetry", and the resultant Jacobian in the lattice theory correctly reproduces the continuum anomaly expressed in terms of the index of the fermion operator. Here we give a unified derivation of such relations for Dirac and Majorana fermions alike in any dimension, and show how these continuous and discrete anomalous symmetries are realized. The connection between GW fermions and extra dimensions is well-established: the first explicit solution to the GW equations being the overlap operator [3, 27, 23, 28, 29] which was derived to describe edge states of domain wall fermions in one higher dimension [2, 30, 31, 32]. It has since been understood that these relativistic systems are equivalent to the topological

This chapter is based on a paper done in collaboration with David B. Kaplan and Hersh Singh [24], with very minor modifications. The author acknowledges their contributions.

insulators and superconductors studied in condensed matter physics, and so the generalized GW relations we derive apply to the massless edge states of the wide variety of topological classes [33, 8] of such materials.[1]

In the following analysis we are interested in the cases of N_F flavors of Dirac or Majorana fermions where (i) the massless theory respects a symmetry G; (ii) a mass term is possible for regulating the theory; (iii) the mass term necessarily breaks the symmetry G. In this set of circumstances we expect the massless theory to have an t'Hooft anomaly involving the G symmetry, a GW relation to exist for the ideally regulated fermion operator, and the existence of an exact G symmetry obeyed by the regulated action, for which the Jacobian reproduces the anomaly of the continuum theory, a generalization of Lüscher symmetry.[2]

2.2 Generalized Ginsparg-Wilson relations for Dirac fermions

2.2.1 Derivation of the relations

Following the logic of the original derivation, we start by considering the continuum theory of a free Dirac fermion Ψ in Euclidean spacetime of arbitrary dimension, possibly in background gauge or gravitational fields, described by the path integral

$$Z = \int d\Psi \, d\bar{\Psi} e^{-S(\bar{\Psi},\Psi)} \ . \tag{2.1}$$

We now do a block transformation, defining a function $f(\mathbf{x})$ whose support lies in a volume a^d about the origin, and our block averaged variables to be

$$\psi_{\mathbf{n}} = \int d^d\mathbf{x} \ \Psi(\mathbf{x}) f(\mathbf{x} - \mathbf{n}a) \tag{2.2}$$

[1]for more on topological classes of relativistic lattice fermions in detail, along with their GW relations, see [34].

[2]Notation: we use upper case Greek letters such as $\Psi(x)$ to denote continuum fields, and lower case, such as $\psi_{\mathbf{n}}$ for lattice variables, generally suppressing indices for the latter. We take Euclidean γ matrices to be Hermitian with $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$; the gauge covariant Dirac operator $\not{D} = \gamma_\mu D_\mu$ is therefore anti-Hermitian with imaginary eigenvalues. For a regulated Dirac operator, such as a generic Ginsparg-Wilson operator, overlap operator, or Pauli-Villars regulated operator, we use the notation $\mathscr{D}_{\rm GW}, \mathscr{D}_{\rm ov}, \mathscr{D}_{\rm PV}$ or simply $\mathscr{D}$. For Majorana fermions, we work with an antisymmetric kinetic and mass operators denoted as D and m. We use the mostly plus convention for our Minkowski metric.

and similarly for $\bar{\psi}_{\mathbf{n}}$. The parameter a will be our lattice spacing, and for the rest of this article we will work in "lattice units" with $a = 1$. Lattice variables $\chi_{\mathbf{n}}$ and a lattice action $S_{\text{lat}} = \bar{\chi}\mathscr{D}\chi$ are defined by

$$e^{-\bar{\chi}\mathscr{D}\chi} = \int d\Psi \, d\bar{\Psi} e^{-S(\bar{\Psi},\Psi)} \, e^{-(\bar{\psi}-\bar{\chi})m(\psi-\chi)} \tag{2.3}$$

so that up to an overall normalization,

$$Z = \int \prod_n d\chi d\bar{\chi} \, e^{-\bar{\chi}\mathscr{D}\chi} \; . \tag{2.4}$$

The parameter m is an invertible Hermitian matrix which we can take to be a real number m times the identity matrix, but we will leave it in matrix form for now so that the identities for Dirac fermions and Majorana fermions (for which m is replaced by m, an imaginary antisymmetric matrix) look similar. Although the subsequent derivations are agnostic about the form of m, it should be local for eq. (2.3) to represent a physically reasonable blocking transformation.

We now assume that the continuum action S is invariant under a global symmetry transformation $\Psi \to \Omega\Psi$, $\bar{\Psi} \to \bar{\Psi}\bar{\Omega}$, where $\bar{\Omega}$ and Ω are some operators. The symmetry transformations of interest are those which are broken by the Gaussian term proportional to m that we have added to the path integral. Examples we will consider include a $U(1)_A$ chiral transformation, a discrete chiral transformation [not contained in $U(1)_A$], and a coordinate reflection:

$$\Omega = \bar{\Omega} = e^{i\alpha\bar{\gamma}} \qquad \text{(chiral symmetry)} \; , \tag{2.5}$$

$$\Omega = \bar{\Omega} = \bar{\gamma} \qquad \text{(discrete chiral symmetry)} \; , \tag{2.6}$$

$$\Omega = -\bar{\Omega} = \varepsilon\mathscr{R}_1\gamma_1 \qquad \text{(reflection symmetry)} \; , \tag{2.7}$$

with $\bar{\gamma}$ being the analog of γ_5 in arbitrary even dimension, where $\mathscr{R}_1$ reflects the sign of the x_1 coordinate; generally $\varepsilon = 1$, but in certain Majorana theories $\varepsilon = i$. Under reflections we assume that background fields are similarly reflected. We will subsequently consider an antilinear symmetry in Euclidean space related to time reversal in Minkowski spacetime. We focus primarily on a single flavor of fermion, and hence do not discuss non-Abelian flavor symmetries, but our analysis can be easily extended to include those. Other symmetries which are directly broken by the discretization function f, such as translation symmetry, spacetime rotations, conformal transformations or supersymmetry transformations do not seem to yield useful relations and we do not consider these (see [35, 36] for interesting attempts in these directions).

While the action is invariant under the $\Omega, \bar{\Omega}$ transformation, the measure generally transforms as $d\Psi d\bar{\Psi} \longrightarrow d\Psi d\bar{\Psi}\, e^{2i\mathcal{A}}$, where $\mathcal{A}$ is called the anomaly and arises from the Jacobian of the transformation [16].

We wish to distinguish between the continuum transformation Ω and the transformation ω of the block-averaged variables,

$$\psi_{\mathbf{m}} \longrightarrow \int \Omega\, \Psi(\mathbf{x}) f(\mathbf{x} - a\mathbf{m})\, d^d\mathbf{x} = \omega_{\mathbf{mn}}\, \psi_{\mathbf{n}} \ . \tag{2.8}$$

The matrices $\omega, \bar{\omega}$ are the lattice-regulated forms of $\Omega, \bar{\Omega}$. They act as ordinary matrices on the lattice variables $\psi_{\mathbf{n}}$, but in the case of reflections, they also reflect the background fields. Defining

$$\mathcal{D}_\omega = \bar{\omega}\mathcal{D}\omega \ , \quad m_\omega = \bar{\omega} m \omega \ , \tag{2.9}$$

it follows that

$$e^{-\bar{\chi}\mathcal{D}_\omega \chi} = \int d\Psi\, d\bar{\Psi}\, e^{2i\mathcal{A}}\, e^{-S(\bar{\Psi},\Psi)}\, e^{-(\bar{\psi}-\bar{\chi})m_\omega(\psi-\chi)} \ . \tag{2.10}$$

Using the relation eq. (A.2) derived in Appendix A, we have

$$e^{-(\bar{\psi}-\bar{\chi})m_\omega(\psi-\chi)} = e^{\mathrm{Tr}\,\ln m_\omega m^{-1}}\, e^{\partial_\chi X_\omega \partial_{\bar{\chi}}}\, e^{-(\bar{\psi}-\bar{\chi})m(\psi-\chi)}, \tag{2.11}$$

where

$$X_\omega = m^{-1} - m_\omega^{-1} \ , \tag{2.12}$$

and so

$$e^{-\bar{\chi}\mathcal{D}_\omega\chi} = e^{2i\mathcal{A}}\, e^{\mathrm{Tr}\,\ln m_\omega m^{-1}}\, e^{\partial_\chi X_\omega \partial_{\bar{\chi}}}\, e^{-\bar{\chi}\mathcal{D}\chi}$$
$$= e^{2i\mathcal{A}}\, e^{\mathrm{Tr}\,\ln m_\omega m^{-1} + \mathrm{Tr}\,\ln Q_\omega}\, e^{-\bar{\chi}\frac{1}{Q_\omega}\mathcal{D}\chi}\ , \tag{2.13}$$

where

$$Q_\omega \equiv (1 - \mathcal{D} X_\omega)\ , \tag{2.14}$$

and in the last step we used the identity eq. (A.2) for a second time.

By equating the χ dependence on both sides of eq. (2.13) we arrive at two equations. The first requires the prefactors of the exponentials to be equal, and we will refer to this as the "anomaly equation":

$$e^{2i\mathcal{A}} = \det\left(m_\omega m^{-1} Q_\omega\right)^{-1} = \det\left(\bar{\omega}\omega Q_\omega\right)^{-1}\ . \tag{2.15}$$

The second equation follows from requiring that the fermion operators in the exponents must be equal,

$$\mathcal{D}_\omega = Q_\omega^{-1}\mathcal{D}\ , \tag{2.16}$$

or equivalently,

$$\mathcal{D}_\omega - \mathcal{D} = \mathcal{D} X_\omega \mathcal{D}_\omega\ , \tag{2.17}$$

and this we call the generalized GW equation. If $\mathcal{D}$ is invertible, the GW equation may be written in the simple form

$$\omega\left(\frac{1}{\mathcal{D}} - \frac{1}{m}\right)\bar{\omega} = \left(\frac{1}{\mathcal{D}} - \frac{1}{m}\right)\ , \tag{2.18}$$

which states that the propagator is symmetric up to a constant local subtraction. Assuming m does not couple neighboring sites, this subtraction is a delta-function in coordinate space. This relation can be further transformed to a yet simpler form by writing

$$\mathcal{D} = m\frac{ih}{1+ih}\ , \tag{2.19}$$

in which case the GW relation eq. (2.18) reduces to the statement that mh is invariant under the ω transformation,

$$\bar{\omega}(mh)\omega = mh \ , \tag{2.20}$$

or if m commutes with $\bar{\omega}$, h itself is invariant. The requirement that $\mathscr{D}$ describes a massless Dirac fermion in the continuum limit means that $\mathscr{D} \to i\not{p}$ for $p^2 \ll m^2$; thus $h \to \not{p}/m$ in that limit, which is Hermitian (assuming for now that m is just a number). If we assume that h both satisfies eq. (2.20) and is Hermitian for *all* momenta, then we can define the unitary matrix $V = -(1 - ih)/(1 + ih)$ and arrive at another useful expression for $\mathscr{D}$,

$$\mathscr{D} = \frac{m}{2}\,(1 + V)\,, \qquad V^\dagger V = 1 \ , \tag{2.21}$$

with

$$V \to -1 + \frac{2i\not{p}}{m} + O\left[\left(\frac{\not{p}}{m}\right)^2\right] \tag{2.22}$$

The eigenvalues of V lie on a unit circle centered at the origin in the complex plane, and those of $\mathscr{D}$ lie on a circle of radius $m/2$ centered at $m/2$. When the theory is gauged, low-lying eigenvalues of $\not{D}$ lie near $V = -1$, while large ones are mapped to the neighborhood of $V = +1$. This is familiar from the discussion in Ref. [3].

2.2.2 Solutions to the Ginsparg-Wilson equation

We now examine solutions to the GW equation, which not only satisfy eq. (2.16), but also satisfy $\mathscr{D} \to \not{D}$ in the continuum limit $m \gg p$, in order to describe a massless Dirac fermion, and which for free fermions only vanish at zero momentum, so as to describe a single flavor in the continuum limit.

The Pauli-Villars solution

Although the GW equation was derived in the context of a lattice regularization, it is in fact more general, and a simple continuum solution to the GW and anomaly equations existed decades before Ginsparg and Wilson wrote their paper; a fermion regulated by a Pauli-Villars (PV) ghost. Examining this case yields insights into the nature of lattice solutions and symmetries.

We have seen that $\mathscr{D} = mih/(1+ih)$ will solve the GW equation and describe a massless Dirac fermion in the low eigenvalue limit if mh obeys the continuum symmetries of a massless Dirac fermion, and $mh \to \not{p}$ for a free fermion at low p. The simplest possible solution to these criteria is to simply set $ih = \not{D}/m$, in which case the GW solution describes a PV regulated fermion:

$$\not{D} \to \mathscr{D}_{\rm PV} = m\frac{\not{D}}{\not{D}+m} = \frac{m}{2}\left(1 - \frac{1-\not{D}/m}{1+\not{D}/m}\right) , \tag{2.23}$$

where we will take $m > 0$ with the "continuum" limit being $m \to \infty$. The operator $\mathscr{D}_{\rm PV}$ is not fully regulated, but the phase of its determinant is, which is where anomalies appear. The unitary matrix V in eq. (2.21) is given by

$$h = -i\not{D}/m \ , \quad V = -\frac{1-\not{D}/m}{1+\not{D}/m} \ . \tag{2.24}$$

We will show that the operator $\mathscr{D}_{\rm PV}$ simply illustrates two general properties of solutions to the GW equation which we discuss below. The first is that the regulated η-invariant of the continuum operator — which describes the phase of the fermion determinant — is realized in terms of $\ln\det V$. The second is that when ghost fields are introduced to represent the PV-regulated fermion, the exact symmetry of the regulated action discovered by Lüscher can be simply related to the symmetry of the unregulated action. The PV solution will also help inform our analysis of massless Majorana fermions in Sec. 2.3.3.

The overlap solution

The first explicit lattice solution to the GW equation was the overlap operator of Neuberger [3], based on the earlier work in conjunction with Narayanan in Refs. [27, 23, 28] and on the domain wall fermion construction in [2]. This solution takes the V matrix to be

$$V = \frac{D_{\rm w}}{\sqrt{D_{\rm w}^\dagger D_{\rm w}}} \ , \tag{2.25}$$

where D_{w} is the lattice operator for a Wilson fermion with mass $-M < 0$ and Wilson coupling $r = M$ [3],

$$D_{\mathrm{w}} = \sum_{\mu} \delta_\mu \gamma_\mu - M - \frac{M}{2}\Delta \ , \tag{2.26}$$

where δ_μ is the covariant symmetric difference operator, and Δ is the covariant lattice Laplacian. Without gauge fields, this gives

$$\begin{aligned} \tilde{D}_{\mathrm{w}}(p) &= \sum_\mu (i\gamma^\mu \sin p_\mu) + M\left[-1 + \sum_\mu (1 - \cos p_\mu)\right] \\ &\to M\left(-1 + i\frac{\not{p}}{M} + O(p^2/M^2)\right) \ . \end{aligned} \tag{2.27}$$

Evidently $V \to \left(-1 + i\frac{\not{p}}{M} + O(p^2/M^2)\right)$ and one can see that near the corners of the Brillouin zone where doublers reside for naive lattice fermions one finds $V = +1$. Therefore this operator behaves correctly as a massless Dirac fermion in the continuum limit.

In even spacetime dimensions, one has chiral symmetry with $\omega = \bar{\omega} = e^{i\alpha\bar{\gamma}}$. Then the GW equation as expressed by eq. (2.20) is equivalent to $\{\bar{\gamma}, h\} = 0$ or $\bar{\gamma}V\bar{\gamma} = V^\dagger$. This latter property readily seen to be satisfied by the overlap solution. In odd spacetime dimensions one is interested in reflection symmetry for which $\omega = -\bar{\omega} = \mathcal{R}_1\gamma_1$ and eq. (2.20) requires $\{h, \omega\} = 0$, implying that $\omega V \omega^{-1} = V^\dagger$, which is also seen to be satisfied by the overlap operator.

2.2.3 *An exact symmetry of the lattice action*

Equation (2.16) together with eq. (2.9) implies that the action $\bar{\chi}\mathcal{D}\chi$ for a GW fermion obeys an exact Lüscher symmetry,

$$\bar{\chi} \to \bar{\chi} Q_\omega \bar{\omega} \ , \quad \chi \to \omega\chi. \tag{2.28}$$

[3] As shown in [30, 31] there is actually an interesting sequence of topological phase transitions as a function of M/r, and taking $M/r = 1$ places the theory in one of several possible topological phases.

This symmetry constrains the Feynman rules for the theory, eliminating the possibility of an additive mass renormalization for χ in perturbation theory since a mass term breaks the symmetry with

$$\bar{\chi}\chi \to \bar{\chi} Q_\omega \bar{\omega}\omega\chi \ , \tag{2.29}$$

where $Q_\omega\bar{\omega}\omega \neq \mathbf{1}$ for the symmetry transformations of interest[4]. The transformation is also not a symmetry of the χ measure, with Jacobian equal to $(\mathbf{1}/\det\bar{\omega}\omega Q_\omega)$, which we see from the anomaly equation eq. (2.15) exactly reproduces the $\exp(\mathbf{2}i\mathscr{A})$ anomaly in the original continuum theory. This symmetry was discovered in the context of infinitesimal chiral transformations in even spacetime dimension by Lüscher [15, 38] with $\omega = \bar{\omega} = \mathbf{1} + i\alpha\bar{\gamma} + O(\alpha^2)$, which we have generalized here to include discrete symmetries.

This symmetry may seem somewhat peculiar, but becomes transparent when considering the PV solution. First one simply adds a Gaussian term for a spinor ghost with Bose statistics,

$$S_\chi \to \bar{\chi}\mathscr{D}_{\rm PV}\chi + m\bar{\phi}\phi = m\left(\bar{\chi}\frac{\not{D}}{\not{D}+m}\chi + \bar{\phi}\phi\right), \tag{2.30}$$

integrating over the ϕ fields, which has no effect other than modifying the normalization of the path integral. The fermion operator $\mathscr{D}_{\rm PV}$ is defined in eq. (2.23). We then make the simultaneous change of variables

$$\bar{\chi} = \bar{\chi}'(\mathbf{1} + \not{D}/m) \ , \qquad \bar{\phi} = \bar{\phi}'(\mathbf{1} + \not{D}/m) \ , \tag{2.31}$$

leaving χ and ϕ unchanged. Because $\bar{\chi}$ and $\bar{\phi}$ have opposite statistics, the Jacobians from these transformations cancel in the integration measure. The action now looks like

$$S_\chi = \left[\bar{\chi}'\not{D}\chi + \bar{\phi}'(\not{D}+m)\phi\right], \tag{2.32}$$

which is the conventional form for PV regularization in perturbative applications with a massless Dirac fermion and a ghost of mass m.

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[4]This symmetry does not protect against finite nonperturbative additive mass renormalizations, such as those that can be generated by instantons as discussed in [37].

Using the identity

$$Q_\omega \bar{\omega} = \frac{1}{(1+\not{D}/m)} \bar{\omega}(1+\not{D}/m) \ . \tag{2.33}$$

the Lüscher symmetry transformation of eq. (2.28) becomes very simple in terms of our new variables,

$$\chi \to \omega\chi \ , \qquad \bar{\chi}' \to \bar{\chi}'\bar{\omega} \ , \tag{2.34}$$

with ϕ and $\bar{\phi}'$ not transforming at all. In other words, the transformations of the χ and $\bar{\chi}'$ fields are just the symmetry transformations that leave the continuum Dirac action invariant. Furthermore, as in the continuum, violation of the symmetry comes from the path integral measure since eq. (2.34) has no compensating transformation of the ghost field. It is clear that since the Feynman rules for χ and $\bar{\chi}'$ in this theory with ghosts respect the ω symmetry, no symmetry-violating operators will be generated by radiative corrections in perturbation theory.

2.2.4 The anomaly equation

The anomaly equation eq. (2.15) states that the continuum anomaly $\exp(2i\mathscr{A}) = 1/\det Q_\omega$ for chiral symmetry transformations (for which $\det\bar{\omega}\omega = 1$), while $\exp(2i\mathscr{A}) = 1/\det(-Q_\omega)$ for reflections (where $\bar{\omega}\omega = -1$), which in both cases equals the Jacobian for the symmetry transformation in eq. (2.28). This relates $\mathscr{A}$, which is a functional of the background fields, to properties of the fermion spectrum. Here we show that in even spacetime dimensions the equation reproduces the Atiyah-Singer index theorem as shown in Ref. [15], while in odd spacetime dimensions it reproduces the relation between the parity anomaly and the η-invariant discovered in Ref. [39]. For recent work on the η-invariant in the context of the overlap operator, see Refs. [40, 41].

We first consider the PV solution in both odd and even dimensions. The phase of the determinant for a massless Dirac fermion may be expressed as $\exp(-i\pi\eta_D(0)/2)$, where η_D is defined as a regulated sum of the signs of eigenvalues of $i\not{D}$, and $\eta_D(0)$ is the universal value as the regulator is removed [42]. The PV solution to the GW equation replaces $\not{D}$ by its regulated form $\mathscr{D}_{\rm PV} = (m/2)(1+V)$ where V is unitary. It follows that

$$\frac{\det \mathscr{D}_{\rm PV}}{\det \mathscr{D}^\dagger_{\rm PV}} = e^{{\rm Tr}\,\ln \frac{1+V}{1+V^\dagger}} = e^{{\rm Tr}\,\ln V}\ . \tag{2.35}$$

The eigenvalues of V are $(-i\lambda/m - 1)/(-i\lambda/m + 1) = -1 - 2i\lambda/m + O(1/m^2)$, and so we have

$${\rm Tr}\,\ln V = -i\pi\sum_\lambda \frac{\lambda}{|\lambda|} + O(1/m) \equiv -i\pi\eta_D(1/m)\ . \tag{2.36}$$

Thus we see that

$$\eta_D(0) = \lim_{m\to\infty}\frac{i}{\pi}\ln\det V \tag{2.37}$$

and the phase of the fermion determinant $\det\mathscr{D}_{\rm PV}$ may be written as $e^{-i\frac{\pi}{2}\eta_D}$. This result applies generally to solutions of the GW equation.

In odd spacetime dimensions with a space reflection transformation as in eq. (2.7) we have $\bar\omega\omega = -1$, $m_\omega = -m$ and $-Q_\omega = -1 + 2\mathscr{D}/m = V$. Therefore the anomaly equation states that $\mathscr{A} = -\frac{1}{2}{\rm Tr}\,\ln V = i\pi\eta_D/2$, correctly realizing the parity anomaly as the regulator is removed [39]. The perturbative expansion of η_D yields the Chern-Simons action, a result also consistent with Ref. [43].

In even spacetime dimensions for a $U(1)_A$ chiral transformation the anomaly equation states $\exp(2i\mathscr{A}) = 1/\det Q_\omega$. In this case it is simplest to expand to linear order in α and one finds

$$Q_\omega = 1 - 2i\alpha/m\mathscr{D}\bar\gamma + O(\alpha^2)\ , \tag{2.38}$$

and the anomaly equation states that

$$2i\mathscr{A} = \frac{2i\alpha}{m}{\rm Tr}\,\bar\gamma\mathscr{D} \tag{2.39}$$

where the continuum anomaly functional $\mathscr{A}$ is proportional to α. The Atiyah-Singer index theorem states that the right side of the above equation should equal $-2i\alpha$ times the index of the Dirac operator, $(n_+ - n_-)$, where $m_\pm$ equals the number of ± 1 chirality zeromodes. This result follows from the analysis by Lüscher [15], after taking into account the relative normalization of $am/2$ between $\mathscr{D}$ and the GW operator analyzed in that paper.

2.2.5 Antilinear symmetry

A theory that possesses an antilinear time-reversal symmetry $\psi(\mathbf{x},t) \to \mathscr{T}\psi(\mathbf{x},-t)$ in Minkowski spacetime will respect a related antilinear symmetry in Euclidean spacetime that does not reverse any coordinates. This is simply because after replacement of t with $-i\tau$, the conjugation of the i in $-i\tau$ has the same effect as $t \to -t$. For this symmetry $\Omega = \bar{\Omega}^\dagger = \hat{\mathscr{T}} T$ where the operator $\hat{\mathscr{T}}$ reverses time in Minkowski spacetime but acts trivially in Euclidean, while T is a unitary matrix satisfying $T^\dagger \gamma_\mu T = \pm\gamma_\mu^T$. When this transformation is a symmetry of the massless theory but is necessarily broken by a fermion mass term, then it will in general be anomalous and there will be corresponding GW relations. A simple example is a massless Dirac fermion in $2+1$ dimensions where we can take the γ matrices to be $\gamma^0 = i\sigma_1$, $\gamma^1 = \sigma_2$, $\gamma^2 = \sigma_3$ and $T = \sigma_2$. Under time reversal the fields transform as $\psi(\mathbf{x},t) \to T\psi(\mathbf{x},-t)$ and $\bar\psi(\mathbf{x},t) \to \bar\psi(\mathbf{x},-t)T$ which is a symmetry of the action for a massless Dirac fermion, but for a massive fermion the transformation flips the sign of the mass term. In Euclidean spacetime the symmetry transformation is identical, $\psi \to T\psi$ and $\bar\psi \to \bar\psi T$, except that there is no change in the coordinates; again one finds that the massless Dirac action is invariant but that a mass term is odd.

Our derivation of the generalized GW relations proceed as above, only now Ω and $\bar\Omega$ are antilinear, while the ω and $\bar\omega$ remain as ordinary matrices. This change results in eq. (2.9) being replaced by

$$\mathscr{D}_\omega = \bar\omega \mathscr{D}^* \omega \ , \quad m_\omega = \bar\omega m^* \omega \ , \tag{2.40}$$

With these changes, the anomaly equation eq. (2.15) and the GW equation eq. (2.16) remain valid. It is evident that $\mathscr{D}_{\rm PV}$ satisfies this antilinear GW equation since $h \propto \not{D}$; one can easily check that $\mathscr{D}_{\rm ov}$ satisfies it as well.

2.3 Generalized Ginsparg-Wilson relations for Majorana fermions

The edge states of topological insulators are typically massless Dirac fermions such as described in the previous section; on the other hand, the edge states of topological superconductors without a conserved fermion number are massless Majorana fermions. Majorana edge states were first discussed in Ref. [44] in the context of simulating gluinos in $d = 3+1$ dimensions, and in Ref. [45] for $d = 1+1$ condensed matter systems. Here we derive the GW relations for Majorana fermions.

2.3.1 Continuum Majorana fermions

We begin by summarizing properties of continuum Majorana fermions in arbitrary d dimensions, and enumerate the symmetries of interest.[5]

The Majorana constraint

To obtain a single flavor of massless Majorana fermion we impose a Lorentz-covariant Majorana constraint on a massless Dirac fermion,

$$\psi = \psi^{\mathcal{K}} \ , \qquad \psi^{\mathcal{K}} \equiv \mathcal{K}^\dagger \bar{\psi}^T \ , \tag{2.41}$$

where for Lorentz invariance and self-consistency of the constraint, $\mathcal{K}$ must equal either an *antisymmetric* $\mathcal{C}$ matrix, or a *symmetric* $\mathcal{T}$ matrix, $\mathcal{C}$ and $\mathcal{T}$ being unitary matrices which satisfy

$$\mathcal{C}\gamma_\mu\mathcal{C}^\dagger = -(\gamma_\mu)^T \ , \qquad \mathcal{T}\gamma_\mu\mathcal{T}^\dagger = (\gamma_\mu)^T \ . \tag{2.42}$$

The Majorana constraint as expressed above is equally valid in Minkowski and Euclidean spacetimes. In Ref. [46] fermions satisfying these constraints are referred to as Majorana ($\mathcal{K} = \mathcal{C}$) or pseudo-Majorana ($\mathcal{K} = \mathcal{T}$); here we will refer to them as $\mathcal{C}$-Majorana and

[5] For a detailed discussion of Majorana fermions in Minkowski and Euclidean spacetimes, see Ref. [46].

$\mathcal{T}$-Majorana respectively when distinguishing between them, and simply by "Majorana" when not. The massless Majorana action can then take the form[6]

$$S = \int d^d x \, \tfrac{1}{2} \psi^T \mathcal{K} \not{D} \psi \ . \tag{2.43}$$

Table 2.1 lists the properties of the $\mathcal{C}$ and $\mathcal{T}$ matrices in different dimensions, and we see that for a single Majorana flavor we can take $\mathcal{K} = \mathcal{C}$ in $d = 2, 3, 4 \mod 8$, and $\mathcal{K} = \mathcal{T}$ in $d = 1, 2, 8 \mod 8$, while there is no solution in $d = 5, 6, 7 \mod 8$. Instead of one flavor, one could consider two flavors and replace $\mathcal{K} \to \mathcal{K} \otimes \tau_2$, where τ_2 is the antisymmetric Pauli matrix in flavor space. Then one requires $\mathcal{K}$ to equal either a *symmetric* $\mathcal{C}$ matrix, or an *antisymmetric* $\mathcal{T}$ matrix. Such fermions are sometimes referred to as symplectic Majorana fermions. In this way one can discuss massless fermions with a reality constraint ($\mathcal{C}$-Majorana, $\mathcal{T}$-Majorana, symplectic Majorana) in any dimension. In this section we will only discuss a single flavor of massless Majorana and are therefore restricted to $d = 2, 3, 4$. We give examples of these theories with discrete symmetry anomalies, as well as an anomalous example of symplectic Majoranas.

In order to follow the GW program we must be able to define a mass term for the Majorana fermion. This can be included in the Euclidean action as $\frac{1}{2} \int \psi^T \mathsf{m} \psi$, where

$$\mathsf{m} = \mu \mathcal{M} = -\mathsf{m}^T \ , \tag{2.44}$$

μ being a number with dimension of mass, while $\mathcal{M}$ is required by Lorentz invariance and fermion statistics to be either an antisymmetric $\mathcal{C}$ or antisymmetric $\mathcal{T}$ matrix. No such matrix exists in $d = 1, 7, 8 \mod 8$. In these cases we can consider symplectic Majoranas (two flavors) in which case μ may be replaced by $\mu \tau_2$ acting in flavor space, and $\mathcal{M}$ must now be a *symmetric* $\mathcal{C}$ or $\mathcal{T}$ matrix[7].

[6]A Majorana fermion may carry gauge charges so long as it is in a (pseudo-)real representation of the gauge group. In that case, $\mathcal{C}$ and $\mathcal{T}$ will have to include the appropriate matrices to effect the similarity transformation from the generators T_a to the conjugate generators $-T_a^T$.

[7]It is stated in Ref. [46] that $\mathcal{T}$-Majorana fermions are necessarily massless, but that assumes that a mass term must have the form $\psi^T \mathcal{T} \psi$. When allowing for a $\psi^T \mathcal{C} \psi$ mass term the statement is no longer true. This can be generated from a Dirac action by applying the $\mathcal{T}$-Majorana constraint to a Dirac mass term of the form $i\bar{\psi}\bar{\gamma}\psi$.

d:	1	2	3	4	5	6	7	8
$\mathcal{T}$	**S**	**S**	·	A	A	A	·	**S**
$\mathcal{C}$	·	**A**	**A**	**A**	·	S	S	S
$\bar{\gamma}$	·	−	·	+	·	−	·	+

Table 2.1: The $\mathcal{C}$ and $\mathcal{T}$ matrices in Euclidean dimensions $d = 1, \ldots, 8 \mod 8$ defined in eq. (2.42). S and A represent whether the corresponding matrix is symmetric or antisymmetric, while a dot indicates it does not exist. The last row denotes whether $\mathcal{C}\bar{\gamma}\mathcal{C}^{-1} = \mathcal{T}\bar{\gamma}\mathcal{T}^{-1} = \pm(\bar{\gamma})^T$, where $\bar{\gamma}$ is the chiral matrix for even d satisfying $\{\bar{\gamma}, \gamma_\mu\} = 0$ for $\mu = 1, \ldots, d$. For a single Majorana flavor, only **bold** entries can play the role of $\mathcal{K}$ in Majorana kinetic terms, and only antisymmetric entries (A) can appear as $\mathcal{M}$ in Majorana mass terms. We refer the reader to Ref. [46] for a pedagogical discussion of this table.

As can be seen from Table 2.1, the requirement that both $\mathcal{K}$ and $\mathcal{M}$ exist still restricts us to discussing $d = 2, 3, 4$ for a single flavor. In $d = 3$ there is the unique choice $\mathcal{K} = \mathcal{M} = \mathcal{C}$. In $d = 2$ we have the single choice $\mathcal{M} = \mathcal{C}$ while $\mathcal{K}$ may equal $\mathcal{C}$ or $\mathcal{T}$. In $d = 4$, the reverse is true: $\mathcal{K} = \mathcal{C}$ while $\mathcal{M}$ may equal $\mathcal{C}$ or $\mathcal{T}$. For the two mixed cases $(\mathcal{K}, \mathcal{M}) = (\mathcal{T}, \mathcal{C})$ in $d = 2$ and $(\mathcal{C}, \mathcal{T})$ in $d = 4$ we have $\mathcal{T}$ equal to $\bar{\gamma}\mathcal{C}$, up to a phase, and Hermiticity in Minkowski spacetime is guaranteed if we take

$$\mathcal{M}^{-1}\mathcal{K} = \begin{cases} 1 & (\mathcal{C}, \mathcal{C}) \\ i\bar{\gamma} & (\mathcal{T}, \mathcal{C}),\ (\mathcal{C}, \mathcal{T}) \end{cases} . \tag{2.45}$$

Symmetries

In dimensions $d = 2, 3, 4$ the massless Dirac action possesses a $U(1)_V$ fermion number, reflection symmetry and charge conjugation symmetries, while in $d = 2, 4$ it also possesses a $U(1)_A$ chiral symmetry. Here we examine what subgroup is left unbroken by the Majorana constraint, and then what is the effect of the regulator.

d:	2		3	4	
$(\mathcal{K},\mathcal{M})$	R	$\bar{\gamma}$	R	R	$e^{i\alpha\bar{\gamma}}$
$(\mathcal{C},\mathcal{C})$	✗	✗	✗	✗	✗
$(\mathcal{C},\mathcal{T})$	·	·	·	✓	✗
$(\mathcal{T},\mathcal{C})$	✓	✗	·	·	·
$(\mathcal{T},\mathcal{T})$	·	·	·	·	·

Table 2.2: Reflection (R) and chiral (discrete or continuous) symmetries for a single massless Majorana flavor in $d = 2, 3, 4$ for different combinations of the $\mathcal{K}$ and $\mathcal{M}$ matrices, where $\mathcal{K}$ defines the kinetic term and $\mathcal{M}$ is used as the regulating mass term. A "✓" indicates a nonanomalous symmetry, an "✗" denotes that the regulator choice $\mathcal{M}$ breaks the symmetry indicating a possible anomaly, and a dot means that the $(\mathcal{K},\mathcal{M})$ combination does not exist. For $d \neq 2, 3, 4$, we need multiple flavors.

In all dimensions $U(1)_V$ fermion number symmetry is broken to a $\mathbf{Z}_2$ subgroup which acts as $(-1)^F$, an element of the Lorentz group. What happens to the $U(1)_A$ chiral symmetry in $d = 2, 4$ depends on the fact that $\mathcal{K}\bar{\gamma}^T\mathcal{K}^{-1} = -\bar{\gamma}$ in $d = 2$ and $+\bar{\gamma}$ in $d = 4$. In $d = 2$ in addition to $(-1)^F$ the Majorana constraint leaves unbroken a $\mathbf{Z}_2$ subgroup of $U(1)_V \times U(1)_A$ corresponding to $\psi \to \bar{\gamma}\psi$, while in $d = 4$ the entire $U(1)_A$ remains unbroken. The latter result should not be surprising since a massless Majorana fermion in $d = 4$ Minkowski spacetime is equivalent to a massless Weyl fermion, whose action possesses a $U(1)$ symmetry; this is not true in $d = 2$.

The charge conjugation symmetry of the Dirac fermion survives the Majorana constraint, but either acts trivially on the Majorana fermion, or as $(-1)^F$.

For reflections we consider transformations of the Dirac field $\psi(x) \to \mathsf{R}\psi(x) = \varepsilon\gamma_1\psi(\tilde{x})$ and $\bar{\psi}(x) \to \mathsf{R}\bar{\psi}(x) = -\varepsilon^*\bar{\psi}(\tilde{x})\gamma_1$, where ε is a phase and $\tilde{x}$ has the sign of x_1 flipped. This is consistent with the Majorana condition eq. (2.41) if $\varepsilon = 1$ when $\mathcal{K} = \mathcal{C}$ and $\varepsilon = i$ when

$\mathcal{K} = \mathcal{T}$ and is therefore always a symmetry for the massless Majorana action. Note that this means that for $\mathcal{C}$-Majoranas we have $\mathsf{R}^2 = 1$ while for $\mathcal{T}$-Majoranas, $\mathsf{R}^2 = (-1)^F$.

When a Majorana mass term m is included the $(-1)^F$ symmetry is not broken, but the discrete chiral symmetry in $d = 2$ and the continuous chiral symmetry in $d = 4$ are; therefore it is reasonable to expect anomalies and GW relations for these transformations. The situation for reflection symmetry is more complicated. Reflection symmetry is broken by the mass term if the $\mathcal{M}$ matrix is the same as the $\mathcal{K}$ matrix, and unbroken if they are unlike (e.g. $(\mathcal{K}, \mathcal{M}) = (\mathcal{C}, \mathcal{T})$ or $(\mathcal{K}, \mathcal{M}) = (\mathcal{T}, \mathcal{C})$). Therefore we should expect reflection symmetry to be anomalous for Majorana fermions in $d = 2, 3$ and in $d = 4$ when $\mathcal{M} = \mathcal{C}$. It will not be anomalous for $\mathcal{T}$-Majorana fermions in $d = 2$ or $\mathcal{C}$-Majorana fermions in $d = 4$ with $\mathcal{M} = \mathcal{T}$. These two cases are quite different from each other, however: in $d = 2$ both $\mathcal{C}$- and $\mathcal{T}$-Majoranas exist with only one way to regulate them (with $\mathcal{M} = \mathcal{C}$), and we find that reflections are anomalous in the former but not the latter. For $d = 4$ we only have a $\mathcal{C}$-Majorana, but two ways to regulate, with $\mathcal{M} = \mathcal{C}$ or $\mathcal{M} = \mathcal{T}$, the former breaking reflections symmetry and the latter not. In this case we would say that choosing $\mathcal{M} = \mathcal{C}$ is a poor choice of regulator, needlessly breaking the symmetry of the massless fermion, and we would not expect the symmetry to be anomalous.[8]

We have summarized the situation with reflection and chiral symmetries in Table 2.2; cases for which GW relations pertain are the entries with the "✗".

2.3.2 Derivation of the relations

Similar to the discussion of Dirac fermions in Sec. 2.2, we can derive a GW relation for Majorana fermions, which we denote as Ξ in the continuum. We follow the same block-spin prescription as for Dirac fermions and perform a transformation $\Xi \to \Omega\Xi$ which is assumed to be a symmetry of the continuum action but not a symmetry of either the block-spin

[8]In some cases, it may be useful to consider regulator choices that break a nonanomalous symmetry. In these cases, the symmetry is not completely lost and a modified form of the symmetry still persists, as has been noted in Refs. [47, 48, 49] for continuous symmetries.

Gaussian or the measure. The analog of eq. (2.10) is

$$e^{-\frac{1}{2}\eta^T \mathsf{D}_\omega \eta} \;\; = \int d\Xi \; e^{i\mathcal{A}} e^{-S[\Xi]-(\eta-\xi)^T \mathsf{m}_\omega (\eta-\xi)} \;, \tag{2.46}$$

where $\xi_{\mathbf{n}}$ are block-averaged lattice fields related to Ξ as in eq. (2.2),

$$\xi_{\mathbf{n}} = \int d^d\mathbf{x} \; \Xi(\mathbf{x}) f(\mathbf{x} - \mathbf{n}a) \tag{2.47}$$

and m is an invertible, imaginary, antisymmetric matrix. We have defined

$$\mathsf{D}_\omega = \omega^T \mathsf{D} \omega \;, \qquad \mathsf{m}_\omega = \omega^T \mathsf{m} \omega \;, \tag{2.48}$$

where ω is related to Ω in analogy with eq. (2.8), and suppress lattice indices as before. The path integral identity we derive in eq. (A.4) allows us to recast this equation as

$$e^{-\frac{1}{2}\eta \mathsf{D}_\omega \eta} = e^{i\mathcal{A}} e^{\frac{1}{2}\mathrm{Tr}\,\ln \frac{\mathsf{m}_\omega}{\mathsf{m}} Q_\omega} e^{-\frac{1}{2}\eta Q_\omega^{-1} \mathsf{D} \eta} \;, \tag{2.49}$$

where

$$Q_\omega = (1 - \mathsf{D}\mathsf{X}_\omega) \,, \qquad \mathsf{X}_\omega = \mathsf{m}^{-1} - \mathsf{m}_\omega^{-1} \;. \tag{2.50}$$

Comparing both sides, we find two equations, the first of which is a generalized GW relation for Majorana fermions

$$\mathsf{D}_\omega = Q_\omega^{-1} \mathsf{D} \;. \tag{2.51}$$

This can be rewritten in a form analogous to the conventional GW relation as

$$\mathsf{D}_\omega - \mathsf{D} = \mathsf{D}\mathsf{X}_\omega \mathsf{D}_\omega . \tag{2.52}$$

If there are no zero modes, then D is invertible and the GW equation is equivalent to

$$\omega^T \left(\frac{1}{\mathsf{D}} - \frac{1}{\mathsf{m}} \right) \omega = \left(\frac{1}{\mathsf{D}} - \frac{1}{\mathsf{m}} \right) \;, \tag{2.53}$$

similar to what we found for the Dirac case in eq. (2.18).

As in the Dirac case, the second equation obtained is the anomaly equation,

$$e^{i\mathcal{A}} = \frac{1}{\sqrt{\det \frac{\mathsf{m}_\omega}{\mathsf{m}} Q_\omega}} . \tag{2.54}$$

As we shall show, the square root is well-defined.

2.3.3 Solutions to the Majorana Ginsparg-Wilson equation

Just as we identified both the PV and overlap solutions to the GW relations for Dirac fermions, we can do the same for Majoranas. The PV solution allows one to easily derive certain useful properties of a solution which generalize.

Pauli-Villars solution

If we write

$$\mathsf{D} = \mathsf{m}\frac{ih}{ih+1} \tag{2.55}$$

then the GW relation in eq. (2.53) is equivalent to the statement

$$\omega^T \mathsf{m} h \omega = \mathsf{m} h \ , \tag{2.56}$$

or that $\mathsf{m}h$ possesses the same symmetry as the continuum operator for a massless Majorana fermion, $\mathcal{K}\not{D}$. Furthermore, the continuum limit requiring that $\mathsf{D} \to i\mathcal{K}\not{p}$ in the low momentum limit for a free fermion implies that $h \to \mathsf{m}^{-1}\mathcal{K}\not{p}$. As in the Dirac example discussed in Sec. 2.2.2, the simplest solution to simply set $\mathsf{m}h = \mathcal{K}\not{D}$, and the interpretation to this solution of the GW equation is a PV regulated Majorana fermion,

$$\mathsf{D}_{\text{PV}} = \mu\mathcal{K}\not{D}\frac{1}{\mathcal{M}^{-1}\mathcal{K}\not{D}+\mu} \ , \tag{2.57}$$

where $\mathcal{M}^{-1}\mathcal{K} = 1$ or $\mathcal{M}^{-1}\mathcal{K} = \pm i\bar{\gamma}$, depending on which of the "✗" cases in Table 2.2 one is discussing, while μ is the PV mass scale. Given that $\mathcal{K}\not{D}$ and $\mathcal{M}$ are antisymmetric, it is easy to show that D_{PV} is antisymmetric as well.

This solution can be written as

$$\mathsf{D}_{\text{PV}} = \frac{\mathsf{m}}{2}\left(1+V_{\mathsf{maj}}\right) \ , \quad V_{\mathsf{maj}} = -\frac{\mu-\mathcal{M}^{-1}\mathcal{K}\not{D}}{\mu+\mathcal{M}^{-1}\mathcal{K}\not{D}} \ , \tag{2.58}$$

where V_{maj} is a unitary matrix. The eigenvalues of V_{maj} lie on a circle, as in the Dirac case, where zero modes of $\not{D}$ are mapped to $V_{\mathsf{maj}} = -1$, while infinite eigenvalues are mapped to

$V_{\mathsf{maj}} = +1$. For the cases where $\mathcal{M} = \mathcal{K} = \mathcal{C}$, V_{maj} is the same matrix we found for Dirac PV solution, eq. (2.24).

Various general properties of V_{maj} can be derived from the expression in eq. (2.58). Antisymmetry of D_{PV} implies that

$$\mathsf{m} V_{\mathsf{maj}} \mathsf{m}^{-1} = \mathcal{M} V_{\mathsf{maj}} \mathcal{M}^{-1} = V_{\mathsf{maj}}^T \tag{2.59}$$

Since V_{maj} is unitary, we can its eigenvalue equation as $V_{\mathsf{maj}}\psi_n = e^{i\theta_n}\psi_n$, while it follows from eq. (2.59) that $V_{\mathsf{maj}}\mathcal{M}^\dagger\psi_n^* = e^{i\theta_n}\mathcal{M}^\dagger\psi_n^*$. Furthermore, ψ_n and $\mathcal{M}^\dagger\psi_n^*$ are mutually orthogonal due to the antisymmetry of $\mathcal{M}$. Therefore it follows that the eigenvalues of V_{maj} are all doubly degenerate. This will be relevant below when we discuss the square root of the determinant of V_{maj}.

Next we show how symmetries impact the eigenvalue spectrum of V_{maj}. In the continuum, reflection symmetry for a Dirac fermion takes $\psi \to (\gamma_1\mathcal{R}_1)\psi$ where $\mathcal{R}_1$ reflects the x_1 coordinate, with $(\gamma_1\mathcal{R}_1)\not{D}(A)(\gamma_1\mathcal{R}_1) = -\not{D}(\tilde{A})$, assuming that background fields A are also suitably reflected to $\tilde{A}$. It follows that since $\mathcal{M}^{-1}\mathcal{K}$ equals one in the $(\mathcal{C},\mathcal{C})$ theories and $i\bar{\gamma}$ in the $(\mathcal{C},\mathcal{T})$ and $(\mathcal{T},\mathcal{C})$ theories that

$$(\gamma_1\mathcal{R}_1)V_{\mathsf{maj}}(\gamma_1\mathcal{R}_1) = \begin{cases} V_{\mathsf{maj}}^\dagger & (\mathcal{C},\mathcal{C}) \\ V_{\mathsf{maj}} & (\mathcal{C},\mathcal{T}),\ (\mathcal{T},\mathcal{C}) \end{cases}, \tag{2.60}$$

again assuming a reflection of background fields in the V_{maj} matrices on the right.

The effect of $\bar{\gamma}$ in $d = 2, 4$ is seen to be the same as seen in the Dirac case, namely

$$\bar{\gamma}V_{\mathsf{maj}}\bar{\gamma} = V_{\mathsf{maj}}^\dagger\ . \tag{2.61}$$

We will be interested in the anomalous symmetries marked by the "✗" in Table 2.2. We see that in each of these cases we have a unitary matrix $\mathcal{U}$ satisfying $\mathcal{U}V_{\mathsf{maj}}\mathcal{U}^\dagger = V_{\mathsf{maj}}^\dagger$. This implies that if $V_{\mathsf{maj}}\psi_n = e^{i\theta_n}\psi_n$, then $V_{\mathsf{maj}}\mathcal{U}^\dagger\psi_n = e^{-i\theta_n}\mathcal{U}^\dagger\psi_n$, and therefore, not only are all eigenvalues of V_{maj} doubly degenerate, but the $V \neq \pm 1$ eigenvalues also come in complex conjugate pairs.[9]

[9]One can relax the assumption that V_{maj} is unitary and still conclude the eigenvalues come in $\{\lambda, \lambda^{-1}\}$

Overlap solution

Armed with insight from the above PV solution, it is straightforward to find a lattice overlap solution to the Majorana GW equation,

$$\mathsf{D}_{\text{ov}} = \frac{\mathsf{m}}{2}(1 + V_{\mathsf{maj}}) \tag{2.62}$$

$$V_{\mathsf{maj}} = \frac{D_{\text{w}}}{\sqrt{D_{\text{w}}^{\dagger} D_{\text{w}}}} \tag{2.63}$$

where

$$D_{\text{w}} = \mathcal{M}^{-1}\mathcal{K}\gamma^{\mu}\delta_{\mu} - \mu(1 + \Delta/2), \tag{2.64}$$

where δ_μ and Δ are the lattice derivative and Laplacian respectively. The overlap solution for V_{maj} obeys the properties we found for the PV solution, Eqs. (2.59) to (2.61). Without gauge fields and in momentum space,

$$\begin{aligned}\tilde{D}_{\text{w}}(p) = \mathcal{M}^{-1}\mathcal{K}\sum_{\mu}\gamma^{\mu} i \sin(p_\mu) \\ + \mu\left[-1 + \textstyle\sum_{\mu}(1 - \cos(p_\mu))\right].\end{aligned} \tag{2.65}$$

Near the origin $p \ll \pi/a$ we have

$$\tilde{D}_{\text{w}}(p) = \mathcal{M}^{-1}\mathcal{K} i\not{p} + O(p^2/\mu^2). \tag{2.66}$$

and thus

$$\begin{aligned} V_{\mathsf{maj}} &= -1 + \frac{\mathcal{M}^{-1}\mathcal{K} i\not{p}}{|\mu|} + O(p^2/\mu^2), \\ \mathsf{D}_{\text{ov}}(p) &= \frac{\mathsf{m}}{2}(1 + V_{\mathsf{maj}}) = \frac{i}{2}\mathcal{K}\not{p} + O(p^2/\mu^2), \end{aligned} \tag{2.67}$$

the correct continuum dispersion relation for a massless Majorana fermion. At the corners of the Brillouin zone, however, $\mu\left[-1 + \sum_{\mu}(1 - \cos(p_\mu))\right] > 0$ and $V_{\mathsf{maj}} \simeq 1$ so that D_{ov} does not have low-lying eigenvalues associated with these states.

pairs for $\lambda \neq \pm 1$.

2.3.4 Exact lattice symmetry for Majorana fermions

As in the Dirac case for the anomalous chiral and parity symmetries, the Majorana GW action respects exact versions of the various anomalous symmetries listed in Table 2.2, with the Jacobians of the transformations reproducing the anomaly $\mathcal{A}$. Here we discuss the exact form respected by the GW operator for each of the symmetries listed in that table. In the next subsection we examine the anomaly equation eq. (2.54) and show how the Jacobians of the exact lattice symmetry transformations correctly reproduce the known continuum anomaly $\mathcal{A}$.

The Majorana GW equation in eq. (2.52) implies an exact Lüscher symmetry for any antisymmetric D which satisfies it. To see this, we can rearrange the Majorana GW relation as

$$\mathsf{D} = \sqrt{Q_\omega}\ \mathsf{D}_\omega\ \sqrt{Q_\omega}^T, \tag{2.68}$$

where $Q_\omega = (1 - \mathsf{D}\mathsf{X}_\omega)$ and $Q_\omega^T = (1 - \mathsf{X}_\omega\mathsf{D})$.

Care must be taken in the definition of the square root. Our convention is to define the square root of Q_ω to be the unique matrix with the same eigenvectors as Q_ω and whose eigenvalues are the square roots of the eigenvalues of Q_ω with non-negative real part. We take the cut for the square root to be along the negative real axis, and for negative real eigenvalues of Q_ω we will either define the corresponding eigenvalues of $\sqrt{Q_\omega}$ to all lie on the positive imaginary or negative imaginary axes, denoting the choice by $\sqrt[\pm]{Q_\omega}$ respectively. We will see in Sec. 2.3.4 that both choices come into play. When giving general arguments we will omit the $\pm$ designation.

Equation (2.68) can be derived by noting that $Q_\omega\mathsf{D} = \mathsf{D}Q_\omega^T$, and so $\sqrt{Q_\omega}\mathsf{D} = \mathsf{D}\sqrt{Q_\omega}^T$. For a discrete symmetry transformation, $Q_\omega = \sqrt{-V_{\mathsf{maj}}^T}$. Therefore, corresponding to the continuum symmetry $\eta \to \omega\eta$, any GW regulated lattice action has an exact Lüscher symmetry

$$\eta \to \omega\sqrt{Q_\omega}^T \eta\ . \tag{2.69}$$

In terms of $\mathsf{D} = \frac{\mathsf{m}}{2}(1 + V_{\mathsf{maj}})$, we can write

$$\begin{aligned}\sqrt{Q_\omega}^T &= [1 - \mathsf{X}_\omega \mathsf{D}]^{1/2} \\ &= \left[\frac{1}{2}(1 + \mathsf{m}_\omega^{-1}\mathsf{m}) - \frac{1}{2}(1 - \mathsf{m}_\omega^{-1}\mathsf{m})V_{\mathsf{maj}}\right]^{1/2}. \end{aligned} \tag{2.70}$$

The low-energy ($\mathsf{m} \to \infty$) limit we have $X_\omega \to 0$ and $Q_\omega \to 1$. The symmetry transformation then reduces to $\eta \to \omega\eta$, as would be expected in the continuum limit.

Although the action is invariant under this symmetry, the fermion measure is, in general, not. The transformation in eq. (2.69) produces a Jacobian $\det(\omega\sqrt{Q_\omega}^T)$. We will see in next subsection that this Jacobian reproduces the correct anomaly, once care is taken with eigenvalues of Q_ω which lie on the cut of the square root. While the exact symmetry in eq. (2.69) is completely general for any (continuous or discrete) symmetry, we will restrict now to the symmetries discussed in Table 2.2 for a single-flavor Majorana. We will also assume V_{maj} is unitary for simplicity, and obeys the properties in Eqs. (2.59) to (2.61), but the arguments can be generalized for the nonunitary case.

Discrete chiral and reflection $\mathbb{Z}_2$ symmetries in $d = 2, 3$

In $d = 2, 3$ a massless $\mathcal{C}$-Majorana has a $\mathbb{Z}_2$ reflection symmetry which is anomalously broken by the regulating mass term. The same is true in $d = 2$ for the discrete chiral symmetry for either type of Majorana.

In all these cases of a $\mathbb{Z}_2$ symmetry broken by the regulator, the mass term flips sign, $\mathsf{m}_\omega \mathsf{m}^{-1} = -1$. In this case $Q_\omega^T = -V_{\mathsf{maj}}$ and the exact symmetry takes the simple form

$$\eta \to \omega\sqrt{-V_{\mathsf{maj}}}\eta. \tag{2.71}$$

where $\omega = \mathcal{R}_1\gamma_1$ for the reflection symmetry and $\omega = \bar{\gamma}$ for the discrete chiral symmetry. We can equally well define the square root as either $\sqrt[\pm]{-V_{\mathsf{maj}}}$ for these discrete symmetries. We will analyze the Jacobian in the next subsection and compare with the continuum anomaly.

The massless $\mathcal{C}$-Majorana in $d = 4$ and the $\mathcal{T}$-Majorana in $d = 2$ also have reflection symmetries R, but they are nonanomalous since a regulating mass term exists which is

R-invariant. In such cases, a GW formulation is trivially invariant under the corresponding continuum symmetry, without any modification.

$U(1)_A$ symmetry in $d=4$

In $d=4$, the continuum $\mathcal{C}$-Majorana fermion has an anomalous continuous $U(1)_A$ symmetry $\eta \to e^{i\alpha\bar{\gamma}}\eta$, since either choice of the regulating mass term breaks this symmetry, as discussed in Table 2.2. Under the $U(1)_A$ transformation $\omega = e^{i\alpha\bar{\gamma}}$, the mass term transforms such that $\mathsf{m}_\omega^{-1}\mathsf{m} = e^{-2i\alpha\bar{\gamma}}$. The exact lattice symmetry of eq. (2.69) can then be simplified to

$$\eta \to e^{i\alpha\bar{\gamma}/2}\left\{\cos\alpha - i\bar{\gamma}V_{\mathsf{maj}}\sin\alpha\right\}^{1/2}\eta. \tag{2.72}$$

In the low-energy limit, $V_{\mathsf{maj}} \to -1$, and this reduces to the continuum symmetry, $\eta \to e^{i\alpha\bar{\gamma}}\eta$.

This continuum $U(1)_A$ for Majorana fermions descends from the anomalous $U(1)_A$ symmetry for Dirac fermions upon imposing a reality condition. However, the Majorana $U(1)_A$ symmetry in eq. (2.72) is distinct from the Dirac case of eq. (2.28). So one might wonder how these two definitions of the symmetry are related. To reconcile this, we note that for Majorana fermions, a straightforward analogy of eq. (2.28) is not possible, since for Dirac fermions we exploited the freedom to transform $\bar{\psi}$ and ψ independently, which is not consistent with the Majorana constraint. However, that choice for how the Dirac fields transform was not unique. To illustrate this, we consider the same example considered in Ref. [15], a Dirac fermion in $d=4$ with $\mathscr{D} = \frac{m}{2}(1+V)$, only assuming that $\mathscr{D}$ obeys the GW equation so that $\bar{\gamma}V\bar{\gamma} = V^{-1}$. The infinitesimal transformation corresponding to eq. (2.28) is

$$\delta\chi = \bar{\gamma}\chi, \qquad \delta\bar{\chi} = \bar{\chi}(-V\bar{\gamma})\ , \tag{2.73}$$

where in the continuum limit ($V \to -1$) this reduces to the conventional chiral symmetry transformation. However the action $\frac{m}{2}\int \bar{\chi}(1+V)\chi$ is invariant under the more general transformation, namely

$$\delta\chi = \bar{\gamma}f(V)\chi, \qquad \delta\bar{\chi} = \bar{\chi}g(V)\bar{\gamma}\ , \tag{2.74}$$

with $f(-1) = g(-1) = 1$, provided that the functions f, g satisfy

$$g(V)V^{-1} = \bar{\gamma} f(V)\bar{\gamma} \tag{2.75}$$

projected on the subspace orthogonal to $V = -1$. Furthermore, one finds that so long as eq. (2.75) is satisfied, the Jacobian of the transformation reproduces the correct anomaly. Equation (2.28) satisfies this with $f = 1$ and $g = -V$; alternatively, a symmetric form compatible with Minkowski spacetime where χ and $\bar{\chi}$ are not independent is $f = g = (1-V)/2$ [15, 50]. It is easily checked that this infinitesimal transformation keeps the Majorana action invariant. This result holds equally well for both $(\mathcal{C}, \mathcal{C})$ and $(\mathcal{C}, \mathcal{T})$ regularizations. However, a drawback with this transformation is that $\bar{\gamma}(1-V)/2$ does not generate a compact $U(1)$ symmetry, its eigenvalues not in general being integer.

Equation (2.75) suggests a different symmetric form consistent with the Majorana constraint, however: $f = g = \sqrt{-V}$, which is precisely eq. (2.69). This choice has the feature that $\bar{\gamma}\sqrt{-V}$ is Hermitian and has ± 1 eigenvalues so that it generates a compact $U(1)$ symmetry; on the other hand, one must take care of the branch cut of the square root, as discussed following eq. (2.68), where we defined $\sqrt[\pm]{-V}$ as $\pm i$ when acting on the eigenstate of V with eigenvalue $V = 1$ which lies on the cut for the square root. Such eigenvalues correspond to the corners of the Brillouin zone for the overlap solution, or infinite momentum for the PV solution. The solution to eq. (2.75) is then $f = \sqrt[+]{-V_{\mathsf{maj}}}$ and $g = \sqrt[-]{-V_{\mathsf{maj}}}$ (or with the $\pm$ reversed). However, this is still not a satisfactory symmetry for the $d = 4$ Majorana fermion because the different treatment of the branch cut for ψ and $\bar{\psi}$ is not consistent with the Majorana constraint, $\psi = \mathcal{C}^\dagger \bar{\psi}^T$.

We are forced then to define the "pseudo-Lüscher symmetry" with $f = g = \sqrt[+]{-V_{\mathsf{maj}}}$ which is consistent with the Majorana constraint, but fails to be a symmetry of the action for $V_{\mathsf{maj}} = +1$ eigenstates. This is a failure at short distance and does not destroy the desirable feature of Lüscher symmetry that chiral symmetry violating operators can only be multiplicatively renormalized. One does lose the feature that the Jacobian of the transformation reproduces the correct anomaly, as there now appears a spurious contribution $2(\bar{n}_+ - \bar{n}_-)$ where $\bar{n}_\pm$ are the number of $\pm$ chirality $V_{\mathsf{maj}} = \mathbf{1}$ modes, but this is exactly compensated by a symmetry violation in the action under such a transformation. Typically, the chiral anomaly comes from a transformation under which the action is invariant but the measure is not, so that the path integral acquires a phase under a transformation which is classically a symmetry. Although this symmetry is violated in $V = \mathbf{1}$ subspace, the path integral acquires the same phase under such a transformation as it would choosing $f = \sqrt[+]{-V_{\mathsf{maj}}}$ and $g = \sqrt[-]{-V_{\mathsf{maj}}}$. Integrating over such modes one recovers the expected anomalous Ward-Takahashi identities, so that this symmetry has the same properties as any anomalous quantum symmetry.

If gauge fields or other parameters in the theory are varied such that an eigenvalue of V_{maj} passes through $+\mathbf{1}$, the operator $\sqrt[+]{-V_{\mathsf{maj}}}$ will be discontinuous. Because of this nonanalyticity, our $U(\mathbf{1})_A$ transformation is nonlocal in spacetime, thereby evading a recent no-go theorem [51]. As we showed at the end of Sec. 2.3.3, however, the eigenvalues of V_{maj} are doubly degenerate, and therefore the determinant of $\sqrt[+]{-V_{\mathsf{maj}}}$ is continuous at such points.

2.3.5 The anomaly equation

We have seen in eq. (2.54) that the anomaly equation gives $e^{i\mathcal{A}} = (\det \frac{\mathsf{m}_\omega}{\mathsf{m}} Q_\omega)^{-1/2}$. On the other hand, the exact symmetry of the GW operator is not symmetry of the path integral measure and gives rise to a Jacobian $\mathbf{1}/\det(\omega\sqrt{Q_\omega}^T)$. The first thing we will show is that these are equivalent. Note that the square of the anomaly from eq. (2.54) is clearly equal to the square of the Jacobian, so these two agree up to a sign. It is easy to see that the anomaly equation and the Jacobian agree for any infinitesimal symmetry transformation, and so it is only the case of discrete symmetries that needs careful examination.

For the anomalous discrete symmetries in Table 2.2 we have $\mathsf{m}_\omega \mathsf{m}^{-1} = -\mathbf{1}$ and so eq. (2.70) gives us $Q_\omega^T = -V_{\mathsf{maj}}$. The matrix V_{maj} has eigenvalues $-e^{i\theta_n}$ with $-\pi < \theta_n \leq \pi$, where the θ_n are doubly degenerate and which occur in $\pm$ pairs for $\theta_n \neq \mathbf{0}, \pi$ (due to reflection and chiral symmetry in odd and even dimensions, respectively). Thus there we can write

$$\dim V_{\mathsf{maj}} = \nu_+ + \nu_- + \nu_c \ , \tag{2.76}$$

where $\nu_\pm$ are the numbers of eigenvalues of V_{maj} equal to ± 1 and ν_c is the number of complex eigenvalues (the $\pm$ is not related to chirality). Here, $\nu_\pm$ are even integers and ν_c is a multiple of 4. The eigenvalues of $\sqrt[+]{-V_{\mathsf{maj}}}$ are then $e^{i\theta_n/2}$ and only the $\theta_n = \pi$ eigenvalues contribute nontrivially to its determinant, so that $\det \sqrt[+]{Q_\omega}^T = i^{\nu_+} = (-1)^{\nu_+/2}$. Since ν_+ is even and $i^{\nu_+} = (-i)^{\nu_+}$, it makes no difference which of the two definitions of the square root $\sqrt[\pm]{-V_{\mathsf{maj}}}$ is used. The matrix ω is traceless and squares to 1, so $\det \omega = (-1)^{\dim V_{\mathsf{maj}}/2}$. Thus we get

$$\det(\omega \sqrt{Q_\omega}^T) = (-1)^{\dim V_{\mathsf{maj}}/2}(-1)^{\nu_+/2} = (-1)^{\nu_-/2} \,, \tag{2.77}$$

where we used eq. (2.76). Since ν_- corresponds to the zeromodes of D, we find that the Jacobian of our exact symmetry yields the mod 2 index of D. In comparison, for our anomaly equation in eq. (2.54) we compute $\sqrt{\det \mathsf{m}_\omega \mathsf{m}^{-1} Q_\omega} = \sqrt{\det V_{\mathsf{maj}}}$ which directly gives the same result, $(-1)^{\nu_-/2}$, since only the -1 eigenvalues of V_{maj} contribute.

2.3.6 Examples

In this section, we present examples of the Majorana anomaly equation in which the GW construction reproduces global anomalies of Majorana fermions. In all the examples below we have $\mathsf{m}_\omega^{-1}\mathsf{m} = -1$ and $\mathsf{X}_\omega = 2\mu^{-1}\mathcal{M}$, so the specification of $(\mathcal{K}, \mathcal{M})$ matrices completely fixes the GW equation and its solutions.

Two dimensions

In two dimensions it is possible to have either a 2-component $\mathcal{C}$- or $\mathcal{T}$-Majorana fermion, but only $\mathcal{C}$ can be chosen as the mass term in the regulator. In this section, we show that the GW formulation reproduces known nonperturbative anomalies for both these theories.

The continuum $\mathcal{T}$-Majorana theory with the action $\int \eta^T \mathcal{T} \not{D} \eta + m\eta^T \mathcal{C} \eta$ corresponds to the field theory of the Kitaev chain. This has an exact (nonanomalous) reflection symmetry with $\mathsf{R}^2 = (-1)^F$ (equivalent to $\mathsf{T}^2 = 1$ in Minkowski space), but the mass term breaks a discrete chiral symmetry: $\eta \to \bar{\gamma}\eta$, suggesting an anomaly for the discrete chiral symmetry. Indeed, the anomaly is given by the mod-2 index of the Dirac operator on modes of one chirality [52, 9, 53]. With the choice $(\mathcal{K}, \mathcal{M}) = (\mathcal{T}, \mathcal{C})$, we can formulate GW equation for the massless $\mathcal{T}$-Majorana fermion and solutions to it. The exact Lüscher symmetry corresponds to $\eta \to \bar{\gamma}\sqrt{-V_{\mathsf{maj}}}$. As shown in the previous section, the Jacobian gives $\det \omega\sqrt{-V_{\mathsf{maj}}} = (-1)^{\nu_-/2}$, where ν_- is the number of modes with $V_{\mathsf{maj}} = -1$, which correspond to exact zeromodes of D. We have seen in Sec. 2.3.3 that $\bar{\gamma} V_{\mathsf{maj}} \bar{\gamma} = V_{\mathsf{maj}}^\dagger$, so the $V_{\mathsf{maj}} = -1$ eigenmodes can be taken to be simultaneous eigenstates of $\bar{\gamma}$. We also showed that the eigenvalues of V_{maj} are doubly degenerate with eigenfunctions ψ and $\mathcal{M}^\dagger \psi^*$. The $d = 2$ relation $\mathcal{M}\bar{\gamma}\mathcal{M}^{-1} = -\bar{\gamma}^T$ then tells use that the eigenvalues of the $V_{\mathsf{maj}} = -1$ eigenmodes come in $\pm$ chiral pairs. Thus, we can write $\nu_- = n_+ + n_- = 2n_+$, where $n_\pm$ are the number of positive and negative chirality zero modes of D. Therefore, the Jacobian of the discrete chiral Lüscher symmetry reduces to $(-1)^{n_+}$, which is precisely the continuum result. On a torus with periodic boundary conditions in both directions, $n_+ = n_- = 1$, and therefore we find a nontrivial anomaly.

Next we consider the case of a single $\mathcal{C}$-Majorana fermion in $d = 2$. This theory has a reflection symmetry $\mathsf{R}\eta(x) = \gamma_1 \eta(\tilde{x})$ with $\mathsf{R}^2 = 1$ and a discrete chiral symmetry, but the $\mathcal{C}$ mass term violates them both. It is known that this theory has a mixed anomaly between R and $(-1)^F$ symmetry which can be detected in the continuum by computing a mod-2 index on a two-dimensional unorientable manifold [9]. In the GW formulation defined with $(\mathcal{K}, \mathcal{M}) = (\mathcal{C}, \mathcal{C})$, this can again be obtained simply from the Jacobian of the exact reflection symmetry for the GW Majorana fermion. The Lüscher symmetry is $\eta \to \gamma_1\sqrt{-V_{\mathsf{maj}}}\eta(\tilde{x})$. By the same argument as before, the Jacobian for this symmetry reduces to $(-1)^{\nu_-/2}$. On a torus with periodic boundary conditions, we have two zero modes. Then $(-1)^{\nu_-/2} = -1$ and therefore the measure acquires a sign under the reflection symmetry.

One dimension

In one dimension, fermi statistics forbid any mass term for a $N = 1$ flavor 1-component Majorana, To apply the GW construction, we therefore need at least $N = 2$ flavors, which allows for the choice $(\mathcal{K}, \mathcal{M}) = (1, \tau_2)$ with the continuum action $S = \int \eta^T \partial_0 \eta + \mu\, \eta^T \tau_2 \eta$, where $\eta^T = (\eta_1, \eta_2)$ and $\eta_{1,2}$ are one-component Majoranas. Note that the kinetic term is invariant under a $\mathsf{R}^2 = (-1)^F$ reflection symmetry which acts as $\mathsf{R}\eta(t) = i\eta(-t)$, but the mass term is odd under this symmetry. Indeed, this system corresponds to the edge modes of the Fidkowski-Kitaev chain and is afflicted by a well-known $\mathbb{Z}_8$ anomaly between R and $(-1)^F$ [54, 9].

With $(\mathcal{K}, \mathcal{M}) = (1, \tau_2)$, we can proceed with the GW construction for $N = 2$ flavors. If n_a is the number of zero modes corresponding to flavor a, the antisymmetric mass matrix $\mathcal{M} = \tau_2$ ensures a doubling of spectrum and $n_1 = n_2$. As before, the Jacobian for the exact reflection symmetry produces a phase of $(-1)^{\nu_-/2}$ and $\nu_- = 2n_1$. Since $n_1 = n_2 = 1$ on a circle with periodic boundary conditions, this represents an anomaly. It is interesting to note that since for two flavors we find a $\mathbb{Z}_2$ anomaly, the GW formulation implies a $\mathbb{Z}_4$ anomaly for a single Majorana flavor, even though a mass term cannot be written in such a theory. The correct answer though is that there should be a $\mathbb{Z}_8$ anomaly. See a discussion in Ref. [55], Eq. (2.26), which suggests that the $\mathbb{Z}_4$ follows from being insensitive to a bosonic anomaly.

2.4 Conclusions

The early work on anomaly descent equations [56, 57, 58] and their embodiment in the bulk/boundary correspondence of gapped fermions [59] has been greatly expanded upon in recent years with the discussions about more general classes of topological materials and a wider variety of anomalies (see, for example, [42]). A parallel development from lattice gauge theory had shown that for the case where the boundary theory is described by a Dirac fermion, one can describe the physics, including chiral anomalies, in terms of a theory that makes no reference to the bulk. Such a theory is governed by the Ginsparg-Wilson equation [4] which has an explicit solution in the form of the overlap operator [3]. In this chapter we have shown how to generalize the GW analysis to encompass a wide range of topological materials that have been classified in the condensed matter literature, focusing on topological superconductors with Majorana edge states, which are less familiar to those working in lattice gauge theory. In each case we have generalized the notion of a Lüscher symmetry: an exact symmetry of the lattice action which becomes identical to the continuum symmetry in the continuum limit, under which the the lattice integration measure transforms by the appropriate phase to account for the anomaly. The class of theories for which we can derive GW relations contain only those for which a fermion mass term can be included, and therefore does not include chiral gauge theories, for example.

Open questions remain. In particular the Dai-Freed anomalies discussed in the literature [60, 9, 61] do not seem apparent in this approach. Thus, for example, one of the results in this work was the derivation of a $\mathbb{Z}_4$ discrete time-reversal anomaly for the Fidkowski-Kitaev Majorana chain, but not the full $\mathbb{Z}_8$ anomaly known to be correct [61]. On the other hand, we know that the overlap operator which solves the GW equation is derived by integrating out bulk modes from a higher-dimension theory [2, 29], which one would expect "knows" about such anomalies.

The solutions presented here for the generalized GW are all formulated in Euclidean spacetime, and are not amenable to a Hamiltonian description of the physics in continuous time. Furthermore, not being ultralocal in Euclidean time makes the derivation of a transfer matrix and Hamiltonian problematic. We note, though, that we defined the anomalous $U(\mathbf{1})_A$ pseudo-Lüscher symmetry that acts on ψ and $\bar{\psi}$ in a way consistent with a Minkowski interpretation, and find that it is not analytic in momentum, and hence not a local operator in spacetime, evading the no-go theorem in Ref. [51]. Pursuing a Hamiltonian formulation of the ideas presented here in order to render the results more applicable to real condensed matter systems seems like another avenue to explore in the future.

Finally, while it has been assumed that the fermions we consider are propagating in smooth, background gauge and gravitational fields, we have not examined in any detail the role played by the role played by unorientable manifolds, which are understood to play an important role in understanding the reflection (time-reversal) anomalies [61].

Chapter 3

A LATTICE HAMILTONIAN FOR GINSPARG-WILSON FERMIONS

3.1 Introduction

There are a number of computations in QCD that require a good realization of chiral symmetry. These include the color-flavor-locking phase [63] and the chiral symmetry restoring phase transition [64], both of which one would hope to be able to study on the lattice. The overlap operator [23, 28] in the Euclidean Lagrangian formulation offers the ideal realization of lattice chiral symmetry in the form of Lüscher symmetry [15], a lattice symmetry which tends toward chiral symmetry in the continuum limit. However, computations using the path integral are afflicted by sign problems. A Hamiltonian approach on a quantum computer might be able to solve these issues, but there currently does not exist a Hamiltonian for GW fermions.

Chiral symmetry can be expected to fail on the lattice because the lattice spacing introduces a mass scale, and masses violate chiral symmetry. This can be made more precise by the Nielsen-Ninomiya no-go theorem [22]; there is no lattice Dirac operator $\mathscr{D}$ in 4 spacetime dimensions which has chiral symmetry, i.e. satisfies

$$\{\gamma_5, \mathscr{D}\} = 0, \tag{3.1}$$

and has other desirable features, namely the correct continuum limit, freedom from doublers, and locality [15]. Ginsparg and Wilson [4] suggested that this should be replaced by:

$$\{\gamma_5, \mathscr{D}\} = a\mathscr{D}\gamma_5\mathscr{D}, \tag{3.2}$$

This work is based on a letter published in Phys. Rev D. [62]

so that exact chiral symmetry fails at the order of the lattice spacing a. The first method for putting chiral fermions on the lattice involved edge states of a domain wall defect in one higher dimension [2]. Neuberger and Narayanan [23, 28] found that this system could be studied in 4 dimensions via the "overlap" operator,

$$\mathscr{D} = \frac{M}{2}\left(1 + V\right), \quad V = \frac{D_w}{\sqrt{D_w^\dagger D_w}}, \tag{3.3}$$

where $M = 1/a$ is the inverse lattice spacing, and D_w is the 4-dimensional Wilson Dirac operator, which, in the absence of gauge fields, can be written in momentum space as:

$$D_w = i\sum_{\mu=1}^{4} \gamma^\mu \sin(p_\mu/M) - 1 + \sum_{\mu=1}^{4}(1 - \cos(p_\mu/M)). \tag{3.4}$$

This operator has the correct continuum limit, and is not hermitian, but is instead "γ_5-hermitian":

$$\gamma_5 \mathscr{D} \gamma_5 = \mathscr{D}^\dagger. \tag{3.5}$$

In fact, it can be quickly checked that any operator of the form

$$\mathscr{D} = \frac{M}{2}\left(1 + V\right); \quad \gamma_5 V \gamma_5 = V^\dagger, \quad V^\dagger V = I, \tag{3.6}$$

satisfies Eq. (3.2) [3]. In this case we say $\mathscr{D}$ is an overlap operator, though in general it may not necessarily be constructed in terms of a state overlap. Lüscher [15] first observed that this operator has the following symmetry:

$$\delta\psi = \gamma_5 \left(\frac{1-V}{2}\right)\psi, \quad \delta\overline{\psi} = \overline{\psi}\left(\frac{1-V}{2}\right)\gamma_5. \tag{3.7}$$

In the continuum limit, this becomes chiral symmetry. Lüscher noted that the Jacobian of this transformation produces the index of $\mathscr{D}$, a lattice version of the Fujikawa calculation [16, 15] of the chiral anomaly. Indeed, there is a good deal of freedom in defining this Lüscher symmetry; hereafter we will refer to any symmetry

$$\delta\psi = \Gamma\psi, \quad \delta\overline{\psi} = \overline{\psi\Gamma}, \tag{3.8}$$

for which

$$\lim_{M\to\infty} \varGamma = \lim_{M\to\infty} \overline{\varGamma} = \gamma_5, \tag{3.9}$$

as a Lüscher symmetry[15], provided its Jacobian reproduces the index of $\mathscr{D}$.

To find a Hamiltonian describing a GW fermion, one may try to compute the transfer matrix of Eq. (3.3) directly, but this involves square roots of the time derivative and is therefore challenging. Creutz et al. [50] considered the following construction. First, define the 3-dimensional overlap operator

$$d = \frac{M}{2}\left(1+v\right), \quad v = \frac{d_w}{\sqrt{d_w^\dagger d_w}}, \tag{3.10}$$

where d_w is the 3-dimensional analogue of Eq. (3.4):

$$d_w = i\sum_{i=1}^{3} \gamma^i \sin(p_i/M) - M + \sum_{i=1}^{3}(1-\cos(p_i/M)), \tag{3.11}$$

and γ^i are 4×4 Clifford algebra matrices. Then by analogy with the continuum Hamiltonian

$$H_\psi^c = \int d^3x \psi^\dagger i\gamma^0\gamma^i D_i\psi, \tag{3.12}$$

(where D_i, H_ψ^c denote the continuum covariant derivative and continuum Hamiltonian, respectively), it is reasonable to identify $\gamma^i D_i$ with the 3-dimensional Dirac operator, and formulate a lattice prescription for a Hamiltonian via the replacement $\gamma^i D_i \to d$:

$$H_\psi \equiv \psi^\dagger i\gamma^0 d\psi. \tag{3.13}$$

This system has the symmetry of Eq. (3.7), and associated to that symmetry is the charge

$$Q_5 = \psi^\dagger\gamma_5\left(\frac{1-V}{2}\right)\psi. \tag{3.14}$$

This chiral charge is conserved with respect to H_ψ, i.e. $[H_\psi, Q_5] = 0$, but upon introduction of the gauge field Hamiltonian

$$H_g = \frac{1}{2}(E^2+B^2), \tag{3.15}$$

one finds $[H_g, Q_5] \neq \mathbf{0}$, since E^2 involves derivatives with respect to the gauge fields in the quantized theory, and the V appearing in Q_5 involves link variables.

It is important to note that the Hamiltonian considered by Creutz et al. is not derived from a GW fermion in the Euclidean Lagrangian; it is simply an ansatz. If it were, it would enjoy a full Lüscher symmetry that descends to the Hamiltonian formulation, even in the presence of gauge fields.

Therefore it is sensible to start at the level of the Lagrangian, with a modified overlap operator which still solves the GW equation, but from which the extraction of a Hamiltonian is considerably easier. It is simpler to consider a theory which is fractional linear in time derivatives, i.e. a rational expression linear in time derivatives. The feasibility of such an approach will become clear by construction of an overlap operator in the continuum with ghosts, namely a Pauli-Villars regulated fermion.

In Section 3.2, we will describe the way in which Pauli-Villars fermions satisfy the GW relation, and the Hamiltonian and Lüscher symmetry associated to them. In Section 3.3.1 we will derive a Lagrangian describing a GW fermion in discrete space and continuous time, and generalizing the arguments of Section 3.2 we will derive a Hamiltonian describing the system. In Section 3.3.2 we will describe the properties of this Hamiltonian.

3.2 Pauli-Villars as Overlap

In a recent paper generalizing the GW relation [24], it was found that the GW equation holds for a Pauli-Villars regulated fermion in the continuum. We will derive the Hamiltonian for this example, as it is instructive for generalization to the lattice.

A Pauli-Villars regulated fermion is equivalent to a Lagrangian with the following Dirac operator:

$$\mathcal{L} = \overline{\psi} \mathcal{D} \psi, \quad \mathcal{D} = M \frac{\not{D}}{\not{D} + M}, \tag{3.16}$$

where $\not{D}$ is the usual Euclidean Dirac operator, $\not{D} = \gamma^\mu D_\mu$. This may be rewritten

$$\mathcal{D} = \frac{M}{\mathbf{2}} \left(\mathbf{1} + V \right), \quad V = \frac{\not{D}/M - \mathbf{1}}{\not{D}/M + \mathbf{1}}. \tag{3.17}$$

This $\mathscr{D}$ satisfies Eq. (3.6), so it is an overlap operator. For reasons that will become clear shortly, is helpful to define $A = \not{D}/M - \mathbf{1}$, and note V is of the form:

$$V = -A^{-1}A^{\dagger}; \quad \gamma_5 A \gamma_5 = A^{\dagger}. \tag{3.18}$$

In order to make this theory look familiar, we introduce ghost fields $\phi, \overline{\phi}$ with opposite statistics to the action, so that the full Lagrangian is

$$\mathcal{L}_{tot} = \overline{\psi}\frac{M}{\mathbf{2}}A^{-1}(A - A^{\dagger})\psi + \overline{\phi}\phi. \tag{3.19}$$

We perform the simultaneous change of variables

$$\overline{\psi}' = \overline{\psi}A^{-1}, \quad \overline{\phi}' = \overline{\phi}A^{-1}. \tag{3.20}$$

This change of variables has trivial Jacobian in the path integral because of the opposite statistics. Under this change of variables the Lagrangian becomes

$$\mathcal{L}_{tot} = \overline{\psi}'\not{D}\psi + \overline{\phi}'(\not{D} + M)\phi. \tag{3.21}$$

Consider how a Lüscher symmetry $\Gamma, \overline{\Gamma}$ on $\psi, \overline{\psi}$ is affected by this change of variables. Γ is unaffected, while the new $\overline{\Gamma}$ is related to the original by

$$\overline{\Gamma}' = A\overline{\Gamma}A^{-1}. \tag{3.22}$$

In particular, consider the choice:

$$\Gamma = \gamma_5, \quad \overline{\Gamma} = -V\gamma_5. \tag{3.23}$$

Since V is of the form Eq. (3.18), Eq. (3.22) becomes

$$\overline{\Gamma}' = A^{-1}A^{\dagger}\gamma_5 A^{-1} = A^{\dagger}\gamma_5 A^{-1} = \gamma_5 A A^{-1} = \gamma_5. \tag{3.24}$$

In summary, the Pauli-Villars fermion described in Eq. (3.16), with the Lüscher symmetry of Eq. (3.23) descends to a massless fermion with ordinary chiral symmetry and a heavy ghost fermion where the symmetry acts trivially. The Hamiltonian of the theory is thus

$$H^c = H^c_{\psi} + H^c_{\phi}, \tag{3.25}$$

where

$$H^c_\psi = \int d^3x \psi^\dagger i\gamma^0\gamma^i D_i \psi, \tag{3.26}$$

$$H^g_\phi = \int d^3x \phi^\dagger i\gamma^0\gamma^i D_i \phi - \phi^\dagger \gamma^0 M \phi. \tag{3.27}$$

In order to study the dynamics of H^c_ψ alone, one must work in the vacuum to vacuum sector of the ghost theory. Excitations with energy less than the regulator mass M (later taken to infinity) do not contribute to scattering amplitudes involving the ψ field. Further discussion on Pauli-Villars fermions can be found in any introductory text on quantum field theory (e.g. [65]). They are not typically dealt with in a Hamiltonian formalism. The Pauli-Villars regularization prescription is simply a helpful tool, as it allows one to regulate the UV divergence that comes from the introduction of a continuous time coordinate, and subsequently describe the low energy modes in a simple effective way.

3.3 Combined Overlap

3.3.1 Overlap Lagrangian

Now we work in continuous time and latticized space. Since the Pauli-Villars and overlap solutions apply to continuum and lattice cases of an overlap operator respectively, it is reasonable to try to write an ansatz for $\mathscr{D}$ which combines the forms of Eqs. (3.17) and (3.3). Such an operator is determined by a choice of unitary and γ_5-hermitian V. Recall the 3-dimensional analogue v in Eq. (3.10). Since the low energy spectrum of v is $-\mathbf{1} + i\not{p}/M$ in the free theory, a reasonable ansatz incorporating Pauli-Villars regularization might be:

$$V = \frac{\gamma^0\partial_t + Mv}{\gamma^0\partial_t - Mv^\dagger}. \tag{3.28}$$

Here when we write the quotient, we mean left-multiplication by the inverse of the denominator, as in Eq. (3.18). Note that the V of Eq. (3.28) also satisfies the relations of Eq. (3.18). Furthermore V is γ_5-hermitian, unitary, and has the correct low energy spectrum. However, this V has doublers: note that zero-modes of $\mathscr{D}$ correspond to $-\mathbf{1}$-modes of V, and therefore generally an overlap operator has doublers if there are any $V = -1$ modes away from the origin in the Brillouin zone. Note that at $v = 1$, $V = -1$, and so the free theory already has doublers at $\vec{p}_i = \pi/a$. This is because at $\partial_t = 0$, $V = -v/v^\dagger$. Since complex conjugation treats the $v = \pm 1$-modes identically, doublers arise at $\partial_t = 0$. Therefore, in order to find a V without doublers, the $v = \pm 1$ modes need to be treated differently under conjugation. One way to do this is to replace

$$v \to -\sqrt{-v} \equiv \xi. \tag{3.29}$$

Then V becomes instead

$$V = \frac{\frac{1}{2}\gamma^0 \partial_t - M\xi}{\frac{1}{2}\gamma^0 \partial_t + M\xi^\dagger}. \tag{3.30}$$

We define the square root $\sqrt{u}$ of a unitary matrix u generally as the unique matrix whose log spectrum lies in the interval $(-i\pi/2, i\pi/2]$, and which squares to u; this can be equivalently defined as the matrix whose eigenvalues are the square root of the eigenvalues of u, with the same eigenvectors. Such a definition involves choosing a branch cut, namely $\sqrt{-1} = i$, and therefore a discontinuity at the edge of the BZ (which introduces non-locality). Note $\sqrt{u^\dagger} \neq \sqrt{u}^\dagger$, but instead

$$\sqrt{u^\dagger} = \sqrt{u}^\dagger + 2iP_{-1}, \tag{3.31}$$

where P_{-1} is the projector onto the -1 modes of u.

It is worth noting the denominator is not invertible at $v = 1$ and $\partial_t = 2iM$. However, it can be seen that $V = 1$ in this limit. In the free case, this guarantees $\mathscr{D}(p)$ is continuous at the edge of the Brillouin zone, but higher order derivatives are discontinuous, so this theory is nonlocal. It should be noted this nonlocality is not so severe that it precludes definition of a gauge theory entirely, as one can still simply replace the free derivative with the covariant derivative in the operator Eq. (3.4) (in the position space representation).

In most cases, the complication which arises in Eq. (3.31) can simply be ignored, and the $V = 1$ case can be treated separately. In particular, we will simply write

$$\gamma^5 \xi \gamma^5 = \gamma^0 \xi \gamma^0 = \xi^\dagger. \tag{3.32}$$

The operator $V = -A^{-1}A^\dagger$ in Eq. (3.30) is, in general, not unitary. Unitarity of V follows from normality of A. However, for time dependent gauge fields,

$$A^\dagger A - AA^\dagger = \frac{1}{2}\gamma^0 \partial_t(\xi + \xi^\dagger) \neq 0 \tag{3.33}$$

Furthermore, V is not in general γ_5-hermitian. Instead, note that the weaker statement

$$\gamma^5 A \gamma^5 = A^\dagger \tag{3.34}$$

still holds, so that for general gauge fields it holds that

$$\gamma^5 V \gamma^5 = V^{-1}. \tag{3.35}$$

As noted in [24], the operator $\mathcal{D} = \frac{1}{2}(1 + V)$ still satisfies the relation Eq. (3.2), and still enjoys Lüscher symmetries, e.g. Eq. (3.23). Under the Lüscher symmetry, the anomalous symmetry violation of the expectation value of an observable $\mathcal{O}$ is given by (cf. [15]):

$$\langle \delta \mathcal{O} \rangle = (-\mathrm{Tr}\,\gamma_5 \mathcal{D} + \mathrm{Tr}\,\gamma_5)\,\langle \mathcal{O} \rangle \tag{3.36}$$

The quantity $\mathrm{Tr}\,\gamma_5 D$ can be evaluated using the arguments of [15];

$$\mathrm{Tr}\,\gamma_5 \mathcal{D} = \mathrm{Tr}\,\gamma_5 + 2\,\mathrm{ind}\,\mathcal{D}, \tag{3.37}$$

therefore,

$$\langle \delta \mathcal{O} \rangle = 2\mathrm{ind}\,\mathcal{D} \langle \mathcal{O} \rangle, \tag{3.38}$$

so this operator has the correct anomaly in the Lagrangian formulation. It is worthwhile to note that evaluation of $\mathrm{Tr}\,\gamma_5 \mathcal{D}$ in [15] relies on the point 0 being an isolated point of the spectrum of $\mathcal{D}$. This can clearly be done when the spacetime lattice is finite, since the operator $\mathcal{D}$ only has a finite number of eigenvalues. In the present case, the eigenvalues ω of ∂_t have no UV cutoff. However, an IR cutoff is sufficient: the eigenvalues ω become discrete. For very large ω, $V \sim 1$. Therefore the lack of a UV cutoff in time adds a dense set of eigenvalues near $\mathcal{D} = 1$. Any eigenvalues at $\mathcal{D} = 0$ remain isolated.

Before proceeding with the prescription of Section 3.2, we integrate out in the path integral the modes at $V = \mathbf{1}$, since they are at high energy and contribute only to overall normalization. They come in chiral pairs and are thus also invariant under the Lüscher symmetry of Eq. (3.23) so cannot contribute to the anomaly.

Therefore, the operator $\mathscr{D} = \frac{1}{2}(\mathbf{1} + V)$, with the choice of V in Eq (3.30) satisfies the Ginsparg-Wilson equation, and has a Lüscher symmetry which correctly realizes the anomaly.

3.3.2 Overlap Hamiltonian

The Hamiltonian following the prescription in Section 3.2 can be seen to be

$$\begin{aligned} &H = H_\psi + H_\phi, \\ &H_\psi = \psi^\dagger h_\psi \psi, \quad H_\phi = \phi^\dagger h_\phi \phi. \end{aligned} \tag{3.39}$$

where

$$\begin{aligned} h_\psi &= M\gamma^0 \left(\xi^\dagger - \xi\right), \\ h_\phi &= \mathbf{2}M\gamma^0\xi^\dagger, \end{aligned} \tag{3.40}$$

and repeated (suppressed) indices are summed over. The ghost fields here have been rescaled to be canonically normalized. The ghost Hamiltonian H_ϕ is clearly gapped; since it is Hermitian and unitary its eigenvalues are all of magnitude $\mathbf{2}M$. Therefore, we consider only ghost vacuum-to-vacuum amplitudes of the combined quantum system at energy scales much lower than the cutoff M, which are described by H_ψ.

Following the analysis in Section 3.2, the Lüscher symmetry in Eq. (3.23) descends once again to ordinary chiral symmetry, so that the chiral charge is

$$Q_5 = \psi^\dagger \gamma^5 \psi. \tag{3.41}$$

It is evident that both h_ψ and h_ϕ are Hermitian matrices. Chiral charge is conserved for the ψ fermions, and violated for the ghost fermions:

$$[\gamma^5, h_\psi] = \mathbf{0}, \quad [\gamma^5, h_\phi] = \mathbf{4}M\gamma^5\gamma^0(\xi^\dagger - \xi). \tag{3.42}$$

This is consistent with anomalous chiral symmetry violation of Pauli-Villars fermions in the continuum, where the mass term in the ghost fermion violates chiral symmetry explicitly. The gauge field Hamiltonian in Eq. (3.15) trivially commutes with γ_5, so that the chiral charge is conserved in the full theory describing the light ψ modes. This is in contrast to the Creutz et al. [50] Hamiltonian of Eq. (3.13), since the chiral charge in Eq. (3.41) no longer involves the overlap operator as in Eq. (3.14), but is in direct analogy with the continuum chiral charge.

It is easily checked that h_ψ has the right continuum limit, since $v \to -1 + i\not{p}$. The same holds true in the presence of gauge fields which are sufficiently smooth.

It is illuminating to consider replicating this construction in $d = 1 + 1$. One finds $v = -e^{-ip\gamma_1/2M}$, and in this case the free Hamiltonian is explicitly

$$h_\psi = 2M\gamma_\chi \sin p/4M, \tag{3.43}$$

where $\gamma_\chi = i\gamma_1\gamma_2$ is the 3-dimensional analogue of γ_5. This matches the continuum Hamiltonian for a free 2-component Dirac spinor in the low energy limit.

3.4 Conclusions

We have derived a Hamiltonian for a massless fermion on a spatial lattice with exact chiral symmetry from a spatial-lattice plus continuous-time Lagrangian for a GW fermion, starting with an overlap operator with Lüscher symmetry and introducing ghosts. This came at the expense of locality.

Chapter 4

CHIRAL DOMAIN WALL FERMIONS ON S^4

4.1 Introduction

Recently it was pointed out [21] that manifolds with a single boundary that host Weyl fermion edge states could serve as a model for simulating chiral gauge theories on the lattice. That similar phenomena could occur on a spacetime lattice follows from the fact that such edge states occur at the boundary between topological phases, and that similar topological phases are found with conventional Wilson fermions. Previously, the geometry of a solid cylinder was considered ($B^1 \times R^{d-1}$, where B^1 represents a solid disk). When formulated on a square lattice, this treats one of the dimensions of the boundary differently than the others, explicitly breaking the underlying hypercubic symmetry of the lattice. In this chapter we consider spherical continuum geometries, with Weyl fermions confined to S^2 or S^4 boundaries. The motivation is that when translated to a lattice system, such a formulation for chiral gauge theories will preserve the underlying hypercubic symmetry of the lattice, which could prove useful when taking the continuum limit.

An interesting feature shown for the disk was that, while the bulk modes have integer orbital angular momentum and half-integer spin, the edge state momenta around the circular boundary are quantized as half-integers, defying the possible intuition that edge momentum should be ℓ/R, where ℓ is the (integer) orbital angular momentum on the disk. The main result of this chapter is that a similar phenomenon results for edge states on all even-dimensional spheres: the edge mode looks like a conventional Weyl fermion on a sphere with double-valued wave functions, even though the bulk fermion is expanded in single-valued normal modes.

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4.2 Fermions on curved manifolds

We begin with a review of fermions on curved manifolds [66], so as to give the domain wall solutions something to be compared to. On a d dimensional manifold with coordinates x^i, one needs a spin connection ω to describe parallel transport of spinors. These require a vielbein, i.e. an orthonormal frame for the cotangent space, which we denote e^i.[1] For example, a vielbein for S^2 would be

$$e^1 = d\theta, \quad e^2 = \sin\theta d\phi, \tag{4.1}$$

and a vielbein for S^1 would simply be $e^1 = d\phi$.[2]

We use the torsion free Levi-Civita spin connection, which is the unique antisymmetric tensor $\omega^a{}_b$ satisfying [66]:

$$de^a + \omega^a{}_b \wedge e^b = 0, \tag{4.2}$$

where d is the differential and $\wedge$ is the wedge product. Then one uses any set of Clifford algebra generators γ^i:

$$\{\gamma^i, \gamma^j\} = 2\delta_{ij}, \tag{4.3}$$

where the γ^i are coordinate-independent matrices. Equipped with these may write the covariant derivative (with gauge fields):

$$D_\mu = \partial_\mu - iA_\mu + \frac{1}{2}\omega^a{}_{b\mu}\Sigma_{ab}, \tag{4.4}$$

with $\Sigma_{ab} = \frac{1}{4i}[\gamma_a, \gamma_b]$. The Dirac operator is then

$$D = \gamma^a E_a^\mu D_\mu. \tag{4.5}$$

[1] in $d = 2$ and $d = 4$, respectively, these are referred to as zweibeins and vierbeins, respectively, which we shall adopt.

[2] This is an abuse of notation, using e^1 to denote a vielbein basis element on both S^1 and S^2.

where E_a is the inverse vielbein.[3]

For example, the spin connection on S^1 vanishes. This may seem unexpected, but recall that the spin connection describes parallel transport, and there is an unambiguous path-independent way of transporting a vector along the circle. The spin connection on the sphere with zweibein $e^1 = d\theta, e^2 = r\sin\theta d\phi$ is given by:

$$\omega^{12} = -\cos\theta d\phi. \tag{4.6}$$

For $d = 4$, the vielbein is:

$$e^1 = d\theta^1, \tag{4.7}$$

$$e^2 = \sin\theta^1 d\theta^2, \tag{4.8}$$

$$e^3 = \sin\theta^1 \sin\theta^2 d\theta^3, \tag{4.9}$$

$$e^4 = \sin\theta^1 \sin\theta^2 \sin\theta^3 d\theta^0. \tag{4.10}$$

where the angles $\theta^1, \theta^2, \theta^3 \in [0, \pi)$ are polar and $\theta^0 \equiv \theta^4 \in [0, 2\pi)$ is azimuthal. The spin connection ω on S^4 is computed in Appendix C, resulting in the Dirac operator:

$$\begin{aligned} D = \gamma_1 &\Big[\frac{\partial}{\partial\theta^1} - i\left(A_1 + \frac{3}{2}\cot\theta^1\right)\Big] \\ &+ \gamma_2 \csc\theta^1 \Big(\frac{\partial}{\partial\theta^2} - i\left(A_2 + \cot\theta^2\right)\Big] \\ &+ \gamma_3 \csc\theta^1 \csc\theta^2 \Big[\frac{\partial}{\partial\theta^3} - i\left(A_3 + \frac{1}{2}\cot\theta^3\right)\Big] \\ &+ \gamma_4 \csc\theta^1 \csc\theta^2 \csc\theta^3 \Big[\frac{\partial}{\partial\theta^4} - iA_4\Big] \end{aligned} \tag{4.11}$$

or, for general d: aside from the csc terms in the definition of the gradient, the effect of the spin connection on S^d is to replace

$$\frac{\partial}{\partial\theta^j} \to \frac{\partial}{\partial\theta^j} + \frac{d-j}{2}\cot\theta^j. \tag{4.12}$$

[3]in the diagonal case, which constitute the cases we consider, these are simply the inverses of the quantities appearing in eq. (4.1). This factor yields, for example, the $1/\sin\theta$ in the definition of the spherical gradient - this factor has nothing to do with the spin connection.

4.3 Domain wall on the disk

We begin with a brief review of disk fermions. In order to obtain a theory of a Weyl fermion in d dimensions, we start with a Dirac fermion ψ in $d+1$ dimensions, where two of the dimensions are special and the other $d-1$ can be ignored in our analysis. The action is

$$S = \int d^{d-1}x r dr d\phi \, \overline{\psi} D \psi, \quad D = \gamma^i \frac{\partial}{\partial x^i} + m(r), \quad i = 1, \cdots, d+1. \tag{4.13}$$

Here (r, ϕ) are the cylindrical coordinates for $(x^d, x^{d+1}) \equiv (x, y)$. The mass term has the profile

$$m(r) = \begin{cases} m & r < R, \\ -M & r > R. \end{cases} \tag{4.14}$$

The insertion of this domain wall doesn't affect the rest of the dimensions, so we focus only on the action containing operators related to disk coordinates. Let us define matrices

$$\Gamma^r = \hat{r} \cdot \vec{\gamma} \ , \qquad \Gamma^\phi = \hat{\phi} \cdot \vec{\gamma}. \tag{4.15}$$

With these we may write the action in eq. (4.13) in the form

$$S = \int r dr d\phi \, \overline{\psi} \left(\Gamma^r \frac{\partial}{\partial r} + \frac{1}{r} \Gamma^\phi \frac{\partial}{\partial \phi} \right) \psi \tag{4.16}$$

Note that

$$\{\Gamma^r, \Gamma^\phi\} = \{\gamma^x, \gamma^y\} \quad \Gamma^r \Gamma^\phi = \gamma^x \gamma^y \equiv 2i \Sigma_{xy} \tag{4.17}$$

which means we may express

$$S = \int r dr d\phi \overline{\psi} \left(\Gamma^r \left(\frac{\partial}{\partial r} + \frac{1}{2r} \right) + \frac{i}{r} \Gamma^\phi \left(-i \frac{\partial}{\partial \phi} + \frac{1}{2} \Sigma_{xy} \right) + m(r) \right) \psi. \tag{4.18}$$

If there were no r in the integration measure of eq. (4.18), as is the case for ordinary domain wall fermions[2], one would find exponentially localized states $\psi \sim e^{m(r)r}$. One should expect to recover this in the limit $R \to \infty$. Therefore to account for the r in the integration measure, localized states are instead of the form

$$\begin{aligned}\psi(r < R) &\sim \frac{e^{mr}}{\sqrt{r}}\chi(\phi) \\ \psi(r > R) &\sim \frac{e^{-Mr}}{\sqrt{r}}\chi(\phi).\end{aligned} \tag{4.19}$$

Such states are annihilated by the operator

$$-\left(\partial_r + \frac{1}{2r}\right) + m(r). \tag{4.20}$$

This implies that modes $\overline{\chi}, \chi$ satisfying the modified Weyl boundary condition

$$\tilde{P}_+\chi = \tilde{P}_-\overline{\chi} = 0, \quad \tilde{P}_\pm = \frac{1 \pm \mathit{\Gamma}^r}{2}, \tag{4.21}$$

can be described by an edge action

$$S = \int d\phi \overline{\chi} i \mathit{\Gamma}^\phi \left(-i\frac{\partial}{\partial\phi} + \frac{1}{2}\mathit{\Sigma}_{xy}\right)\chi. \tag{4.22}$$

While this action describes a particle on the circle, the matrix $\mathit{\Gamma}^\phi$ is coordinate-dependent. We may perform a so called "straightening" transformation on the matrices, i.e. one which satisfies

$$V\mathit{\Gamma}^\phi V^\dagger = \gamma^\phi, \tag{4.23}$$

where γ^ϕ is a coordinate-independent gamma matrix. In order to make this consistent with the rest of the bulk action and the Weyl condition, we should really find a transformation satisfying

$$V\mathit{\Gamma}^a V^\dagger = \gamma^a, \tag{4.24}$$

for each $a = 1, \cdots, d-1, \phi, r$, where $\mathit{\Gamma}^a \equiv \gamma^a$ for $a = 1, \cdots, d-1$, and the γ^a are all *coordinate-independent* matrices satisfying eq. (4.3). One such transformation, satisfying $\gamma^\phi = \gamma^x, \gamma^r = \gamma^y$ is given by

$$V(\phi) = e^{i\phi\mathit{\Sigma}_{xy}}. \tag{4.25}$$

Note that this matrix is double-valued, and thus $V\chi$ is double valued. It immediately follows that

$$\overline{\chi}' P_{-} = P_{+}\chi = 0, \quad P_{\pm} = \frac{1+\gamma^r}{2}. \tag{4.26}$$

Under the local Lorentz transformation

$$\chi \to V\chi \equiv \chi' \quad \overline{\chi} \to \overline{\chi} V^{\dagger} \equiv \overline{\chi}', \tag{4.27}$$

the action becomes, after reinserting the first $d-1$ coordinates,

$$S' = \int d^{d-1}x d\phi \overline{\chi}' \left(\gamma^{\phi} \frac{\partial}{\partial \phi} + \gamma^{i} \frac{\partial}{\partial x^{i}} \right) \chi', \quad i = 1, \cdots, d-1, \tag{4.28}$$

and the path integral

$$Z = \int d\chi' d\overline{\chi}' e^{-S'} \tag{4.29}$$

is evaluated only over Weyl modes. The straightening matrix eliminates the inhomogenous term Σ_{xy} in eq. (4.22), and produces the expected result for a circle, which has vanishing spin connection. This comes at a cost: while the bulk action admits solutions periodic in ϕ, the edge action after the local Lorentz transformation V describes an antiperiodic Weyl fermion on $S^1 \times R^{d-1}$, i.e. one which satisfies

$$P_{-}\overline{\chi}' = P_{+}\chi' = 0, \quad P_{\pm} = \frac{1 \pm \gamma^r}{2}, \quad \chi'(\phi+\pi) = -\chi'(\phi), \quad \overline{\chi}'(\phi+\pi) = -\overline{\chi}'(\phi). \tag{4.30}$$

We will see that an analagous result holds generally for spheres. In following sections we reserve the notation $P_{\pm} = (1 \pm \gamma^r)/2$ for coordinate-independent projectors and and $\tilde{P}_{\pm} = (1 \pm \Gamma^r)/2$ for the coordinate-dependent counterparts.

4.4 Domain walls on S^2

We begin with the action for a domain wall fermion with spherical edge state:

$$S = \int d^3x\, \overline{\psi} D \psi, \quad D = \gamma^j \partial_j + m(r), \quad j = 0, 1, 2. \tag{4.31}$$

with the γ^j satisfying eq. (4.3) and m satisfying eq. (4.14) in spherical coordinates. We define analagous $\mathit{\Gamma}$-matrices

$$\mathit{\Gamma}^r = \hat{r}\cdot\vec{\gamma}, \quad \mathit{\Gamma}^\theta = \hat{\theta}\cdot\vec{\gamma}, \quad \mathit{\Gamma}^\phi = \hat{\phi}\cdot\vec{\gamma}, \tag{4.32}$$

so that the action may be written

$$S = \int dr d\theta d\phi\, r^2 \sin\theta \overline{\psi}\left[\mathit{\Gamma}^r\left(\partial_r + \frac{1}{r}\right) + \frac{\mathit{\Gamma}^r}{r}\left(\vec{\gamma}\cdot\vec{J} - \frac{1}{2}\right) + m\right]\psi, \tag{4.33}$$

where $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum operator. Edge modes localized to $r = R$ can be found and solved for exactly [67]. However, using the same arguments of the previous section, dimensional analysis yields

$$\begin{aligned}\psi(r<R) &\sim \frac{e^{mr}}{r}\chi(\theta,\phi),\\ \psi(r>R) &\sim \frac{e^{-Mr}}{r}\chi(\theta,\phi),\end{aligned} \tag{4.34}$$

where χ is a 2-component spinor. Solutions of this form are annihilated by the operator

$$-\left(\partial_r + \frac{1}{r}\right) + m(r), \tag{4.35}$$

which implies that modes $\chi, \overline{\chi}$ satisfying the modified Weyl boundary condition in eq. (4.21) with the new $\mathit{\Gamma}^r$ can be described by an edge action

$$S = \frac{1}{R}\int_{S^2} \sin\theta d\theta d\phi\, \overline{\chi}\left(\vec{\mathit{\Gamma}}\cdot\vec{J} - \frac{1}{2}\right)\chi, \tag{4.36}$$

or equivalently

$$S = \frac{1}{R}\int_{S^2} \sin\theta d\theta d\phi\, \overline{\chi}\left(\mathit{\Gamma}^\theta\partial_\theta + \frac{\mathit{\Gamma}^\phi}{\sin\theta}\frac{\partial}{\partial\phi} - 1\right)\chi. \tag{4.37}$$

This may be compared to the ordinary formulation of fermions on spheres[68, 66, 69] via a straightening transformation similar to eq. (4.24):

$$V^\dagger\mathit{\Gamma}^\theta V = \gamma^\theta, V^\dagger\mathit{\Gamma}^\phi V = \gamma^\phi, \quad V^\dagger\mathit{\Gamma}^r V = \gamma^r \tag{4.38}$$

with $\gamma^\theta, \gamma^\phi, \gamma^r$ are again matrices satisfying eq. (4.3). Note it immediately follows that $\chi' = V\chi$ and $\overline{\chi}' = \overline{\chi}V^\dagger$ satisfy the Weyl condition eq. (4.26). Here, we will take the convention $\gamma^\phi = \gamma^0, \gamma^\theta = \gamma^1$, and $\gamma^r = \gamma^2$. Taking $\gamma^i = \sigma^{i+1}$ in Cartesian coordinates, one such straightening transformation is given by

$$V(\theta, \phi) = e^{i\theta\sigma_1/2} e^{i\phi\sigma_3/2}. \tag{4.39}$$

This transformation is anti-periodic in ϕ and thus produces double valued spinors. Under this transformation, the action of eq. (4.37) is transformed to

$$S = \frac{1}{R}\int_{S^2} \sin\theta d\theta d\phi \overline{\chi} \left(\sigma_2 \left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2}\right) + \sigma_1 \frac{\partial}{\partial\phi}\right)\chi, \tag{4.40}$$

which is the usual action for fermions on the sphere [68]. The term $\sigma_1 \cot\theta/2$ is recognizable as the spin connection as discussed in the previous section, whose origin in the present context follows from the transformation eq. (4.39) on eq. (4.37) introducing derivative terms $\sim \gamma_\theta P \partial_\theta P$. The double-valued spinors correctly describe the theory of spherical spinors [68].

4.5 Domain walls on S^4

We denote the coordinates on the d-sphere by $\Theta = (\theta^j)$, $j = 0, \cdots d-1$, with $\theta^0 \in [0, 2\pi)$ and the remaining $\theta^j \in [0, \pi)$. For now d is arbitrary but later we will take $d = 4$. The bulk integration measure in spherical coordinates can be written

$$d^{d+1}x = r^d dr d\Theta, \tag{4.41}$$

where

$$d\Theta \equiv \prod_{j=0}^{d-1} (\sin\theta^j)^j d\theta^j. \tag{4.42}$$

The arguments of the previous section can be applied to the action

$$S = \int d^{d+1}x \, \overline{\psi} D\psi, \; D = \gamma^\mu \partial_\mu + m(r), \; \mu = 0, \cdots d. \tag{4.43}$$

Once again we define matrices

$$\Gamma^{\theta^j} = \hat{\theta}^j \cdot \vec{\gamma},\ j < d, \quad \Gamma^r \equiv \Gamma^d = \hat{r} \cdot \vec{\gamma}, \tag{4.44}$$

and an explicit form for these unit vectors is provided in Appendix B. The asymptotic domain wall solutions of the form for $r < R$ in eq. (4.34) are replaced via the same dimensional analysis arguments with

$$\psi(r < R) \sim \frac{e^{mr}}{\sqrt{r^d}} \chi(\Theta). \tag{4.45}$$

These states[4] are annihilated by the operator

$$-\left(\partial_r + \frac{d}{2r}\right) + m(r). \tag{4.46}$$

The action on the edge for fermions satisfying eq. (4.21) can then be written

$$S = \frac{1}{R} \int_{S^d} d\Theta\, \overline{\chi} \left(\Gamma^{d-1} \frac{\partial}{\partial \theta^{d-1}} + \frac{\Gamma^{d-2}}{\sin \theta^{d-1}} \frac{\partial}{\partial \theta^{d-2}} + \cdots - \frac{d}{2} \right) \chi. \tag{4.47}$$

This can be straightened as in the previous section via successive transformations

$$V = V_d V_{d-1} \cdots V_1, \tag{4.48}$$

where

$$V_j = e^{i\theta_j \Sigma_{j,j+1}}, \quad j = 0, \cdots, d-1. \tag{4.49}$$

This transformation has the effect of rotating Γ^0 into γ^0, Γ^1 into γ^1, and so on, where γ^i is any choice of matrices for the Cartesian coordinates x^i. Taking $d = 4$, this straightening operation can be seen to produce the spin correction for S^4 correctly. This is, once again, double-valued in $\theta^0 \equiv \phi$, and the modes $\chi' = V\chi$ and $\overline{\chi}' = \overline{\chi}' V^\dagger$ once again satisfy eq. (4.26). The action resulting from the straightening transformation (with gauge fields) may be written as $S = \int \overline{\chi}' D \chi'$, with D given in eq. (4.11). Once again the path integral is done only over Weyl modes (satisfying eq. (4.26)).

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[4] In fact, the exact solutions are $I_\nu(\kappa r)/\sqrt{r^d}$, where κ is a parameter related to the mass m and ν is related to the angular momentum. See Ref. [21] for an exact solutions on the disk.

4.6 Conclusions

The theory of a Weyl fermion on the surface of even d-dimensional spheres, with an emphasis on $d = 4$, has been derived via the introduction of a radial domain wall in $d + 1$ dimensions. The double-valued nature of Weyl fermions on even dimensional spheres has been clarified, and is consistent with single valued fermions in bulk. If this theory is to be realized on the lattice, it may evade the doubling problem typical of the usual domain wall approach[20, 10].

Chapter 5

CONCLUSIONS

Recent advances in the theory of topological insulators and superconductors [8, 7] have demonstrated that the existence of bulk topological phases of fermions depends on dimension and symmetries. At the interface between these bulk phases lies a gapless edge state with an anomalous symmetry [70, 9], much like domain wall fermions with chiral symmetry on the edge [2]. A recent no-go theorem [51] has shown that there does not exist a local lattice Hamiltonian description for such edge states in 3 spatial dimensions with exact chiral symmetry. One might expect that an extra dimension is therefore necessary in order to describe fermions with chiral symmetry on the lattice. However, overlap fermions [3] provide a local Dirac operator for the action for a lattice theory in $3+1$ spacetime dimensions, provided one relaxes the requirement of chiral symmetry and replaces it with Lüscher symmetry, which these operators have because they satisfy the Ginsparg-Wilson relation. Motivated by the classification of topological insulators and superconductors, we discuss generalizations of GW fermions in Chapter 2 to other types of relativistic fermions and their anomalous symmetries.

Lüscher symmetries and generalized Ginsparg-Wilson fermions may prove helpful in describing the edge states of various topological phases of matter in contexts outside relativistic field theories. This would require a Hamiltonian description of Ginsparg-Wilson fermions. In attempting to find the Hamiltonian for a generalized Ginsparg-Wilson fermion in Chapter 3, we found that while the fermions had Lüscher symmetry in the spacetime action, the Hamiltonian theory describing the low energy modes had exact chiral symmetry and was nonlocal, consistent with Ref. [51]. A Hamiltonian description would allow these edge states to be simulated on a quantum computer, and would avoid the sign problems that afflict path integral simulations of lattice fermions with chiral symmetry on classical computers.

The primary importance of this program is a realization of the lattice as a non-perturbative regulator for chiral gauge theories. In Chapter 4 we discuss the possibility of a Weyl fermion confined to a spherical domain wall. This fermion may be helpful in defining a chiral gauge theory, similar to the construction of disk Weyl fermions in Ref. [21]. The added benefit of the spherical Weyl fermions is the preservation of hypercubic symmetry which is violated by the mass term of the disk fermion. Recent work has suggested [71] these may suffer the same issues that ordinary domain wall fermions do, but further investigation is warranted.

Appendix A

DERIVATION OF PATH INTEGRAL IDENTITIES

Here we derive two identities used in this thesis. For Dirac fermions and an invertible hermitian operator A we write

$$e^{-\bar{\chi}A\chi} = \det A \int d\psi d\bar{\psi}\, e^{\bar{\psi}A^{-1}\psi+\bar{\psi}\chi+\bar{\chi}\psi}\ . \tag{A.1}$$

It follows that

$$\begin{aligned} e^{\partial_\chi B\partial_{\bar{\chi}}}\, e^{-\bar{\chi}A\chi} &= \det A \int d\psi d\bar{\psi}\, e^{-\bar{\psi}\left(A^{-1}-B\right)\psi+\bar{\psi}\chi+\bar{\chi}\psi} \\ &= \det\left(1-AB\right)\, e^{-\bar{\chi}\left(\frac{1}{1-AB}A\right)\chi} \\ &= e^{\mathrm{Tr}\ \log(1-AB)}\, e^{-\bar{\chi}\left(\frac{1}{1-AB}A\right)\chi}\ . \end{aligned} \tag{A.2}$$

The above result extends to noninvertible A.

An analogous identity can be derived for Majorana fermions. Assuming an invertible imaginary antisymmetric operator A we have

$$e^{\frac{1}{2}\eta\mathsf{A}\eta} = \frac{1}{\mathsf{Pf}\left(\mathsf{A}^{-1}\right)} \int d\nu\, e^{\frac{1}{2}\nu\mathsf{A}^{-1}\nu+\nu\eta}\ . \tag{A.3}$$

From this one derives for antisymmetric B

$$\begin{aligned} e^{\frac{1}{2}\partial_\eta\mathsf{B}\partial_\eta}e^{-\frac{1}{2}\eta\mathsf{A}\eta} &= \frac{1}{\mathsf{Pf}(-\mathsf{A})^{-1}} \int d\nu\, e^{\frac{1}{2}\nu\left(-\mathsf{A}^{-1}+\mathsf{B}\right)\nu+\ \nu\eta} \\ &= \mathsf{Pf}\left(\mathsf{A}\right)\ \mathsf{Pf}\left(-\mathsf{A}^{-1}+\mathsf{B}\right)\ e^{-\frac{1}{2}\eta\left(\frac{1}{1-\mathsf{AB}}\mathsf{A}\right)\eta}\ , \\ &= e^{\frac{1}{2}\mathrm{Tr}\ \ln(1-\mathsf{AB})}e^{-\frac{1}{2}\eta\left(\frac{1}{1-\mathsf{AB}}\mathsf{A}\right)\eta}\ , \end{aligned} \tag{A.4}$$

where for the last line we used the identity $\mathsf{Pf}\left(\mathsf{A}\right)\ \mathsf{Pf}\left(\mathsf{B}\right) = \exp\frac{1}{2}\mathrm{Tr}\ \ln(-\mathsf{AB})$. The above result also extends to noninvertible A.

The Majorana result of eq. (A.4) can be seen to be consistent with the Dirac result of eq. (A.2) by writing a Dirac fermion as a Majorana one with

$$\eta = \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} \qquad \mathsf{A} = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \ . \tag{A.5}$$

Then the left and right sides of eq. (A.4) are equal to

$$e^{\frac{1}{2}\partial_\eta \mathsf{B} \partial_\eta} e^{-\frac{1}{2}\eta \mathsf{A} \eta} = e^{\partial_\chi B \partial_{\bar{\chi}}} e^{-\bar{\chi} A \chi} \ , \tag{A.6}$$

$$e^{\frac{1}{2}\mathrm{Tr}\ \ln(1-\mathsf{AB})} e^{-\frac{1}{2}\eta\left(\frac{1}{1-\mathsf{AB}}\mathsf{A}\right)\eta} = e^{\mathrm{Tr}\ \log(1-AB)}\ e^{-\bar{\chi}\left(\frac{1}{1-AB}A\right)\chi} \quad , \tag{A.7}$$

which match the two sides of eq. (A.2).

Appendix B
UNIT VECTORS ON HYPERSPHERES

Spherical unit vectors in higher dimensions can be produced inductively, starting in $d+1=2$.

$$\hat{\phi}_{(2)} = (-\sin\phi, \cos\phi) \tag{B.1}$$

$$\hat{r}_{(2)} = (\cos\phi, \sin\phi), \tag{B.2}$$

Given the unit vectors in spherical coordinates in $d+1$ dimensions, it is straightforward to construct angular unit vectors in $d+2$ dimensions ($\theta^0 = \phi$):

$$\hat{\theta}^j_{(d+2)} = (\hat{\theta}^j_{(d+1)}, 0), \quad j < d \tag{B.3}$$

$$\hat{\theta}^d_{(d+2)} = (-\sin\theta^d \hat{r}_{(d+1)}, \cos\theta^d) \tag{B.4}$$

$$\hat{r}_{(d+2)} = (\cos\theta^d \hat{r}_{(d+1)}, \sin\theta^d) \tag{B.5}$$

Appendix C

SPIN CONNECTION ON S^4

We use α, β, γ to denote angles $\theta^1, \theta^2, \theta^3$ and $\phi \equiv \theta^0$. The vierbein in eq. (4.7) is, in this notation:

$$e^1 = d\alpha, \tag{C.1}$$

$$e^2 = \sin\alpha d\beta, \tag{C.2}$$

$$e^3 = \sin\alpha \sin\beta d\gamma, \tag{C.3}$$

$$e^4 = \sin\alpha \sin\beta \sin\gamma d\phi. \tag{C.4}$$

and corresponding differentials

$$de^1 = 0, \tag{C.5}$$

$$de^2 = \cos\alpha d\alpha \wedge d\beta, \tag{C.6}$$

$$de^3 = \cos\alpha \sin\beta d\alpha \wedge d\gamma + \sin\alpha \cos\beta d\beta \wedge d\gamma, \tag{C.7}$$

$$de^4 = \cos\alpha \sin\beta \sin\gamma d\alpha \wedge d\phi \tag{C.8}$$

$$+ \sin\alpha \cos\beta \sin\gamma d\beta \wedge d\phi \tag{C.9}$$

$$+ \sin\alpha \sin\beta \cos\gamma d\gamma \wedge d\phi. \tag{C.10}$$

So the Cartan equations are:

$$\omega^{12} \wedge \sin\alpha d\beta + \omega^{13} \wedge \sin\alpha \sin\beta d\gamma + \omega^{14} \wedge \sin\alpha \sin\beta \sin\gamma d\phi = \mathbf{0}, \tag{C.11}$$

$$\omega^{21} \wedge d\alpha + \omega^{23} \wedge \sin\alpha \sin\beta d\gamma + \omega^{24} \wedge \sin\alpha \sin\beta \sin\gamma d\phi = \tag{C.12}$$

$$-(\cos\alpha d\alpha \wedge d\beta), \tag{C.13}$$

$$\omega^{31} \wedge d\alpha + \omega^{32} \wedge \sin\alpha d\beta + \omega^{34} \wedge \sin\alpha \sin\beta \sin\gamma d\phi = \tag{C.14}$$

$$-(\cos\alpha \sin\beta d\alpha \wedge d\gamma + \sin\alpha \cos\beta d\beta \wedge d\gamma), \tag{C.15}$$

$$\omega^{41} \wedge d\alpha + \omega^{42} \wedge \sin\alpha d\beta + \omega^{43} \wedge \sin\alpha \sin\beta d\gamma = \tag{C.16}$$

$$-(\cos\alpha \sin\beta \sin\gamma d\alpha \wedge d\phi \tag{C.17}$$

$$+ \sin\alpha \cos\beta \sin\gamma d\beta \wedge d\phi \tag{C.18}$$

$$+ \sin\alpha \sin\beta \cos\gamma d\gamma \wedge d\phi). \tag{C.19}$$

The spin connection components $(\omega^{ab})_\mu$ can be easily solved for using this linear system of equations. One finds that the components of the covariant derivative in eq. (4.4) are:

$$D_\alpha = \frac{\partial}{\partial\alpha}, \tag{C.20}$$

$$D_\beta = \frac{\partial}{\partial\beta} - \cos\alpha \Sigma_{\alpha\beta}, \tag{C.21}$$

$$D_\gamma = \frac{\partial}{\partial\gamma} - \cos\alpha \sin\beta \Sigma_{\alpha\gamma} - \cos\beta \Sigma_{\beta\gamma}, \tag{C.22}$$

$$D_\phi = \frac{\partial}{\partial\phi} - \cos\alpha \sin\beta \sin\gamma \Sigma_{\alpha\phi} - \cos\beta \sin\gamma \Sigma_{\beta\phi} - \cos\gamma \Sigma_{\gamma\phi}. \tag{C.23}$$

BIBLIOGRAPHY

[1] H.B. Nielsen and M. Ninomiya. A no-go theorem for regularizing chiral fermions. *Physics Letters B*, 105(2):219–223, 1981.

[2] David B. Kaplan. A method for simulating chiral fermions on the lattice. *Phys. Lett. B*, 288:342–347, 1992.

[3] Herbert Neuberger. Exactly massless quarks on the lattice. *Phys. Lett. B*, 417:141–144, 1998.

[4] Paul H. Ginsparg and Kenneth G. Wilson. A Remnant of Chiral Symmetry on the Lattice. *Phys. Rev. D*, 25:2649, 1982.

[5] Martin Lüscher. Chiral gauge theories revisited. *Theory and Experiment Heading for New Physics; World Scientific: Singapore*, pages 41–89, 2001.

[6] AD Kennedy. Algorithms for dynamical fermions. In *Perspectives in Lattice QCD*, pages 15–81. World Scientific, 2008.

[7] Shinsei Ryu, Andreas P Schnyder, Akira Furusaki, and Andreas W W Ludwig. Topological insulators and superconductors: tenfold way and dimensional hierarchy. *New Journal of Physics*, 12(6):065010, jun 2010.

[8] Alexei Kitaev. Periodic table for topological insulators and superconductors. *AIP Conf. Proc.*, 1134(1):22–30, 2009.

[9] Edward Witten. Fermion path integrals and topological phases. *Reviews of Modern Physics*, 88(3):035001, jul 2016.

[10] Vadim Furman and Yigal Shamir. Axial symmetries in lattice QCD with Kaplan fermions. *Nuclear Physics B*, 439(1–2):54–78, April 1995.

[11] Stephen R. Sharpe. Future of chiral extrapolations with domain wall fermions, 2007.

[12] Michael Creutz. Aspects of chiral symmetry and the lattice. *Reviews of Modern Physics*, 73(1):119–150, January 2001.

[13] David B. Kaplan. Chiral Symmetry and Lattice Fermions. In *Les Houches Summer School: Session 93: Modern perspectives in lattice QCD: Quantum field theory and high performance computing*, pages 223–272, 12 2009.

[14] E. Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1918:235–257, 1918.

[15] Martin Lüscher. Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation. *Phys. Lett. B*, 428:342–345, 1998.

[16] Kazuo Fujikawa. Path-integral measure for gauge-invariant fermion theories. *Physical Review Letters*, 42(18):1195, 1979.

[17] Michael Francis Atiyah and Isadore Manuel Singer. The index of elliptic operators: I. *Annals of mathematics*, pages 484–530, 1968.

[18] Heinz J. Rothe. *Lattice Gauge Theories : An Introduction (Fourth Edition)*, volume 43. World Scientific Publishing Company, 2012.

[19] Kenneth G. Wilson. Confinement of quarks. *Phys. Rev. D*, 10:2445–2459, Oct 1974.

[20] Yigal Shamir. Chiral fermions from lattice boundaries. *Nuclear Physics B*, 406(1–2):90–106, September 1993.

[21] David B. Kaplan. Chiral gauge theory at the boundary between topological phases, 2024.

[22] Holger Bech Nielsen and M. Ninomiya. Absence of Neutrinos on a Lattice. 1. Proof by Homotopy Theory. *Nucl. Phys. B*, 185:20, 1981. [Erratum: Nucl.Phys.B 195, 541 (1982)].

[23] Rajamani Narayanan and Herbert Neuberger. Chiral fermions on the lattice. *Phys. Rev. Lett.*, 71(20):3251, 1993.

[24] Michael Clancy, David B. Kaplan, and Hersh Singh. Generalized Ginsparg-Wilson relations. *Phys. Rev. D*, 109:014502, Jan 2024.

[25] The index theorem in QCD with a finite cut-off, author=Hasenfratz, Peter and Laliena, Victor and Niedermayer, Ferenc. *Physics Letters B*, 427(1-2):125–131, 1998.

[26] Wolfgang Bietenholz and Jun Nishimura. Ginsparg-Wilson fermions in odd dimensions. *Journal of High Energy Physics*, 2001(07):015, 2001.

[27] Rajamani Narayanan and Herbert Neuberger. Infinitely many regulator fields for chiral fermions. *Phys. Lett. B*, 302:62–69, 1993.

[28] Rajamani Narayanan and Herbert Neuberger. Chiral determinant as an overlap of two vacua. *Nucl. Phys. B*, 412:574–606, 1994.

[29] Herbert Neuberger. Vectorlike gauge theories with almost massless fermions on the lattice. *Physical Review D*, 57(9):5417, 1998.

[30] Karl Jansen and Martin Schmaltz. Critical momenta of lattice chiral fermions. *Phys. Lett. B*, 296:374–378, 1992.

[31] Maarten F. L. Golterman, Karl Jansen, and David B. Kaplan. Chern-simons currents and chiral fermions on the lattice. *Phys. Lett. B*, 301:219–223, 1993.

[32] David B Kaplan and Martin Schmaltz. Domain wall fermions and the η-invariant. *Physics Letters B*, 368(1-2):44–52, 1996.

[33] Andreas P Schnyder, Shinsei Ryu, Akira Furusaki, and Andreas WW Ludwig. Classification of topological insulators and superconductors in three spatial dimensions. *Physical Review B*, 78(19):195125, 2008.

[34] Taro Kimura and Masataka Watanabe. Band Flattening and Overlap Fermion. 9 2023.

[35] Georg Bergner, Falk Bruckmann, Yoshio Echigo, Yuji Igarashi, Jan M. Pawlowski, and Sebastian Schierenberg. Blocking-inspired supersymmetric actions: A status report. *Phys. Rev. D*, 87(9):094516, 2013.

[36] Georg Bergner, Falk Bruckmann, and Jan M. Pawlowski. Generalising the Ginsparg-Wilson relation: Lattice Supersymmetry from Blocking Transformations. *Phys. Rev. D*, 79:115007, 2009.

[37] David B. Kaplan and Aneesh V. Manohar. Current Mass Ratios of the Light Quarks. *Phys. Rev. Lett.*, 56:2004, 1986.

[38] Martin Lüscher. Abelian chiral gauge theories on the lattice with exact gauge invariance. *Nuclear Physics B*, 549(1-2):295–334, 1999.

[39] Luis Alvarez-Gaume, S Della Pietra, and Gregory Moore. Anomalies and odd dimensions. *Annals of Physics*, 163(2):288–317, 1985.

[40] Hidenori Fukaya, Naoki Kawai, Yoshiyuki Matsuki, Makito Mori, Katsumasa Nakayama, Tetsuya Onogi, and Satoshi Yamaguchi. The atiyah–patodi–singer index on a lattice. *Progress of Theoretical and Experimental Physics*, 2020(4):043B04, 2020.

[41] Hidenori Fukaya, Mikio Furuta, Yoshiyuki Matsuki, Shinichiroh Matsuo, Tetsuya Onogi, Satoshi Yamaguchi, and Mayuko Yamashita. Mod-two APS index and domain-wall fermion. *Lett. Math. Phys.*, 112(2):16, 2022.

[42] Edward Witten and Kazuya Yonekura. Anomaly Inflow and the η-Invariant, September 2020.

[43] Antoine Coste and Martin Lüscher. Parity Anomaly and Fermion-Boson Transmutation in 3-Dimensional Lattice QED. *Nucl. Phys. B*, 323:631–659, 1989.

[44] David B. Kaplan and Martin Schmaltz. Supersymmetric Yang-Mills theories from domain wall fermions. *Chin. J. Phys.*, 38:543–550, 2000.

[45] A Yu Kitaev. Unpaired majorana fermions in quantum wires. *Physics-Uspekhi*, 44(10S):131, 2001.

[46] Michael Stone. Gamma matrices, Majorana fermions, and discrete symmetries in Minkowski and Euclidean signature. dec 2021.

[47] Peter Hasenfratz and Reto von Allmen. Towards Weyl fermions on the lattice without artefacts. *Journal of High Energy Physics*, 2008(02):079–079, February 2008.

[48] Christof Gattringer and Markus Pak. New overlap construction of Weyl fermions. *Nuclear Physics B*, 801(3):353–360, October 2008.

[49] Peter Hasenfratz, Ferenc Niedermayer, and Reto von Allmen. Lattice Regularization and Symmetries. *Journal of High Energy Physics*, 2006(10):010–010, October 2006.

[50] Michael Creutz, Ivan Horvath, and Herbert Neuberger. A New fermion Hamiltonian for lattice gauge theory. *Nucl. Phys. B Proc. Suppl.*, 106:760–762, 2002.

[51] Lukasz Fidkowski and Cenke Xu. A no-go result for implementing chiral symmetries by locality-preserving unitaries in a 3 dimensional hamiltonian lattice model of fermions. 2023.

[52] Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang. Fermionic Symmetry Protected Topological Phases and Cobordisms. *Journal of High Energy Physics*, 2015(12):1–21, December 2015.

[53] Andreas Karch, David Tong, and Carl Turner. A Web of 2d Dualities: $\mathbb{Z}_2$ Gauge Fields and Arf Invariants. *SciPost Physics*, 7(1):007, July 2019.

[54] Lukasz Fidkowski and Alexei Kitaev. The effects of interactions on the topological classification of free fermion systems. *Phys. Rev. B*, 81:134509, 2010.

[55] Justin Kaidi, Julio Parra-Martinez, Yuji Tachikawa, and Arun Debray. Topological superconductors on superstring worldsheets. *SciPost Physics*, 9(1):010, 2020.

[56] MF Atiyah and Isadore M Singer. Dirac operators coupled to vector potentials. *Proceedings of the National Academy of Sciences*, 81(8):2597–2600, 1984.

[57] Bruno Zumino. Chiral anomalies and differential geometry. *Current Algebra and Anomalies, World Scientific*, pages 361–391, 1985.

[58] Raymond Stora. Algebraic structure and topological origin of anomalies. In *Progress in gauge field theory*, pages 543–562. Springer, 1984.

[59] Curtis G. Callan, Jr. and Jeffrey A. Harvey. Anomalies and fermion zero modes on strings and domain walls. *Nucl. Phys. B*, 250:427–436, 1985.

[60] Xianzhe Dai and Daniel S Freed. η-invariants and determinant lines. *Journal of Mathematical Physics*, 35(10):5155–5194, 1994.

[61] Edward Witten. The "parity" anomaly on an unorientable manifold. *Physical Review B*, 94(19):195150, 2016.

[62] Michael Clancy. Toward a Ginsparg-Wilson lattice Hamiltonian. *Phys. Rev. D*, 110:L011502, Jul 2024.

[63] Mark Alford, Krishna Rajagopal, and Frank Wilczek. Color-flavor locking and chiral symmetry breaking in high density QCD. *Nuclear Physics B*, 537(1–3):443–458, January 1999.

[64] Robert D Pisarski. Review of the chiral phase transition. In *Fundamental Interactions*, pages 90–110. World Scientific, 2003.

[65] Daniel V. Schroeder and Michael Peskin. *An introduction to quantum field theory*. Addison-Wesley, 1995.

[66] Tohru Eguchi, Peter B. Gilkey, and Andrew J. Hanson. Gravitation, Gauge Theories and Differential Geometry. *Phys. Rept.*, 66:213, 1980.

[67] Shoto Aoki, Hidenori Fukaya, and Naoto Kan. A lattice formulation of Weyl fermions on a single curved surface, 2024.

[68] A. A. Abrikosov. Fermion states on the sphere S2. *International Journal of Modern Physics A*, 17(06n07):885–889, March 2002.

[69] Christian Bär. The Dirac operator on space forms of positive curvature. *Journal of The Mathematical Society of Japan*, 48, 01 1996.

[70] Andreas W W Ludwig. Topological phases: classification of topological insulators and superconductors of non-interacting fermions, and beyond. *Physica Scripta*, T168:014001, dec 2015.

[71] Maarten Golterman and Yigal Shamir. Conserved currents in five-dimensional proposals for lattice chiral gauge theories, 2024.