Variation of Instability in Invariant Theory

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Abstract

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When a reductive group acts on an algebraic variety, a linearized ample line bundle induces a stratification on the variety where the strata are ordered by the degrees of instability. In this thesis, we study variation of stratifications caused by different choices of linearized ample line bundles. This serves as a refinement of variation of GIT quotients, a subject well studied in the 90's by Doglachev, Hu and Thaddeus. For a representation of a reductive group, linearized line bundles correspond to the characters of the group. We provide sufficient conditions for two characters to induce the same stratification. We also formulate two types of walls that completely capture two kinds of wall crossing behaviours. We then compute an example coming from the moduli of ordered points on the projective line. Finally, we explore variation of stratifications that occur in the GIT quotient construction for projective toric varieties. We prove that the variation is intrinsic to the primitive collections and the relations among ray generators of the fans.

VARIATION OF INSTABILITY IN INVARIANT THEORY

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Abstract. When a reductive group acts on an algebraic variety, a linearized ample line bundle induces a stratification on the variety where the strata are ordered by the degrees of instability. In this thesis, we study variation of stratifications caused by different choices of linearized ample line bundles. This serves as a refinement of variation of GIT quotients, a subject well studied in the 90's by Doglachev, Hu and Thaddeus. For a representation of a reductive group, linearized line bundles correspond to the characters of the group. We provide sufficient conditions for two characters to induce the same stratification. We also formulate two types of walls that completely capture two kinds of wall crossing behaviours. We then compute an example coming from the moduli of ordered points on the projective line. Finally, we explore variation of stratifications that occur in the GIT quotient construction for projective toric varieties. We prove that the variation is intrinsic to the primitive collections and the relations among ray generators of the fans.

Contents

1. Introduction

Geometric Invariant Theory (abbreviated as GIT) was developed in [Mum65] by Mumford to construct quotients for group actions on algebraic varieties. The local pieces of an algebraic variety are affine varieties. An affine variety is the spectrum of a ring. Elements in the ring are algebraic functions defined on the affine variety. Constructing varieties that parametrize orbits is not immediate even for group actions on affine varieties. While it is natural to consider the spectrum of the subring of functions that are constant on the orbits, namely, the invariant functions, there might not be enough invariants to separate the orbits. Mumford realized some orbits are so exceptional, that they must be left out of the quotient. The notion of stability was introduced by Mumford to specify an invariant open subset, known as the semistable locus that allows for a good quotient. Since then GIT has led to fruitful results in algebraic geometry, especially constructions of various moduli spaces. Examples include representations of quivers [Kin94], vector bundles on a curve [Mum63], sequence of linear subspaces [Mum63], and coherent sheaves on a projective variety [Sim94].

GIT stability however, depends on the choice of a linearized line bundle. Variation of Geometric Invariant Theory (abbreviated as VGIT) quotients coming from different choices of linearized line bundles were well studied by [DH98] and [Tha96] in the 90's. The main result was that when a reductive group G over the complex numbers acts on a normal complex projective variety, the space of G-linearizations has a finite decomposition into connected open chambers and closed codimension one walls such that

(1) stability and the GIT quotient do not change inside each chamber,
(2) the variation of GIT quotients after wall crossing is described by a flip.

It is worthwhile pointing out that VGIT has applications to birational geometry. It not only provides examples of flips, but also realizes Mori theory as an instance of VGIT. In [HK00] it was shown that if a projective variety is a Mori dream space, it is a GIT quotient of an affine variety by a torus. In this case the Mori chambers correspond to VGIT chambers.

Our work takes on the theme of variations. We study the variation of the stratifications of a variety resulting from different choices of linearized ample line bundles. The idea is to include the complement of the semistable locus, known as the unstable locus, into the picture. This is mainly motivated by a theorem due to Kempf and as a tribute to his paper [Kem78], we adopt the title Variation of Instability in Invariant Theory (abbreviated as VIIT) for this thesis. In [Kem78], it was shown that every unstable point is maximally destabilized by an indivisible one parameter subgroup, unique up to conjugacy, in a numerical sense that comes from the Hilbert-Mumford index. As a consequence, a linearized ample line bundle L on a variety X induces the following stratification

$$X = X^{\mathrm{ss}}(L) \cup \bigcup_{\lambda,d} S_{[\lambda],d}.$$

Here $X^{\mathrm{ss}}(L)$ is the semistable locus for L, $[\lambda]$ is the conjugacy class of the one parameter subgroup λ that maximally destabilizes points in the stratum $S_{[\lambda],d}$, and d is some negative real number indicating the degree of the instability of the stratum. The idea is that the more negative the number d is, the more unstable the points in the stratum $S_{[\lambda],d}$ are. The stratum $X^{\mathrm{ss}}(L)$ can be thought of as having instability zero. In [Hes79], Hesselink showed that each stratum is a locally closed invariant subvariety with the property that $\partial S_{[\lambda],d} \cap S_{[\lambda'],d'} \neq \emptyset$ only if $d' < d$. Later as an application, Kirwan related the stratification of a variety to the cohomology of its quotients in [Kir84].

We see that the variation of stratifications comes from the following two possibilities:

(1) The set of the strata changes, or
(2) the ordering of the strata changes.

Namely, two points may be in the same stratum under a linearized ample line bundle but in different strata under another linearized ample line bundle. Similarly, the order of a pair of strata may be flipped under different choices of linearized ample line bundles. The major task of VIIT is to nail down for which linearized line bundles the stratification undergoes the two changes described above. We attempt to explore VIIT in three contexts: The affine case, the projective case and the toric variety case.

For the affine case, we consider affine spaces with linear actions by a reductive group G. The G-linearized line bundles correspond to the group of characters

$$\boldsymbol{\chi}(G) := \{\chi : G \to \mathbf{G}_m\}.$$

Let

$$\boldsymbol{\chi}(G)_{\mathbf{R}} := \boldsymbol{\chi}(G) \otimes_{\mathbf{Z}} \mathbf{R}$$

be the associated finite dimensional vector space. We then identify two types of proper closed subsets in the character space $\boldsymbol{\chi}(G)_{\mathbf{R}}$, called type one and type two critical subsets (Definition 5.3.1 and Definition 5.3.4). Type one critical subsets capture the characters that induce different set of strata and type two critical subsets capture the characters that induce different orderings to the set of strata. We show in Proposition 5.3.8 that a critical subset is of codimension at least one and is defined by some homogeneous polynomial of degree at most two with rational coefficients. The complement of the union of these critical subsets can be decomposed into a disjoint union of open semi-chambers (Definition 5.3.9). The main theorem (Theorem 5.4.4) states that

Theorem A. *Let G be a reductive group acting linearly on an affine space X. The character space $\boldsymbol{\chi}(G)_{\mathbf{R}}$ has a finite decomposition into open semi-chambers and closed critical subsets of codimension at least 1 such that whenever two characters share the same unstable locus and are in the same semi-chamber, they induce the same stratification of X.*

For an affine variety Y equipped with a an action by a reductive group G, it has a G-equivariant embedding into an affine space X on which G acts linearly (Proposition 3.1.3). We can then apply techniques developed for X to Y via the embedding. We conjecture that Theorem A holds for arbitrary affine varieties equipped with actions by reductive groups.

For the projective case, we compute an example coming from the moduli of ordered points on the projective line. The projective variety is a product of projective lines equipped with the diagonal action of SL(2). We summarize the results as Theorem 6.1.10. In this case the space of linearized line bundles is exactly the Picard group. We describe a finer decomposition of each VGIT chamber into VIIT chambers and VIIT walls. We compute the stratification in each VIIT chamber and we describe the VIIT wall crossing behaviours for each VIIT wall.

The toric variety case is motivated by the fact that every projective toric variety can be realized as a GIT quotient of an affine space by a subgroup of a torus. Moreover, toric varieties are famous for being a good testing ground in algebraic geometry due to their concreteness and computability. In fact, our main result, Theorem A was motivated by phenomena we saw when working on VIIT for the toric variety case.

Our toric varieties are assumed to be normal. In this case toric varieties are constructed using the combinatorial objects that are known as fans (see [CLS11]). It was shown in [CLS11] that every projective toric variety is a GIT quotient of an affine space by a subgroup of a torus with respect to linearized line bundles that correspond to ample divisors on the projective toric variety. Hence it is natural to consider VIIT in the ample cone of a projective toric variety. In the ample cone of a projective toric variety, the unstable locus in the GIT quotient construction can be described by primitive collections of the fan (Definition 7.2.1 and Proposition 7.2.6). In Section 7.4, we relate several topological properties of the strata to the primitive collections of the fan. The main theorem (Theorem 7.3.10) we have in the toric case is:

Theorem B. *For a projective toric variety, the variation of stratifications induced by different ample divisors is intrinsic to the primitive collections and the relations among ray generators of the fan.*

Finally, due to the availability of numerous computer packages for toric varieties, we wrote a computer program to generate rich examples. Section 8 explains how my computer program works, and in Section 8.7, we supply some examples computed by the program. The examples from Section 8.7 are to address the following comparison:

VIIT vs VGIT. We define two linearized ample line bundles on a variety to be IIT-equivalent if they induce the same stratification of the variety. Roughly speaking, two linearized ample line bundles on a variety induce the same stratification if both the set of strata and the partial ordering on the strata is the same (Definition 3.3.6). On the other hand, in VGIT two linearized ample line bundles are defined to be GIT-equivalent if they share the same semistable locus. We see that IIT-equivalence is a refinement of GIT-equivalence. While [DH98] showed that each VGIT chambers corresponds to a GIT-equivalence class, there is no similar correspondence established for VIIT. Theorem A only implies that the intersection of a VGIT chamber and a semi-chamber is contained in an IIT-equivalence class. In addition, it is shown in [Res00], that each GIT-equivalence class spans the interior of a convex rational polyhedral cone in the space of linearizations. Again there is no similar description for IIT-equivalence classes. The heart of Section 8.7 is to address the fundamental questions: In a VGIT chamber, does each semi-chambers correspond to an IIT-equivalence class? What might an IIT-class look like? We will see in Example 8.7.3 that there are more semi-chambers than IIT-equivalence classes. In view of this, we conclude that the decomposition of the space of characters by semi-chambers and critical subsets is only an approximation to the decomposition by IIT-equivalence classes. In Example 8.7.5, we will have a glimpse of what semi-chambers look like. They do not have to be convex.

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2. Linear programming

We start this paper by recalling some key facts from linear programming. This enables us to find those one parameter subgroups that maximally destabilize unstable points in Section 3. The key references for this section would be [Kem78] and [Nes79]. The optimization technique presented here is different but very similar to the aforementioned two papers. Roughly speaking what we look at here is the shortest distance problem to a polyhedral cone while [Kem78] and [Nes79] dealt with shortest distance problem to a polytope. The measure of distance comes from an inner product. The existence of extrema is easy to establish. The non-trivial parts are on the uniqueness and rationality of the extremal point where rationality essentially gives rise to the existence of maximally destabilizing one parameter subgroups.

We will first derive results for finite dimensional inner product spaces and specialize to inner product spaces with some rationality assumptions. We also discuss distance to linear subspaces in Section 2.2, which will be essential for describing critical subsets in Section 5.

2.1. **The shortest distance problem.** We are concerned with the relative maximum of a linear functional on the intersection of the unit sphere and a polyhedral cone. While existence is obvious from compactness, we establish uniqueness of the extremal point if f takes positive values on the polyhedral cone. We also reinterpret this problem as a shortest distance problem in Theorem 2.1.6.

Let V be a finite dimensional real vector space with positive definite inner product $(-,-) : V \times V \to \mathbf{R}$. Let $|| - || : V \to \mathbf{R}$ be the induced norm on V. The space V and therefore any subspace W have an induced metric topology. We recall some basic facts from linear algebra.

Lemma 2.1.1. *(Cauchy-Schwarz inequality) For any $u, v \in V$, it is true that*

$$(u, v)^2 \leq (u, u) \cdot (v, v).$$

The equality holds if and only if u and v are linearly dependent.

Definition 2.1.2. For any vector $v \in V$ and any sub-space $W \subset V$, we set $\operatorname{Proj}_W v$ to be *the projection of v onto W along the orthogonal complement $W^\perp$*. Namely, $\operatorname{Proj}_W v \in W$ and

$$(v - \operatorname{Proj}_W v, w) = 0 \text{ for all } w \in W.$$

If $w \in V$ is a vector, $\operatorname{Proj}_w v$ means *the projection of v onto the sub-space spanned by w.*

Obviously the Pythagorean law holds: If $(v, w) = 0$, then

$$||v + w||^2 = ||v||^2 + ||w||^2.$$

In particular,

$$||v||^2 = ||v - \operatorname{Proj}_w v||^2 + ||\operatorname{Proj}_w v||^2 \text{ for any } w, v \in V. \tag{2.1}$$

Lemma 2.1.3. *For any nonzero $w \in v$ and any $v \in V$, the following formula is true:*

$$\operatorname{Proj}_w v = \frac{(v, w)}{||w||^2} \cdot v. \tag{2.2}$$

Proof. Since $\operatorname{Proj}_w v$ is in the sub-space spanned by v, there is a real number r such that $\operatorname{Proj}_w v = r \cdot w$. Then

$$0 = (w, v - \operatorname{Proj}_w v) = (w, v - r \cdot w) = (w, v) - r \cdot ||w||^2.$$

Hence $r = \frac{(v,w)}{||w||^2}$ and the lemma is proved. □

Also recall that if $f : V \to \mathbf{R}$ is a linear functional, there is a unique vector $f^* \in V$ such that

$$f(v) = (v, f^*) \text{ for all } v \in V. \tag{2.3}$$

Indeed, if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V, we have

$$f^* = \sum_i f(e_i) \cdot e_i.$$

Together with formula (2.2), we have

$$\operatorname{Proj}_v f^* = \frac{f(v)}{||v||^2} \cdot v.$$

In particular, if v is a unit vector, we have

$$\operatorname{Proj}_v f^* = f(v) \cdot v. \tag{2.4}$$

We are now ready to establish the following result due to Kempf ([Kem78]).

Lemma 2.1.4. *Let f be a non-zero linear functional on V and $S \subset V$ be the unit sphere. Then f has only two antipodal critical points on S, one of which gives rise to a unique positive relative maximum $||f^*||$ and the other gives rise to a unique negative relative minimum $-||f^*||$.*

Proof. The set S is compact. Hence f always achieves global maximum and minimum. In particular, f achieves relative maximum and relative minimum on S. If $\dim V = 1$, $S = \{p_1\} \coprod \{p_2\}$. Since f is non-trivial we may assume $f(p_1) > 0$. Then $f(p_2) = -f(p_1) < 0$. So the only thing we need to show is $f(p_1) = ||f^*||$. Since p_1 provides an orthonormal basis of V, we have $f^* = f(p_1) \cdot p_1$ so that $||f^*|| = f(p_1)$. If $\dim V = 2$, by choosing an orthonormal basis we may assume $S = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}$. The kernel of f is a line, separating S into S_+ and S_- where $f \geq 0$ on S_+ and $f \leq 0$ on S_-. Parametrizing by $(x, y) = (\cos(\theta), \sin(\theta))$, we consider the function $g(\theta) := f(\cos(\theta), \sin(\theta))$. Then the double derivative $g''(\theta)$ is $-g(\theta)$. Since $f > 0$ on the interior of S_+, we see that $g'' < 0$ there. Hence there is only one point $s \in S$ where f attains local maximum on S. Same technique applies to the statement about uniqueness of relative minimum. When

$\dim V > 2$, if s_1 and s_2 are two points on S giving relative maxima, then form the space W spanned by s_1 and s_2. Then $\dim W \leq 2$. Restricting f and S to W, we get a contradiction. Finally, to show that the relative maximum is $||f^*||$ and the relative minimum is $-||f^*||$, note the Cauchy-Schwarz inequality has that $f(v) = (f^*, v) \leq ||f^*||$ for any $v \in S$. Taking $v = \frac{f^*}{||f^*||}$, the right hand side of the inequality is achieved. Same technique applies to the statement about the value of relative minimum. □

For any linear functional f on V, there are the associated *half space*

$$H_f^+ := \{v \in V | f(v) \geq 0\},$$

and the linear subspace

$$H_f := \{v \in V | f(v) = 0\}.$$

A *polyhedral cone* in V is a finite intersection of half spaces. Let σ be a polyhedral cone. The hyperplane H_f is a *supporting hyperplane for* σ if $\sigma \subset H_f^+$. A *face* F of σ is $H_f \cap \sigma$ for some supporting hyperplane H_f of σ. Notation-wise we write

$$F \preceq \sigma.$$

Obviously any face of a polyhedral cone is a polyhedral cone. The *relative interior* of a polyhedral cone σ is the interior of σ in the subspace W spanned by σ and is written as

$$\text{Relint}(\sigma).$$

We note without proof that for a polyhedral cone σ, its collection of faces is finite and

$$\sigma = \coprod_{F \preceq \sigma} \text{Relint}(F). \tag{2.5}$$

Using the same technique from Lemma 2.1.4, we now state the

Proposition 2.1.5. *Let S be the unit sphere in V, $\sigma \subset V$ a polyhedral cone, and $f : V \to \mathbf{R}$ be a linear functional. Suppose f assumes a positive value at a point on σ, then there is a unique $s \in S \cap \sigma$ such that $f(s)$ is the absolute maximum on $\sigma \cap S$.*

Proof. We can always restrict to the linear sub-space spanned by σ so let us assume σ is full dimensional. Note that $S \cap \sigma$ is a compact set so f always achieves absolute maximum there. We now attempt to establish uniqueness. When V is one dimensional, $S = \{p_1\} \coprod \{p_2\}$ consists of two points. Then one of them say p_1 is the desired point. If V has dimension two, then f is concave down on the semi-circle S^+ where f is non-negative. Since $S^+ \cap \sigma$ is a closed arc contained in S^+, we see that f attains maximum at a unique point $s \in S \cap \sigma$. When σ has higher dimension, suppose s_1 and s_2 are two distinct points on $S \cap \sigma$ where f attains maximum, then we may form the two dimensional subspace W spanned by s_1 and s_2. Then the polyhedral cone $\sigma \cap W$ is of dimension at most two. As s_1 and s_2 are on $\sigma \cap W$, we get a contradiction to previous results. □

It is well known that in a Euclidean space, every point is closest to a unique point on a closed convex set. The following theorem describes this geometric meaning of proposition 2.1.5.

Theorem 2.1.6. *Suppose f takes positive values on σ. Then the unique $s \in S$ coming from Proposition 2.1.5 satisfies*

$$||f^* - v|| \geq ||f - f(s) \cdot s|| \text{ for all } v \in \sigma.$$

Namely, the point $f(s) \cdot s = \operatorname{Proj}_s f^*$ is the unique point on σ closest to f^*.

Proof. Let $v \in \sigma$. There are two cases:

(1) $f(v) \leq 0$, and
(2) $f(v) > 0$.

For case (1), (2.2) implies $\operatorname{Proj}_{f^*} v = r \cdot f^*$ for some $r \leq 0$. Since $\operatorname{Proj}_{f^*} v - v$ is orthogonal to $f^* - \operatorname{Proj}_{f^*} v$, we have

$$\begin{aligned}||f^* - v|| &= ||f^* - \operatorname{Proj}_{f^*} v + \operatorname{Proj}_{f^*} v - v|| = (||f^* - \operatorname{Proj}_{f^*} v||^2 + ||\operatorname{Proj}_{f^*} v - v||^2)^{1/2} \\ &\geq ||f^* - \operatorname{Proj}_{f^*} v|| = (1-r) \cdot ||f^*|| \geq ||f^*|| \geq ||f^* - \operatorname{Proj}_s f^*||.\end{aligned}$$

For case (2), (2.2) implies $\operatorname{Proj}_v f^* \in \sigma$. Then $||\operatorname{Proj}_v f^*|| = f(\frac{\operatorname{Proj}_v f^*}{||\operatorname{Proj}_v f^*||}) \leq f(s) = ||\operatorname{Proj}_s f^*||$. Then as $f^* - \operatorname{Proj}_v f^*$ is orthogonal to $\operatorname{Proj}_v f^* - v$, we have

$$\begin{aligned}||f^* - v|| &= ||f^* - \operatorname{Proj}_v f^* + \operatorname{Proj}_v f^* - v|| = (||f^* - \operatorname{Proj}_v f^*||^2 + ||\operatorname{Proj}_v f^* - v||^2)^{1/2} \\ &\geq ||f^* - \operatorname{Proj}_v f^*|| = (||f^*||^2 - ||\operatorname{Proj}_v f^*||^2)^{1/2} \\ &\geq (||f^*||^2 - ||\operatorname{Proj}_s f^*||^2)^{1/2} = ||f^* - \operatorname{Proj}_s f^*||.\end{aligned}$$

□

Having established the existence and uniqueness of the point s, we would like to see how to find s in practice. For a face $F \preceq \sigma$, let $\operatorname{Sp}(F)$ be the sub-space spanned by F. In particular, if $F = \sigma$, $\operatorname{Sp}(\sigma)$ is the subspace spanned by σ. Locating s is made easier by the

Proposition 2.1.7. *Let $\sigma \subset V$ be a polyhedral cone and $f : V \to \mathbf{R}$ be a linear functional that assumes a positive value on σ. Let s be the unique point on the unit sphere where f achieves the relative maximum on σ. If $\frac{f^*}{||f^*||} \in \sigma$, then $s = \frac{f^*}{||f^*||}$. On the other hand, if s is in the relative interior of σ, then*

$$s = \frac{\operatorname{Proj}_{\operatorname{Sp}(\sigma)} f^*}{||\operatorname{Proj}_{\operatorname{Sp}(\sigma)} f^*||}.$$

Proof. By Lemma 2.1.4, f attains absolute maximum at $\frac{f^*}{||f^*||}$ on the whole sphere S. The first part follows. For the second part, since $(f|_{\operatorname{Sp}(\sigma)})^* = \operatorname{Proj}_{\operatorname{Sp}(\sigma)} f^*$, we may assume σ is full dimensional, and s is in the interior of σ. Then the goal is to prove that $s = \frac{f^*}{||f^*||}$. If $\dim \sigma = 1$, then since $\{s\}$ is an orthonormal basis of V, we have $f^* = f(s) \cdot s$. Then

$$\frac{f^*}{||f^*||} = \frac{f(s) \cdot s}{f(s)} = s$$

since $f(s) > 0$. When $\dim \sigma = 2$, let S^+ be the semi-circle where f is non-negative. If $s \neq \frac{f^*}{||f^*||}$, then

(1) there is an arc $C \subset S^+$ joining s and $\frac{f^*}{||f^*||}$, and
(2) $\frac{f^*}{||f^*||} \notin \sigma$.

Since s is in the interior of σ, one of the rays of σ say ρ would cross C. Since f is strictly increasing on the arc joining s and $\frac{f^*}{||f^*||}$, we have $f(\rho \cap C) > f(s)$. This is a contradiction. The case $\dim \sigma = 2$ is proved. If $\dim \sigma > 2$ and $s \neq \frac{f^*}{||f^*||}$, we can form the two dimensional sub-space W spanned by s and $\frac{f^*}{||f^*||}$ and by the previous argument we get a contradiction. □

Corollary 2.1.8. *Let $\sigma \subset V$ be a polyhedral cone and f a linear functional that assumes a positive value on σ. Let s be the unique point on the unit sphere where f achieves the relative maximum on σ. Then*

$$s = \frac{\operatorname{Proj}_{\operatorname{Sp}(F)} f^*}{|| \operatorname{Proj}_{\operatorname{Sp}(F)} f^*||} \text{ for some face } F \preceq \sigma.$$

Proof. Section 2.1 implies that $s \in \operatorname{Relint}(F)$ for some $F \preceq \sigma$. The corollary then follows from Proposition 2.1.7 applied to $\operatorname{Sp}(F) \cap \sigma$ and $f|_{\operatorname{Sp}(F)}$. □

Now s is completely computable. Since σ has finitely many faces, one can compute $\operatorname{Proj}_{\operatorname{Sp}(F)} f^*$ for all faces F and then compare

$$f(\frac{\operatorname{Proj}_{\operatorname{Sp}(F)} f^*}{|| \operatorname{Proj}_{\operatorname{Sp}(F)} f^*||}) = \frac{(\operatorname{Proj}_{\operatorname{Sp}(F)} f^*, f^*)}{|| \operatorname{Proj}_{\operatorname{Sp}(F)} f^*||} = || \operatorname{Proj}_{\operatorname{Sp}(F)} f^*||$$

for all faces F where $\operatorname{Proj}_{\operatorname{Sp}(F)} f^* \in F$. Due to the usefulness of this observation, we state it as

Corollary 2.1.9. *Let $\sigma \subset V$ be a polyhedral cone and f a linear functional that assumes a positive value on σ. Let s be the unique point on the unit sphere where f achieves the relative maximum on σ. Define*

$$\Lambda_\sigma^f = \{\operatorname{Proj}_{\operatorname{Sp}(F)} f^* \big| \operatorname{Proj}_{\operatorname{Sp}(F)} f^* \in \sigma, F \preceq \sigma\}.$$

Then $s = \frac{v}{||v||}$ where $v \in \Lambda_\sigma^f$, and $||v|| \geq ||v'||$ for all $v' \in \Lambda_\sigma^f$.

Remark 2.1.10. The assumption that f takes a positive value on σ can not be dropped to conclude that s is unique. For example, if H_f is a supporting hyperplane of σ so that the face $F = H_f \cap \sigma$ is a non-trivial cone, then every non-zero points on face $F \cap S$ yield maximum for $-f$.

The following is a nice integrality result which is the key to the development of instability in invariant theory. Fix a full dimensional lattice $M \subset V$. We say a polyhedral cone is a *rational polyhedral cone* if all the half spaces are defined by linear functionals that take integral values on M. A point $m \in M$ is *indivisible* if there are no $m' \in M$ and an integer $k \geq 2$ such that $m = k \cdot m'$. A point $v \in V$ is a *rational point* if some positive multiple of it lies in M. For example, $(1, \sqrt{2}) \in \mathbf{R}^2$ is not a rational point with respect to the standard lattice $\mathbf{Z}^2 \subset \mathbf{R}^2$.

Theorem 2.1.11. *Let $M \subset V$ be a full dimensional lattice, $\sigma \subset V$ be a rational polyhedral cone and $S \subset V$ be the unit sphere. Suppose the inner product $(-,-)$ takes integral values on $M \times M$ and $f : V \to \mathbf{R}$ is a linear functional that takes integral values on M. Then there is a rational point $v \in S$ such that $f(v)$ attains absolute maximum on $\sigma \cap S$. In particular, if f takes positive values on σ, the unique point $s \in S$ where $f(s)$ achieves maximum is rational.*

Proof. The statement is trivial for $f = 0$. Assume f is non-constant. Let $s \in S \cap \sigma$ be a point where the maximum is attained. There are three cases:

(1) $f(s) < 0$,
(2) $f(s) = 0$, or
(3) $f(s) > 0$.

For case (1), let $s \in \mathrm{Relint}(F)$ for some face $F \preceq \sigma$. Then s is a relative maximum for f on $F \cap S$. If $\dim F \geq 2$, then by Lemma 2.1.4, $f(s) > 0$, contradicting the assumption that $f(s) < 0$. Hence $\dim F \leq 1$. Since $s \neq 0$ is on F, $\dim F \geq 1$. We see that F is a ray. Since σ is rational polyhedral, F is spanned by a lattice point $m \in M$. Hence $s = m/||m|| \in S$ is rational. For case (2), note that $F = H_f \cap \sigma$ is a face of σ. Hence F is also rational. Since $s \neq 0$ is on F, $\dim F \geq 1$. If $\dim F = 1$, then s is a ray of F, which is rational. If $\dim F \geq 2$, there are nonzero lattice points on F and any normalized lattice point on F will do. For case (3), by Corollary 2.1.8, it is enough to show $\mathrm{Proj}_{\mathrm{Sp}(F)} f^*$ is rational for every face $F \preceq \sigma$. Let F be a face. The assumption that σ is rational implies that the subspace $\mathrm{Sp}(F)$ has an $\mathbf{R}$-basis in M. Hence the lattice $\mathrm{Sp}(F) \cap M$ is of full rank in $\mathrm{Sp}(F)$. Since $(f|_{\mathrm{Sp}(F)})^* = \mathrm{Proj}_{\mathrm{Sp}(F)} f^*$, we may restrict f to $\mathrm{Sp}(F)$ and the inner product to $\mathrm{Sp}(F) \times \mathrm{Sp}(F)$, taking integral values on the lattice $(\mathrm{Sp}(F) \cap M) \times (\mathrm{Sp}(F) \cap M)$. Therefore it is enough to prove rationality of f^*. Let $\{m_1, \dots, m_n\}$ be a $\mathbf{Z}$-basis of M. Then applying the Gram-Schmidt process, we see that there is an orthonormal basis $\{v_1, \dots, v_n\}$ of V where each v_i is a rational point. This implies $f^* = \sum_i f(v_i) \cdot v_i$ is a rational point. Hence $\mathrm{Proj}_{\mathrm{Sp}(F)} f^*$ is a rational point for all $F \preceq \sigma$. The theorem is proved. □

Remark 2.1.12. We will use the dual version of Theorem 2.1.11 when we discuss the affine instability. With the same assumption, there is also a rational point $v \in S$ such that $f(v)$ attains absolute minimum on $\sigma \cap S$. In the case that f takes negative values on σ, f attains absolute minimum at a unique rational point. This can be verified by maximizing $-f$.

2.2. **Distance to linear subspaces.** This section is essential to understand critical subsets in Section 5. Let $v \in V$ be a vector and $W \subset V$ be a subspace. We first note that the vector $\mathrm{Proj}_W v \in W$ satisfies

$$||v - \mathrm{Proj}_W v|| \leq ||v - w|| \text{ for all } w \in W.$$

Indeed, the vector $v - \mathrm{Proj}_W v$ is in $W^\perp$ so for any $w \in W$, we have

$$||v-w|| = ||v-\mathrm{Proj}_W v+\mathrm{Proj}_W v-w|| = \sqrt{||v - \mathrm{Proj}_W v||^2 + ||\mathrm{Proj}_W v - w||^2} \geq ||v-\mathrm{Proj}_W v||.$$

Hence $\mathrm{Proj}_W v \in W$ has the shortest distance to v and $||v - \mathrm{Proj}_W v||$ is the distance from v to W. We have an explicit formula to compute the distance. Let $\mathscr{B} = \{f_1^*, \dots, f_\epsilon^*\}$ be a basis for $W^\perp$. Equivalently, W is cut out by the linear functionals $\{f_1, \dots, f_\epsilon\}$ corresponding to $\mathscr{B}$. Applying the Gram-Schmidt process, we can assume $\mathscr{B}$ is orthonormal. Then

$$v - \mathrm{Proj}_W v = \mathrm{Proj}_{W^\perp} v = \sum_{i=1}^{\epsilon} f_i(v) \cdot f_i^*.$$

Therefore,

$$||v - \mathrm{Proj}_W v|| = \sqrt{\sum_{i=1}^{\epsilon} f_i(v)^2}. \tag{2.6}$$

Given two subspaces $W_1, W_2 \subset V$, we are interested in the equi-distance problem: Describe points $v \in V$ that are equi-distant to W_1 and W_2, namely, the collection

$\{v \in V \mid ||v - \mathrm{Proj}_{W_1} v|| = ||v - \mathrm{Proj}_{W_2} v||\}$. By the Pythagorean law, Equation (2.1), this is equivalent to the collection $\{v \in V \mid ||\,\mathrm{Proj}_{W_1} v|| = ||\,\mathrm{Proj}_{W_2} v||\}$. Before we attempt to describe this collection, here are some comments on change of basis and change of coordinates. Suppose $\dim V = N$. Each ordered basis $\mathscr{B} = \{b_1, \ldots, b_N\}$ of V corresponds to a coordinate system $X = \{x_1, \ldots, x_N\}$ on V in the sense that every point $v \in V$ can be expressed uniquely as $v = \sum_{i=1}^{N} x_i b_i$. Suppose $X' = \{x'_1, \ldots, x'_N\}$ is another coordinate system on V with respect to $\mathscr{B}'$. Let A be the transition matrix from $\mathscr{B}'$ to $\mathscr{B}$, namely, for every j, $b'_j = \sum_i a_{ij} b_i$, then X' and X is related by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = A \cdot \begin{pmatrix} x'_1 \\ \vdots \\ x'_N \end{pmatrix}. \tag{2.7}$$

This allows us to go between change of basis and change of coordinates. Sometimes it is easier to change coordinates instead of basis to obtain a simpler expression for quadratic forms as we will see in the lemma below.

Lemma 2.2.1. *Let W_1 and W_2 be two linear subspaces of V. Let m and n be the codimensions of W_1 and W_2 in $W_1 + W_2$ respectively. Then up to a change of coordinates, the collection Z of points $v \in V$ that are equi-distant to W_1 and W_2 is the vanishing locus of a homogeneous quadratic polynomial of the form $x_1^2 + \cdots x_m^2 - y_1^2 - \cdots y_n^2$. In particular, if both W_1 and W_2 are of codimension 1 in $W_1 + W_2$, Z is the union of two hyperplanes.*

Proof. Let $\{f_1^*, \ldots, f_\epsilon^*\}$ be a basis for $W_1^\perp \cap W_2^\perp = (W_1 + W_2)^\perp$. This basis extends to basis for $W_1^\perp$ and $W_2^\perp$, say $\mathscr{A} = \{f_1^*, \ldots, f_\epsilon^*, g_1^*, \ldots, g_m^*\}$ and $\mathscr{B} = \{f_1^*, \ldots, f_\epsilon^*, h_1^*, \ldots, h_n^*\}$ respectively. Then

$$\mathscr{A} \cup \mathscr{B} = \{f_1^*, \ldots, f_\epsilon^*, g_1^*, \ldots, g_m^*, h_1^*, \ldots, h_n^*\}$$

is a basis for $W_1^\perp + W_2^\perp$. Applying the Gram-Schmidt processes, we may assume $\mathscr{A}$ and $\mathscr{B}$ are orthonormal. Then by the distance formula (Equation (2.6)), the condition that $||v - \mathrm{Proj}_{W_1} v|| = ||v - \mathrm{Proj}_{W_2} v||$ yields $\sum_{i=1}^{\epsilon} f_i(v)^2 + \sum_{i=1}^{m} g_i(v)^2 = \sum_{i=1}^{\epsilon} f_i(v)^2 + \sum_{i=1}^{n} h_i(v)^2$ which in turn gives

$$\sum_{i=1}^{m} g_i(v)^2 = \sum_{i=1}^{n} h_i(v)^2. \tag{2.8}$$

Extending $\mathscr{A} \cup \mathscr{B}$ to a basis of V, say $\mathscr{C}$, we obtain a coordinate system on V say X. Then each element $r_i \in \mathscr{C}$, viewed as a linear functional on V, gives a linear expression of X, say $r_i(X)$. Then the following change of coordinate

$$r_i(X) = \begin{cases} x_i, & \text{if } r_i = g_i, \\ y_i, & \text{if } r_i = h_i, \\ x'_i, & \text{otherwise.} \end{cases}$$

gives 2.8 the desired form

$$x_1^2 + \cdots + x_m^2 = y_1^2 + \cdots + y_n^2.$$

□

To describe the equi-distant collections in V, we recall some terminologies from real manifolds. Let $F : N \to M$ be a C^∞ map between real manifolds. We say $c \in M$ is a *regular value of F* if either c is not in the image of F or at every point $p \in F^{-1}(c)$, the differential $F_*, p : T_pN \to T_{F(p)}M$ is surjective. The inverse image $F^{-1}(c)$ of a regular value c is called a *regular level set*. We now recall the regular level set theorem from [Tu11]:

Theorem 2.2.2 (Regular level set theorem)**.** *Let $F : N \to M$ be a C^∞ map of manifolds, with $\dim N = n$ and $\dim M = m$. Then a nonempty regular level set $F^{-1}(c)$, where $c \in M$, is a regular submanifold of N of dimension equal to $n - m$.*

We may now describe the equi-distant collection in V.

Proposition 2.2.3. *Let W_1 and W_2 be two sub-spaces of V and let Z be the collection of points $v \in V$ that are equi-distant to W_1 and W_2. If there is a containment, say $W_1 \subset W_2$, then Z is a linear subspace of codimension equal to the codimension of W_1 in W_2. If there is no containment between W_1 and W_2, then Z, away from a subspace of codimension at least 2, is a regular submanifold of V of codimension 1.*

Proof. Suppose $W_1 \subset W_2$. Then by Lemma 2.2.1, Z is defined by the equation $x_1^2 + \cdots + x_m^2 = 0$, which is a codimension m subspace of V. If there are no containment between W_1 and W_2, then Z is defined by $x_1^2 + \cdots + x_m^2 - y_1^2 - \cdots - y_n^2 = 0$ with both $m, n > 0$. Let $F : V \to \mathbf{R}$ be the C^∞ function $x_1^2 + \cdots + x_m^2 - y_1^2 - \cdots - y_n^2$. Let $V' \subset V$ be the subspace $V(x_1, \cdots, x_m, y_1, \cdots, y_n)$. Then $0 \in \mathbf{R}$ is a regular value for the restriction $F : V - V' \to \mathbf{R}$ and $F^{-1}(0)$ is obviously a nonempty regular level set. By the regular level set theorem, Theorem 2.2.2, $Z - V'$ is a regular submanifold of codimension 1. □

3. Instability - the affine case

Set up. We refer the reader to section 3, chapter 1 in [Mum65] for the definition of linearization of line bundles and section 4 for stability with respect to a linearization of a line bundle. We let k be an algebraically closed field and G be a reductive group over k. Namely, G is a smooth linear algebraic group such that for every finite dimensional representation M and any $m \neq 0$ that is fixed by G, there is a G-invariant, non-constant polynomial $f \in \operatorname{Sym} M^\vee$ such that $f(m) \neq 0$. By a variety we mean a reduced scheme that is finite type and separated over k. By a G-variety we mean a variety X with a left action $\sigma : G \times X \to X$. If X is an affine variety, then $k[X]$ denotes the ring $\mathrm{H}^0(X, \mathscr{O}_X)$. If V is a finite representation of G, then the affine space $\operatorname{Spec}(\operatorname{Sym}^* V)$ is equipped with an action G. Let X be an affine space. We say *G acts on X linearly* if there is a G-equivariant isomorphism $X \simeq \operatorname{Spec}(\operatorname{Sym}^* V)$ for some representation V of G.

Outline of the section. We are concerned with affine G-varieties with linearizations of the trivial line bundle that come from the characters of G. We first recall stability with respect to a character χ of G and the affine Hilbert-Mumford criterion (Theorem 3.1.7). This numerical criterion dictates that for every χ-unstable point, there is a non-trivial one parameter subgroup of G that destabilizes it. In Section 3.2 we then attempt to find those one parameter subgroups that are most responsible for instability (Theorem 3.2.9, [Kem78]). Then in Section 3.3 we establish the stratification (Theorem 3.3.4, [Hes79]) induced by χ, and formulate IIT-equivalence classes (Definition 3.3.7) for characters. Finally, in Section 3.4, Theorem 3.4.4 describes the structure of strata in the special case when G is a torus and X is an affine space on which G acts linearly. This result will be useful to describe strata in the toric variety case (Theorem 7.4.1).

3.1. **Definition and a numerical criterion.** Let X be an affine G-variety and $\sigma : G \times X \to X$ be the action morphism. A character $\chi : G \to \mathbf{G}_m$ gives a linearization of the trivial bundle

$$\Sigma : G \times X \times \mathbf{A}^1_k \to X \times \mathbf{A}^1_k$$

defined by the formula

$$\Sigma(g, x, z) = (\sigma(g, x)), \chi^{-1}(g) \cdot z).$$

We translate the definition of stability and invariants in the sense of [Mum65] with respect to χ.

Definition 3.1.1. Let X be an affine G-variety and $\chi : G \to \mathbf{G}_m = \operatorname{Spec} k[t]_t$ be a character. Let

(1) $\hat{\sigma} : k[X] \to k[G] \otimes_k k[X]$ be the co-action, and
(2) $\chi^\sharp : k[t]_t \to k[G]$ be the map that corresponds to χ.

An element $f \in k[X]$ is χ-*invariant of weight* d if $\hat{\sigma}(f) = \chi^\sharp(t)^d \otimes f$.

We explain how the collection of χ-invariants is naturally the subring of invariants of the linearized action. Suppose X is an affine G-variety and let χ be a character of G. For every $d \geq 0$, we let $k[X]_{\chi,d}$ be the collection of χ-invariant functions of weight d. Then the direct sum $\oplus_{d\geq 0} k[X]_{\chi,d}$ has a graded ring structure. Give $\mathbf{A}^1_k$ coordinate z. The linearized action $\Sigma : G \times X \times \mathbf{A}^1_k \to X \times \mathbf{A}^1_k$ corresponds to the co-action $\hat{\Sigma} : k[X, z] \to k[G] \otimes k[X, z]$ defined by $\hat{\Sigma}(f) = \hat{\sigma}(f)$ if $f \in k[X]$ and $\hat{\Sigma}(z) = \chi^\sharp(t)^{-d} \cdot z$. Then it is a straightforward computation to verify that the map

$$\bigoplus_{d\geq 0} k[X]_{\chi,d} \to k[X, z]^G$$

defined by

$$f \mapsto f \cdot z^d \text{ for } f \in k[X]_{\chi,d}$$

is an isomorphism of k-algebras. This shows that the ring $\oplus_{d\geq 0} k[X]_{\chi,d}$ is finitely generated over k. Hence the scheme

$$X/\!\!/_\chi G := \operatorname{Proj}(\oplus_{d\geq 0} k[X]_{\chi,d}),$$

which is known as the GIT quotient of X with respect to χ, is a quasi-projective variety.

Definition 3.1.2. Let X be an affine G-variety and χ be a character of G.

(1) A point $x \in X$ is χ-*semistable* if there is a χ-invariant f of positive weight such that $x \in X_f$.
(2) A point $x \in X$ is χ-*stable* if there is a χ-invariant f of positive weight such that $x \in X_f$, the stabilizer G_x is finite and for any point $y \in X_f$, the orbit $G \cdot y$ is closed in X_f.
(3) A point $x \in X$ is χ-*unstable* if it is not χ-semistable

The χ-semistable locus and the χ-stable locus will be written as $X^{ss}(\chi)$ and $X^s(\chi)$ respectively. The χ-unstable locus will be written as $X^{\mathrm{us}}(\chi)$.

We can linearize stability with respect to characters of G via the

Proposition 3.1.3. *Let G be a reductive group and X be an affine G-variety. Then there is a finite dimensional representation V of G that allows for a G-equivariant embedding $X \hookrightarrow \mathbf{V} := \operatorname{Spec}(\operatorname{Sym}^* V)$ such that $X^{ss}(\chi) = \mathbf{V}^{ss}(\chi) \cap X$ for all characters χ of G.*

Proof. It is not difficult to construct a G-equivariant embedding into an affine space on which G acts linearly. The coordinate ring $k[X]$ is a representation of G via the co-action $\hat{\sigma} : k[X] \to k[G] \otimes_k k[X]$. We may consider the vector space W spanned by a finite set of generators of $k[X]$ as a k-algebra. Then W being finite dimensional, is contained in a finite dimensional subrepresentation V of $k[X]$. Then the inclusion $V \subset k[X]$ as k-vector spaces induces a surjection $\operatorname{Sym}^* V \to k[X]$ and a closed immersion $X \hookrightarrow \mathbf{V}$. So far we have not used the fact that G is reductive. As $X \to \mathbf{V}$ is G-equivariant, for every d the map $k[\mathbf{V}] \to k[X]$ induces a map

$$k[\mathbf{V}]_{\chi,d} \to k[X]_{\chi,d}.$$

This implies $\mathbf{V}^{\mathrm{ss}}(\chi) \cap X \subset X^{\mathrm{ss}}(\chi)$. On the other hand we have isomorphisms

$$\bigoplus_d k[X]_{\chi,d} \simeq k[X,z]^G \text{ and } \bigoplus_d k[\mathbf{V}]_{\chi,d} \simeq k[\mathbf{V},z]^G$$

that are compatible with $k[\mathbf{V},z] \to k[X,z]$ and $k[\mathbf{V}] \to k[X]$. When k has characteristic zero, G is linearly reductive so the restrictions

$$k[\mathbf{V}]_{\chi,d} \to k[X]_{\chi,d}$$

are epimorphism for all d. If k has positive characteristic, then Lemma A.1.2 in [Mum65] ensures that for every $f \in k[X,z]^G$, there is a positive power f^N that lifts to an element in $k[V,z]^G$. In any case we have the inclusion $X^{\mathrm{ss}}(\chi) \subset \mathbf{V}^{\mathrm{ss}}(\chi) \cap X$. □

Example 3.1.4. Take $G = \mathbf{G}_m = \operatorname{Spec} k[t]_t$ and $A = k[x_1, \dots, x_n]$ with the co-action $\hat{\sigma} : k[x_1, \dots, x_n] \to k[t]_t \otimes k[x_1, \dots, x_n]$ where $x_i \mapsto t \otimes x_i$. The ring A^G of invariants consists of constant polynomials. We take $\chi : \mathbf{G}_m \to \mathbf{G}_m$ to be

(1) The identity. In this case, the χ-invariants of weight n is exactly homogeneous polynomials of degree n and $X^{ss}(\chi) = \mathbf{A}^n_k - \{0\}$. Stabilizers at each semistable point is the identity $e \in \mathbf{G}_m$. We see that in this case $X^{ss}(\chi) = X^s(\chi)$.
(2) The trivial character. In this case, χ-invariants of each weight is the same as $A^G = k$. Hence $X^{ss}(\chi) = \mathbf{A}^n_k$ and $\oplus_{n \geq 0} A^G_{\chi,n} \simeq A^G[x] = k[x]$. For stable points, note that the only point with closed orbit is the origin. Hence the action $\mathbf{G}_m \times \mathbf{A}^n_k \to \mathbf{A}^n_k$ is not closed. Therefore $X^s(\chi) = \emptyset$.
(3) The inverse $t \mapsto t^{-1}$. In this case there are no χ-invariants of positive weight. Hence $X^{ss}(\chi) = X^s(\chi) = \emptyset$.

Note that all positive multiples of these three characters do not give us new variation of $X^{ss}(\chi)$ and $X^s(\chi)$.

We will introduce a numerical criterion which allows us to determine if a point is semistable without knowing the invariants. Let

$$\boldsymbol{\chi}(G) := \{\chi : G \to \mathbf{G}_m\}$$

be the group of characters of G and

$$\boldsymbol{\Gamma}(G) := \{\lambda : \mathbf{G}_m \to G\}$$

be the set of one parameter subgroups of G.

There is a pairing

$$\langle -, - \rangle : \boldsymbol{\chi}(G) \times \boldsymbol{\Gamma}(G) \to \mathbf{Z}$$

defined by the formula

$$\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle} \text{ for all } k\text{-points } t \in \mathbf{G}_m.$$

Remark 3.1.5. While the set $\boldsymbol{\chi}(G)$ has the natural structure of an abelian group, the set $\boldsymbol{\Gamma}(G)$ in general does not. However, if $G = (k^\times)^m = \{(t_1, \ldots, t_m) | t_i \in k^\times\}$ is a torus, $\boldsymbol{\Gamma}(G)$ can be identified as $\mathbf{Z}^m$ where $(b_1, \ldots, b_m) \in \mathbf{Z}^m$ induces a one parameter subgroup λ

$$t \mapsto (t^{b_1}, \ldots, t^{b_m}).$$

Likewise $\boldsymbol{\chi}(G)$ is identified as $\mathbf{Z}^m$ where each $(a_1, \ldots, a_m) \in \mathbf{Z}^m$ defines a character χ

$$(t_1, \ldots, t_m) \mapsto t_1^{a_1} \ldots t_m^{a_m}.$$

and $\langle \chi, \lambda \rangle = \sum_i a_i b_i$. In fact the natural pairing $\langle -, - \rangle$ identifies $\boldsymbol{\chi}(G)$ as the dual of $\boldsymbol{\Gamma}(G)$ and vice versa.

The group G acts on $\boldsymbol{\Gamma}(G)$ by conjugation. Namely, for $g \in G$ and $\lambda \in \boldsymbol{\Gamma}(G)$,

$$(g * \lambda)(t) = g \cdot \lambda(t) \cdot g^{-1} \text{ for all } k\text{-points } t \in \mathbf{G}_m.$$

We also note that any two maximal tori in a reductive group are conjugate to each other. For each maximal torus T in a reductive group G, the inclusion $T \subset G$ induces an inclusion $\boldsymbol{\Gamma}(T) \subset \boldsymbol{\Gamma}(G)$. Since the image of a one parameter subgroup of G is contained in a maximal torus, we get

$$\boldsymbol{\Gamma}(G) = \bigcup_{T \text{ maximal torus}} \boldsymbol{\Gamma}(T).$$

If we fix a maximal torus T, then

$$\boldsymbol{\Gamma}(G) = \bigcup_{g \in G} \boldsymbol{\Gamma}(gTg^{-1}) = \bigcup_{g \in G} g\boldsymbol{\Gamma}(T)g^{-1}.$$

For a k-point point $x \in X$ and $\lambda \in \boldsymbol{\Gamma}(G)$, there is a map

$$\lambda_x : \mathbf{G}_m \to X$$

defined by the composition

$$\mathbf{G}_m \xrightarrow{\lambda} G \xrightarrow{\cdot x} X.$$

We say the limit $\lim_{t \to 0} \lambda(t) \cdot x$ exists in X if λ_x extends to a map

$$\mathbf{A}^1_k \to X.$$

We list two useful facts whose proofs are immediate and are omitted:

Lemma 3.1.6. *Let X be a G-variety, $x \in X$ be a k-point, $\chi : G \to \mathbf{G}_m$ be a character and $\lambda : \mathbf{G}_m \to G$ be a one parameter subgroup. Then*

- $\langle \chi, g\lambda g^{-1} \rangle = \langle \chi, \lambda \rangle$ *for all $g \in G$.*
- *The limit $\lim_{t \to 0} \lambda(t) \cdot x$ exists in X if and only if $\lim_{t \to 0}(g * \lambda)(t) \cdot (g \cdot x)$ exists in X for some (and hence for all) $g \in G$.*

Here is the numerical criterion:

Theorem 3.1.7. *(Affine-Hilbert Mumford Criterion.* [Kin94], [Mum65]*) Let G be a reductive group and X be a G-variety. A k-point $x \in X$ is χ-semistable if and only if for all one parameter subgroup $\lambda : \mathbf{G}_m \to G$ such that the limit $\lim_{t \to 0} \lambda(t) \cdot x$ exists, we have $\langle \chi, \lambda \rangle \geq 0$.*

As an application, we get the following

Corollary 3.1.8.
$$X^{ss}(\chi) = \bigcap_{T \text{ maximal torus}} X^{ss}(\chi_T)$$
where χ_T is the character χ restricted to T.

Here is an example how the numerical criterion can be used.

Example 3.1.9. Let $G = \mathrm{GL}_d(k)$, $V = k^{d\times n}$ be the representation of G where G acts on V by left multiplication and $\chi : G \to \mathbf{G}_m$ be the determinant. The χ-unstable locus is the set of matrices not of full rank. To see this, let A be a rank d matrix and λ be a one parameter subgroup with $\lim_{t\to 0}\lambda(t)\cdot A$ exists. By Lemma 3.1.6 and the fact that the rank of A is G-invariant, we can assume λ is contained in the diagonal torus. Since no row of A is a zero row, all weights of λ is non-negative. The number $\langle \chi, \lambda\rangle$ is the sum of weights and therefore is non-negative. We see that A is χ-semistable. If A is of smaller rank, say rank $d-i$. Then by the above bullets we can assume A is in reduced row-echelon form with the last i rows consisting of zeros. Then the one parameter subgroup
$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & t^{-1} & & \\ & & & & \ddots & \\ & & & & & t^{-1} \end{pmatrix}$$
destabilizes A.

For a fixed character $\chi \in \boldsymbol{\chi}(G)$, the function $\langle \chi, -\rangle : \mathbf{\Gamma}(G) \to \mathbf{Z}$ serves as a measure of instability. For any one parameter subgroup λ and any positive integer N, we have
$$\langle \chi, \lambda^N\rangle = N \cdot \langle \chi, \lambda\rangle.$$
We see that multiples of a one parameter subgroup does not give new information. Some adjustments are needed. The idea is to find a real-valued norm $||-||$ on $\mathbf{\Gamma}(G)$ and consider the normalized function $\frac{\langle \chi, -\rangle}{||-||} : \mathbf{\Gamma}(G) \to \mathbf{R}$. We make the norm precise in the next section.

3.2. **Numerical analysis for instability.** Judging from the numerical criterion, Theorem 3.1.7, it is natural to ask for a χ-unstable point if there is a one parameter subgroup that is the most responsible for instability. We will define a norm on $\mathbf{\Gamma}(G)$ and analyze the normalized function $\frac{\langle \chi, -\rangle}{||-||} : \mathbf{\Gamma}(G) \to \mathbf{R}$. We follow [Mum65] to set up a norm on $\mathbf{\Gamma}(G)$.

Let $T \subset G$ be a maximal torus. Note that $\mathbf{\Gamma}(T)$ is a lattice of finite rank. The *Weyl Group* $N(T)/T$ is finite where $N(T)$ is the normalizer of T. G acts on $\mathbf{\Gamma}(G)$ by conjugation. The action descends to an action of the Weyl group on $\mathbf{\Gamma}(T)$. There exists one positive definite symmetric bilinear form on $\mathbf{\Gamma}(T)_{\mathbf{R}}$, assuming integral values on $\mathbf{\Gamma}(T)$ and invariant under the Weyl group. Fix such a form $(-,-) : \mathbf{\Gamma}(T)_{\mathbf{R}} \times \mathbf{\Gamma}(T)_{\mathbf{R}} \to \mathbf{R}$. Since any two maximal tori are conjugate to each other, for any one parameter subgroup $\lambda \in \mathbf{\Gamma}(G)$, there is a $g \in G$ such that $g\cdot\lambda\cdot g^{-1} \in \mathbf{\Gamma}(T)$. We then put
$$||\lambda|| = (g\cdot\lambda\cdot g^{-1}, g\cdot\lambda\cdot g^{-1})^{\frac{1}{2}}.$$
This does not depend on the choice of g due to lemma 2.8 from [Mum65]:

Lemma 3.2.1. *Let $\lambda \in \mathbf{\Gamma}(T)$ and $g \in G$ such that $g\lambda g^{-1} \in \mathbf{\Gamma}(T)$, then there exists a $\gamma \in N(T)$ such that*

$$\gamma\lambda\gamma^{-1} = g\lambda g^{-1}.$$

We therefore obtain a norm $||-||$ on $\mathbf{\Gamma}(G)$ satisfying the following two properties:

(1) $||g * \lambda|| = ||\lambda||$ for all $g \in G$ and for all $\lambda \in \mathbf{\Gamma}(G)$,
(2) for any maximal torus $T' \subset G$, there is a positive definite symmetric bilinear form $(-,-)'$ on $\mathbf{\Gamma}(T')_{\mathbf{R}} \times \mathbf{\Gamma}(T')_{\mathbf{R}}$, invariant under its Weyl group and takes integral values on $\mathbf{\Gamma}(T') \times \mathbf{\Gamma}(T')$, and such that $(\lambda, \lambda)' = ||\lambda||^2$ for any $\lambda \in \mathbf{\Gamma}(T')$.

Definition 3.2.2. Let G be a reductive group and X be an affine G-variety. For a point $x \in X$, set

$$C_x = \{\lambda \in \mathbf{\Gamma}(G) | \lim_{t\to 0} \lambda(t) \cdot x \text{ exits}\}.$$

If χ is a character of G and x is χ-unstable, we set

$$M^\chi(x) = \inf_{\lambda \in C_x} \frac{\langle \chi, \lambda\rangle}{||\lambda||}.$$

In this case, we say a one parameter subgroup λ is χ-*adapted to* x if $\frac{\langle\chi,\lambda\rangle}{||\lambda||} = M^\chi(x)$. We write $\Lambda^\chi(x)$ as the collection of all indivisible one parameter subgroups λ that are χ-adapted to x. Moreover, we say λ is χ-*adapted to a subset* $S \subset X^{\mathrm{us}}(\chi)$ if λ is χ-adapted to every point in S.

In the case that G is a torus and X is an affine space on which G acts linearly, the set C_x is rather explicit. From Remark 3.1.5, $\mathbf{\Gamma}(G)_{\mathbf{R}} := \mathbf{\Gamma}(G) \otimes_{\mathbf{Z}} \mathbf{R}$ is of dimension $\dim G$, containing $\mathbf{\Gamma}(G)$ as a full dimensional lattice.

Lemma 3.2.3. *Let G be a torus acting linearly on an affine space X. For each $x \in X$, $\sigma_x := C_x \otimes_{\mathbf{Z}} \mathbf{R} \subset \mathbf{\Gamma}(G)_{\mathbf{R}}$ is a rational polyhedral cone.*

Proof. We may assume $X = \mathbf{A}_k^n$ and that there are n characters $\chi_1, \ldots, \chi_n$ of G such that

$$g \cdot (x_1, \ldots, x_n) = (\chi_1(g) \cdot x_1, \ldots, \chi_n(g) \cdot x_n)$$

for all $g \in G$ and $(x_1, \ldots, x_n) \in X$. If $\lambda : \mathbf{G}_m \to G$ is a one parameter subgroup and if $t \in \mathbf{G}_m$, we get

$$\lambda(t) \cdot (x_1, \ldots, x_n) = (t^{\langle\chi_1,\lambda\rangle} x_1, \ldots, t^{\langle\chi_n,\lambda\rangle} x_n).$$

For each $i = 1, \ldots, n$, χ_i induces a $\mathbf{Z}$-linear map

$$\langle \chi_i, -\rangle : \mathbf{\Gamma}(G) \to \mathbf{Z}$$

which extends to a linear functional on $\mathbf{\Gamma}(G)_{\mathbf{R}}$ over $\mathbf{R}$. The cone

$$\sigma_x := \bigcap_{i, x_i \neq 0} \{v \in \mathbf{\Gamma}(G)_{\mathbf{R}} | \langle \chi_i, v\rangle \geq 0\}$$

is a rational polyhedral cone and obviously $\sigma_x = C_x \otimes_{\mathbf{Z}} \mathbf{R}$. ☐

Theorem 3.2.4. *Let G be a reductive group, X be an affine G-variety, and $\chi : G \to \mathbf{G}_m$ be a character. For a k-point $x \in X^{us}(\chi)$, we have*

(1) the value $M^\chi(x)$ is finite and $\Lambda^\chi(x)$ is not empty,
(2) for any $g \in G$, $\Lambda^\chi(g \cdot x) = g\Lambda^\chi(x)g^{-1}$,
(3) the function $M^\chi(-) : X^{us}(\chi) \to \mathbf{R}$ assumes finitely many values,

(4) $M^\chi(-)$ *is constant on* G*-orbits. Namely, for each* $g \in G$ *and* $x \in X^{us}(\chi)$*, we have*

$$M^\chi(x) = M^\chi(g \cdot x).$$

Moreover, if G *is a torus,*

(5) $\Lambda^\chi(x)$ *is a singleton.*

Proof. We apply Proposition 3.1.3 to get a G-equivariant embedding of X into an affine space on which G acts linearly. The stability of any k-point in X is invariant under the embedding. Moreover, C_x is invariant, too. Hence we may replace X by some affine space where G acts linearly. We will prove the theorem for the case $G = T$ is a torus first. Since tori are abelian groups, statement (2) becomes $\Lambda^\chi(x) = \Lambda^\chi(g \cdot x)$. By the numerical criterion, theorem 3.1.7, a point $x \in X$ is χ-unstable if and only if the function $\langle \chi, - \rangle : \mathbf{\Gamma}(G) \to \mathbf{Z}$ achieves negative values on C_x. The rationality result of theorem 2.1.11 implies that the relative minimum of $\frac{\langle \chi, - \rangle}{||-||}$ on σ_x is attained at a unique indivisible one parameter subgroup. Namely, $\Lambda^\chi(x)$ is a singleton and $M^\chi(x)$ is finite. This proves (1) and (5). To see that there are only finitely values taken by $M^\chi(-)$, simply note that there are only finitely many rational polyhedral cones $C_x \otimes_{\mathbf{Z}} \mathbf{R}$ as x runs through X. For each $g \in G$, the two equalities $M^\chi(x) = M^\chi(g \cdot x)$ and $\Lambda^\chi(x) = \Lambda^\chi(g \cdot x)$ hold because $C_x = C_{g \cdot x}$. Hence the theorem is proved when G is a torus. For general reductive group G, consider a maximal torus $T \subset G$ and let χ_T be the restriction of χ to T. Consider the subset

$$H_x = \{g \in G \mid g \cdot x \in X^{\text{us}}(\chi_T)\}$$

of k-points of G. This is non-empty by Corollary 3.1.8. I claim that

$$M^\chi(x) = \inf_{g \in H_x} M^{\chi_T}(g \cdot x). \tag{3.1}$$

It is immediate that $M^\chi(x) \geq \inf_{g \in H_x} M^{\chi_T}(g \cdot x)$. To prove the other inequality, note that $M^{\chi_T}(-) : X^{\text{us}}(\chi_T) \to \mathbf{R}$ takes only finitely many finite values. It follows that the right hand side is finite and achievable by some nonzero indivisible one parameter subgroup of T, say λ' at the point $g \cdot x$. Then $g^{-1}\lambda' g \in C_x$ so that

$$\frac{\langle \chi_T, \lambda' \rangle}{||\lambda'||} = \frac{\langle \chi, g^{-1}\lambda' g \rangle}{||g^{-1}\lambda' g||} \geq M^\chi(x).$$

Therefore $\lambda' \in \Lambda^\chi(x)$. This proves (1). Moreover, Equation (3.1) makes it apparent that $M^\chi(x)$ is constant on G-orbits and that

$$\{M^\chi(x) | x \in X^{\text{us}}(\chi)\} \subset \{M^{\chi_T}(x) | x \in X^{\text{us}}(\chi_T)\}.$$

Since the right hand side is a finite set, so is the left hand side. This proves (3). Finally, (2) is a consequence of (4) and Lemma 3.1.6. The theorem is proved. □

Equation (3.1) is so useful later in Section 5 that it deserves a separate proposition.

Proposition 3.2.5. *Let* G *be a reductive group and* $T \subset G$ *be a maximal torus. Let* X *be an affine* G*-variety and* $x \in X^{us}(\chi)$ *for some character* χ *of* G*. Set*

$$H_x = \{g \in G \big| g \cdot x \in X^{us}(\chi_T)\}.$$

Then H_x *is non-empty, and*

$$M^\chi(x) = \inf_{g \in H_x} M^{\chi_T}(g \cdot x).$$

Moreover, if $M^{\chi}(x) = M^{\chi_T}(g \cdot x)$ for some $g \in H_x$, and $\lambda = \Lambda^{\chi_T}(g \cdot x) \in \mathbf{\Gamma}(T)$, we have $g^{-1}\lambda g \in \Lambda^{\chi}(x)$.

To fully describe $\Lambda^{\chi}(x)$ for χ-unstable x, we need to recall the following

Definition 3.2.6. For every $\lambda \in \mathbf{\Gamma}(G)$, $P(\lambda)$ is the subgroup of G where $p \in P(\lambda)$ if and only if

$$\lim_{t \to 0} \lambda(t) p \lambda(t)^{-1}$$

exists in G.

It is shown in proposition 2.6 in [Mum65] that

(1) for any $\gamma \in P(\lambda)$, the limit $\lim_{t\to 0} \lambda(t) \cdot \gamma \cdot \lambda^{-1}(t)$ centralizes λ, and
(2) $P(\lambda)$ is a parabolic subgroup of G.

Example 3.2.7. Let $G = \mathrm{GL}_d(k)$ and T be the maximal torus of G consisting diagonal matrices. If $\lambda \in \mathbf{\Gamma}(T)$, we may assume λ looks like

$$\begin{pmatrix} \boxed{N_1} & & & \\ & \boxed{N_2} & & \\ & & \ddots & \\ & & & \boxed{N_r} \end{pmatrix}$$

where each N_i is a block diagonal matrix with weight n_i with

$$n_1 > n_2 > \cdots > n_r.$$

Let us compute $P(\lambda)$. Suppose $A \in G$, then

$$\lambda(t) A \lambda^{-1}(t) = \begin{pmatrix} 0 & M_{12} & \cdots & M_{1r} \\ M_{21} & 0 & \cdots & M_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & 0 \end{pmatrix}$$

where each M_{ij} is the multiplication of the corresponding block matrix in A by $t^{n_i - n_j}$ and the block zero matrix is the multiplication by t^0. We see that the lower triangular part of $\lambda(t)A\lambda^{-1}(t)$ has to be zero for the limit to exist. But there are no constraints on the upper triangular part so we see that

$$P(\lambda) = \begin{pmatrix} * & * & \cdots & * \\ & * & \cdots & * \\ & & \ddots & \vdots \\ & 0 & & * \end{pmatrix}.$$

Lemma 3.2.8. *Let X be an affine G-variety and $x \in X$ be the k-point. Then if $\lambda \in C_x$, $\lambda \in C_{p\cdot x}$ for any $p \in P(\lambda)$ as well.*

Proof. Applying Proposition 3.1.3, we may replace X by $\mathbf{V}$ where V is a finite dimensional representation of G. In this case, $\lim_{t\to 0} \big(\lambda(t) p \lambda^{-1}(t)\big) \cdot x$ exists. In fact

$$\lim_{t\to 0} \big(\lambda(t) p \lambda^{-1}(t)\big) \cdot x = \big(\lim_{t\to 0} \lambda(t) p \lambda^{-1}(t)\big) \cdot x.$$

The space V has the decomposition $V = \oplus_{i \in \mathbf{Z}} V_i$ where $v' \in V_i$ if and only if $\lambda(t) \cdot v' = t^i \cdot v'$ for all $t \in k^{\times}$. Let v be the vector corresponding to x. Writing $v = \sum_i v_i$ for $v_i \in V_i$, we see that in order for the limit $\big(\lim_{t\to 0} \lambda(t) p \lambda^{-1}(t)\big) \cdot x$ to exists, $p \cdot v_i \in \oplus_{j \geq i} V_i$ for all $v_i \in V_i$ and for all $i \in \mathbf{Z}$. Now by assumption

$\lim_{t\to 0}\lambda(t)x$ exists so $v \in \oplus_{i\geq 0}V_i$. By what we just showed $pv \in \oplus_{i\geq 0}V_i$ still. Hence $\lim_{t\to 0}\lambda(t)\cdot(pv)$ exists. The lemma is proved. □

Theorem 3.2.9. *(Kempf, [Kem78]) Let G be a reductive group, X be an affine G-variety, χ be a character of G and $x \in X$ be a χ-unstable point. Then*

(1) $\Lambda^\chi(x)$ is not empty;
(2) for each maximal torus $T \subset G$, $\mathbf{\Gamma}(T)$ intersects $\Lambda^\chi(x)$ in at most one point;
(3) for any $g \in G$, $\Lambda^\chi(g\cdot x) = g\Lambda^\chi(x)g^{-1}$;
(4) there is a parabolic subgroup $P(\chi, x)$ such that for all $\lambda \in \Lambda^\chi(x)$, $P(\lambda) = P(\chi, x)$;
(5) all elements of $\Lambda^\chi(x)$ are conjugate to each other by elements of $P(\chi, x)$. In particular, if G is abelian, $\Lambda^\chi(x)$ is a singleton.

Proof. Statement (1) and (3) follow from Theorem 3.2.4. For statement (2), let λ be a one parameter subgroup in $\Lambda^\chi(x)$ that is also in $\mathbf{\Gamma}(T)$. Then as $\langle\chi,\lambda\rangle = \langle\chi_T,\lambda\rangle < 0$, x is χ_T-unstable. Obviously $\lambda \in \Lambda^{\chi_T}(x)$ as well so by the last statement in Theorem 3.2.4, $\Lambda^\chi(x)\cap\mathbf{\Gamma}(T)$ is a singleton. (4) and (5) can be proved simultaneously. Let λ_1 and λ_2 be two elements in $\Lambda^\chi(x)$. There are two maximal tori T_1 and T_2 such that

$$\operatorname{Im}\lambda_i \subset T_i \subset P(\lambda_i) \text{ for } i = 1, 2.$$

By a theorem from [Tit62], the intersection $P(\lambda_1)\cap P(\lambda_2)$ contains a maximal torus T. Hence T and T_i are both maximal tori for $P(\lambda_i)$. We may find $p_i \in P(\lambda_i)$ so that

$$T = p_i T_i p_i^{-1}.$$

Then by Lemma 3.2.8, we have $p_i\lambda_i p_i \in C_x$ and

$$p_i^{-1}\lambda_i p_i \in \Lambda^\chi(x)\cap\mathbf{\Gamma}(T).$$

Hence by (2), $p_1^{-1}\lambda_1 p_1 = p_2^{-1}\lambda_2 p_2$. It follows that

$$P(\lambda_1) = p_1^{-1}P(\lambda_1)p_1 = P(p_1^{-1}\lambda_1 p_1) = P(p_2^{-1}\lambda_2 p_2) = P(\lambda_2) = P(\chi, x)$$

and λ_1, λ_2 are conjugates by $P(\chi, x)$. □

3.3. **Stratification of the null cone.** Let G be a reductive group and X be an affine G-variety. For each $\lambda \in \mathbf{\Gamma}(G)$, define $[\lambda]$ to be the conjugacy class and

$$S_{[\lambda]} = \{x \in X^{us}(\chi) | \Lambda^\chi(x)\cap[\lambda] \neq \emptyset\}.$$

Let

$$\Lambda^\chi = \{[\lambda] | \lambda \in \Lambda^\chi(x) \text{ for some } x \in X^{\text{us}}(\chi)\}.$$

We define a strict partial ordering on Λ^χ by

$$[\lambda] > [\lambda'] \text{ if } \frac{\langle\chi,\lambda\rangle}{||\lambda||} < \frac{\langle\chi,\lambda'\rangle}{||\lambda'||}.$$

Definition 3.3.1. Let X be an affine G-variety and λ be a one parameter subgroup. We define

$$X_\lambda = \{x \in X \mid \lim_{t\to 0}\lambda(t)\cdot x \text{ exists in } X\}.$$

Remark 3.3.2. In [Tha96], X_λ has a nice description. The one parameter subgroup λ induces a $\mathbf{Z}$-grading on $k[X]$. Then X_λ is cut out by the ideal generated by $\oplus_{d<0}k[X]_d$.

Lemma 3.3.3. *For any $\lambda \in \mathbf{\Gamma}(G)$, $G\cdot X_\lambda$ is closed in X.*

Proof. Consider the maps

$$G \times X_\lambda \xrightarrow{\gamma} G \times X \xrightarrow{\delta} G/P(\lambda) \times X$$

where $\gamma(g, x) = (g, gx)$ and $\delta(g, y) = (gP(\lambda), y)$. Let $M = \delta\gamma(G \times X_\lambda)$. Since X_λ is $P(\lambda)$-invariant by Lemma 3.2.8, $\delta^{-1}(M) = \{(g, y)|g^{-1}y \in X_\lambda\}$, which is closed in $G \times X$. Since δ is a quotient map, $\delta(\delta^{-1}(M))$ is closed in $G/P(\lambda) \times X$. $G \cdot X_\lambda$ is the image of M under the second projection $G/P(\lambda) \times X \to X$. By completeness of $G/P(\lambda)$, $G \cdot X_\lambda$ is closed. □

Theorem 3.3.4. *(Hesselink, [Kir84]) Let G be a reductive group and X an affine G-variety. Fix a character χ of G. Then Λ^χ is finite and*

$$X = X^{ss}(\chi) \cup \bigcup_{[\lambda] \in \Lambda^\chi} S_{[\lambda]}$$

is a disjoint union of G-invariant, locally closed subvariety of X. Moreover, $S_{[\lambda]} \cap \partial S_{[\lambda']} \neq \emptyset$ only if $[\lambda] > [\lambda']$.

Proof. That $X^{\mathrm{us}}(\chi) = \bigsqcup_{[\lambda] \in \Lambda^\chi} S_{[\lambda]}$ and that $S_{[\lambda]}$ is G-invariant are direct results of Theorem 3.2.9. We will prove the last statement first and locally closedness of each stratum can be carried out by induction. Let $x \in S_{[\lambda]} \cap \partial S_{[\lambda']}$. By Lemma 3.3.3, $\partial S_{[\lambda']} \subset G \cdot X_{\lambda'}$. Hence there is some $g' \in G$ such that $g' * \lambda \in C_x$. Obviously $S_{[\lambda]} \subset G \cdot X_\lambda$ so there is a $g \in G$ such that $g^* \lambda \in C_x$. Then by assumption we have

$$\frac{\langle \chi, g' * \lambda' \rangle}{||g' * \lambda'||} \geq \frac{\langle \chi, g * \lambda \rangle}{||g * \lambda||}.$$

This cannot be an equality unless $[\lambda] = [\lambda']$. Hence we have proved the last statement. For locally closedness, the statement we just proved implies that

$$\partial S_{[\lambda]} \subset \bigcup_{[\lambda'] > [\lambda]} S_{[\lambda']}.$$

Locally closedness now follows from the observation that

(1) $S_{[\lambda]} = \overline{S_{[\lambda]}} - \cup_{[\lambda'] > [\lambda]} S_{[\lambda']}$, and
(2) induction on the order of $[\lambda] \in \Lambda^\chi$.

□

Remark 3.3.5. The proof of Theorem 3.3.4 also implies that the function $M^\chi(-) : X \to \mathbf{R}$ is lower semicontinuous on the unstable locus. Namely, for every $d \in \mathbf{R}$, the set

$$X_d = \{x \in X^{us}(X) | M^\chi(x) \geq d\}$$

is closed.

We define stratification on a G-variety and when two stratifications are equivalent.

Definition 3.3.6. Let G be a reductive group and X be a G-variety. A *stratification of* X is a finite collection $\{X_a\}_{a \in \mathscr{A}}$ of mutually disjoint G-invariant locally closed subvarieties of X such that

(1) $\mathscr{A}$ is a strictly partially ordered set,
(2) $\cup_{a \in \mathscr{A}} X_a = X$, and
(3) $X_{a'} \cap \partial X_a \neq \emptyset$ only if $a' > a$.

Two stratifications $\{X_a\}_{a \in \mathscr{A}}$ and $\{X_\beta\}_{\beta \in \mathscr{B}}$ of X are equivalent if there is an order preserving bijection $\Phi : \mathscr{A} \to \mathscr{B}$ such that $X_a = X_{\Phi(a)}$ as sets.

We refer to the stratification in Theorem 3.3.4 as *the stratification induced by* χ and each stratum as a χ-*stratum*. The convention for us is to index the χ-semistable locus by the zero one parameter subgroup, having order zero. Doing this also ensures that condition (3) in Definition 3.3.6 is satisfied for the boundary of the semistable locus.

Obviously the stratification depends on the choice of a character.

Definition 3.3.7. Let G be a reductive group and X an affine G-variety. We say χ_1 and χ_2 are *IIT equivalent* (with repsect to the action of G on X) if χ_1 and χ_2 induce equivalent stratifications.

3.4. **The structure of strata.** In this section we describe the structure of strata for representations of tori. For a torus T acting linearly on $X = \mathbf{A}^n_k$ defined diagonally by n-characters $\chi_1, \dots, \chi_n \in \chi(T)$, we set the following notations:

- $[n] = \{1, \dots, n\}$, and for any subset $B \subset [n]$,
- $V(B) = V(x_i | i \in B)$,
- $D(B) = D(\prod_{i\in B} x_i)$, namely, the principal open subset in X define by the monomial $\prod_{i\in B} x_i$,
- $L(B) = V([n] - B) \cap D(B)$,
- $\sigma_B = \{v \in \mathbf{\Gamma}(T)_{\mathbf{R}} | \langle \chi_i, v\rangle \geq 0 \text{ for } i \in B\}$, and if $A \subset B \subset [n]$, we also set
- $L(A; C) = \cup_B L(B)$ where the union is over all B with $A \subset B \subset C$,

Let $\chi \in \chi(T)$ be a character. In this case all χ-strata, if any, are indexed by a single one parameter subgroup of T. Let $\lambda \in \mathbf{\Gamma}(G)$ be a one parameter subgroup indexing the stratum S_λ. It is obvious that S_λ is a disjoint union of $L(B)$ for various subsets B. This section intends to give a more detailed description of what S_λ looks like. This is related to the subspace $X_\lambda \subset X$ defined earlier by

$$X_\lambda := \{x \in X | \lim_{t\to 0} \lambda(t) \cdot x \text{ exists}\}.$$

Equivalently,

$$X_\lambda = V(x_i | \langle \chi_i, \lambda\rangle < 0).$$

Obviously $S_\lambda \subset X_\lambda$. We will show that S_λ is open in X_λ. Let $|| - || : \mathbf{\Gamma}(T)_{\mathbf{R}} \to \mathbf{R}$ be the standard norm. We first show

Lemma 3.4.1. *For every one parameter subgroup λ of T, if λ minimize $\frac{\langle \chi, -\rangle}{||-||}$ on σ_S for some subset $S \subset [n]$, then there is only one maximal subset $C \subset [n]$ such that λ minimizes the function $\frac{\langle \chi, -\rangle}{||-||}$ on σ_C. Moreover, an element $i \in [n]$ is in C if and only if $\langle \chi_i, \lambda\rangle \geq 0$.*

Proof. Suppose λ minimize $\frac{\langle \chi, -\rangle}{||-||}$ on σ_S for some subset S, let $C' = \{i \in [n] | \langle \chi_i, \lambda\rangle \geq 0\}$. Then clearly $S \subset C'$. In particular, $C \subset C'$ for any maximal C such that λ minimizes $\frac{\langle \chi, -\rangle}{||-||}$ on σ_C. If $C' = \emptyset$, then $C = C'$ and we are done. If $C' \neq \emptyset$ and $C \subsetneq C'$, we observe that $\lambda \in \sigma_{C'}$ by definition of C'. Then by maximality of C,there is a $\lambda' \in \sigma_{C'}$ such that

$$\frac{\langle \chi, \lambda'\rangle}{||\lambda'||} < \frac{\langle \chi, \lambda\rangle}{||\lambda||}.$$

However, $\sigma_{C'} \subset \sigma_C$, contradicting the fact that λ minimizes $\frac{\langle \chi, -\rangle}{||-||}$ on σ_C. □

Lemma 3.4.2. *If $A \subset C$ are subsets of $[n]$ such that both $L(A)$ and $L(C)$ are contained in $S_{[\lambda]}$, then $L(B) \subset S_{[\lambda]}$ for all subsets B with $A \subset B \subset C$.*

Proof. That $\sigma_C \subset \sigma_B \subset \sigma_A$ implies the function $\frac{\langle \chi, - \rangle}{||-||}$ takes the same minimum value on the three cones. The statement now follows from Theorem 2.1.11 applied to σ_A. □

We may now describe the stratum $S_{[\lambda]}$.

Proposition 3.4.3. *Let C be the maximal subset of $[n]$ as in Lemma 3.4.1. If each A_j for $j = 1, \ldots, N$ is a minimal subset of $[n]$ such that λ minimizes $\frac{\langle \chi, - \rangle}{||-||}$ on σ_{A_j}, then*

$$S_\lambda = V([n] - C) \bigcap \Big(\bigcup_{j=1}^{N} D(A_j) \Big) = X_\lambda \bigcap \Big(\bigcup_{j=1}^{N} D(A_j) \Big).$$

In particular, S_λ is irreducible, connected, smooth and locally closed as an open subset of $V([n] - C) = X_\lambda$. Moreover, the zariski closure of the stratum $\overline{S_\lambda}$ is the linear subspace X_λ.

Proof. By Lemma 3.4.2, $L(A_j; C) \subset S_\lambda$ and we have

$$S_\lambda = \bigcup_{j=1}^{N} L(A_j; C).$$

It comes down to the simple calculation that $L(A_j; C) = V([n] - C) \bigcap D(A_j)$. □

The above discussion extends to the following setting.

Theorem 3.4.4. *Let a torus T act diagonally on $X = \mathbf{A}^n_k$ with characters χ_i for $i = 1, \ldots, n$. Let $G \subset T$ be a closed subgroup of T and G act on X by restricting the action of T on X. Let χ be a character of G. Give a norm on $\mathbf{\Gamma}(G)$ by restricting the standard norm of $\mathbf{\Gamma}(T)$ to $\mathbf{\Gamma}(G)$. Then for every $x \in X^{us}(\chi)$, there is a unique indivisible one parameter subgroup that is χ-adapted to x. Moreover, each χ-stratum is of the form*

$$S_\lambda = X_\lambda \bigcap \Big(\bigcup_{j=1}^{N} D(A_j) \Big)$$

where each A_j is a minimal subset of $[n]$ such that λ minimizes the function $\frac{\langle \chi, - \rangle}{||-||}$ on $\sigma_{A_j} \cap \mathbf{\Gamma}(G)$.

Proof. Since G is abelian, we already know that each χ-stratum is indexed by a single one parameter subgroup of G by Theorem 3.2.9. There is an alternative and more down to earth explanation in which the proof of the entire theorem becomes clear. Let $x \in X^{\mathrm{us}}(\chi)$ and let $\mathbf{\Gamma}(G)_{\mathbf{R}} \subset \mathbf{\Gamma}(T)_{\mathbf{R}}$ be the subspace generated by the image $\mathbf{\Gamma}(G) \to \mathbf{\Gamma}(T)_{\mathbf{R}}$. Then to find the one parameter subgroups that are χ-adapted to x is to minimize the function $\frac{\langle \chi, - \rangle}{||-||}$ on $\mathbf{\Gamma}(G)_{\mathbf{R}} \cap \sigma_{S_x}$ where $S_x \subset [n]$ is defined by $i \in S_x$ if and only if $x_i \neq 0$. Since $\mathbf{\Gamma}(G)_{\mathbf{R}} \cap \sigma_{S_x}$ is a rational polyhedral cone in $\mathbf{\Gamma}(G)_{\mathbf{R}}$, there is a unique indivisible one parameter subgroup that is χ-adapted to x by the dual version of Theorem 2.1.11. Hence each χ-stratum is indexed by a single one parameter subgroup. We end up dealing with minimizing a linear functional on rational polyhedral cones whose supporting hyperplanes are indexed by subsets of $[n]$. The rest of the theorem can be proved similarly as before. □

4. Instability - the projective case

In this section, our group G is still reductive over an algebraically closed field k and our G-varieties will be proper over k. We begin with the Hilbert-Mumford criterion for stability on a projective variety with respect to linearized ample line bundles. Next, we introduce a measure of instability in Section 4.2 and recall Kempf's results (Theorem 4.3.1) on maximally destabilizing one parameter subgroups for unstable points. We then reduce the space of linearizations by algebraic equivalence (Definition 4.2.3). Finally, for each algebraic equivalence class of a linearized ample line bundle, we introduce the stratification induced by it, and its IIT-equivalence class in Section 4.3.

Most results in this section were proved by considering linear actions on projective spaces. This is due to a combination of proposition 1.7, section 3, chapter 1 and theorem 1.19, section 5, chapter 1 from [Mum65]:

Theorem 4.0.1. *Let X be a G-variety proper over k and L be a very ample G-linearized line bundle. Then $\mathrm{H}^0(X, L)$ is a G-representation and the closed embedding $\iota : X \hookrightarrow \mathbf{P}(\mathrm{H}^0(X, L))$ is G-equivariant such that*

(1) $\iota*(\mathscr{O}(1)) \simeq L$ as G-linearized invertible sheaves, and
(2) $\mathbf{P}(\mathrm{H}^0(X, L))^{ss}(\mathscr{O}(1)) \cap X = X^{ss}(L)$

4.1. **A numerical criterion.** Let X be a G-variety proper over k. The group of G-linearized line bundles on X will be written as $\mathrm{Pic}^G(X)$. For each $x \in X$ and $\lambda \in \mathbf{\Gamma}(G)$, the limit $\lim_{t\to 0} \lambda(t) \cdot x$ always exists in X due to properness. Let the limit be x_0. Then x_0 is a fixed point of λ. We therefore have a sequence of group homomorphisms

$$\mathrm{Pic}^G(X) \to \mathrm{Pic}^{\mathbf{G}_m}(X) \to \mathrm{Pic}^{\mathbf{G}_m}(x_0) \simeq \boldsymbol{\chi}(\mathbf{G}_m) \simeq \mathbf{Z}. \tag{4.1}$$

The map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^{\mathbf{G}_m}(X)$ comes from restricting the G-action to $\mathbf{G}_m$ via λ. The map $\mathrm{Pic}^{\mathbf{G}_m}(X) \to \mathrm{Pic}^{\mathbf{G}_m}(x_0)$ comes from the $\mathbf{G}_m$-euqivariant embedding $x_0 \to X$. Finally, the linearizations of the structure sheaf of a single point are exactly given by the group of characters. Then for any $L \in \mathrm{Pic}^G(X)$, we set

$$\mu^L(x, \lambda) = -r$$

where r is the image of L under sequence (4.1).

Fact 4.1.1. For a G-variety X proper over k, we have the following key facts:

- Functoriality: If $f : X \to Y$ is a G-equivariant map between G-varieties proper over k, then
$$\mu^{f^*L}(x, \lambda) = \mu^L(f(x), \lambda)$$
for any k-point $x \in X$, $L \in \mathrm{Pic}^G(X)$ and $\lambda \in \mathbf{\Gamma}(G)$.
- Translation: $\mu^L(g \cdot x, g\lambda g^{-1}) = \mu^L(x, \lambda)$ for all k-points $x \in X$, $g \in G$ and $\lambda \in \mathbf{\Gamma}(G)$.
- For any $\lambda \in \mathbf{\Gamma}(G)$ and any $p \in P(\lambda)$, $\mu^L(x, \lambda) = \mu^L(p \cdot x, \lambda)$.

Recall the following numerical criterion:

Theorem 4.1.2. *Let X be a G-variety proper over k and let L be an ample G-linearized line bundle on X. Then*

(1) a k-point $x \in X$ is semistable with respect to L if and only if $\mu^L(x, \lambda) \geq 0$ for all one parameter subgroup $\lambda \in \mathbf{\Gamma}(G)$,
(2) a k-point $x \in X$ is stable with respect to L if and only if $\mu^L(x, \lambda) > 0$ for all non-trivial one parameter subgroups λ.

4.2. **Numerical analysis of instability.** Choosing a norm $||-||$ on the set of one parameter subgroups of G as in Section 3, if X is a G-variety proper over k and $L \in \mathrm{Pic}^G(X)$ is ample, we set

$$M^L(x) := \inf_{\lambda \in \mathbf{\Gamma}(G)} \frac{\mu^L(x, \lambda)}{||\lambda||}.$$

In [Nes79], a finiteness result similar to Theorem 3.2.4 states:

Theorem 4.2.1. *Let X be a projective G-variety and $L \in \mathrm{Pic}^G(X)$ be ample. Then for all $x \in X$, there is a one parameter subgroup λ such that $M^L(x) = \frac{\mu^L(x,\lambda)}{||\lambda||}$. In particular, $M^L(x)$ is finite. Moreover,*

$$M^L(x) = \inf_{g \in G} M^{L_T}(g \cdot x)$$

where L_T means L restricted to the linearization given by a maximal torus $T \subset G$. Finally, $M^L(-) : X \to \mathbf{R}$ takes only finitely many values.

In the above situation, we say a one parameter subgroup λ is L-adapted to x if $M^L(x) = \frac{\mu^L(x,\lambda)}{||\lambda||}$. We let $\Lambda^L(x)$ be the collection of indivisible one parameter subgroups that are L-adapted to x.

Theorem 4.2.2. *(Kempf) Let X be a G-variety proper over k and $L \in \mathrm{Pic}^G(X)$ be ample. Then for any $x \in X^{us}(L)$, we have*

(1) $\Lambda^L(g \cdot x) = g\Lambda^L(x)g^{-1}$ for all $g \in G$,
(2) there exists a parabolic subgroup $P(L, x)$ of G such that $P(\lambda) = P(L, x)$ for all $\lambda \in \Lambda^L(x)$, and
(3) any two elements in $\Lambda^L(x)$ are conjugate by $P(L, x)$.

For any G-variety, the group $\mathrm{Pic}^G(X)$ may not be finitely generated. The notion of G-algebraic equivalence introduced in [Res00] and [Tha96] reduces $\mathrm{Pic}^G(X)$ to a finitely generated abelian group and stability of a G-linearized line bundle depends only on its G-algebraic equivalence class. Recall

Definition 4.2.3. Let X be a G-variety. We say two G-linearized line bundle L_1, L_2 are G-algebraically equivalent if there are a connected variety S, a G-linearized line bundle L on $S \times X$ where G acts on $S \times X$ via X, and two points $s_1, s_2 \in S$ such that $L_1 \simeq L_{s_1}$ and $L_2 \simeq L_{s_2}$ as G-linearized line bundles. We write $\mathrm{NS}^G(X)$ as the group of $\mathrm{Pic}^G(X)$ modulo G-algebraic equivalence.

In [Res00], the author showed

Proposition 4.2.4. *Let X be a G-variety proper over k and L_1, L_2 be two G-algebraically equivalent invertible sheaves. Then for any one parameter subgroup λ and any k-point $x \in X$, we have $\mu^{L_1}(x, \lambda) = \mu^{L_2}(x, \lambda)$.*

Another fundamental result from [Tha96] states that

Theorem 4.2.5. *Let X be a normal projective variety over k and $\mathrm{NS}(X)$ be the group $\mathrm{Pic}(X)$ modulo algebraic equivalence. Then the kernel of the map $\mathrm{NS}^G(X) \to \mathrm{NS}(X)$ is $\boldsymbol{\chi}(G)$ modulo a torsion subgroup. In particular, $\mathrm{NS}^G(X)$ is finitely generated.*

Therefore, for a normal projective variety X, there is an exact sequence of finite dimensional vector spaces

$$0 \longrightarrow \boldsymbol{\chi}(G)_{\mathbf{R}} \longrightarrow \mathrm{NS}^G(X)_{\mathbf{R}} \longrightarrow \mathrm{NS}(X)_{\mathbf{R}} \longrightarrow 0.$$

We define $\mathrm{NS}^G(X)^+$ to be the preimage of ample line bundles under the forgetful map $\mathrm{NS}^G(X) \to \mathrm{NS}(X)$. Note that for $L \in \mathrm{NS}^G(X)^+$, $X^{\mathrm{ss}}(L)$ is well defined by Theorem 4.1.2 and Proposition 4.2.4. Also note that given $L_1, L_2 \in \mathrm{NS}^G(X)^+$, we have $L_1 \otimes L_2 \in \mathrm{NS}^G(X)^+$. Hence $\mathrm{NS}^G(X)^+$ spans a convex cone in $\mathrm{NS}^G(X)_{\mathbf{R}}$.

4.3. **Stratification of the null cone.** The unstable locus is stratified by the conjugacy classes of one parameter subgroups from Theorem 4.2.2. We need some notations first. Let $L \in \mathrm{NS}^G(X)^+$. For a one parameter subgroup $\lambda \in \mathbf{\Gamma}(G)$, $[\lambda]$ means the conjugacy class of λ under the action of G. We let

$$S^L_{d,[\lambda]} = \{x \in X^{\mathrm{us}}(L) \mid M^L(x) = d \text{ and } \Lambda^L(x) \cap [\lambda] \neq \emptyset\}.$$

We also let Λ^L to be the collection of conjugacy classes of indivisible one parameter subgroups that are L-adapted to some L-unstable points.

Theorem 4.3.1 (Hesselink)**.** *Let X be a G-variety proper over k and $L \in \mathrm{NS}^G(X)^+$. Then*

$$X = X^{ss}(L) \bigcup_{d<0,[\lambda]\in\Lambda^L} S^L_{d,[\lambda]}$$

is a finite disjoint union of locally closed G-invariant subvarieties of X. Moreover, $\partial S^L_{d,[\lambda]} \cap S^L_{d',[\lambda']} \neq \emptyset$ only if $d' < d$.

We define a strict partial order on the finite set of L-stratum by letting

$$S^L_{d',[\lambda']} > S^L_{d,[\lambda]} \text{ if } d' < d.$$

We now make the following definition

Definition 4.3.2. Let X be a normal G-variety proper over k. We say $L_1, L_2 \in \mathrm{NS}^G(X)^+$ are IIT-equivalent if they induce stratifications of X that are equivalent in the sense of Definition 3.3.6.

5. Variation of instability - the affine case

We let G be a reductive group over an algebraically closed field k, acting linearly on an affine space X. Here we investigate variation of stratifications of X caused by different choices of characters of G. We start with an example at Section 5.2 where we compute the unstable locus and its stratification for every character. The summary is included in Table 2 where the space of characters is divided by regions that correspond to different stratifications. We then formulate critical subsets, semi-chambers and establish various properties similar but different to walls and chambers in VGIT (Section 5.3). Finally, we state and prove the main result, Theorem 5.4.4.

5.1. **Set up and notations.** Let G be a reductive group acting linearly on an affine space X. Fix a maximal torus $T \subset G$. We fix a norm

$$||-|| : \mathbf{\Gamma}(G) \to \mathbf{R}$$

such that the restriction of $||-||$ to $\mathbf{\Gamma}(T)$ is a norm that comes from an inner product

$$(-,-) : \mathbf{\Gamma}(T)_{\mathbf{R}} \times \mathbf{\Gamma}(T)_{\mathbf{R}} \to \mathbf{R}.$$

For every $\chi \in \boldsymbol{\chi}(G)_{\mathbf{R}}$, let $\chi_T \in \boldsymbol{\chi}(T)_{\mathbf{R}}$ be the restriction. Then under the perfect pairing

$$\langle -,- \rangle : \boldsymbol{\chi}(T)_{\mathbf{R}} \times \mathbf{\Gamma}(T)_{\mathbf{R}} \to \mathbf{R},$$

there is a corresponding vector $\chi_T^* \in \mathbf{\Gamma}(T)_{\mathbf{R}}$ such that

$$(\chi_T^*, v) = \langle \chi_T, v\rangle \text{ for all } v \in \mathbf{\Gamma}(T)_{\mathbf{R}}.$$

If $W \subset \mathbf{\Gamma}(T)_{\mathbf{R}}$ is a subspace, the projection of χ_T^* onto W along the orthogonal complement $W^\perp$ in $\mathbf{\Gamma}(T)_{\mathbf{R}}$ is denoted by

$$\operatorname{Proj}_W \chi_T^*.$$

The subspaces of $\mathbf{\Gamma}(T)_{\mathbf{R}}$ which we want to project χ_T^* onto come from the the action of T on X. Assume T acts on X diagonally by the characters $\chi_1, \dots, \chi_n \in \boldsymbol{\chi}(T)$. For any subset $S \subset [n]$, we define the subspace

$$W_S := \{v \in \mathbf{\Gamma}(T)_{\mathbf{R}} | \langle \chi_i, v\rangle = 0, i \in S\}$$

and the rational polyhedral cone

$$\sigma_S := \{v \in \mathbf{\Gamma}(T)_{\mathbf{R}} | \langle \chi_i, v\rangle \geq 0, i \in S\}.$$

Remark 5.1.1. With these notations, we note without proof that each face of σ_S is of the form $W_Z \cap \sigma_S$ for some subset $Z \subset S$ and the subspace spanned by a face of σ_S is of the form W_Z for some $Z \subset S$.

Given a point $x = (x_1, \dots, x_n) \in X$, write the T-states of x as

$$S_x := \{i \in [n] \big| x_i \neq 0\}.$$

The lattice points of the cone

$$\sigma_x := \sigma_{S_x}$$

are exactly the one parameter subgroups in T that have limits at x. The interest we have in the projections $\operatorname{Proj}_{W_Z} \chi_T^*$ for $Z \subset [n]$ is the following: Suppose x is χ_T-unstable (notice the restriction to T) and let $\lambda \in \sigma_x \cap \mathbf{\Gamma}(T)$ be χ_T-adapted to x. Then by Corollary 2.1.8 and Remark 5.1.1, λ is on the ray $\mathbf{R}_{>0} \cdot (-\operatorname{Proj}_{W_Z} \chi_T^*)$ for some subset $Z \subset S_x$. Now if x is χ-unstable, then by Proposition 3.2.5, the one parameter subgroups that are χ-adapted to x are conjugate to one parameter subgroups on the ray $\mathbf{R}_{>0} \cdot (-\operatorname{Proj}_{W_Z} \chi_T^*) \subset \sigma_{g\cdot x} \subset \mathbf{\Gamma}(T)_{\mathbf{R}}$ for some $Z \subset S_{g\cdot x}$ and for some $g \in G$. Hence for each χ, the collection of rays $\{\mathbf{R}_{>0} \cdot (-\operatorname{Proj}_{W_Z} \chi_T^*) \big| Z \subset [n]\}$ includes conjugacy classes of one parameter subgroups indexing the χ-strata. In Section 5.3, we will study the variation of the projections $\{\operatorname{Proj}_{W_Z} \chi_T^* \big| Z \subset [n]\}$ with respect to $\chi \in \boldsymbol{\chi}(G)_{\mathbf{R}}$ and establish the main result, Theorem 5.4.4 in Section 5.4. First, let us see an example.

5.2. **An elementary example.**

5.2.1. *The set up.* Let $G = \mathbf{G}_m \times \mathrm{GL}(2)$ act on $\mathbf{A}_k^2 \times \mathbf{A}_k^2$ by

$$(t, A) \cdot (x, y, z, w) = (t \cdot x, t^{-1} \cdot y, A \cdot \begin{bmatrix} z \\ w \end{bmatrix}).$$

We have $\boldsymbol{\chi}(G) \simeq \boldsymbol{\chi}(\mathbf{G}_m) \oplus \boldsymbol{\chi}(\mathrm{GL}(2) \simeq \mathbf{Z}^2$ where each $(a, b) \in \mathbf{Z}^2$ defines the character

$$(t, A) \mapsto t^a \cdot \det(A)^b \text{ for } t \in \mathbf{G}_m, A \in \mathrm{GL}(2).$$

Embed G as a subgroup of GL(3) as two block diagonals and let $T \simeq \mathbf{G}_m^3$ be the maximal torus of G in the diagonal of GL(3). Then the restriction $\boldsymbol{\chi}(G) \to \boldsymbol{\chi}(T)$ is a map of free $\mathbf{Z}$-modules $\mathbf{Z}^2 \to \mathbf{Z}^3$ represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The action when restricted to T, is already diagonalized:

$$(t_1, t_2, t_3) \cdot (x, y, z, w) \mapsto (t_1 \cdot x, t_1^{-1} \cdot y, t_2 \cdot z, t_3 \cdot w).$$

Next, we will choose a norm on the set $\mathbf{\Gamma}(G)$. The normalizer $N(T) \subset G$ is the product of $\mathbf{G}_m$ and the generalized permutation matrix of GL(2). Hence the Weyl group $N(T)/T$ acts on $\mathbf{\Gamma}(T)$ by permuting the entries coming from GL(2) while the entry from $\mathbf{G}_m$ is fixed. Therefore the standard inner product on $\mathbf{\Gamma}(T)_{\mathbf{R}}$ is invariant under $N(T)/T$. It follows that the standard norm on $\mathbf{\Gamma}(T)$ is extendable to $\mathbf{\Gamma}(G)$. If $\chi = (a, b)$ is a character of G, and $\lambda \in \mathbf{\Gamma}(T)$ where $\lambda(t) = \begin{bmatrix} t^\alpha & 0 & 0 \\ 0 & t^\beta & 0 \\ 0 & 0 & t^\gamma \end{bmatrix}$, we have

$$\langle \chi, \lambda \rangle = \langle \chi_T, \lambda \rangle = a \cdot \alpha + b \cdot \beta + b \cdot \gamma.$$

It follows that

$$\chi_T^* = (a, b, b). \tag{5.1}$$

5.2.2. *Analysis of stability.*

Lemma 5.2.1. *Let* GL(2) *act on* $\mathbf{A}_k^2$ *via the identity group homomorphism* GL(2) $\to$ GL(2) *and let* $b \in \mathbf{Z}$ *be a character of* GL(2)*. Then*

$$(\mathbf{A}_k^2)^{us}(b) = \begin{cases} \mathbf{A}_k^2 & \text{if } b \neq 0, \\ \emptyset & \text{if } b = 0. \end{cases} \tag{5.2}$$

Proof. The action of GL(2) on $\mathbf{A}_k^2$ has two orbits, the origin $\{0\}$ and the complement $\mathbf{A}_k^2 - \{0\}$. The origin is obviously unstable with respect to any non-trivial character. For the orbit $\mathbf{A}_k^2 - \{0\}$, consider the point $(1, 0) \in \mathbf{A}_k^2 - \{0\}$. Then with respect to the character

$$A \mapsto \det(A)^b \text{ with } b \neq 0,$$

the point $(1, 0)$ is destabilized by the one parameter subgroup

$$\lambda(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} & \text{if } b > 0, \\ \\ \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} & \text{if } b < 0. \end{cases}$$

We see that $(1, 0)$ can be destabilized as long as $b \neq 0$. Therefore the entire orbit $\mathbf{A}^2 - \{0\}$ is unstable whenever $b \neq 0$. □

Lemma 5.2.2. *Let* $\mathbf{G}_m$ *act on* $\mathbf{A}_k^2$ *via* $t \cdot (x, y) \mapsto (tx, t^{-1}y)$*. Let* $a \in \mathbf{Z}$ *be a character of* $\mathbf{G}_m$*, then*

$$(\mathbf{A}_k^2)^{us}(a) = \begin{cases} V(x) & \text{if } a > 0, \\ V(y) & \text{if } a < 0, \\ \emptyset & \text{if } b = 0. \end{cases} \tag{5.3}$$

Proof. This is a straightforward application of the numerical criterion, Theorem 3.1.7. □

Proposition 5.2.3. *Let $G = \mathbf{G}_m \times \mathrm{GL}(2)$ act on $\mathbf{A}^2_k \times \mathbf{A}^2_k$ as was set up in the beginning of this section. Let $\chi = (a, b)$ be a character of G. Then*

$$(\mathbf{A}^2_k \times \mathbf{A}^2_k)^{us}(\chi) = \begin{cases} \mathbf{A}^2_k \times \mathbf{A}^2_k \text{ if } b \neq 0, \\ V(x) \times \mathbf{A}^2_k \text{ if } a > 0,\ b = 0, \\ V(y) \times \mathbf{A}^2_k \text{ if } a < 0,\ b = 0, \text{ and} \\ \emptyset \text{ if } a, b = 0. \end{cases} \tag{5.4}$$

Proof. Using the numerical criterion, Theorem 3.1.7, one finds that if $\chi = (a, b)$ is a character of G, then

$$(\mathbf{A}^2_k \times \mathbf{A}^2_k)^{\mathrm{us}}(\chi) = \big((\mathbf{A}^2_k)^{\mathrm{us}}(a) \times \mathbf{A}^2_k\big) \bigcup \big(\mathbf{A}^2_k \times (\mathbf{A}^2_k)^{\mathrm{us}}(b)\big).$$

By Lemma 5.2.1 and Lemma 5.2.2, we get the result. □

5.2.3. *Analysis of instability.* Now that we know the unstable loci for all characters, we want to compute the stratifications of $\mathbf{A}^2_k \times \mathbf{A}^2_k$. The main tool is Proposition 3.2.5. Let $\chi = (a, b)$ be a character of G. Give the first factor of $\mathbf{A}^2_k \times \mathbf{A}^2_k$ coordinate x, y. Suppose $b = 0$. Then $-\chi^*_T = (-a, 0, 0)$. If $a > 0$, then Equation (5.4) tells us that the unstable locus is $V(x) \times \mathbf{A}^2_k$. It can be checked that $-\chi^*_T \in \sigma_p$ for all $p \in V(x) \times \mathbf{A}^2_k$. Namely, $-\chi^*_T$ has limit at p for all $p \in V(x) \times \mathbf{A}^2_k$. By Proposition 2.1.7, $-\chi^*_T$ is χ_T-adapted to $V(x) \times \mathbf{A}^2_k$. By Proposition 3.2.5, $-\chi^*_T$ is χ-adapted to $V(x) \times \mathbf{A}^2_k$. When $a < 0$, the same argument shows that $-\chi^*_T$ is χ-adapted to $V(y) \times \mathbf{A}^2_k$. Notice that

$$-\chi^*_T \text{ is parallel to } \begin{cases} (-1, 0, 0) \text{ if } a > 0, \\ (1, 0, 0) \text{ if } a < 0. \end{cases}$$

We therefore have the short summary:

character (a, b)	Stratum	Indexing one parameter subgroup
$a > 0, b = 0$	$V(x) \times \mathbf{A}^2_k$	$(-1, 0, 0)$
$a < 0, b = 0$	$V(y) \times \mathbf{A}^2_k$	$(1, 0, 0)$

When $b \neq 0$, the entire $\mathbf{A}^2_k \times \mathbf{A}^2_k$ is unstable, and a more devoted analysis is needed. For any subset $S \subset \{x, y\}$, define

$$L_S = D(s \mid s \in S) \cap V(s \mid s \notin S).$$

The first step is a reduction.

Proposition 5.2.4. *Decompose the space $\mathbf{A}^2_k \times \mathbf{A}^2_k$ into eight G-invariant subvarieties:*

(1) $L_{xy} \times (\mathbf{A}^2_k - \{0\})$,
(2) $L_x \times (\mathbf{A}^2_k - \{0\})$,
(3) $L_y \times (\mathbf{A}^2_k - \{0\})$,
(4) $\{0\} \times (\mathbf{A}^2_k - \{0\})$,
(5) $L_{xy} \times \{0\}$,
(6) $L_x \times \{0\}$,
(7) $L_y \times \{0\}$,
(8) $\{0\} \times \{0\}$.

Let $\chi = (a, b)$ be a character with $b \neq 0$. Then each subvariety is contained in a χ-stratum.

Proof. Note that only the first and the fifth subvarieties are not G-orbits. Hence it is sufficient to consider the first and the fifth. Let p be a point in $L_{xy} \times (\mathbf{A}_k^2 - \{0\})$. We see that the T- states of $g \cdot p$ as g runs through G are exhausted by those of the three points $(1,1,1,0),(1,1,0,1),(1,1,1,1)$ in $L_{xy} \times (\mathbf{A}_k^2 - \{0\})$. Now these three points are in a G-orbit. This fact combined with Proposition 3.2.5, we see that the conjugacy class of the one parameter subgroups that are χ-adapted to any point in $L_{xy} \times (\mathbf{A}_k^2 - \{0\})$ is the same. Therefore the entire $L_{xy} \times (\mathbf{A}_k^2 - \{0\})$ is in a χ-stratum. The proof for $L_{xy} \times \{0\}$ is similar. □

The implication is that the stratification induced by χ is a grouping of the eight subvarieties. This reduces computing the stratification of $\mathbf{A}_k^2 \times \mathbf{A}_k^2$ to computing the one parameter subgroups that are χ-adapted to each subvariety listed in Proposition 5.2.4. The stratification of the χ-unstable locus for every character χ is summarized in Table 2. For interested readers, in the rest of the section before Table 2, we supply the computation of the one parameter subgroups that are χ-adapted to $L_y \times (\mathbf{A}_k^2 - \{0\})$ when

(1) χ is in the first quadrant. Namely, $a \geq 0, b > 0$, and
(2) χ is in the second quadrant. Namely, $a \leq 0 < b$.

If p is a point in $L_y \times (\mathbf{A}_k^2 - \{0\})$, the possible T-states of $g \cdot p$ are exhausted by the three points $(0,1,1,0),(0,1,1,1),(0,1,0,1)$ in $L_y \times (\mathbf{A}_k^2 - \{0\})$, which are in a G-orbit (in fact the entire $L_y \times (\mathbf{A}_k^2 - \{0\})$ is a G-orbit). By Proposition 3.2.5, we only need to compare $M^{\chi_T}((0,1,1,0)), M^{\chi_T}((0,1,1,1))$, and $M^{\chi_T}((0,1,0,1))$ to figure out the conjugacy class of the one parameter subgroups that are χ-adapted to $L_y \times (\mathbf{A}_k^2 - \{0\})$.

Proposition 5.2.5 (The first quadrant)**.** *Let $\chi = (a,b)$ be a character of G with $a \geq 0, b > 0$. Then the one parameter subgroups that are χ-adapted to $L_y \times (\mathbf{A}^2 - \{0\})$ are conjugate to the one parameter subgroup $(-a,0,-b)$ in T.*

Proof. For the point $(0,1,1,1) \in L_y \times (\mathbf{A}_k^2 - \{0\})$, a one parameter subgroup $\lambda = (\alpha,\beta,\gamma) \in \mathbf{\Gamma}(T)$ has limit at $(0,1,1,1)$ if and only if $-\alpha,\beta,\gamma \geq 0$. If $a = 0$, $\langle \chi_T, \lambda \rangle = b(\beta+\gamma) \geq 0$. Hence $(0,1,1,1)$ is not χ_T-unstable. If $a > 0$, then $(0,1,1,1)$ is χ_T-unstable. However, the one parameter subgroup $-\chi_T^* = (-a,-b,-b)$ does not have limit at $(0,1,1,1)$ as $\beta = \gamma = -b < 0$. Using Corollary 2.1.8, we then project $-\chi_T^*$ to the subspace defined by $\beta = \gamma = 0$, resulting in $(-a,0,0) \parallel (-1,0,0)$. This means $M^{\chi_T}((0,1,1,1))$ is achieved by the one parameter subgroup $(-1,0,0)$ if $a > 0$. On the other hand, λ has limit at $(0,1,1,0)$ if and only if $\alpha \leq 0, \beta \geq 0$. There is no constraint on γ so we can take $(-a,0,-b)$ to attain $M^{\chi_T}((0,1,1,0))$ whether $a = 0$ or not. Similarly $M^{\chi_T}((0,1,0,1))$ can be achieved by $(-a,-b,0)$ whether $a = 0$ or not. Even when $a > 0$,

$$M^{\chi_T}((0,1,1,1)) = -a > M^{\chi_T}((0,1,1,0)) = M^{\chi_T}((0,1,0,1)) = -\sqrt{a^2+b^2}.$$

Since $(-a,0,-b)$ is conjugate to $(-a,-b,0)$, we conclude that $L_y \times (\mathbf{A}_k^2 - \{0\})$ is in the stratum indexed by the conjugacy class of the one parameter subgroup $(-a,0,-b)$. □

Proposition 5.2.6 (The second quadrant)**.** *Let $\chi = (a,b)$ be a character of G with $a \leq 0 < b$. Then the one parameter subgroups that are χ-adapted to $L_y \times (\mathbf{A}^2 - \{0\})$ are conjugate to the one parameter subgroup $(0,0,-1)$ in T.*

Proof. As before, a one parameter subgroup $\lambda = (\alpha, \beta, \gamma) \in \mathbf{\Gamma}(T)$ has limit at $(0,1,1,1)$ if and only if $-\alpha, \beta, \gamma \geq 0$. Now $a \leq 0$ implies $\langle \chi_T, \lambda \rangle \geq 0$. Hence $(0,1,1,1)$ is never χ_T-unstable. On the other hand, λ has limit at $(0,1,1,0)$ if and only if $-\alpha, \beta \geq 0$. If $a < 0$, $-\chi_T^* = (-a,-b,-b)$ does not have limit at $(0,1,1,0)$ as $\alpha = -a > 0$ and $\beta = -b < 0$. Using Corollary 2.1.8, we then project $-\chi_T^*$ onto the subspace defined by $\alpha = \beta = 0$, resulting in $(0,0,-b) \parallel (0,0,-1)$. Hence $M^{\chi_T}((0,1,1,0))$ can only be achieved by $(0,0,-1)$. Similarly $M^{\chi_T}((0,1,0,1))$ can be achieved by the one parameter subgroup $(0,-1,0)$. If $a = 0$, the one parameter subgroup $(0,-b-b)$ still does not have limit at $(0,1,1,0)$ nor at $(0,1,0,1)$. One has to project $(0,-b,-b)$ to $(0,-1,0)$ or $(0,0,-1)$. In any case, since $(0,-1,0)$ is conjugate to $(0,0,-1)$, $L_y \times (\mathbf{A}_k^2 - \{0\})$ is in the stratum indexed by the conjugacy class of $(0,0,-1)$. □

We have

Table 1. The list of conjugacy classes of one parameter subgroups that are χ-adapted to the eight subvarieties when $\chi = (a,b)$ is in the first and the second quadrants.

Subvariety	$a \geq 0, b > 0$	$a \leq 0 < b$
$L_{xy} \times (\mathbf{A}_k^2 - \{0\})$	$(0,0,-1)$	$(0,0,-1)$
$L_x \times (\mathbf{A}_k^2 - \{0\})$	$(0,0,-1)$	$(-a,0,-b)$
$L_y \times (\mathbf{A}_k^2 - \{0\})$	$(-a,0,-b)$	$(0,0,-1)$
$\{0\} \times (\mathbf{A}_k^2 - \{0\})$	$(-a,0,-b)$	$(-a,0,-b)$
$L_{xy} \times \{0\}$	$(0,-1,-1)$	$(0,-1,-1)$
$L_x \times \{0\}$	$(0,-1,-1)$	$(-a,-b,-b)$
$L_y \times \{0\}$	$(-a,-b-b)$	$(0,-1,-1)$
$\{0\} \times \{0\}$	$(-a,-b,-b)$	$(-a,-b-b)$

Note that

(1) Setting $a = 0$, the vector $(-a,0,-b)$ (resp. $(-a,-b,-b)$) becomes parallel to $(0,0,-1)$ (resp. $(0,-1,-1)$). Hence when $a = 0$, the stratum indexed by the conjugacy class of $(-a,0,-b)$ (resp. $(-a,-b,-b)$) is identified with the stratum indexed by the conjugacy class of $(0,0,-1)$ (resp. $(0,-1,-1)$).
(2) When $a^2 + b^2 = 2b^2$, the order of the stratum indexed by the conjugacy class of $(0,-1,-1)$, which is parallel to $(0,-b,-b)$, becomes the same as the order of the stratum indexed by the conjugacy class of $(-a,0,-b)$

5.2.4. *Summary.* Here is a complete description of the variation of stratifications inside the character space, namely, the a,b-plane. The plane is divided into twelve regions.

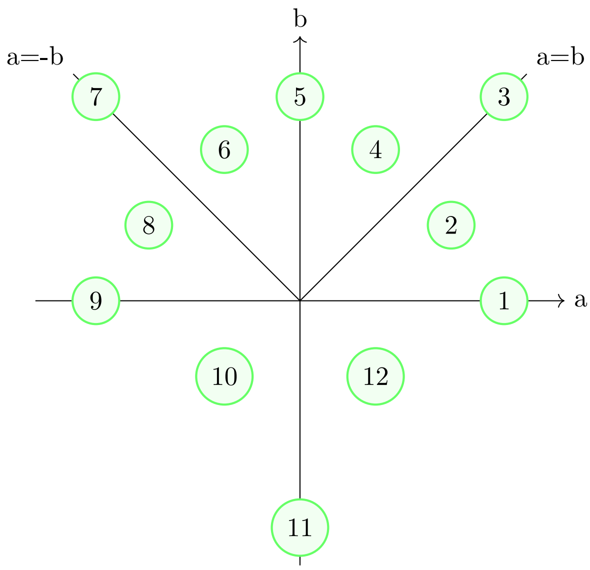


TABLE 2. Stratification with respect to any character, listed counterclockwise in the a, b-plane, starting from the positive a-axis.

Region	Stratification
1	A single stratum $V(x) \times \mathbf{A}_k^2$.
2	$V(x) \times \{0\} > V(x) \times (\mathbf{A}_k^2 - \{0\}) > D(x) \times \{0\} > D(x) \times (\mathbf{A}_k^2 - \{0\})$.
3	Order of $V(x) \times (\mathbf{A}_k^2 - \{0\}), D(x) \times \{0\}$ come together.
4	$V(x) \times \{0\} > D(x) \times \{0\} > V(x) \times (\mathbf{A}_k^2 - \{0\}) > D(x) \times (\mathbf{A}_k^2 - \{0\})$.
5	$\mathbf{A}_k^2 \times \{0\} > \mathbf{A}_k^2 \times (\mathbf{A}_k^2 - \{0\})$.
6	$V(y) \times \{0\} > D(y) \times \{0\} > V(y) \times (\mathbf{A}_k^2 - \{0\}) > D(y) \times (\mathbf{A}_k^2 - \{0\})$.
7	Order of $D(y) \times \{0\}, V(y) \times (\mathbf{A}_k^2 - \{0\})$ come together.
8	$V(y) \times \{0\} > V(y) \times (\mathbf{A}_k^2 - \{0\}) > D(y) \times \{0\} > D(y) \times (\mathbf{A}_k^2 - \{0\})$.
9	A single stratum $V(y) \times \mathbf{A}_k^2$
10	$V(y) \times \mathbf{A}_k^2 > D(y) \times \mathbf{A}_k^2$.
11	A single stratum $\mathbf{A}_k^2 \times \mathbf{A}_k^2$.
12	$V(x) \times \mathbf{A}_k^2 > D(x) \times \mathbf{A}_k^2$.

Four remarkable phenomena are captured by this example.

(1) Crossing the line $b = 0$, the semistable locus changes,
(2) crossing the line $a = 0$, some strata divide or collapse (row 4, 5, 6 and row 10, 11, 12 in Table 2).
(3) there is a rational quadratic wall defined by $a^2 + b^2 = 2b^2$, such that crossing the wall in the upper half plane, the ordering of two strata are flipped (row 2, 3, 4 and row 6, 7, 8 in Table 2), and finally,
(4) only the parts of the wall $a^2 + b^2 = 2b^2$ in the upper half plane is interfering the strata.

5.3. **Critical subsets and semi-chambers.** We now define critical subsets and semi-chambers in the space $\boldsymbol{\chi}(G)_{\mathbf{R}}$. In Section 5.1, we have seen that if G acts on $\mathbf{A}_k^n$ linearly, then for every $\chi \in \boldsymbol{\chi}(G)$, the collection of rays

$$\{\mathbf{R}_{>0} \cdot (-\operatorname{Proj}_{W_S} \chi_T^*) \big| S \subset [n]\}$$

includes all one parameter subgroups whose conjugacy classes index some χ-strata. Therefore, the study the variation of stratification (Definition 3.3.6) caused by different choices of characters of G is related to the following question: For what character χ does a pair $\operatorname{Proj}_{W_{S_1}} \chi_T^*, \operatorname{Proj}_{W_{S_2}} \chi_T^*$ satisfy

(1) $\operatorname{Proj}_{W_{S_1}} \chi_T^* = \operatorname{Proj}_{W_{S_2}} \chi_T^*$, or
(2) $|| \operatorname{Proj}_{W_{S_1}} \chi^* || = || \operatorname{Proj}_{W_{S_2}} \chi^* ||$.

In Proposition 5.3.7, we will nail down conditions on the character χ such that either condition described above holds. When condition (1) holds, certain strata may come together. When condition (2) holds, the order of some strata may come together. The story begins with torus actions.

Definition 5.3.1. Let T be a torus acting linearly on $\mathbf{A}_k^n$. Let $H \subset \boldsymbol{\chi}(T)_{\mathbf{R}}$ be a subset. We say H is a *type one wall* (with respect to the action) if there are subsets $S_1, S_2 \subset [n]$ such that

(1) there is a containment, say $W_{S_2} \subset W_{S_1}$ and W_{S_2} is a codimension one subspace of W_{S_1}, and
(2) $H = \{\chi \in \boldsymbol{\chi}(T)_{\mathbf{R}} \mid \operatorname{Proj}_{W_{S_1}} \chi^* = \operatorname{Proj}_{W_{S_2}} \chi^*\}$.

We say H is a *type two wall* (with respect to the action) if there are subsets $S_1, S_2 \subset [n]$ such that

(1) there is no containment between W_{S_1} and W_{S_2}, and
(2) $H = \{\chi \in \boldsymbol{\chi}(T)_{\mathbf{R}} \mid || \operatorname{Proj}_{W_{S_1}} \chi^* || = || \operatorname{Proj}_{W_{S_2}} \chi^* ||\}$.

Remark 5.3.2. For a torus T, walls are always proper subsets of $\boldsymbol{\chi}(T)_{\mathbf{R}}$. To see this, the map $\boldsymbol{\chi}(T)_{\mathbf{R}} \to \boldsymbol{\Gamma}(T)_{\mathbf{R}}$ defined by $\chi \mapsto \chi^*$ is an isomorphism. Hence for any two linear subspaces $W_1, W_2 \subset \boldsymbol{\Gamma}(T)_{\mathbf{R}}$, the following statements are equivalent:

(1) $\operatorname{Proj}_{W_1} \chi^* = \operatorname{Proj}_{W_2} \chi^*$ for all $\chi \in \boldsymbol{\chi}(T)_{\mathbf{R}}$,
(2) $|| \operatorname{Proj}_{W_1} \chi^* || = || \operatorname{Proj}_{W_2} \chi^* ||$ for all $\chi \in \boldsymbol{\chi}(T)_{\mathbf{R}}$,
(3) $W_1 = W_2$.

Since in the definition of walls the pair of subspaces W_{S_1}, W_{S_2} are never equal, walls are always proper.

Remark 5.3.3. The non-containment assumption required for a type two wall is to avoid including all type one walls as type two walls. If $W_{S_2} \subset W_{S_1}$, then for any character χ, $|| \operatorname{Proj}_{W_{S_1}} \chi^* || = || \operatorname{Proj}_{W_{S_2}} \chi^* ||$ is equivalent to $\operatorname{Proj}_{W_{S_1}} \chi^* = \operatorname{Proj}_{W_{S_2}} \chi^*$. In particular, if W_{S_2} is a codimension one subspace of W_{S_1}, the condition $|| \operatorname{Proj}_{W_{S_1}} \chi^* || = || \operatorname{Proj}_{W_{S_2}} \chi^* ||$ defines nothing other than a type one wall. The point of defining two types of walls is to distinguish two types of wall crossing behaviour.

We now define critical subsets in $\boldsymbol{\chi}(G)_{\mathbf{R}}$ for representations of arbitrary reductive groups.

Definition 5.3.4. Suppose G is a reductive group acting linearly on an affine space X. Let T be a maximal torus and $\phi : \boldsymbol{\chi}(G)_{\mathbf{R}} \to \boldsymbol{\chi}(T)_{\mathbf{R}}$ be the restriction. Let T act on X by restricting the action of G to T. A *type one* (resp. *type two*)*critical subset of* $\boldsymbol{\chi}(G)_{\mathbf{R}}$ (with respect to the action of G) is the preimage $\phi^{-1}(H)$

of a type one (resp. type two) wall $H \subset \mathbf{\Gamma}(T)_{\mathbf{R}}$ where we require that $\phi^{-1}(H)$ is a proper subset of $\boldsymbol{\chi}(G)_{\mathbf{R}}$.

Example 5.3.5. Let us take a look at the example from Section 5.2 where $G = \mathbf{G}_m \times \mathrm{GL}(2)$ acts on $\mathbf{A}_k^2 \times \mathbf{A}_k^2$. The map $\boldsymbol{\chi}(G)_{\mathbf{R}} \to \boldsymbol{\chi}(T)_{\mathbf{R}}$ is given by

$$(a, b) \mapsto (a, b, b).$$

Moreover, T acts on $\mathbf{A}_k^2 \times \mathbf{A}_k^2$ by the characters $\chi_1 = (1, 0, 0)$, $\chi_2 = (-1, 0, 0)$, $\chi_3 = (0, 1, 0)$, and $\chi_4 = (0, 0, 1)$. Let $\chi = (a, b)$ be a character of G. For brevity, for a subset $\{i_1, \ldots, i_k\} \subset [n]$, instead of writing $W_{\{i_1,\ldots,i_k\}} \subset \mathbf{\Gamma}(T)_{\mathbf{R}}$, we write $W_{i_1 \ldots i_k}$. Then

(1) $\chi_T^* = (a, b, b)$,
(2) $\mathrm{Proj}_{W_1} \chi_T^* = \mathrm{Proj}_{W_2} \chi_T^* = (0, b, b)$,
(3) $\mathrm{Proj}_{W_3} \chi_T^* = (a, 0, b)$, and
(4) $\mathrm{Proj}_{W_{23}} \chi_T^* = \mathrm{Proj}_{W_{13}} \chi_T^* = (0, 0, b)$.

Now, the type one critical subset defined by

(1) $\chi_T^* = \mathrm{Proj}_{W_1} \chi_T^* (= \mathrm{Proj}_{W_2} \chi_T^*)$ with respect to the pair $(\emptyset, \{1\})$, and
(2) $\mathrm{Proj}_{W_3} \chi_T^* = \mathrm{Proj}_{W_{23}} \chi_T^* (= \mathrm{Proj}_{W_{13}} \chi^*)$ with respect to the pair $(\{3\}, \{2, 3\})$.

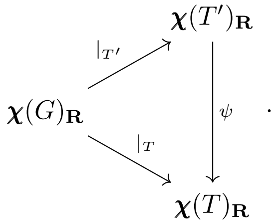
is the linear subspace $a = 0$ in $\boldsymbol{\chi}(G)_{\mathbf{R}}$. We have seen in row 4, 5, 6 and row 10, 11, 12 in Table 2 that crossing the linear subspace $a = 0$ in $\boldsymbol{\chi}(G)_{\mathbf{R}}$ collapses certain strata. On the other hand, the condition $||\,\mathrm{Proj}_{W_1} \chi_T^*|| = ||\,\mathrm{Proj}_{W_3} \chi_T^*||$ defines a type two critical subset with respect to the pair $(\{1\}, \{3\})$. The equation of this type two critical subset is $a^2 + 2b^2 = b^2$. We have seen in row 2, 3, 4 and row 6, 7, 8 in Table 2 that crossing this critical subset swaps orderings of certain strata.

Proposition 5.3.6. *Critical subsets are independent of the choices of maximal tori.*

Proof. Let T and T' be two maximal tori. Since all maximal tori are conjugate to each other, there exists an element $g \in G$ such that $T' = gTg^{-1}$. The isomorphism $T \simeq T'$ defined by $t \mapsto gtg^{-1}$ induces an isomorphism

$$\psi : \boldsymbol{\chi}(T')_{\mathbf{R}} \simeq \boldsymbol{\chi}(T)_{\mathbf{R}}$$

such that the following diagram commutes



Hence it is sufficient to show that walls in $\boldsymbol{\chi}(T')_{\mathbf{R}}$ are mapped to walls in $\boldsymbol{\chi}(T)_{\mathbf{R}}$ under ψ. First, note that $T \simeq T'$ also induces an isomorphism

$$\varphi : \mathbf{\Gamma}(T)_{\mathbf{R}} \simeq \mathbf{\Gamma}(T')_{\mathbf{R}}.$$

dual to ψ. Moreover, the norm on $\mathbf{\Gamma}(G)$ is constructed in a way so that φ is an isomorphism of inner product spaces. It follows that for any $\chi' \in \boldsymbol{\chi}(T')_{\mathbf{R}}$,

$$\psi(\chi')^* = \varphi^{-1}\big((\chi')^*\big). \tag{5.5}$$

Next, suppose

$$V = \bigoplus_{\chi' \in \boldsymbol{\chi}(T')} V_{\chi'}$$

is the weight decomposition for T', then a straightforward computation shows that

$$g^{-1} \cdot V_{\chi'} = V_{\psi(\chi')}.$$

Therefore,

$$V = \bigoplus_{\chi'} g^{-1} \cdot V_{\chi'} = \bigoplus_{\psi(\chi')} V_{\psi(\chi')}$$

is the weight decomposition for T. For any subset $S \subset [n]$, define

$$W'_S = \{v' \in \mathbf{\Gamma}(T')_{\mathbf{R}} \big| \langle \chi', v' \rangle = 0 \text{ for all } \chi' \in S\},$$

and similarly

$$W_S = \{v \in \mathbf{\Gamma}(T)_{\mathbf{R}} \big| \langle \psi(\chi'), v \rangle = 0 \text{ for all } \chi' \in S\}.$$

Then

$$\varphi(W_S) = W'_S. \tag{5.6}$$

Equation (5.6) implies that for any $v \in \mathbf{\Gamma}(T)_{\mathbf{R}}$ and any $S \subset [n]$,

$$\varphi(\operatorname{Proj}_{W_S} v) = \operatorname{Proj}_{W'_S} \varphi(v). \tag{5.7}$$

Equation (5.5) and Equation (5.7) imply that

$$\varphi(\operatorname{Proj}_{W_S} \psi(\chi')^*) = \operatorname{Proj}_{W'_S} (\chi')^* \text{ for any } S \subset [n].$$

Since φ is an isomorphism, it is clear that type one walls are sent to type one walls under ψ. That type two walls are sent to type two walls follow from the fact that φ preserves inner products. □

Proposition 5.3.7. *Let G be a reductive group acting linearly on $\mathbf{A}^n_k$ and $T \subset G$ be a maximal torus. Let S_1 and S_2 be two subsets of $[n]$. Then the collection $Z = \{\chi \in \boldsymbol{\chi}(G)_{\mathbf{R}} | \operatorname{Proj}_{W_{S_1}} \chi^*_T = \operatorname{Proj}_{W_{S_2}} \chi^*_T\}$ is either the whole $\boldsymbol{\chi}(G)_{\mathbf{R}}$, or is an intersection of type one critical subsets. On the other hand, the collection $Z' = \{\chi \in \boldsymbol{\chi}(G)_{\mathbf{R}} \mid ||\operatorname{Proj}_{W_{S_1}} \chi^*_T|| = ||\operatorname{Proj}_{W_{S_2}} \chi^*_T||\}$ is either the whole $\boldsymbol{\chi}(G)_{\mathbf{R}}$, is an intersection of type one critical subsets, or is a type two critical subset.*

Proof. We first prove the proposition for a torus T. The condition that $\operatorname{Proj}_{W_{S_1}} \chi^* = \operatorname{Proj}_{W_{S_2}} \chi^*$ is equivalent to

$$\operatorname{Proj}_{W_{S_1}} \chi^* = \operatorname{Proj}_{W_{S_1 \cup S_2}} \chi^* = \operatorname{Proj}_{W_{S_2}} \chi^*.$$

There exists a sequence $l_1, \ldots, l_k$ in S_2 such that

$$W_{S_1} \supsetneq W_{S_1 \cup l_1} \supsetneq \ldots \supsetneq W_{S_1 \cup l_1 \cup \ldots \cup l_k} = W_{S_1 \cup S_2}$$

where each space is of codimension one of the preceding one. Hence the condition that $\operatorname{Proj}_{W_{S_1}} \chi^* = \operatorname{Proj}_{W_{S_1 \cup S_2}} \chi^*$ corresponds to an intersection of type one walls. Similarly $\operatorname{Proj}_{W_{S_2}} \chi^* = \operatorname{Proj}_{W_{S_1 \cup S_2}} \chi^*$ corresponds to an intersection of type one walls. Therefore Z is an intersection of type one walls. For Z', if there is a containment $W_{S_1} \subset W_{S_2}$, then $Z = Z'$. If there is no containment between W_{S_1} and W_{S_2}, then Z' is a type two wall. For an arbitrary reductive group G, the subset Z is an intersection of the preimages of type one walls. If Z is proper, then one of the preimage of type one wall is proper. In this case Z is an intersection of type one critical subsets. The result for Z' is proved similarly. □

Hence the idea for critical subsets is this:

- Type one critical subsets capture the characters χ where some χ-strata come together.
- Type two critical subsets capture the characters χ where the order of some χ-strata come together.

We will establish a series of results analogous to VGIT ([DH98], or [Tha96]). Specifically, we

(1) prove that each wall is of codimension 1;
(2) establish rationality of the defining equations of critical subsets;
(3) define semi-chambers, and show that
  (a) there are finitely critical subsets and semi-chambers;
  (b) each critical subset and semi-chamber is a cone (possibly not convex), and finally,
  (c) invariance of stratification inside each semi-chamber.

Notice that when G is torus, there is no distinction between critical subsets and walls. While there are properties we establish specifically for walls, any property satisfied by critical subsets holds for walls as well. For example, the rationality of critical subsets that we describe in the following:

Proposition 5.3.8. *All walls are of codimension one and all critical subsets are of codimension at least 1. More specifically,*

(1) A type one wall is a codimension 1 linear subspace.
(2) A type two wall is a codimension 1 regular submanifold away from a linear subspace of codimension at least 2.
(3) A type one critical subset is a codimension 1 linear subspace.
(4) A type two critical subset is either a subspace of codimension at least 1, or a regular submanifold of codimension 1 away from a subspace of codimension at least 2.

Moreover, the defining equations of critical subsets satisfy the following rationality property: Let $M \subset \boldsymbol{\chi}(G)$ be a free abelian group with $M_{\mathbf{R}} = \boldsymbol{\chi}(G)_{\mathbf{R}}$. If $\{m_1, \ldots, m_r\}$ is a $\mathbf{Z}$-basis for M, then

- *the condition that a point $\sum_i a_i m_i \in \boldsymbol{\chi}(G)_{\mathbf{R}}$ is in a type one critical subset corresponds to a linear equation of a_i's with rational coefficients,*
- *the condition that a point $\sum_i a_i m_i \in \boldsymbol{\chi}(G)_{\mathbf{R}}$ is in a type two critical subset corresponds to a homogeneous quadratic equation of a_i's with rational coefficients.*

In particular, a type one critical subset intersects $M \subset M_{\mathbf{R}} = \boldsymbol{\chi}(G)_{\mathbf{R}}$ non-trivially.

Proof. For (1), let T be a torus acting linearly on $\mathbf{A}^n_k$. Let

$$\langle -, - \rangle : \boldsymbol{\chi}(T)_{\mathbf{R}} \times \boldsymbol{\Gamma}(T)_{\mathbf{R}} \to \mathbf{R}$$

be the natural pairing. Suppose $\chi \in \boldsymbol{\chi}(T)_{\mathbf{R}}$ is in a type one wall. Then there are subsets $S_1, S_2 \subset [n]$ such that W_{S_2} is a codimension one subspace of W_{S_1} and $\operatorname{Proj}_{W_{S_1}} \chi^* = \operatorname{Proj}_{W_{S_2}} \chi^*$. There exists some $l \in S_2 - S_1$ such that $W_{S_2} = W_{S_1 \cup \{l\}}$. Let $\varrho : \boldsymbol{\chi}(T)_{\mathbf{R}} \to \mathbf{R}$ be defined by the composition

$$\boldsymbol{\chi}(T)_{\mathbf{R}} \xrightarrow{\ *\ } \boldsymbol{\Gamma}(T)_{\mathbf{R}} \xrightarrow{\operatorname{Proj}_{W_{S_1}}(-)} W_{S_1} \xrightarrow{\langle \chi_l, - \rangle} \mathbf{R}\ .$$

Namely,

$$\varrho(\chi) = \langle \chi_l, \operatorname{Proj}_{W_{S_1}} \chi^* \rangle.$$

The map ϱ is clearly a linear functional and

$$\operatorname{Proj}_{W_{S_1}} \chi^* = \operatorname{Proj}_{W_{S_2}} \chi^* \Leftrightarrow \operatorname{Proj}_{W_{S_1}} \chi^* \in W_{S_2} \Leftrightarrow \langle \chi_l, \operatorname{Proj}_{W_{S_1}} \chi^* \rangle = \varrho(\chi) = 0.$$

Hence the type one wall which χ belongs to is the kernel of ϱ. Since χ_l cuts out a proper subspace of W_{S_1} and $*$ is an isomorphism, ϱ is not a trivial linear functional. Therefore $\ker \varrho$ is a codimension one hyperplane in $\boldsymbol{\chi}(T)_{\mathbf{R}}$.

For (2), notice that the condition $|| \operatorname{Proj}_{W_{S_1}} \chi^*|| = || \operatorname{Proj}_{W_{S_2}} \chi^*||$ is equivalent to saying that χ^* is equi-distant to W_{S_1} and W_{S_2}. Since the map $\chi \mapsto \chi^*$ is an isomorphism, the result follows from Proposition 2.2.3.

For (3), the pull-back of a codimension one subspace under the restriction $\boldsymbol{\chi}(G)_{\mathbf{R}} \to \boldsymbol{\chi}(T)_{\mathbf{R}}$ is either the whole $\boldsymbol{\chi}(G)_{\mathbf{R}}$ or a codimension 1 subspace. Hence every type one critical subset is a codimension 1 subspace.

For (4), note that a type two critical subset is defined by the quadratic form q associated to the bilinear form on $\boldsymbol{\chi}(G)_{\mathbf{R}} \times \boldsymbol{\chi}(G)_{\mathbf{R}}$ defined by

$$(5.8) \qquad (\chi, \chi') \mapsto (\operatorname{Proj}_{W_{S_1}} \chi_T^*, \operatorname{Proj}_{W_{S_1}} {\chi'_T}^*) - (\operatorname{Proj}_{W_{S_2}} \chi_T^*, \operatorname{Proj}_{W_{S_2}} {\chi'_T}^*).$$

By Sylvester's law of inertia, q looks like $x_1^2 + \cdots + x_m^2 - y_1^2 - \cdots y_n^2$. If one of m or n is zero, then q defines a linear subspace. If both m and n are nonzero, then q defines a regular submanifold of codimension 1 away from the linear subspace $V(x_1, \ldots, x_m, y_1, \ldots, y_n)$. This is a linear subspace of codimension at least 2.

We now deal with rationality of the equations of critical subsets. Let $M \subset \boldsymbol{\chi}(G)$ be a free abelian subgroup with $M_{\mathbf{R}} = \boldsymbol{\chi}(G)_{\mathbf{R}}$, and $\{m_1, \ldots, m_r\}$ be a $\mathbf{Z}$-basis for M. Suppose $\chi = \sum_i a_i m_i$. There is an explicit way to calculate the equations of critical subsets when integral basis of W_S is known for each $S \subset [n]$. Let $\{\lambda_1, \ldots, \lambda_q\} \subset \boldsymbol{\Gamma}(T)$ be an $\mathbf{R}$-basis of W_S. Then there exist $b_1, \ldots, b_q$ such that $\sum_i b_i \lambda_j = \operatorname{Proj}_{W_S} \chi_T^*$. For each $i = 1, \ldots, q$, we have

$$\langle \chi_T, \lambda_i \rangle = (\operatorname{Proj}_{W_S} \chi_T^*, \lambda_i) = \sum_{j=1}^{q} b_j (\lambda_i, \lambda_j).$$

Letting A be the $q \times q$ invertible matrix with integral entries (λ_i, λ_j), we see that

$$A \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix} = \begin{pmatrix} \langle \chi_T, \lambda_1 \rangle \\ \vdots \\ \langle \chi_T, \lambda_q \rangle \end{pmatrix}$$

Since each λ_i is in T, $\langle \chi_T, \lambda_i \rangle = \langle \chi, \lambda_i \rangle$ is an integral combination of a_j's. It follows that that each b_i is a $\mathbf{Q}$-combination of a_j's. Therefore, the condition that $\langle \chi_l, \operatorname{Proj}_{W_S} \chi_T^* \rangle = \sum_{i=1}^q b_i \langle \chi_l, \lambda_i \rangle = 0$ for any $l \in [n]$ corresponds to a linear equation of a_i's with rational coefficients. The rationality description about a type one critical subset is proved. For a type two critical subset, it is obvious that $(\operatorname{Proj}_{W_S} \chi_T^*, \operatorname{Proj}_{W_S} \chi_T^*)$ is a $\mathbf{Z}$-combination of $b_i b_j$'s. It follows that a type two wall corresponds to a quadratic homogeneous polynomial of a_i's with rational coefficients. □

We now have seen that each critical subset is defined by a polynomial. It is apparent that there only finitely many critical subsets: If G acts linearly on $\mathbf{A}_k^n$, then there are only finitely many subsets of the finite set $[n]$. Let $\{F_i\}$ be the finite collection of defining polynomials for all critical subsets in $\boldsymbol{\chi}(G)_{\mathbf{R}}$. Then the complement of the union of all critical subsets is a disjoint union of the open semi-algebraic sets of the form

$$\{\chi \in \boldsymbol{\chi}(G)_{\mathbf{R}} \mid \pm F_i(\chi) > 0 \text{ for all } i\}$$

where the sign of each F_i is either positive or negative (but not both). These open semi-algebraic subset need not be connected. We hence adopt the notion of semi-chamber instead of chamber:

Definition 5.3.9. Let G be a reductive group, and $\{F_i\}$ be the defining equations for critical subsets in $\boldsymbol{\chi}(G)_{\mathbf{R}}$. A *semi-chamber* is a non-empty open semi-algebraic set of the form

$$\{\chi \in \boldsymbol{\chi}(G)_{\mathbf{R}} \mid \pm F_i(\chi) > 0 \text{ for all i}\}.$$

Proposition 5.3.10. *There are only finitely many critical subsets and semi-chambers. Moreover, each critical subset H and each semi-chamber C is a cone in the following sense: If a point χ is in H (resp. in C), then $r \cdot \chi$ is in H (resp. in C) for any $r > 0$.*

Proof. The finiteness part is clear. The cone part follows from the fact that the defining equations for critical subsets are homogeneous polynomials. □

5.4. **The main result.** For a point $\chi \in \boldsymbol{\chi}(G)_{\mathbf{R}}$ and a subset $S \subset [n]$, we define the set

$$\Lambda_S^\chi = \{\mathrm{Proj}_{W_Z} \chi_T^* \big| - \mathrm{Proj}_{W_Z} \chi_T^* \in \sigma_S, Z \subset S\} \subset \boldsymbol{\Gamma}(T)_{\mathbf{R}}.$$

For any $g \in G$ and any $v \in \boldsymbol{\Gamma}(T)_{\mathbf{R}}$, $g^{-1}vg$ denotes the image of v under the $\mathbf{R}$-extension of the conjugation $\boldsymbol{\Gamma}(T) \to \boldsymbol{\Gamma}(g^{-1}Tg)$. We have the following IIT-version of Corollary 2.1.9:

Proposition 5.4.1. *Let G act on an affine space X linearly. Then for any character χ and any point $x \in X^{us}(\chi)$, we have $M^\chi(x) = -||v||$ for some $v \in \cup_{g\in G}\Lambda_{S_{g\cdot x}}^\chi$ where*

$$||v|| \geq ||v'|| \text{ for all } v' \in \cup_{g\in G}\Lambda_{S_{g\cdot x}}^\chi.$$

Moreover, let g be an element of G such that $v \in \Lambda_{S_{g\cdot x}}^\chi$. Then the ray

$$\mathbf{R}_{>0} \cdot (-g^{-1}vg) \subset \boldsymbol{\Gamma}(g^{-1}Tg)_{\mathbf{R}}$$

contains a one parameter subgroup that is χ-adapted to x.

Proof. This is a direct application of Proposition 3.2.5. □

The following lemma ensures the description of the one parameter subgroups that are χ-adapted to a point x is consistent in a semi-chamber (Proposition 5.4.3).

Lemma 5.4.2. *Suppose G is a reductive group acting linearly on $\mathbf{A}_k^n$. Let $T \subset G$ be a maximal torus and $S_1, \ldots, S_l$ be a collection of subsets of $[n]$. Order the vectors in $\boldsymbol{\Gamma}(T)_{\mathbf{R}}$ by their norms. If χ_a and χ_b are in a semi-chamber, then there is an order preserving bijection*

$$\Xi : \cup_{i=1}^l \Lambda_{S_i}^{\chi_a} \to \cup_{i=1}^l \Lambda_{S_i}^{\chi_b}$$

defined by

$$\mathrm{Proj}_{W_{Z_i}}(\chi_a|_T)^* \mapsto \mathrm{Proj}_{W_{Z_i}}(\chi_b|_T)^*$$

for all subsets $Z_i \subset S_i$ such that $-\mathrm{Proj}_{W_{Z_i}}(\chi|_a)^* \in \Lambda_{S_i}^{\chi_a}$, and for all $i = 1 \ldots l$.

Proof. Let χ_a and χ_b be in a semi-chamber. We first show the following:

(1) For any $Z, Z' \subset [n]$, $\mathrm{Proj}_{W_Z}(\chi_a|_T)^* = \mathrm{Proj}_{W_{Z'}}(\chi_a|_T)^*$ if and only if $\mathrm{Proj}_{W_Z}(\chi_b|_T)^* = \mathrm{Proj}_{W_{Z'}}(\chi_b|_T)^*$, and

(2) for any $Z \subset S \subset [n]$, $-\operatorname{Proj}_{W_Z}(\chi_a|_T)^* \in \sigma_S \Leftrightarrow -\operatorname{Proj}_{W_Z}(\chi_b|_T)^* \in \sigma_S$

Notice that (1) implies the map Ξ is well defined and injective. (2) implies that the image of Ξ lands inside $\cup_{i=1}^{l}\Lambda^{\chi_b}_{S_i}$ and that Ξ is surjective. For (1), Proposition 5.3.7 says the collection of characters χ with $\operatorname{Proj}_{W_Z}(\chi_T)^* = \operatorname{Proj}_{W_{Z'}}(\chi_T)^*$ is either $\boldsymbol{\chi}(G)_{\mathbf{R}}$ or is an intersection of type one critical subsets. Since χ_a, χ_b are in a semi-chamber, (1) holds. For (2), let $m \in S$ and $\varrho_m : \boldsymbol{\chi}(G)_{\mathbf{R}} \to \mathbf{R}$ be the map defined by $\chi \mapsto \langle \chi_m, \operatorname{Proj}_{W_Z} \chi_T^* \rangle$. Then the kernel of ϱ_m is either $\boldsymbol{\chi}(G)_{\mathbf{R}}$ or a type one critical subset with respect to the subsets Z and $Z \cup \{m\}$. If $\varrho_m(\chi_a) = 0$, then as χ_a is not in a critical subset, $\ker \varrho_m = \boldsymbol{\chi}(G)_{\mathbf{R}}$. This implies $\varrho_m(\chi_b) = 0$ also. If $\varrho(\chi_a) < 0$, then χ_a is in a semi-chamber where ϱ_m is negative. Hence $\varrho_m(\chi_b) < 0$ also. Same argument works for the case $\varrho_m(\chi_a) > 0$. Hence we see that $\varrho_m(\chi_a) \leq 0$ if and only if $\varrho_m(\chi_b) \leq 0$ for all $m \in S$. This exactly means $-\operatorname{Proj}_{W_Z}(\chi_a|_T)^* \in \sigma_S$ if and only if $-\operatorname{Proj}_{W_Z}(\chi_b|_T)^* \in \sigma_S$.

Let us prove that Ξ preserves order. It is enough to show that for any $Z, Z' \subset [n]$,

$$||\operatorname{Proj}_{W_Z}(\chi_a|_T)^*|| < ||\operatorname{Proj}_{W_{Z'}}(\chi_a|_T)^*|| \Leftrightarrow ||\operatorname{Proj}_{W_Z}(\chi_b|_T)^*|| < ||\operatorname{Proj}_{W_{Z'}}(\chi_b|_T)^*||.$$

Suppose $||\operatorname{Proj}_{W_Z}(\chi_a|_T)^*|| < ||\operatorname{Proj}_{W_{Z'}}(\chi_a|_T)^*||$. We consider two cases where in one case there is a containment between W_Z and $W_{Z'}$, and in the other there is no containment. If there is a containment between W_Z and $W_{Z'}$, since $||\operatorname{Proj}_{W_Z}(\chi_a|_T)^*|| < ||\operatorname{Proj}_{W_{Z'}}(\chi_a|_T)^*||$, $W_Z \subsetneq W_{Z'}$. Hence $||\operatorname{Proj}_{W_Z}(\chi_b)^*|| \leq ||\operatorname{Proj}_{W_{Z'}}(\chi_b)^*||$. If $||\operatorname{Proj}_{W_Z}(\chi_b|_T)^*|| = ||\operatorname{Proj}_{W_{Z'}}(\chi_b|_T)^*||$, then we would have $\operatorname{Proj}_{W_Z}(\chi_b|_T)^* = \operatorname{Proj}_{W_{Z'}}(\chi_b|_T)^*$. (1) implies $\operatorname{Proj}_{W_Z}(\chi_a|_T)^* = \operatorname{Proj}_{W_{Z'}}(\chi_a|_T)^*$, a contradiction. Hence $||\operatorname{Proj}_{W_Z}(\chi_b|_T)^*|| < ||\operatorname{Proj}_{W_{Z'}}(\chi_b|_T)^*||$. Suppose there is no containment between W_Z and $W_{Z'}$, the condition that $||\operatorname{Proj}_{W_Z} \chi_T^*|| = ||\operatorname{Proj}_{W_{Z'}} \chi_T^*||$ is either true throughout $\boldsymbol{\chi}(G)_{\mathbf{R}}$ or defines a type two critical subset. Since $||\operatorname{Proj}_{W_Z}(\chi_a|_T)^*|| < ||\operatorname{Proj}_{W_{Z'}}(\chi_a|_T)^*||$, the condition $||\operatorname{Proj}_{W_Z} \chi_T^*|| = ||\operatorname{Proj}_{W_{Z'}} \chi_T^*||$ defines a type two critical subset. As χ_a, χ_b are in a semi-chamber, $||\operatorname{Proj}_{W_Z}(\chi_b|_T)^*|| < ||\operatorname{Proj}_{W_{Z'}}(\chi_b|_T)^*||$. The argument can be reversed. The lemma is proved. □

Recall in Section 3.3 we defined $\Lambda^{\chi}(x)$ to be the collection of indivisible one parameter subgroups that are χ-adapted to x.

Proposition 5.4.3. *Let G be a reductive group acting linearly on an affine space X. Suppose χ_a, χ_b are two characters in a semi-chamber. Let $x \in X$ be a point and Z be a subset of S_x. Then the one parameter subgroups on the ray $\mathbf{R}_{>0} \cdot (-\operatorname{Proj}_{W_Z}(\chi_a|_T)^*) \cap \boldsymbol{\Gamma}(T)$ have limits at x if and only if the one parameter subgroups on the ray $\mathbf{R}_{>0} \cdot (-\operatorname{Proj}_{W_Z}(\chi_b|_T)^*) \cap \boldsymbol{\Gamma}(T)$ have limits at x. Moreover, assume x is both χ_a, χ_b-unstable. Let $g \in G$ be a k-point and Z' some subset of $S_{g\cdot x}$. Then $\mathbf{R}_{>0} \cdot (-g \operatorname{Proj}_{W_{Z'}}(\chi_a|_T)^* g) \cap \Lambda^{\chi_a}(x) \neq \emptyset$ if and only if $\mathbf{R}_{>0} \cdot (-g \operatorname{Proj}_{W_{Z'}}(\chi_b|_T)^* g) \cap \Lambda^{\chi_b}(x) \neq \emptyset$.*

Proof. Recall that the points in $\boldsymbol{\Gamma}(T) \cap \sigma_{S_x}$ are the one parameter subgroups in T that have limits at x. The statement that the one parameter subgroups on the ray $\mathbf{R}_{>0} \cdot (-\operatorname{Proj}_{W_Z}(\chi_a|_T)^*)$ have limits at x is equivalent to the condition that $-\operatorname{Proj}_{W_Z}(\chi_a|_T)^* \in \sigma_{S_x}$. This condition by definition is precisely the condition that $-\operatorname{Proj}_{W_Z}(\chi_a|_T)^* \in \Lambda^{\chi_a}_{S_x}$. By Lemma 5.4.2 applied to the single subset S_x, we get the first part of the proposition. The second part of the proposition is the direct result of Proposition 5.4.1, and Lemma 5.4.2 applied to the finite collection $\{S_{g\cdot x} | g \in G\}$. □

Theorem 5.4.4. *Let G be a reductive group acting linearly on an affine space X. The space of characters $\boldsymbol{\chi}(G)_{\mathbf{R}}$ has a finite decomposition into open semi-chambers and closed codimension at least one critical subsets such that whenever two characters of G share the same unstable locus and are in a semi-chamber, then they induce the same stratification of the unstable locus in X.*

Proof. We have seen that there are finitely many critical subsets and semi-chambers. It remains to prove that if two characters share the same unstable locus and are in a semi-chamber, then they induce the same stratification of the unstable locus. Let T be a maximal torus, and χ_a, χ_b be in a semi-chamber. Recall that each point $x \in X$ gives rise to a finite collections of subsets $\{S_{g \cdot x} | g \in G\}$. Suppose x is χ_a-unstable. Then by Proposition 5.4.1, there exists a $g_x \in G$ such that the ray $\mathbf{R}_{>0} \cdot (-g_x^{-1} \operatorname{Proj}_{W_{Z_x}}(\chi_a|_T)^* g_x) \cap \Lambda^{\chi_a}(x) \neq \emptyset$ for some subset $Z_x \subset [n]$. Let y be another χ_a-unstable point. Then there is a $g_y \in G$ such that the ray $\mathbf{R}_{>0} \cdot (-g_y^{-1} \operatorname{Proj}_{W_{Z_y}}(\chi_a|_T)^* g_y) \cap \Lambda^{\chi_a}(y) \neq \emptyset$ for some $Z_y \subset [n]$. Suppose x, y are in the same stratum under χ_a, we need to show x, y are still in the same stratum under χ_b. Since x, y are in the same stratum under χ_a, there is an $h \in G$ such that

$$g_y^{-1} \operatorname{Proj}_{W_{Z_y}}(\chi_a|_T)^* g_y = h^{-1} g_x^{-1} \operatorname{Proj}_{W_{Z_x}}(\chi_a|_T)^* g_x h$$

where the equality is up to a positive multiple. Since one parameter subgroups on the ray $\mathbf{R}_{>0} \cdot (-g_y^{-1} \operatorname{Proj}_{W_{Z_y}}(\chi_a|_T)^* g_y)$ have limits at y, the one parameter subgroups on the ray $\mathbf{R}_{>0} \cdot (\operatorname{Proj}_{W_{Z_x}}(\chi_a|_T)^*)$ have limits at $(g_x h) \cdot y$. By Proposition 5.4.3 applied to the point $(g_x h) \cdot y$, the one parameter subgroups on the ray $\mathbf{R}_{>0} \cdot (-\operatorname{Proj}_{W_{Z_x}}(\chi_b|_T)^*)$ have limits at $(g_x h) \cdot y$. This implies one parameter subgroups on the ray $\mathbf{R}_{>0} \cdot (h^{-1} g_x^{-1} \operatorname{Proj}_{W_{Z_x}}(\chi_b|_T)^* g_x h)$ have limits at y. By Proposition 5.4.3, $\mathbf{R}_{>0} \cdot (-g_y^{-1} \operatorname{Proj}_{W_{Z_y}}(\chi_b|_T)^* g_y) \cap \Lambda^{\chi_b}(y) \neq \emptyset$, and $\mathbf{R}_{>0} \cdot (-g_x \operatorname{Proj}_{W_{Z_x}}(\chi_b|_T)^* g_x) \cap \Lambda^{\chi_b}(x) \neq \emptyset$. Hence if x, y are in different strata under χ_b, we have

$$|| \operatorname{Proj}_{W_{Z_y}}(\chi_b|_T)^* || > || \operatorname{Proj}_{W_{Z_x}}(\chi_b|_T)^* ||.$$

Since χ_a, χ_b are in a semi-chamber, we have

$$|| \operatorname{Proj}_{W_{Z_y}}(\chi_a|_T)^* || > || \operatorname{Proj}_{W_{Z_x}}(\chi_a|_T)^* ||$$

also. This contradicts the assumption that x, y are in the same stratum under χ_a. The argument is reversible so we get that x, y are in the same stratum under χ_a if and only x, y are in the same stratum under χ_b. That the partial ordering of the strata is preserved is the result of Lemma 5.4.2, applied to the collection of subsets $\{S_x | x \in X^{\mathrm{us}}(\chi_a) = X^{\mathrm{us}}(\chi_b)\}$. □

So far we only considered affine spaces on which G acts linearly. For an arbitrary affine G-variety Y, we apply Proposition 3.1.3 to embed Y equivariantly in an affine space $X = \mathbf{A}_k^n$ on which G acts linearly. Then by considering subsets of $S_y \subset [n]$ for $y \in Y$ instead of all $x \in X$, we may still define critical subsets and semi-chambers in $\boldsymbol{\chi}(G)_{\mathbf{R}}$ for Y. The conjecture is that

(1) critical subsets for Y should not depend on the embedding, and
(2) we may derive Theorem 5.4.4 for Y.

The reasoning to back up statement (1) is that whether a one parameter subgroup has limit at y is indifferent to the embedding. Hence the polyhedral cone $\sigma_y \subset \boldsymbol{\Gamma}(T)_{\mathbf{R}}$ and its faces are invariant under different embeddings for any $y \in Y$.

6. Variation of instability - the projective case

We provide an elementary example where SL(2) acts diagonally on a product of the projective lines. This is the moduli of ordered points on the projective line. The picture here is simple enough that there are clearly defined hyperplanes in the space of SL(2)-linearized ample line bundles such that away from these hyperplanes, the stability and the stratification of the unstable locus (in the sense of Theorem 4.3.1) does not change in the sense of Definition 3.3.6. We shall refer to the hyperplanes as VGIT walls if crossing them results in changes in stability. We refer to the hyperplanes as VIIT walls if crossing them results in changes of stratification of the unstable locus. Theorem 6.1.10 summarizes the computations. To make sense of the statement, see notations defined in Definition 6.1.6, and Definition 6.1.7 for VGIT chambers.

6.1. **An elementary example.** We consider a much studied scenario: the moduli of ordered points in projective spaces. For simplicity let us consider ordered m points on the projective line $\mathbf{P}^1_k$. Let G be the special linear group SL(2) and X be the m-fold product $(\mathbf{P}^1_k)^m$. The group G acts on $\mathbf{P}^1_k$ by the natural representation so $\mathscr{O}(1)$ is G-linearized. In this case, we have

$$\mathrm{NS}^G(X) \simeq \mathrm{Pic}^G(X) \simeq \mathrm{Pic}(X) \simeq \mathbf{Z}^m.$$

If $\lambda \in \mathbf{\Gamma}(G)$, we write $\mu(p,\lambda)$ instead of $\mu^{\mathscr{O}(1)}(p,\lambda)$ (Section 4.1) for any $p \in \mathbf{P}^1_k$. We let G act diagonally on X. Namely, the projection maps $\mathrm{pr}_i : X \to \mathbf{P}^1_k$ for $i = 1, \dots m$ are G-equivariant. Let the m-tuple $a = (a_1, \dots, a_m)$ be the line bundle $\mathscr{O}(a_1) \boxtimes \cdots \boxtimes \mathscr{O}(a_m)$ on X. Then the G-ample cone is the interior of the first octant, where $a_i > 0$ for all i. As the projections are G-equivariant, for any one parameter subgroup $\lambda \in \mathbf{\Gamma}(G)$ and $x = (x_1, \dots, x_m) \in X$, we have

$$\mu^a(x,\lambda) = \sum_{i=1}^m a_i \cdot \mu(x_i,\lambda). \tag{6.1}$$

Lemma 6.1.1. *For any indivisible non-trivial one parameter subgroup $\lambda \in \mathbf{\Gamma}(G)$, the Hilbert-Mumford index $\mu(p,\lambda)$ is either 1 or -1 for all $p \in \mathbf{P}^1_k$.*

Proof. Since $\mu(p,\lambda) = \mu(g\cdot p, g\lambda g^{-1})$ for any $g \in G$, we can assume λ is in the diagonal maximal torus in G. Then λ is either $\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$ or $\begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}$. Suppose λ is $\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$. Then

$$\mu(p,\lambda) = \begin{cases} -1, & \text{if } p = [1:0] \\ 1, & \text{else} \end{cases}. \tag{6.2}$$

The case when $\lambda = \begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}$ can be checked similarly. ∎

Lemma 6.1.2. *The entire $\mathbf{P}^1_k$ is unstable with respect to $\mathscr{O}(1)$. Moreover, every one parameter subgroup destabilizes at most one point in $\mathbf{P}^1_k$.*

Proof. Since G acts transitively on $\mathbf{P}^1_k$, it is enough to destabilize $p = [1:0]$. Obviously $\lambda = \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$ destabilizes p. For the second part of the lemma, suppose λ is indivisible and destabilizes some point $p \in \mathbf{P}^1_k$. Then $\mu(p,\lambda) = -1$ as we have seen. For any $g \in G$, we have

(1) $p \neq q \Leftrightarrow g \cdot p \neq g \cdot q$, and
(2) $\mu(p, \lambda) = \mu(g \cdot p, g\lambda g^{-1})$.

Therefore, we may assume λ is in the diagonal. Suppose $\lambda = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, then $p = [1 : 0]$. Hence if $p \neq q$, we have seen in Lemma 6.1.1 that $\mu(q, \lambda) = 1 > 0$. The case that $\lambda = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$ can be proved similarly. □

Remark 6.1.3. The calculation in Lemma 6.1.1 actually shows that the maximal torus in the diagonal does not destabilize p in $\mathbf{P}^1_k$ as long as $p \neq [1 : 0]$ or $[0 : 1]$ although the entire $\mathbf{P}^1_k$ is unstable. On the other hand, as the action is transitive, there is only one stratum, then entire $\mathbf{P}^1_k$. The worst destabilizing one parameter subgroups for any point in $\mathbf{P}^1_k$ are conjugate to $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

Theorem 6.1.4. *Let $a = (a_1, \dots, a_m)$ be a G-linearized ample invertible sheaf on X. A point $x \in X$ is a-semistable if and only if*

$$\sum_{x_i = p} a_i \leq \frac{1}{2} \sum_{i=1}^{m} a_i \text{ for all } p \in \mathbf{P}^1_k. \tag{6.3}$$

A point $x \in X$ is a-stable if and only if $\leq$ is replaced by $<$.

Proof. This is a direct consequence of Lemma 6.1.1 and Lemma 6.1.2. The bad cases are when one of $p = x_i$ is destabilized by some one parameter subgroup. In this case we have

$$\mu^a(x, \lambda) = -\sum_{x_j = p} a_j + \sum_{x_j \neq p} a_j = \sum_{j=1}^{m} a_j - 2 \sum_{x_j = p} a_j$$

assuming λ is indivisible. Inequality Equation (6.3) guarantees that the number $\mu^a(x, \lambda)$ is non-negative for all λ. □

Next we recall the G-effective ample cone for X.

Proposition 6.1.5. *Let a be a G-linearized ample invertible sheaf on X. Then $X^{ss}(a) \neq \emptyset$ if and only if for all $k = 1, \dots, m$,*

$$a_k \leq \frac{1}{2} \sum_{i=1}^{m} a_i.$$

Proof. Suppose $a_k \leq \frac{1}{2} \sum_{i=1}^{m} a_i$ for all k. Then the collection of points

$$U = \{(x_1, \dots, x_m) \mid x_i \neq x_j \text{ for all } 1 \leq i < j \leq m\}$$

is obviously a-semistable. Conversely, if $X^{\mathrm{ss}}(a) \neq \emptyset$, then $X^{\mathrm{ss}}(a)$ being an open subset of X, has non-empty intersection with the open set U. Then in order for any point in U to be a-semistable, we must have $a_k \leq \frac{1}{2} \sum_{i=1}^{m} a_i$ for all k. □

We now introduce a few notations.

Definition 6.1.6. Let I be a subset of $[m]$ and a be a G-linearized ample invertible sheaf on X. By a_I we mean the sum $\sum_{i \in I} a_i$. We set

$$\Delta_I = \{x \in X \mid x_{i_1} = x_{i_2} \text{ for all } i_1, i_2 \in I\}$$

and

$$L_I := \Delta_I - \cup_{I' \supsetneq I} \Delta_{I'}.$$

Obviously both Δ_I and L_I are G-invariant. The symbol Δ is inspired by diagonal closed immersions and the letter L stands for locally closed. We will see that these L_I corresponds to IIT-equivalence classes of G-linearized ample invertible sheaves.

Also recall from [DH98] that the VGIT walls are of the form

$$a_{I^c} = a_I \text{ for some proper, non-empty subsets of } [m]. \tag{6.4}$$

Indeed, for every proper subset I, a point $x \in (\Delta_I \cap \Delta_{I^c}) - \Delta_{[m]}$ is strictly semistable with respect to a where $a_{I^c} = a_I$. In [DH98], VGIT chambers in the G-effective ample cone. Since we look at instability rather than stability, let us include chambers where the semistable locus is empty as VGIT chambers as well.

Definition 6.1.7. Let X be the m-fold product $(\mathbf{P}^1_k)^m$ where $G = \mathrm{SL}(2)$ acts diagonally X. *A VGIT chamber for* X is a connected component of the complement of the union of hyperplanes $\{a_I^c = a_I\}_{I \subset [m]}$ in the G-ample cone of X.

We see that if a is in a VGIT chamber inside the G-ample cone, then for every subset $I \subset [m]$, a_I is either greater than or less than a_{I^c}. Hence a VGIT chamber $\mathcal{C}$ corresponds to a system of strict inequalities

$$\{a_{I^c} < a_I\}_{I \in S_{\mathcal{C}}}$$

where $S_{\mathcal{C}}$ is a collection of subsets of $[m]$ and the list $\{I^c, I \mid I \in S_{\mathcal{C}}\}$ exhausts all subsets of $[m]$. We emphasize that we include the dumb inequality

$$0 < a_1 + \cdots + a_m$$

to make notations work later. However, $S_{\mathcal{C}}$ is not a random collection. We describe some conditions on $S_{\mathcal{C}}$ necessary for a VGIT chamber $\mathcal{C}$.

Lemma 6.1.8. *Let $\mathcal{C}$ be a VGIT chamber in the G-ample cone. Then the following conditions hold for $S_{\mathcal{C}}$:*

(1) If $I \in S_{\mathcal{C}}$, then $J \notin S_{\mathcal{C}}$ for any $J \subset I^c$,
(2) if $I \in S_{\mathcal{C}}$, then $I' \in S_{\mathcal{C}}$ for any $I' \supset I$, and
(3) for any $I, I' \in S_{\mathcal{C}}$, $I \cap I'$ is never empty.

Proof. For condition (1), let $I \in S_{\mathcal{C}}$ and $J \subset I^c$. Since we are in the interior of the first octant, we have

$$a_J \leq a_{I^c} < a_I \leq a_{J^c}.$$

Hence J is never in $S_{\mathcal{C}}$. The second condition can be proved similarly. For the third condition, suppose $I \cap I' = \emptyset$ for some $I, I' \in S_{\mathcal{C}}$, then $I' \subset I^c$. This contradicts condition (1). □

Proposition 6.1.9. *Let $\mathcal{C}$ be a VGIT chamber in the G-ample cone with the system of inequalities $\{a_{I^c} < a_I\}_{I \in S_{\mathcal{C}}}$. Then*

(1) for each $I \in S_{\mathcal{C}}$, for each invertible sheaf $a \in \mathcal{C}$ and any point $x \in L_I$,

$$\min_{\lambda \text{ indivisible}} \mu^a(x, \lambda) = -a_I + a_{I^c}.$$

(2) the L_I's are mutually disjoint for $I \in S_{\mathcal{C}}$,
(3)

$$\bigcup_{I \in S_{\mathcal{C}}} L_I = \bigcup_{I \in S_{\mathcal{C}}} \Delta_I,$$

and

(4) the unstable locus in $\mathcal{C}$ is

$$\bigcup_{I\in S_{\mathcal{C}}} \Delta_I = \bigcup_{I \text{ minimal in } S_{\mathcal{C}}} \Delta_I.$$

Proof. For (1), let $x \in L_I$ and J be a subset of $[m]$ so that $x \in \Delta_J$ with $J \subset I^c$. By Lemma 6.1.8, $J \notin S_{\mathcal{C}}$ so we get $a_J < a_{J^c}$. Hence if λ is indivisible and destabilizes x_j for some (and hence for all) $j \in J$, we have $\mu^a(x, \lambda) = -a_J + a_{J^c} > 0$. This shows that $\mu^a(x, \lambda) > 0$ for any one parameter subgroup λ that does not destabilize any x_i for $i \in I$. Statement (1) is proved. Statement (2) would follow if for any $I, I' \in S_{\mathcal{C}}$, we have $I \cap I' \neq \emptyset$. This holds by Lemma 6.1.8. For statement (3), note that

$$\Delta_I = \cup_{I' \supset I} L_{I'} \text{ for each } I \in S_{\mathcal{C}}.$$

By Lemma 6.1.8, all subsets I' that contain I are in $S_{\mathcal{C}}$. Hence $\cup_{I\in S_{\mathcal{C}}} \Delta_I \subset \cup_{I\in S_{\mathcal{C}}} L_I$. The other inclusion is obvious. For statement (4), statement (1) and (3) imply $\cup_{I\in S_{\mathcal{C}}} \Delta_I \subset X^{\mathrm{us}}(a)$. Conversely, suppose $x \notin \cup_{I\in S_{\mathcal{C}}}$. Let J be some subset of $[m]$ so that $x \in \Delta_J$. Then $J \notin S_{\mathcal{C}}$ and $J \neq [m]$. Hence $a_J < a_{J^c}$. This shows that destabilizing any x_i for $i = 1, \ldots, m$ does not yield negative Hilbert-Mumford index. Hence x is a-semistable. Obviously the union over all $I \in S_{\mathcal{C}}$ can be reduced to the union over all minimal ones. Each minimal $I \in S_{\mathcal{C}}$ actually corresponds to an irreducible component of the unstable locus. Statement (4) is proved. □

Theorem 6.1.10. *Let $G = \mathrm{SL}(2)$ act diagonally on the m-fold product $X = (\mathbf{P}^1_k)^m$ via the natural representation. Let $\mathcal{C}$ be a VGIT chamber in the G-ample cone defined by the system of strict inequalities $\{a_{I^c} < a_I\}_{I\in S_{\mathcal{C}}}$. If an ample G-linearized line bundle $\mathscr{L}$ is in a VGIT chamber $\mathcal{C}$ defined by the system of strict inequalities $\{a_{I^c} < a_I\}_{I\in S_{\mathcal{C}}}$, then the unstable locus*

$$X^{us}(\mathscr{L}) = \bigcup_{I\in S_{\mathcal{C}}} \Delta_I.$$

The VGIT chamber $\mathcal{C}$ has a VIIT wall and chamber decomposition where the VIIT walls are of the form

$$H_{I,J} := \{a \mid a_{J\setminus I} = a_{I\setminus J} \text{ for } I, J \in S_{\mathcal{C}} \text{ with no containments.}\}$$

Inside each VIIT chamber the unstable locus is stratified by L_I's for $I \in S_{\mathcal{C}}$, having order $-a_I + a_{I^c}$. Crossing a VIIT wall $H_{I,J}$ swaps the orderings between any $L_{I'}$ and $L_{J'}$ with $I'\setminus J' = I\setminus J$ and $J'\setminus I' = J\setminus I$. On a VIIT wall $H_{I,J}$ the strata $L_{I'}$ and $L_{J'}$ come together for any $I'\setminus J' = I\setminus J$ and $J'\setminus I' = J\setminus I$.

Proof. We have seen that L_I has order $-a_I + a_{I^c}$. So the VIIT walls are of the form $-a_I + a_{I^c} = -a_J + a_{J^c}$, which can be rearranged to the form as is stated in the theorem. The non-containment assumption either comes from the fact that we are in the interior of the first octant or from a slightly more sophisticated fact that

$$L_J \subset \partial L_I \text{ if } J \supset I.$$

Hence according to the theorem of Hesselink, Theorem 4.3.1, L_J and L_I will never be in the same stratum if there is a containment between I and J. □

Example 6.1.11. Take $m = 3$ and the VGIT chamber defined by

(1) $a_2 + a_3 < a_1$,
(2) $a_2 < a_1 + a_3$,

(3) $a_3 < a_1 + a_2$, and
(4) $0 < a_1 + a_2 + a_3$.

This means a_1 is large. By Proposition 6.1.5, the entire X is unstable. Indeed, with Proposition 6.1.9, we have that the unstable locus is

$$\Delta_1 \cup \Delta_{13} \cup \Delta_{12} = \Delta_1 = X.$$

Using Proposition 6.1.9, L_{123} has the highest order $-a_1 - a_2 + a_3$, L_{12} has order $-a_1 - a_2 + a_3$, L_{13} has order $-a_1 - a_3 + a_2$ and L_1 has order $-a_1 + a_2 + a_3$. There is a VIIT wall $a_2 - a_3 = 0$. On the VIIT wall L_{13}, L_{12} comes together and crossing the VIIT wall L_{13}, L_{12} swaps the ordering between L_{13} and L_{12}.

Example 6.1.12. Take $m = 4$ and the VGIT chamber defined by

(1) $a_k < \sum_{i \neq k} a_i$ for $k = 1, \ldots 4$,
(2) $a_1 + a_2 < a_3 + a_4$,
(3) $a_1 + a_3 < a_2 + a_4$,
(4) $a_1 + a_4 < a_2 + a_3$, and
(5) $0 < a_1 + a_2 + a_3 + a_4$.

This means a_1 is not large. For example, $(1, 2, 2, 2)$ is in the VGIT chamber. Note that we are in the G-effective ample cone. The unstable locus for any line bundle in this chamber is

$$\Delta_{34} \cup \Delta_{24} \cup \Delta_{23}.$$

Take $I = \{1, 2, 3\}$ and $J = \{1, 2, 4\}$, we have the VIIT wall $a_3 = a_4$. Let $I' = \{2, 3\}$ and $J' = \{2, 4\}$, we have the same VIIT wall. Hence the VIIT wall $a_3 = a_4$ interferes L_{123}, L_{124} and L_{23}, L_{24}. On the VIIT wall the four strata come together and crossing the wall swaps the orderings between L_{123}, L_{124} and the ordering between L_{23}, L_{24}.

For arbitrary reductive group G and a product of projective spaces on which G acts diagonally, we conjecture that a result similar to Theorem 5.4.4 holds. Namely, there are finitely many closed codimension at least one subsets in the G ample cone such that away from these closed subsets, if two line bundles share the same unstable locus, they induce the same stratification of the unstable locus.

7. The toric variety case

The main motivation behind this section comes from the fact that every projective toric variety is a GIT quotient of an affine space by a reductive group with respect to linearizations that come from ample divisors (Theorem 7.2.3, and Proposition 7.2.6). Hence the theme of this section is to investigate the variation of stratification inside the ample cone of a projective toric variety. We start by fixing notations for toric varieties, and in Section 7.2 we construct each projective toric variety as a GIT quotient. The unstable locus can be described by primitive collections (Proposition 7.2.6). Section 7.3 contains the main theorem (Theorem 7.3.10) which states that VIIT in the ample cone of a projective toric variety depends only on the primitive collections and the relations among ray generators of the fan. Moreover, we formulate critical subsets and semi-chambers in the ample cone for a projective toric variety. We then derive Theorem 7.3.16, which states that in a semi-chamber, the stratification stays constant. This is analogous to Theorem 5.4.4. Finally, we describe what a strata looks like (Theorem 7.4.1) and various other combinatorial results in Section 7.4.

7.1. **Set up.** The main reference for this section is [CLS11]. We refer the reader for the definition of a fan and how to construct a toric variety from a fan to chapter 3 in [CLS11]. Let N be a lattice and $M = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ be the dual. Let $\langle -, - \rangle : M \times N \to \mathbf{Z}$ be the perfect pairing where $\langle m, n \rangle = m(n)$. Set $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$ to be $N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M \otimes_{\mathbf{Z}} \mathbf{R}$ respectively. Let $\Sigma \subset \mathbf{N_R}$ be a fan. By $\Sigma(i)$ we mean the collection of i-dimensional cones in Σ. For each ray $\rho \in \Sigma(1)$, we write $u_\rho \in N$ as the indivisible lattice point that generates ρ. Let X_Σ be the toric variety constructed from the fan Σ. In this case $T_N := N \otimes_{\mathbf{Z}} \mathbf{C}^\times$ is the torus embedded in X_Σ. The lattice N is realized as the group of one parameter subgroups of T_N in the following way: given an element $n \in N$, the assignment $\mathbf{C}^\times \to T_N$ defined by

$$t \mapsto n \otimes t$$

is a one parameter subgroup of T_N. The lattice M is realized as the group of characters of T_N in the following way: given an element $m \in M$ and an element $n \otimes t \in T_N$, the assignment $T_N \to \mathbf{C}^\times$ defined by

$$n \otimes t \mapsto t^{\langle m, n \rangle}$$

is a character of T_N. By the orbit-cone correspondence (theorem 3.2.6 in [CLS11]), each ray ρ in Σ corresponds to a T_N-invariant divisor $D_\rho \subset X_\Sigma$. Let

$$\mathbf{Z}^{\Sigma(1)} = \bigoplus_{\rho \in \Sigma(1)} \mathbf{Z} \cdot D_\rho$$

be the free abelian group generated by the divisors D_ρ for $\rho \in \Sigma(1)$. Each $m \in M$, viewed as a character χ^m of T_N, is a rational function on X_Σ. The principal divisor $\mathrm{div}(\chi^m)$ satisfies the formula

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle \cdot D_\rho. \tag{7.1}$$

It is proved in theorem 4.1.3 in [CLS11] that the natural map $\mathbf{Z}^{\Sigma(1)} \to \mathrm{Cl}(X_\Sigma)$ is surjective with kernel $M \to \mathbf{Z}^{\Sigma(1)}$ given by $m \mapsto \mathrm{div}(\chi^m)$. Namely, we have an exact sequence

$$M \longrightarrow \mathbf{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0\ . \tag{7.2}$$

Moreover, it is exact on the left if Σ is complete. Let $\mathrm{CDiv}_{T_N}(X_\Sigma)$ be the group of torus invariant Cartier divisors on X_Σ. Then there is the exact sequence

$$M \longrightarrow \mathrm{CDiv}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0 \tag{7.3}$$

that fits into the commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} M & \longrightarrow & \mathrm{CDiv}_{T_N}(X_\Sigma) & \longrightarrow & \mathrm{Pic}(X_\Sigma) & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \\ M & \longrightarrow & \mathbf{Z}^{\Sigma(1)} & \longrightarrow & \mathrm{Cl}(X_\Sigma) & \longrightarrow & 0 \end{array}.$$

When Σ is smooth, there is no distinction between sequence (7.2) and sequence (*Equation* (7.3)). When Σ is complete, as $\mathbf{C}^\times$ is divisible, we get another short exact sequence by applying $\mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{C}^\times)$ to sequence (7.2):

$$1 \longrightarrow G \longrightarrow (\mathbf{C}^\times)^{\Sigma(1)} \longrightarrow T_N \longrightarrow 1 \tag{7.4}$$

where $G = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Cl}(X_\Sigma), \mathbf{C}^\times)$. Facts about the group G are summarized in the

Proposition 7.1.1. *Let Σ be a complete fan and $G = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Cl}(X_\Sigma), \mathbf{C}^\times)$. Then*

(1) $G \simeq T \times H$ for some torus T and some finite group H. In particular, G is abelian and reductive.

(2) The inclusion $T \subset G$ induces $\mathbf{\Gamma}(T) \simeq \mathbf{\Gamma}(G)$ and $\boldsymbol{\chi}(G)_{\mathbf{R}} \simeq \boldsymbol{\chi}(T)_{\mathbf{R}}$.

(3) $Cl(X_\Sigma) \simeq \boldsymbol{\chi}(G)$.

(4) An element $t \in (\mathbf{C}^\times)^{\Sigma(1)}$ is in G if and only if

$$\prod_{\rho\in\Sigma(1)} t_\rho^{\langle m,u_\rho\rangle} = 1 \text{ for all } m \in M.$$

(5) There is an exact sequence

$$0 \longrightarrow \mathbf{\Gamma}(G) \longrightarrow \mathbf{Z}^{\Sigma(1)} \xrightarrow{\varphi} N \tag{7.5}$$

dual to (7.2), where if $\{e_\rho\}$ is the standard basis for $\mathbf{Z}^{\Sigma(1)}$, φ is defined by $e_\rho \mapsto u_\rho$. In particular, $\mathbf{\Gamma}(G)$ records the relation among ray generators of $\Sigma(1)$.

Proof. For (1), $\mathrm{Cl}(X_\Sigma) \simeq A \oplus B$ for some lattice A and finite group B by (7.2). Hence $G \simeq T \times H$ for some torus T and some finite group H. (2) is a direct consequence of (1). (3) follows from the definition of G. For (4), see lemma 5,1,1 in [CLS11]. For (5), the inclusion $G \hookrightarrow (\mathbf{C}^\times)^{\Sigma(1)}$ induces an inclusion $\mathbf{\Gamma}(G) \hookrightarrow \Gamma((\mathbf{C}^\times)^{\Sigma(1)}) \simeq \mathbf{Z}^{\Sigma(1)}$. By (4), a one parameter subgroup $b \in \mathbf{Z}^{\Sigma(1)}$ is in $\mathbf{\Gamma}(G)$ if and only if

$$\prod_{\rho\in\Sigma(1)} t^{b_\rho\cdot\langle m,u_\rho\rangle} = t^{\langle m,\sum_\rho b_\rho u_\rho\rangle} = 1 \text{ for all } t \in \mathbf{C}^\times \text{ and for all } m \in M.$$

Hence $\langle m, \sum_{\rho\in\Sigma(1)} b_\rho u_\rho\rangle = 0$ for all $m \in M$. Since the pairing between M and N is perfect, we have

$$\sum_{\rho\in\Sigma(1)} b_\rho u_\rho = 0.$$

Namely, $\mathbf{\Gamma}(G)$ is exactly the relation among ray generators. Moreover, the dual of $M \to \mathbf{Z}^{\Sigma(1)}$ defined in (7.2) is φ. Hence (7.5) is the dual of (7.2) is □

Hence we would write χ_D as a character of G when $D \in \mathrm{Cl}(X_\Sigma)$. The group G acts on $\mathbf{C}^{\Sigma(1)}$ by restricting the natural action of $(\mathbf{C}^\times)^{\Sigma(1)}$ to G. One sees that the group G acts diagonally on $\mathbf{C}^{\Sigma(1)}$ with the characters $\{\chi_{D_\rho}\}_{\rho\in\Sigma(1)}$. Next, we would like to understand the group of one parameter subgroups of G.

7.2. **Quotient construction for projective toric varieties.** For each fan Σ, we define the affine space

$$\mathbf{C}^{\Sigma(1)} = \mathrm{Spec}\,\mathbf{C}[x_\rho \mid \rho \in \Sigma(1)].$$

Let $\Sigma_{\max}$ be the collection of maximal cones in Σ. For each cone $\sigma \in \Sigma_{\max}$, define the monomial

$$x^{\hat{\sigma}} := \prod_{\rho\notin\sigma(1)} x_\rho.$$

Then define the ideal

$$B(\Sigma) = (x^{\hat{\sigma}} | \sigma \in \Sigma_{\max}) \subset \mathbf{C}[x_\rho \mid \rho \in \Sigma(1)].$$

We then have the vanishing locus

$$Z(\Sigma) = V(B(\Sigma)) \subset \mathbf{C}^{\Sigma(1)}.$$

There is a cleaner description of $B(\Sigma)$ and $Z(\Sigma)$, using the notion of *primitive collections*.

Definition 7.2.1. A subset $C \subset \Sigma(1)$ is a *primitive collection* if:

(1) $C \not\subset \sigma(1)$ for all $\sigma \in \Sigma$;
(2) for any proper subset C' of C, there is a cone $\sigma \in \Sigma$ such that $C' \subset \sigma(1)$.

Proposition 7.2.2. *There is a primary decomposition $B(\Sigma) = \bigcap_C (x_\rho | \rho \in C)$, which induces $Z(\Sigma) = \bigcup_C V(x_\rho | \rho \in C)$ where both the union and intersection are taken over all primitive collections C of $\Sigma(1)$.*

Here is part of theorem 5.1.11 in [CLS11]:

Theorem 7.2.3. *Let X_Σ be a complete toric variety and $G = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Cl}(X_\Sigma), \mathbf{C}^\times)$. There is a map $\pi : \mathbf{C}^{\Sigma(1)} \backslash Z(\Sigma) \to X_\Sigma$ that realizes X_Σ as an almost geometric quotient of $\mathbf{C}^{\Sigma(1)} \backslash Z(\Sigma)$ by G.*

Since G acts on $\mathbf{C}^{\Sigma(1)}$ and $\mathrm{Cl}(X_\Sigma)$ is the group of characters of G, whenever we are given a divisor class $D \in \mathrm{Cl}(X_\Sigma)$, we can associate the GIT quotient $\mathbf{C}^{\Sigma(1)} /\!\!/_{\chi_D} G$. It is already proved in [CLS11] that

$$\mathbf{C}^{\Sigma(1)} /\!\!/_{\chi_D} G \simeq X_\Sigma$$

whenever X_Σ is projective and D is an ample divisor. Here we give an alternative proof. Note that a GIT quotient is a good categorical quotient of the semistable locus. Hence it is sufficient to show that $(\mathbf{C}^{\Sigma(1)})^{ss}(\chi_D) = \mathbf{C}^{\Sigma(1)} \backslash Z(\Sigma)$ whenever D is ample. We recall a numerical criterion for ampleness.

Theorem 7.2.4. *Let Σ be a fan and $D = \sum_\rho a_\rho D_\rho$ be a divisor on X_Σ. Then D is Cartier if and only if for all cone $\sigma \in \Sigma_{max}$, there is an $m_\sigma \in M$ such that*

$$\langle m, u_\rho \rangle = -a_\rho \text{ for all } \rho \in \sigma(1).$$

Such m_σ is unique modulo $\sigma^\perp \cap M$. Moreover, if Σ is complete, and $D = \sum_\rho a_\rho D_\rho$ is Cartier with $m_\sigma \in M$ for each $\sigma \in \Sigma_{max}$ as in the first part of the theorem, then D is ample if and only if

$$\langle m_\sigma, u_\rho \rangle > -a_\rho \text{ for all } \sigma \in \Sigma_{max} \text{ and for all } \rho \notin \sigma(1).$$

Corollary 7.2.5. *Suppose there is full dimensional cone in Σ. Then $\mathrm{Pic}(X_\Sigma)$ is free of finite rank. In particular, this holds when Σ is complete.*

Proof. Because of the short exact sequence (7.3), we know that $\mathrm{Pic}(X_\Sigma)$ is finitely generated. Hence it is sufficient to prove that $\mathrm{Pic}(X_\Sigma)$ is torsion free. Let σ be a full dimensional cone in Σ. Let $D = \sum_\rho a_\rho D_\rho$ be a torus invariant Cartier divisor and $K \cdot D$ is linearly equivalent to 0 for some $K > 0$. We need to prove that D is linearly equivalent to 0. By Theorem 7.2.4, there exists an $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$. By the short exact sequence (7.3), there exists an $m \in M$ such that $\langle m, u_\rho \rangle = -K \cdot a_\rho$ for all $\rho \in \Sigma(1)$. Hence $\langle K \cdot m_\sigma, u_\rho \rangle = \langle m, u_\rho \rangle$ for all $\rho \in \sigma(1)$. Since σ is full dimensional, $K \cdot m_\sigma = m$. As $\mathbf{Z}^{\Sigma(1)}$ is torsion free, we also have $\mathrm{div}(\chi^{m_\sigma}) = D$. Hence D is zero in $\mathrm{Pic}(X_\Sigma)$. ☐

Proposition 7.2.6. *Let X_Σ be a projective toric variety and $D \in Cl(X_\Sigma)$ be an ample divisor. Then*

$$(\mathbf{C}^{\Sigma(1)})^{ss}(\chi_D) = \mathbf{C}^{\Sigma(1)} \backslash Z(\Sigma).$$

Proof. We will prove this using purely the Hilbert Mumford criterion, Theorem 3.1.7. We first show that a point $x \notin Z(\Sigma)$ is χ_D-semistable for an ample divisor $D = \sum_\rho a_\rho D_\rho$. Let $\lambda \in \mathbf{Z}^{\Sigma(1)}$ be a one parameter subgroup of G such that $\lim_{t\to 0} \lambda(t)\cdot x$ exists. For each $\rho \in \Sigma(1)$, we have

$$(\lambda(t)\cdot x)_\rho = t^{\lambda_\rho}\cdot x_\rho.$$

By definition of $Z(\Sigma)$, there is a maximal cone σ in the fan such that $x^{\hat{\sigma}}(x) \neq 0$. Hence $\lim_{t\to 0}\lambda(t)\cdot x$ exists only if $\lambda_\rho \geq 0$ for all $\rho \notin \sigma(1)$. Recall we have $m_\sigma \in M$ from Theorem 7.2.4. Now

$$\begin{aligned}
\langle \chi_D, \lambda\rangle &= \sum_\rho a_\rho\lambda_\rho = \sum_{\rho\in\sigma(1)} a_\rho\lambda_\rho + \sum_{\rho\notin\sigma(1)} a_\rho\lambda_\rho \\
&= -\sum_{\rho\in\sigma(1)} \langle m_\sigma, u_\rho\rangle\lambda_\rho + \sum_{\rho\notin\sigma(1)} a_\rho\lambda_\rho \\
&\geq -\sum_{\rho\in\sigma(1)} \langle m_\sigma, u_\rho\rangle\lambda_\rho - \sum_{\rho\notin\sigma(1)} \langle m_\sigma, u_\rho\rangle\lambda_\rho \\
&= -\langle m_\sigma, \sum_\rho \lambda_\rho u_\rho\rangle = 0
\end{aligned}$$

by Theorem 7.2.4 and the fact that $\mathbf{\Gamma}(G)$ is the relation among ray generators. Conversely, let $x \in Z(\Sigma)$, we need to establish a one parameter subgroup λ of G such that

- $\lim_{t\to 0}\lambda(t)\cdot x$ exists, and
- $\langle \chi_D, \lambda\rangle < 0$.

For this, there is a primitive collection C such that $x \in V(x_\rho | \rho \in C)$. Let

$$u = \sum_{\rho\in C} u_\rho.$$

Then since Σ is complete, there is a maximal cone $\sigma \in \Sigma_{\max}$ such that $u \in \sigma$. By proposition 11.1.7 in [CLS11], there is a subcollection $S \subset \sigma(1)$ such that $u \in \mathrm{Cone}(S)$ and such that $\mathrm{Cone}(S)$ is a simplicial cone. Hence there are $b_\rho \in \mathbf{Q}_{\geq 0}$ for $\rho \in \sigma(1)$ such that $u = \sum_{\rho\in\sigma(1)} b_\rho u_\rho$. Let K be a positive integer large enough so that $K\cdot b_\rho \in \mathbf{N}$ for all $\rho \in \sigma(1)$. Then we have a relation

$$K\cdot u = \sum_{\rho\in\sigma(1)} (K\cdot b_\rho)u_\rho.$$

Let

$$c_\rho = \begin{cases} -K & \text{if } \rho \in C\backslash\sigma(1), \\ K\cdot(b_\rho - 1) & \text{if } \rho \in C\cap\sigma(1), \\ K\cdot b_\rho & \text{if } \rho \in \sigma(1)\backslash C, \\ 0 & \text{otherwise.}\end{cases}$$

Since each c_ρ is an integer and $\sum_\rho c_\rho u_\rho = 0$, we see that the collection $\{c_\rho\}_{\rho\in\Sigma(1)}$ induces a one parameter subgroup λ of G. Furthermore $c_\rho \geq 0$ for all $\rho \notin C$. Hence the limit $\lim_{t\to 0}\lambda(t)\cdot x$ exists. It remains to show that $\langle \chi_D, \lambda\rangle = \sum_\rho a_\rho c_\rho < 0$. Since C is a primitive collection, there is at least one $\rho' \in C\backslash\sigma(1)$ and $c_{\rho'} = -K <$

0. Therefore, we have

$$\begin{aligned}\sum_{\rho} a_\rho c_\rho &= \sum_{\rho\in\sigma(1)} a_\rho c_\rho + \sum_{\rho\notin\sigma(1)} a_\rho c_\rho \\ &< -\sum_{\rho\in\sigma(1)} \langle m_\sigma, u_\rho\rangle c_\rho - \sum_{\rho\notin\sigma(1)} \langle m_\sigma, u_\rho\rangle c_\rho \\ &= -\langle m_\sigma, \sum_\rho c_\rho u_\rho\rangle = 0,\end{aligned}$$

where the m_σ comes from Theorem 7.2.4. □

Therefore, $Z(\Sigma)$ is the χ_D-unstable locus whenever D is ample. The aim is to describe how the stratification of $Z(\Sigma)$ varies when the ample divisor varies (referred as *toric VIIT*). To do this we first need a norm on $\mathbf{\Gamma}(G)$. Since G is a subgroup of $(\mathbf{C}^\times)^{\Sigma(1)}$, we may take the natural norm on $\mathbf{\Gamma}((\mathbf{C}^\times)^{\Sigma(1)})$ and restrict it to $\mathbf{\Gamma}(G)$. Since G is abelian, the restricted norm on G obviously satisfies the conditions imposed in Section 3.2. We write

$$\mathrm{Amp}(X_\Sigma)$$

as the collection of ample divisors on X_Σ. The *ample cone* for X_Σ is the real cone over $\mathrm{Amp}(X_\Sigma)$ in $\mathrm{Pic}(X_\Sigma)_\mathbf{R}$ and is denoted by

$$\mathrm{Amp}(X_\Sigma)_\mathbf{R}.$$

The ample cone of projective toric varieties are well understood. It is the interior of the nef cone. To see this, we recall theorem 6.3.22 from [CLS11].

Theorem 7.2.7. *Let X_Σ be a projective toric variety. The Nef cone Nef(X_Σ) is a full dimensional, strongly convex rational polyhedral cone in Pic(X_Σ)$_\mathbf{R}$ and a Cartier divisor D is ample if and only if its class in Pic(X_Σ) is in the interior of Nef(X_Σ).*

Also note that

Lemma 7.2.8. *Let σ be a full dimensional rational polyhedral cone. Then int(σ) is the cone over lattice points in int(σ).*

Proof. Let $v = (\epsilon_1, \dots, \epsilon_n)$ again be any point in the interior of σ. First, let us consider the case when $\epsilon_1 \neq 0$. Then choose a large integer a_1 such that the point

$$\frac{a_1}{\epsilon_1}(\epsilon_1, \dots, \epsilon_n) := (a_1, \dots, a_n)$$

has the rectangle $\{(t_1, \dots, t_n) | [a_i] \leq t_i \leq [a_i] + 1\}$ contained in the interior of σ. Then there exists $0 \leq \lambda_i \leq 1$ such that $\lambda_i[a_i] + (1-\lambda_i)([a_i]+1) = a_i$ for all $i \geq 2$. Then we have

$$(a_1, \dots, a_n) = \lambda_2(a_1, [a_2,]a_3, \dots, a_n) + (1-\lambda_2)(a_1, [a_2]+1, a_3, \dots, a_n).$$

We also have

$$(a_1, [a_2], a_3, \dots, a_n) = \lambda_3(a_1, [a_2], [a_3], a_4, \dots, a_n) + (1-\lambda_3)(a_1, [a_2], [a_3]+1, a_4, \dots, a_n).$$

Similarly $(a_1, [a_2]+1, a_3, \dots, a_4)$ is

$$\lambda_3(a_1, [a_2]+1, [a_3], a_4, \dots, a_n) + (1-\lambda_3)(a_1, [a_2]+1, [a_3]+1, a_4, \dots, a_n).$$

Continuing, we realize $(\epsilon_1, \dots, \epsilon_n)$ as a sum of non-negative multiple of interior lattice points in σ. Now if ϵ_1 is zero, we may proceed to the largest index i with $\epsilon_i = 0$ and do the same trick from ϵ_{i+1}. □

Corollary 7.2.9. *Let X_Σ be a projective toric variety. Then $\operatorname{Amp}(X_\Sigma)_{\mathbf{R}}$ is the interior of the nef cone.*

7.3. **Toric VIIT.** In this section we will prove the following two statements. For a projective toric variety,

(1) the variation of stratifications induced by different ample divisors depends only on the primitive collections and relations among the ray generators of the fan, and
(2) there is a finite decomposition of the ample cone by critical subsets and semi-chambers such that the stratification stays constant in each semi-chamber.

For statement (1), we start by some combinatorics of complete fans that will be needed.

Lemma 7.3.1. *Let N be a lattice of rank n and $\Sigma \subset N_{\mathbf{R}}$ be a complete fan. Then $N_{\mathbf{R}}$ is the union of n-dimensional cones.*

Proof. The union of n-dimensional cones is a closed subset of $N_{\mathbf{R}}$. Hence if the union of n-dimensional cones is not the whole $N_{\mathbf{R}}$, an open subset U is left out. Since Σ is complete, the union of cones of lesser dimensions contains U. But $\dim U = n$, which is not contained in a finite union of cones of smaller dimensions. □

Corollary 7.3.2. *Let N be a lattice of rank n and $\Sigma \subset N_{\mathbf{R}}$ be a complete fan. Then for every cone σ in Σ, there is an n-dimensional cone τ containing σ as a face.*

Proof. If $\dim \sigma = n$, we can take $\tau = \sigma$. If $\dim \sigma < n$ then by Lemma 7.3.1,

$$\sigma = \cup_{\dim \sigma' = n} \sigma \cap \sigma'.$$

Suppose σ is not contained in any n-dimensional cones, then each $\sigma \cap \sigma'$ is a proper face of σ. This means σ is a union of its proper faces, an impossibility. □

Corollary 7.3.3. *Let N be a lattice of rank n and $\Sigma \subset N_{\mathbf{R}}$ be a complete fan. Then a cone $\sigma \in \Sigma$ is maximal if and only if σ is of dimension n.*

Proof. Let $\sigma \in \Sigma$ be a maximal cone. By Corollary 7.3.2, σ is contained in an n-dimensional cone τ. Then since σ is maximal, $\tau = \sigma$, which is n-dimensional. Conversely, if σ is of dimension n, then there is a maximal cone τ that contains σ. By dimension count τ is of dimension n. If σ is a proper face of σ, the dimension of σ is at most $n-1$, an impossibility. Hence $\sigma = \tau$, which is maximal. □

Corollary 7.3.4. *Let $\Sigma \subset N_{\mathbf{R}}$ be a complete fan. Then the union of primitive collections is $\Sigma(1)$.*

Proof. Suppose N is of rank n. Let $\rho \in \Sigma(1)$ be a ray. We will construct a primitive collection that contains ρ. Since Σ is complete, Lemma 7.3.1 ensures that there exists a $\sigma \in \Sigma(n)$ such that $-u_\rho$ is in σ. Then since σ is strongly convex, Cone $(u_\rho) \cap \sigma = \{0\}$. Hence $\tilde{\sigma} :=$ Cone (u_ρ, σ) is a cone properly containing σ. By Corollary 7.3.3, $\tilde{\sigma}$ is not contained in σ' for any $\sigma' \in \Sigma(n)$. This implies $\{\rho\} \cup \sigma(1)$ is not contained in any $\sigma'(1)$. We make take $S \subset \sigma(1)$ to be a subset minimal with respect to this property: $\{\rho\} \cup S$ is not contained in any $\sigma'(1)$ for any $\sigma' \in \Sigma(n)$. It follows from the definition of primitive collections that $\{\rho\} \cup S$ is a primitive collection. □

Next we observe that a complete fan is completely determined, or generated by its maximal cones.

Corollary 7.3.5. *Let N be a lattice of rank n and Σ, Σ' be two complete fans in $N_{\mathbf{R}}$. Then $\Sigma = \Sigma'$ if and only if $\Sigma(n) = \Sigma'(n)$.*

Proof. Only one direction is non-trivial. Suppose $\Sigma(n) = \Sigma'(n)$ and let $\sigma \in \Sigma$ be a cone. Then by Lemma 7.3.1, there exists a maximal cone $\tau \in \Sigma(n)$ such that

$$\sigma \preceq \tau \in \Sigma(n) = \Sigma'(n) \subset \Sigma'.$$

Then by definition of a fan, σ is in Σ'. This shows $\Sigma \subset \Sigma'$. The converse follows from exactly the same argument. □

We now prove that up to a rational transformation, a complete fan is determined by its primitive collections and the relations among ray generators.

Proposition 7.3.6. *Let N_1 and N_2 be two free abelian groups of rank n. Let $\Sigma_1 \subset (N_1)_{\mathbf{R}}, \Sigma_2 \subset (N_2)_{\mathbf{R}}$ be two complete fans. Suppose there is an index set L for both $\Sigma_1(1)$ and $\Sigma_2(1)$ such that the following two properties hold:*

1. *a subset $C \subset L$ indexes a primitive collection in $\Sigma_1(1)$ if and only if C indexes a primitive collection in $\Sigma_2(1)$, and*
2. *if $\{u_\rho\}_{\rho \in L}$ are the ray generators in Σ_1 and $\{v_\rho\}_{\rho \in L}$ are the ray generators in Σ_2, a tuple of integers $\{a_\rho\}_{\rho \in L}$ satisfies $\sum_\rho a_\rho u_\rho = 0$ if and only $\{a_\rho\}_{\rho \in L}$ satisfies $\sum_\rho a_\rho v_\rho = 0$.*

Then there is a $\mathbf{Q}$-linear isomorphism $\Phi : (N_1)_{\mathbf{Q}} \to (N_2)_{\mathbf{Q}}$ such that

1. *$\Phi(u_\rho) = v_\rho$ for all $\rho \in L$, and*
2. *if $\Phi_{\mathbf{R}}$ is the extension of Φ over $\mathbf{R}$, then a cone $\sigma_1 \subset (N_1)_{\mathbf{R}}$ is a cone in Σ_1 if and only if $\Phi_{\mathbf{R}}(\sigma_1)$ is a cone in Σ_2.*

Proof. Let $\{u_1, \dots, u_n\}$ be a collection of $\mathbf{Q}$-linearly independent ray generators from Σ_1, which exists as Σ_1 is complete. Let $\{v_1, \dots, v_n\}$ be the corresponding ray generators in Σ_2. Then there is a $\mathbf{Q}$-linear map $\Phi : (N_1)_{\mathbf{Q}} \to (N_2)_{\mathbf{Q}}$ defined by sending u_i to v_i.By the assumption on the relation among ray generators, $\{v_1, \dots, v_n\}$ is independent over $\mathbf{Q}$ as well. The map Φ is therefore an isomorphism. For any other u_ρ, since $\{u_1, \dots, u_n\}$ is a $\mathbf{Q}$-basis of $(N_1)_{\mathbf{Q}}$, there exist a nonzero $K \in \mathbf{Z}$ and integers a_i such that $K \cdot u_\rho = \sum_i a_i u_i$. By the assumption on the relation among ray generators, we also have $K \cdot v_\rho = \sum_i a_i v_i$. Therefore, we get

$$K \cdot \Phi(u_\rho) = \Phi(K \cdot u_\rho) = \sum_i a_i \Phi(u_i) = \sum_i a_i v_i = K \cdot v_\rho.$$

Since M_2 is torsion free, we have $\Phi(u_\rho) = v_\rho$ for all $\rho \in L$. For statement on the cones, let σ_1 be a maximal cone in Σ_1 and $L_1 \subset L$ be the index set for rays in σ_1. Then L_1 and any subset is not a primitive collection. I claim that $\Phi(\sigma_1)$ is a cone in Σ_2 with ray generators $\{v_\rho\}_{\rho \in L}$. For this, since L_1 is the list of ray generators of the cone σ_1, each subset of L_1 (including L_1) does not index a primitive collection in Σ_1(and therefore in Σ_2). Therefore, either

(1) $\{v_\rho\}_{\rho \in L_1}$ is contained the list of ray generators of some cone in Σ_2, or
(2) there is a proper subset $L_2 \subsetneq L_1$ and $\{v_\rho\}_{\rho \in L_2}$ is not contained in the list of ray generators of any cone in Σ_2.

Suppose (1) fails for L_1. As L_2 does not index a primitive collection in Σ_2, the same two conditions can be said about L_2. Since (1) fails for L_2 also, there is a proper subset $L_3 \subsetneq L_2$ such that (1) fails for L_3. This process continues but breaks as any $\rho \in L_1$ indexes a ray in Σ_2. Therefore, (1) holds for L_1. Let σ_2 be a cone in Σ_2 whose list of ray generators contains $\{v_\rho\}_{\rho \in L_1}$. Then clearly $\Phi(\sigma_1) \subset \sigma_2$. I claim that $\Phi(\sigma_1) = \sigma_2$. If this is not the case, then there exists an index $\rho' \notin L_1$ so that $v_{\rho'} \in \sigma_2 - \Phi(\sigma_1)$. Then $u_{\rho'} \notin \sigma_1$. To see this, suppose on the contrary that $u_{\rho'} \in \sigma_1$, then by proposition 11.1.7 in [CLS11], there are non-negative rational numbers q_ρ for $\rho \in L_1$ such that $u_{\rho'} = \sum_{\rho \in L_1} q_\rho u_\rho$. By the assumption on relation among ray generators, we have $v_{\rho'} = \sum_{\rho \in L_1} q_\rho v_\rho$, contradicting the condition that $v_{\rho'}$ is not in $\Phi(\sigma_1)$. Hence $u_{\rho'} \notin \sigma_1$. Since σ_1 is maximal, $\{u_\rho | \rho \in L \cup \{\rho'\}\}$ is not contained in the list of ray generators of any cone in Σ_1. But $L \cup \{\rho'\}$ indexes a subset of ray generators of σ_2. So $L \cup \{\rho'\}$ and any proper subset do not index primitive collection in Σ_1. As before, this will break so $\Phi(\sigma_1) = \sigma_2$. The argument can clearly be reversed so we see that Φ induces a bijection between maximal cones in Σ_1 and Σ_2. The result now follows from Corollary 7.3.5. □

Definition 7.3.7. Let N_1, N_2 be two lattices of the same rank. We say two complete fans $\Sigma_1 \subset (N_1)_{\mathbf{R}}, \Sigma_2 \subset (N_2)_{\mathbf{R}}$ are *rationally equivalent* if there is a bijection between $\Sigma_1(1)$ and $\Sigma_2(1)$ that satisfies the two conditions on primitive collections and relations among ray generators in Proposition 7.3.6.

Corollary 7.3.8. *Let $\Sigma_1 \subset (N_1)_{\mathbf{R}}, \Sigma_2 \subset (N_2)_{\mathbf{R}}$ be two rationally equivalent complete fans with Σ_1 smooth. If $\Phi : (N_1)_{\mathbf{Q}} \to (N_2)_{\mathbf{Q}}$ is the $\mathbf{Q}$-linear isomorphism constructed in Proposition 7.3.6, its restriction to N_1 factors through N_2 as a $\mathbf{Z}$-linear monomorphism $N_1 \hookrightarrow N_2$. Moreover, Σ_2 is simplicial and the following statements are equivalent:*

(1) One of the cones in Σ_2 is smooth.
(2) The restriction of Φ to N_1 factors through N_2 as a $\mathbf{Z}$-isomorphism $N_1 \simeq N_2$.
(3) Σ_2 is smooth.

Proof. Since Σ_1 is smooth and complete, there is a maximal cone $\sigma_1 \in \Sigma_1$ whose rays form a $\mathbf{Z}$-basis for N_1. Hence the restriction of Φ to N_1 factors through N_2. Since $\Phi : (N_1)_{\mathbf{Q}} \to (N_2)_{\mathbf{Q}}$ is an isomorphism, the factorization φ from N_1 to N_2 is injective. That Σ_2 is simplicial follows from the injectivity of φ. It is also clear that φ is an isomorphism if and only if for some (or for all) maximal cone $\sigma_1 \in \Sigma_1$, $\Phi_{\mathbf{R}}(\sigma_1)$ is a smooth cone in Σ_2. The rest of the corollary follows from this observation and that Φ induces a bijection between cones in Σ_1 and cones in Σ_2. □

We formalize two adjunctions, one specifically for the pairings of reductive groups obtained from finitely generated abelian groups and the other for perfect pairings between vector spaces. The first adjunction will not be used in the sequel but is interesting enough to introduce. The second adjunction for vector spaces will be used to prove Theorem 7.3.10.

Let A be a finitely generated abelian group. Suppose the free part of A is of rank r. Let $G := \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{C}^\times)$. Then $G \simeq (\mathbf{C}^\times)^r \times H$ where H is a finite group. Hence G is abelian, reductive and $(\mathbf{C}^\times)^r$ is the unique maximal torus in G. For any $a \in A$, the element a can be viewed as a character χ_a of G by setting

$$\chi_a(g) = g(a) \text{ for } g \in G.$$

Let

$$\langle -, - \rangle : \boldsymbol{\chi}(G) \times \boldsymbol{\Gamma}(G) \to \mathbf{Z}$$

be the natural pairing as before. Then

- the assignment $a \mapsto \chi_a$ induces an isomorphism $A \simeq \boldsymbol{\chi}(G)$,
- the inclusion $\boldsymbol{\Gamma}((\mathbf{C}^\times)^r) \subset \boldsymbol{\Gamma}(G)$ induced by the inclusion $(\mathbf{C}^\times)^r \subset G$ is actually an equality,
- the assignment $\lambda \mapsto \langle -, \lambda \rangle : \boldsymbol{\chi}(G) \to \mathbf{Z}$ induces an isomorphism
$$\boldsymbol{\Gamma}(G) \simeq \boldsymbol{\chi}(G)^\vee,$$
- the pairing $\langle -, - \rangle$ is perfect after extending to $\mathbf{Q}$.

In particular, $\boldsymbol{\Gamma}(G)$ is a free abelian group. Next, we discuss the functorial behavior of the above construction. Let $\psi : A_1 \to A_2$ be a homomorphism of finitely generated abelian groups. Let $G_i := \mathrm{Hom}_{\mathbf{Z}}(A_i, \mathbf{C}^\times)$ and $\psi^* : G_2 \to G_1$ be the group homomorphism induced by ψ. Then for any $a_1 \in A_1$,

$$\psi(\chi_{a_1}) = \chi_{a_1} \circ \psi^*$$

as a character on G_2. Abusing the notation, let us also set $\psi^* : \boldsymbol{\Gamma}(G_2) \to \boldsymbol{\Gamma}(G_1)$ to be the group homomorphism induced by ψ^*. Let

$$\langle -, - \rangle_{G_i} : \boldsymbol{\chi}(G_i) \times \boldsymbol{\Gamma}(G_i) \to \mathbf{Z}$$

be the natural pairings. Then for any $\lambda_2 \in \boldsymbol{\Gamma}(G_2)$ and $a_1 \in A_1$, the following adjunction holds:
(Adjunction Formula)
$$\langle \psi(\chi_{a_1}), \lambda_2 \rangle_2 = (\chi_{a_1} \circ \psi^*) \circ \lambda_2 = \chi_{a_1} \circ (\psi^* \circ \lambda_2) = \langle \chi_{a_1}, \psi^*(\lambda_2) \rangle_1.$$

Therefore, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{\chi}(G_2)^\vee & \xrightarrow{\psi^\vee} & \boldsymbol{\chi}(G_1)^\vee \\ \sim\uparrow & & \sim\uparrow \\ \boldsymbol{\Gamma}(G_2) & \xrightarrow{\psi^*} & \boldsymbol{\Gamma}(G_1) \end{array}.$$

Hence $\psi^* : \boldsymbol{\Gamma}(G_2) \to \boldsymbol{\Gamma}(G_1)$ can also be thought of as the map dual to $\psi : \boldsymbol{\chi}(G_1) \to \boldsymbol{\chi}(G_2)$.

In the case of a toric variety X_Σ, we take $A = \mathrm{Cl}(X_\Sigma)$. However in many cases for two toric varieties X_{Σ_1} and X_{Σ_2}, there might not be a map between their divisor class groups that comes naturally from the structures of their fans, even under the assumption of Proposition 7.3.6. In this case we look at the $\mathbf{Q}$-vector spaces $\mathrm{Cl}(X_{\Sigma_1})_{\mathbf{Q}}$ and $\mathrm{Cl}(X_{\Sigma_2})_{\mathbf{Q}}$. Here is an adjunction in the case of vector spaces whose proof is tautological and omitted.

Lemma 7.3.9. *Let k be a field. Suppose V_i, W_i for $i = 1, 2$ are vector spaces over k with a perfect pairing $\langle -, - \rangle_i : V_i \times W_i \to k$ that identifies V_i and W_i as duals respectively. Then for any k-linear map $f : W_1 \to W_2$ and $v \in V_2$, $w \in W_1$, we have*

$$\langle v_2, f(w) \rangle_2 = \langle f^*(v_2), w \rangle_1.$$

Theorem 7.3.10. *Suppose Σ_1 and Σ_2 are two complete fans that are rationally equivalent. Then there is a bijection of IIT-equivalence classes between* $\mathrm{Amp}(X_{\Sigma_1})$ *and* $\mathrm{Amp}(X_{\Sigma_2})$.

Proof. Let us write $\Sigma(1)$ as the indexing set for the rays in both Σ_1 and Σ_2. Let $\Phi^* : (M_2)_{\mathbf{Q}} \to (M_1)_{\mathbf{Q}}$ be the dual of the isomorphism $\Phi : (N_1)_{\mathbf{Q}} \to (N_2)_{\mathbf{Q}}$

constructed in Proposition 7.3.6. If u_ρ is the ray generator for a ray $\rho \in \Sigma_1$, we write v_ρ as $\Phi(u_\rho)$. Then we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (M_2)_{\mathbf{Q}} & \longrightarrow & \mathbf{Q}^{\Sigma(1)} & \longrightarrow & (\mathrm{Cl}(X_{\Sigma_2}))_{\mathbf{Q}} & \longrightarrow & 0 \\ & & \downarrow_{\Phi^*} & & \| & & \downarrow_{\psi} & & \\ 0 & \longrightarrow & (M_1)_{\mathbf{Q}} & \longrightarrow & \mathbf{Q}^{\Sigma(1)} & \longrightarrow & (\mathrm{Cl}(X_{\Sigma_1}))_{\mathbf{Q}} & \longrightarrow & 0 \end{array}$$

where ψ sends a class of $\mathbf{Q}$-Weil divisor $\sum_\rho q_\rho D_\rho$ in $(\mathrm{Cl}(X_{\Sigma_2}))_{\mathbf{Q}}$ to its class in $(\mathrm{Cl}(X_{\Sigma_1}))_{\mathbf{Q}}$. I first claim that ψ restricts to an isomorphism $\mathrm{Pic}(X_{\Sigma_2})_{\mathbf{Q}} \simeq \mathrm{Pic}(X_{\Sigma_1})_{\mathbf{Q}}$ with $\psi(\mathrm{Amp}(X_{\Sigma_2})_{\mathbf{Q}}) = \mathrm{Amp}(X_{\Sigma_1})_{\mathbf{Q}}$. For this, let $D = \sum_\rho q_\rho D_\rho$ be a nonzero $\mathbf{Q}$-Cartier divisor on X_{Σ_2}. We need to construct an integer $K > 0$ such that $\psi(K \cdot D)$ is Cartier. For this, first let us choose an integer $K' > 0$ such that

(1) $K' \cdot q_\rho \in \mathbf{Z}$ for all ρ, and
(2) $K' \cdot D$ is Cartier on X_{Σ_2}.

Let σ_1 be a maximal cone in Σ_1. Then by Proposition 7.3.6, $\Phi(\sigma_1)$ is a maximal cone in Σ_2. Let $\langle -, - \rangle_i : (M_i)_{\mathbf{Q}} \times (N_i)_{\mathbf{Q}} \to \mathbf{Q}$ be the natural perfect pairing. Then by Theorem 7.2.4, there exists an $m_{\Phi(\sigma_1)} \in M_2$ such that $\langle m_{\Phi(\sigma_1)}, v_\rho \rangle_2 = -K' \cdot q_\rho$ for all $\rho \in \sigma_1(1)$. By the adjunction, Lemma 7.3.9,

$$\langle \Phi^*(m_{\Phi(\sigma_1)}), u_\rho \rangle_1 = \langle m_{\Phi(\sigma_1)}, v_\rho \rangle_2 = -K \cdot q_\rho \text{ for all } \rho \in \sigma(1).$$

Now choose an integer $K'' > 0$ such that $\Phi^*(K'' \cdot m_{\Phi(\sigma_1)}) \in M_1$. Let $K = K'' \cdot K'$. Then by Theorem 7.2.4, the divisor$\psi(K \cdot D)$ is Cartier. The argument can be reversed so ψ restricts to an isomorphism $\mathrm{Pic}(X_{\Sigma_2})_{\mathbf{Q}} \to \mathrm{Pic}(X_{\Sigma_1})_{\mathbf{Q}}$. The statement on ample cones can be proved similarly. For the behavior of stratifications, note that since Σ_1 and Σ_2 have the same set of primitive collections, the bad locus in their quotient constructions are the same. We will denote the bad locus as $Z(\Sigma)$. By 7.5, the dual of the earlier diagram on short exact sequences is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbf{\Gamma}(G_1))_{\mathbf{Q}} & \longrightarrow & \mathbf{Q}^{\Sigma(1)} & \longrightarrow & (N_1)_{\mathbf{Q}} & \longrightarrow & 0 \\ & & \downarrow_{\psi^*} & & \| & & \downarrow_{\Phi} & & \\ 0 & \longrightarrow & (\mathbf{\Gamma}(G_2))_{\mathbf{Q}} & \longrightarrow & \mathbf{Q}^{\Sigma(1)} & \longrightarrow & (N_2)_{\mathbf{Q}} & \longrightarrow & 0 \end{array} .$$

Let $||-||_i$ be the norm on $\mathbf{\Gamma}(G_i)$, extended over $\mathbf{Q}$. Recall that the norms on $\mathbf{\Gamma}(G_i)$ comes from restricting the standard norm on $\mathbf{\Gamma}((C^\times)^{\Sigma(1)})$. Therefore, for every $\lambda \in (\mathbf{\Gamma}(G_1))_{\mathbf{Q}}$, we have $||\lambda||_1 = ||\psi^*(\lambda)||_2$. Because of this and Lemma 7.3.9, we get

$$\text{(7.6)} \quad \frac{\langle \psi(\chi_D), \lambda \rangle_{G_1}}{||\lambda||_1} = \frac{\langle \chi_D, \psi^*(\lambda) \rangle_{G_2}}{||\psi^*(\lambda)||_2} \text{ for all } \chi_D \in \mathrm{Cl}(X_{\Sigma_2})_{\mathbf{Q}} \text{ and } \lambda \in (\mathbf{\Gamma}(G_1))_{\mathbf{Q}}$$

where $\langle -, - \rangle_{G_i} : \boldsymbol{\chi}(G_i)_{\mathbf{Q}} \times \mathbf{\Gamma}(G_i)_{\mathbf{Q}} \to \mathbf{Q}$ denotes the $\mathbf{Q}$-extension of the natural pairing. By Corollary 7.2.5, $\mathrm{Pic}(X_{\Sigma_i})$ are lattices. For any $\mathbf{Q}$-Cartier divisor D on X_{Σ_i}, we let $[D]$ to be the indivisible Cartier divisor $(\mathbf{Q}_{>0} \cdot D) \cap \mathrm{Pic}(X_{\Sigma_i})$. Similarly for any element $\lambda \in (\mathbf{\Gamma}(G_i))_{\mathbf{Q}}$, we let $[\lambda]$ be the indivisible one parameter subgroup $(\mathbf{Q}_{>0} \cdot \lambda) \cap \mathbf{\Gamma}(G_i)$. With Equation (7.6), we see that for any $x \in Z(\Sigma)$, we have

$$[\Phi(\lambda)] = \Lambda^{[\chi_D]}(x) \Leftrightarrow [\lambda] = \Lambda^{[\psi^*(\chi_D)]}(x) \text{ for all } \chi_D \in \mathrm{Amp}(X_{\Sigma_2})_{\mathbf{Q}}, \lambda \in \mathbf{\Gamma}(G_1)_{\mathbf{Q}}.$$

Moreover, for an element $\chi_D \in \mathrm{Amp}(X_{\Sigma_2})$, if $\lambda, \lambda' \in \mathbf{\Gamma}(G_1)$ are two elements in $\Lambda^{[\psi^*(\chi_D)]}$, then $\lambda > \lambda'$ if and only if $[\Phi(\lambda)] > [\Phi(\lambda')]$ in $\Lambda^{[\chi_D]}$. Hence $[D]$ and $[\psi(D)]$ induces the same stratification. In particular, two characters D_1 and D_2 are IIT-equivalent in $\mathrm{Amp}(X_{\Sigma_1})$ if and only if $[\psi(D_1)]$ and $[\psi(D_2)]$ are IIT-equivalent in $\mathrm{Amp}(X_{\Sigma_2})$. Namely, ψ induces a bijection of IIT-equivalence classes between $\mathrm{Amp}(X_{\Sigma_1})$ and $\mathrm{Amp}(X_{\Sigma_2})$. $\square$

Remark 7.3.11. If in addition one assumes Σ_1 is smooth, then $\Phi : (N_1)_{\mathbf{Q}} \to (N_2)_{\mathbf{Q}}$ factors through $\Phi : N_1 \to N_2$. Hence $\Phi^* : M_2 \to M_1$ induces a morphism $\psi : \mathrm{Cl}(X_{\Sigma_2}) \to \mathrm{Cl}(X_{\Sigma_1})$. Let $f : X_{\Sigma_1} \to X_{\Sigma_2}$ be the toric morphism induced by Φ. The restriction $\psi : \mathrm{Cl}(X_{\Sigma_2}) \to \mathrm{Cl}(X_{\Sigma_1})$ to $\mathrm{Pic}(X_{\Sigma_2})$ factors through $\mathrm{Pic}(X_{\Sigma_1})$ and is just the pull back $f^* : \mathrm{Pic}(X_{\Sigma_2}) \to \mathrm{Pic}(X_{\Sigma_1})$. Moreover,

$$\psi \text{ is a } \mathbf{Z}\text{-isomorphism } \Leftrightarrow \Phi^* \text{ is a } \mathbf{Z}\text{-isomorphism } \Leftrightarrow \Sigma_2 \text{ is smooth.}$$

In the rest of the section, we describe a decomposition of the ample cone of a projective toric variety by semi-chambers and critical subsets analogous to Section 5.3. We then state Theorem 7.3.16 which says the stratification stays constant in a semi-chamber. The proof is similar to Theorem 5.4.4. For this reason we skip the proof for Theorem 7.3.16. First, due to Proposition 7.1.1, we do not distinguish between χ and χ_T. Second, we had $Z(\Sigma) = \cup_C V(x_\rho | \rho \in C)$ where the union is taken over primitive collections C. Hence if x is unstable with respect to some ample divisor, the state $S_x \subset \Sigma(1) - C$ for some primitive collection C. Therefore, unlike Section 5, we only compare subsets of $\Sigma(1)$ that are contained in the complement of some primitive collection. Because of this, we define $\mathcal{L}$ to be the union of power sets of $\Sigma(1) - C$ over all primitive collections C.

Definition 7.3.12. Suppose X_Σ is a projective toric variety. Let $\emptyset \neq H \subsetneq \mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$ be a proper subset. Define We say H is a *type one critical subset for* X_Σ if there are subsets $S_1, S_2 \in \mathcal{L}$ such that

(1) there is a containment, say $W_{S_2} \subset W_{S_1}$ and W_{S_2} is a codimension 1 subspace of W_{S_1}, and
(2) $H = \{\chi \in \mathrm{Amp}(X_\Sigma)_{\mathbf{R}} \mid \mathrm{Proj}_{W_{S_1}} \chi^* = \mathrm{Proj}_{W_{S_2}} \chi^*\}$.

We say H is a *type two critical subset for* X_Σ if there are two subsets $S_1, S_2 \in \mathcal{L}$ such that

(1) there is no containment between W_{S_1} and W_{S_2}, and
(2) $H = \{\chi \in \mathrm{Amp}(X_\Sigma)_{\mathbf{R}} \mid ||\,\mathrm{Proj}_{W_{S_1}} \chi^*|| = ||\,\mathrm{Proj}_{W_{S_2}} \chi^*||\}$.

Analogous to Proposition 5.3.8, we have

Proposition 7.3.13. *Let X_Σ be a projective toric variety.*

(1) A type one critical subset for X_Σ is the intersection of a codimension 1 subspace of $\mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$ with $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$.
(2) A type two critical subset for X_Σ is either the intersection of a codimension at least 1 subspace of $\mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$ with $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$ or the intersection of a regular codimension 1 submanifold of $\mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$ away from a subspace of codimension at least 2 with $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$.
(3) If X_Σ is simplicial, then a type two critical subset for X_Σ is the intersection of a regular codimension 1 submanifold of $\mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$ away from a subspace of codimension 2 with $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$.

Moreover, the defining equations of critical subsets satisfy the following rationality property:

- *the condition that a point $\sum_\rho a_\rho D_\rho \in \mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$ is in a type one critical subset corresponds to a linear equation of a_ρ's with rational coefficient,*

- *the condition that a point* $\sum_\rho a_\rho D_\rho \in \mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$ *is in a type two wall corresponds to a homogeneous quadratic equation of* a_ρ*'s with rational coefficients.*

Proof. Due to the similarity to Proposition 5.3.8, we only prove the statement on critical subset of type two when X_Σ is simplicial. First we have

$$\mathrm{Pic}(X_\Sigma)_{\mathbf{R}} \simeq \mathrm{Cl}(X_\Sigma)_{\mathbf{R}} \simeq \boldsymbol{\chi}(G)_{\mathbf{R}}$$

by Proposition 8.1.3. Then as $\boldsymbol{\chi}(G)_{\mathbf{R}} \simeq \boldsymbol{\chi}(T)_{\mathbf{R}}$ where $T \subset G$ is the unique maximal torus. The result follows from Proposition 5.3.8. □

Definition 7.3.14. Let X_Σ be a projective toric variety. Let $\{F_i\}$ be the defining equations for critical subsets in $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$. A *semi-chamber* is a non-empty open semi-algebraic set of the form

$$\{\chi \in \mathrm{Amp}(X_\Sigma)_{\mathbf{R}} \mid \pm F_i(\chi) > 0 \text{ for all } i\}.$$

Proposition 7.3.15. *Let* X_Σ *be a projective toric variety. There are finitely many critical subsets and semi-chambers for* X_Σ*.*

The following result can be proved using similar arguments from Theorem 5.4.4.

Theorem 7.3.16. *Let* X_Σ *be a projective toric variety. Inside each semi-chamber for* X_Σ*, the stratification of* $Z(\Sigma)$ *does not change.*

7.4. **The structure of strata.** In this section we supply some topological properties of the strata.

Theorem 7.4.1. *Let* X_Σ *be a projective toric variety,* $D \in Cl(X_\Sigma)$ *be an ample divisor and* G *be the group in the quotient construction for* X_Σ*. Then for every point* $x \in Z(\Sigma)$*, there is a unique indivisible one parameter subgroup* $\lambda \in \boldsymbol{\Gamma}(G)$ *that is* χ_D*-adapted to* x*. For each* $\lambda \in \Lambda^{\chi_D}$*, the strata* S_λ *is smooth, irreducible and open inside its closure* $\overline{S_\lambda}$*. More precisely,*

$$S_\lambda = V(x_\rho | \rho \in S) - \cup_R V(x_\rho | \rho \in R)$$

where S *is a subset of* $\Sigma(1)$ *containing some primitive collection and* R *runs through some collection of subsets of* $\Sigma(1) - S$*.*

Proof. The group G acts on $X = \mathbf{A}_{\mathbf{C}}^{\Sigma(1)}$ by restricting the action of $(\mathbf{C}^\times)^{\Sigma(1)}$ to G. Since G is abelian, Theorem 3.4.4 implies that for every $x \in Z(\Sigma)$, there exists a unique indivisible one parameter subgroup of G that is χ_D-adapted to x. Let λ be an element in Λ^{χ_D}. By Theorem 3.4.4, there exists a unique minimal subset S of $\Sigma(1)$ such that $L(\Sigma(1) - S) \subset S_\lambda$. Since $L(\Sigma(1) - S) \subset S_\lambda \subset Z(\Sigma)$, we have $\overline{L(\Sigma(1) - S)} = V(x_\rho | \rho \in S) \subset \overline{S_\lambda} \subset Z(\Sigma)$. Since $V(x_\rho|\rho \in S)$ is irreducible, by Proposition 7.2.2, there exists a primitive collection $C \subset \Sigma(1)$ such that $V(x_\rho|\rho \in S) \subset V(x_\rho|\rho \in C)$. Then S contains C. Now the minimal subsets A_j of $\Sigma(1)$ such that $L(A_j) \subset S_\lambda$ is contained in $\Sigma(1) - S$. The result follows from Theorem 3.4.4. □

Corollary 7.4.2. *Let* D *be an ample divisor of the projective toric variety* X_Σ*. Then for every primitive collection* $C \subset \Sigma(1)$ *, there is a unique* $\lambda_C \in \Lambda^{\chi_D}$ *such that* $\overline{S_{\lambda_C}} = V(x_\rho|\rho \in C)$*.*

Proof. By Proposition 7.2.2, each primitive collection C corresponds to the irreducible component $V(x_\rho|\rho \in C)$ of $Z(\Sigma)$. We also have $Z(\Sigma) = \cup_{\lambda \in \Lambda^{\chi_D}} S_\lambda = \cup_{\lambda \in \Lambda^{\chi_D}} \overline{S_\lambda}$. By Theorem 7.4.1, each $\overline{S_\lambda}$ is irreducible. Hence the maximal $\overline{S_\lambda}$'s correspond to the irreducible components of $Z(\Sigma)$. Namely, for each primitive collection C, there is a $\lambda_C \in \Lambda^{\chi_D}$ such that

$$\overline{S_{\lambda_C}} = V(x_\rho \mid \rho \in C).$$

The uniqueness of λ_C follows from the fact that the boundary of a stratum only contains higher strata. □

We can also derive the

Corollary 7.4.3. *For every ample divisor D on X_Σ and every primitive collection $C \subset \Sigma(1)$, let λ_C be as in Corollary 7.4.2. We have*

$$\{0\} = \bigcap_{\lambda \in \Lambda^{\chi_D}} \overline{S_\lambda} = \bigcap_{primitive\ collection\ C} \overline{S_{\lambda_C}}$$

where $\{0\}$ stands for the origin in $\mathbf{A}_{\mathbf{C}}^{\Sigma(1)}$.

Proof. By Corollary 7.3.4, the intersection $\bigcap_C V(x_\rho|\rho \in C)$ taken over all primitive collections C of $\Sigma(1)$ is the origin. By Corollary 7.4.2, we have $\bigcap_C \overline{S_{\lambda_C}} = \bigcap_C V(x_\rho|\rho \in C)$. By Theorem 7.4.1, each $\overline{S_\lambda}$ is a linear subspace so $\{0\} \in \bigcap_{\lambda \in \Lambda^{\chi_D}} \overline{S_\lambda}$. In sum, we have

$$\{0\} \in \bigcap_{\lambda \in \Lambda_D^{\chi}} \overline{S_\lambda} \subset \bigcap_C \overline{S_{\lambda_C}} = \bigcap_C V(x_\rho|\rho \in C) = \{0\}.$$

The corollary is proved. □

8. The computer program

Given a projective toric variety, the computer program works out

(1) its ample cone (Section 8.1, Section 8.2),
(2) potential one parameter subgroups that index the strata of the unstable locus in the GIT quotient construction (Section 8.3),
(3) critical subsets in the ample cone (Section 8.4),
(4) stratification of the unstable locus with respect to a particular ample divisor (Section 8.5), and
(5) visualization of the critical subset and semi-chamber decomposition of ample cone if the ample cone is less than three dimensional (Section 8.6).

The computer program is solely sageMath where a number of packages for toric varieties are already available, such as constructing fans and enumerating their primitive collections. It is worthwhile pointing out that for a projective toric variety X_Σ, the program works with any divisors $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ without referencing to any basis of $\mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$. This is due to the following two reasons. First, computing the basis of the Picard group can be challenging. Second, the output has more flexibility. Users may specialize the output to any basis they have. We will formulate a class of projective toric varieties that the program works with whose basis of the Picard group is easy to describe in Section 8.1. In the last section, Section 8.7, we supply a few examples and counter examples produced by the computer program. Among them Example 8.7.3 is the richest one.

We will walk through the computer program with the easiest non-trivial example from the projective toric variety $\mathbf{P}_{\mathbf{C}}^1 \times \mathbf{P}_{\mathbf{C}}^1$. Let Σ_{ex} be the fan in $\mathbf{R}^2$ consisting of the four rays

(1) $\rho_0 = \text{Cone}\ (e_1)$,

(2) $\rho_1 = \text{Cone}\,(-e_1)$,
(3) $\rho_2 = \text{Cone}\,(e_2)$,
(4) $\rho_3 = \text{Cone}\,(-e_2)$,

and the four maximal cones

(1) Cone (ρ_0, ρ_2),
(2) Cone (ρ_1, ρ_2),
(3) Cone (ρ_1, ρ_3), and
(4) Cone (ρ_0, ρ_3).

Then $X_{\Sigma_{\text{ex}}} \simeq \mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$.

At the start of the program, the user is asked to enter a fan as a list of the coordinates of rays, followed by list of ambient indicies for each maximal cone in the fan. In the case of Σ_{ex}, the list of rays is $(1,0), (-1,0), (0,1), (0,-1)$ and the list of ambient indicies is $(0,2), (1,2), (1,3), (0,3)$.

8.1. **Computing the Picard group.** This section serves as the preparation for computing the ample cones. Although there are computer programs in SageMath and Macaulay2 that compute ample cones, the output is difficult to utilize internally in my program. Hence it becomes more desirable that I design the ample cone computation by myself. Since ample divisors are by definition Cartier, to compute the ample cone of a toric variety, we must know what divisors are Cartier first. However, even with concrete combinatorial data that comes with a toric variety, computing the basis of the Picard group of a toric variety can be a challenging task. Again SageMath and Macualay2 do compute the basis of the Picard group, but their output is not useful for my purpose. In view of this, we restrict to a certain class of projective toric varieties whose Picard group and ample cones are easier to compute. We consider projective toric varieties X_Σ whose Picard groups satisfy

$$\text{Pic}(X_\Sigma)_{\mathbf{R}} \simeq \bigoplus_{\rho \notin \sigma(1)} \mathbf{R} \cdot D_\rho \text{ for some maximal cone } \sigma \in \Sigma. \tag{8.1}$$

The computer program does not detect if condition (8.1) holds when a fan is given. The results may not be correct if condition (8.1) fails. However, condition (8.1) includes all simplicial projective toric varieties as we will see in Corollary 8.1.4. Users may use the characterization, Corollary 8.1.2 to determine if their X_Σ satisfies condition (8.1). First, we need

Proposition 8.1.1. *Let Σ be a fan and σ_0 be a full dimensional cone in Σ. Then the natural map $\text{CDiv}_{T_N}(X_\Sigma) \to \text{Pic}(X_\Sigma)$ induces an isomorphism*

$$\varphi : \{D = \sum_\rho a_\rho D_\rho \in \text{CDiv}_{T_N}(X_\Sigma) \mid a_\rho = 0 \textit{ for all } \rho \in \sigma_0(1)\} \simeq \text{Pic}(X_\Sigma).$$

Proof. Suppose $D = \sum_\rho a_\rho D_\rho \in \text{CDiv}_{T_N}(X_\Sigma)$ is Cartier. There exists $m_{\sigma_0} \in M$ as in Theorem 7.2.4. Then $D \sim D - \text{div}(\chi^{m_{\sigma_0}})$ in $\text{Pic}(X_\Sigma)$ and the later is in the image of φ. This shows φ is surjective. To prove that φ is injective, suppose $\varphi(D) = 0$ where $D = \sum_{\rho \notin \sigma_0(1)} a_\rho D_\rho$. Then by the short exact sequence (7.3), there exists an $m \in M$ such that $D = \text{div}(\chi^m)$. In particular, $\langle m, u_\rho \rangle = 0$ for all $\rho \in \sigma_0(1)$. Since σ_0 is full dimensional, $m = 0$, This proves injectivity. □

Here is a characterization about when (8.1) holds:

Corollary 8.1.2. *Let X_Σ be a toric variety and let $\sigma_0 \in \Sigma$ be a full dimensional cone. Then the following statements are equivalent:*

(1) *The inclusion* $\mathrm{Pic}(X_\Sigma) \subset \mathbf{Z}^{\Sigma(1)}$ *induces* $\mathrm{Pic}(X_\Sigma)_{\mathbf{R}} \simeq \bigoplus_{\rho\notin\sigma_0(1)} \mathbf{R}\cdot D_\rho$,
(2) *the inclusion* $\mathrm{Pic}(X_\Sigma) \subset \mathbf{Z}^{\Sigma(1)}$ *induces* $\mathrm{Pic}(X_\Sigma)_{\mathbf{Q}} \simeq \bigoplus_{\rho\notin\sigma_0(1)} \mathbf{Q}\cdot D_\rho$, *and*
(3) D_ρ *is* $\mathbf{Q}$*-Cartier for all* $\rho \notin \sigma_0(1)$.

Proof. There is an inclusion $\mathrm{Pic}(X_\Sigma) \subset \mathbf{Z}^{\Sigma(1)}$ due to Proposition 8.1.1. Statement (1) and (2) are equivalent because $\mathbf{R}$ is faithfully flat over $\mathbf{Q}$. The equivalence of (2) and (3) follows from Proposition 8.1.1. □

To see why the class of toric varieties that satisfies (8.1) includes projective simplicial toric varieties, we need the

Proposition 8.1.3. *Let* X_Σ *be a toric variety. The following statements are equivalent:*

(1) X_Σ *is simplicial.*
(2) *Every* $D \in \mathrm{Cl}(X_\Sigma)$ *is* $\mathbf{Q}$*-Cartier. Namely, for every* $D \in \mathrm{Cl}(X_\Sigma)$*, there exists an integer* K *such that* $K\cdot D \in \mathrm{Pic}(X_\Sigma)$.
(3) $\mathrm{Pic}(X_\Sigma)$ *is of finite index in* $\mathrm{Cl}(X_\Sigma)$*. Namely, there exists an integer* K *such that* $K\cdot D \in \mathrm{Pic}(X_\Sigma)$ *for all* $D \in \mathrm{Cl}(X_\Sigma)$.
(4) *The inclusion* $\mathrm{Pic}(X_\Sigma) \subset \mathrm{Cl}(X_\Sigma)$ *induces* $\mathrm{Pic}(X_\Sigma)_{\mathbf{Q}} \simeq \mathrm{Cl}(X_\Sigma)_{\mathbf{Q}}$.

Proof. Statements (2),(3),(4) are equivalent because $\mathrm{Cl}(X_\Sigma)$ is finitely generated and tensoring $\mathbf{Q}$ is flat and kills torsion. The only non-trivial part is to prove (2) is equivalent to (1). For this, let σ be a cone in Σ. Viewing σ itself as a fan, we have the following short exact sequence

$$M \longrightarrow \mathbf{Z}^{\sigma(1)} \longrightarrow \mathrm{Cl}(X_\sigma) \longrightarrow 0$$

whose dual is

$$0 \longrightarrow \mathrm{Cl}(X_\sigma)^\vee \longrightarrow \mathbf{Z}^{\sigma(1)} \xrightarrow{\psi_\sigma} N$$

by (7.5). We see that X_Σ is simplicial if and only if ψ_σ is injective. This is equivalent to requiring that $\mathrm{Cl}(X_\sigma)^\vee = 0$, which in turn is equivalent to $\mathrm{Cl}(X_\sigma)$ is torsion. Because there are only finitely many σ and $\{X_\sigma\}_{\sigma\in\Sigma}$ covers X_Σ, (1) implies (2). For the converse, it is enough to show that $\mathrm{Cl}(X_\Sigma)$ is torsion for all $\sigma \in \Sigma$. Since the restriction $\mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(X_\sigma)$ is surjective, we may write a divisor in $\mathrm{Cl}(X_\sigma)$ as $D|_{X_\sigma}$ for some divisor D on X_Σ. By assumption, there exists a $K > 0$ such that $K\cdot D$ is Cartier. Then $K\cdot D|_{X_\sigma}$ is also Cartier. The result follows from the fact that affine normal toric varieties does not have non-trivial Picard group. See proposition 4.2.2 in [CLS11]. □

Corollary 8.1.4. *Let* Σ *be simplicial and* $\sigma_0 \in \Sigma$ *be a full dimensional cone. We have that* $\mathrm{Pic}(X_\Sigma)_{\mathbf{Q}} \simeq \mathrm{Cl}(X_\Sigma)_{\mathbf{Q}}$ *has* $\mathbf{Q}$*-basis* $\{D_\rho\}_{\rho\notin\sigma_0(1)}$*. If* Σ *is smooth,* $\mathrm{Pic}(X_\Sigma) \simeq \mathrm{Cl}(X_\Sigma)$ *is free with* $\mathbf{Z}$*-basis* $\{D_\rho\}_{\rho\notin\sigma_0(1)}$.

Proof. By Proposition 8.1.3, every D_ρ is $\mathbf{Q}$-Cartier. Using statement (3) from Corollary 8.1.2, we get the result. □

Hence we get that the class of toric varieties that satisfies (8.1) includes simplicial projective toric varieties. For the fan Σ_{ex}, condition (8.1) is satisfied for the cone $\mathrm{Cone}(\rho_1, \rho_3)$. For brevity, for a ray ρ_i, instead of writing D_{ρ_i} (resp. a_{ρ_i}), we write D_i (resp. a_i). With these notations, $\mathrm{Pic}(X_{\Sigma_{\mathrm{ex}}})_{\mathbf{R}}$ has basis D_0, D_2. In fact, $D_0 = \mathscr{O}(1,0)$ and $D_2 = \mathscr{O}(0,1)$ on $\mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$ so that $a_0D_0 + a_2D_2 = \mathscr{O}(a_0, a_2)$.

8.2. **Computing the ample cone.** Recall that $\Sigma_{\max}$ means the collection of maximal cones in Σ. The logic behind the computation of ample cones is mainly the following:

Proposition 8.2.1. *Let Σ be a complete fan and $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor. For every $\sigma \in \Sigma_{max}$ and every $\rho' \notin \sigma(1)$, fix a relation $\sum_{\rho\in\sigma(1)} b_\rho u_\rho = u_{\rho'}$ with $b_\rho \in \mathbf{Q}$ for all $\rho \in \sigma(1)$. Then D is ample if and only if $\sum_{\rho\in\sigma(1)} b_\rho a_\rho < a_{\rho'}$ for all $\sigma \in \Sigma_{max}$ and $\rho' \notin \sigma(1)$.*

Proof. Let Σ be a fan in $N_{\mathbf{R}}$ and M be the dual lattice of M. Since Σ is complete, all maximal cones in Σ are full dimensional by Corollary 7.3.3. Hence we can solve for rational relations for every ray outside of a maximal cone. Fix a relation (not necessarily unique) $\sum_{\rho\in\sigma(1)} b_\rho u_\rho = u_{\rho'}$ for each maximal cone σ and each $\rho' \notin \sigma(1)$. Since D is Cartier, for every $\sigma \in \Sigma_{\max}$, there exists an unique $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho\rangle = -a_\rho$ for all $\rho \in \sigma(1)$ by Theorem 7.2.4. In order for D to be ample, we also need $\langle m_\sigma, u_{\rho'}\rangle > -a_{\rho'}$ for all $\rho' \notin \sigma(1)$. But

$$\langle m_\sigma, u_{\rho'}\rangle = \sum_{\rho\in\sigma(1)} \langle m_\sigma, b_\rho u_\rho\rangle = -\sum_{\rho\in\sigma(1)} a_\rho b_\rho.$$

The result follows. □

Therefore, computing the ample cone breaks down into the following steps:

(1) Loop through each maximal cone σ in the fan and solves relations $u_{\rho'} = \sum_{\rho\in\sigma(1)} b_\rho u_\rho$ for each $\rho' \notin \sigma(1)$.

(2) Construct a polyhedral cone $P \subset \mathbf{R}^{\Sigma(1)}$ defined by the list of inequalities

$$\{a_{\rho'} - \sum_{\rho\in\sigma(1)} b_\rho a_\rho > 0\}_{\sigma \text{ max},\ \rho'\notin\sigma(1)}.$$

(3) Return a non-redundant list, call it L_{gen}, of normal vectors of the supporting hyperplanes of P.

Now we know when a Cartier divisor $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ is ample. Next,

(4) Intersect P with $\text{Pic}(X_\Sigma)_{\mathbf{R}}$.

The *gen* in step (3) stands for generic as we look at divisors $\sum_\rho a_\rho D_\rho$ without restricting to any basis of $\text{Pic}(X_\Sigma)_{\mathbf{R}}$. To go from step (3) to step(4), if condition (8.1) is satisfied for a maximal cone σ, then it is simply setting the ρ-th component of vectors in L_{gen} to be 0 for $\rho \in \sigma(1)$. In the program we simply remove the ρ-th component of the vectors in L_{gen} for $\rho \in \sigma(1)$. We now have a reduced list of normal vectors. Call it L_{red}. The vectors in L_{red} can be either thought of as coefficients of strict inequalities for a divisor $\sum_{\rho\notin\sigma(1)} a_\rho D_\rho$ to be ample, or as the normal vectors of the supporting hyperplanes for $\text{Nef}(X_\Sigma)$ by Corollary 7.2.9.

Take Σ_{ex} for example, the fan is smooth so all divisors are Cartier. The only relations we have are

$$u_{\rho_0} + u_{\rho_1} = 0 \text{ and } u_{\rho_2} + u_{\rho_3} = 0.$$

Hence the divisor $\sum_{i=0}^3 a_i D_i$ is ample if and only if

$$a_0 + a_1 > 0 \text{ and } a_2 + a_3 > 0.$$

The output of L_{gen} is the list $(1,1,0,0),(0,0,1,1)$ of normal vectors. We also know that $D_0 = \mathscr{O}(1,0)$ and $D_2 = \mathscr{O}(0,1)$ form a basis for $\text{Pic}(X_\Sigma)$ so (8.1) is satisfied for the cone $\sigma = \text{Cone}\ (\rho_1, \rho_3)$. In this case $L_{\text{red}} = (1,0),(0,1)$, corresponding to the fact that $a_0D_0 + a_2D_2$ is ample if and only if $a_0, a_2 > 0$.

8.3. **Potential one parameter subgroups.** Let us look at the second task of the program: computing potential one parameter subgroups that index the χ_D-strata for ample divisors D. This is the core part of the program as its output would be used internally to compute stratifications, enumerate critical subsets and finally, plot the ample cone decomposition by critical subsets.

Recall that we defined $\mathcal{L}$ to be the union of the power sets of $\Sigma(1) - C$ over all primitive collections C. For any $S \in \mathcal{L}$, we have the subspace

$$W_S := \{\lambda \in \mathbf{\Gamma}(G)_{\mathbf{R}} \mid \langle \chi_{D_\rho}, \lambda \rangle = 0 \text{ for } \rho \in S\}.$$

Remark 8.3.1. The description of W_S is rather simple: A point $v \in \mathbf{R}^{\Sigma(1)}$ is in W_S if and only if $v \in \mathbf{\Gamma}(G)_{\mathbf{R}}$ and $v_\rho = 0$ for all $\rho \in S$.

The goal is to compute $-\operatorname{Proj}_{W_S} \chi_D^* \in \mathbf{\Gamma}(G)_{\mathbf{R}}$ for all $S \in \mathcal{L}$ and for all $D \in \operatorname{Amp}(X_\Sigma)_{\mathbf{R}}$ where $W_S \neq \{0\}$. This breaks down roughly into the following steps:

(1) Enumerate primitive collections of the fan Σ, and compute the set $\mathcal{L}$.
(2) Compute a rational basis for W_S and its complement $W_S^\perp$ for each $S \in \mathcal{L}$.
(3) Compute $-\operatorname{Proj}_{W_S} \chi_D^*$ for every $D = \sum_\rho a_\rho D_\rho \in \mathbf{R}^{\Sigma(1)}$.

Each output $-\operatorname{Proj}_{W_S} \chi_D^*$ is a vector in $\mathbf{R}^{\Sigma(1)}$ whose components are $\mathbf{Q}$-combination of the a_ρ's.

(4) Then the user may specialize the results to the ample cone by specializing to $\operatorname{Pic}(X_\Sigma)_{\mathbf{R}}$ first. For this,
    (a) identify a cone σ for which condition (8.1) is satisfied, then
    (b) set $a_\rho = 0$ for all $\rho \in \sigma(1)$ in the output of $-\operatorname{Proj}_{W_S} \chi_D^*$.
(5) Then use the ample cone description obtained earlier.

Let us see how this is carried out. For step (1), the primitive collections can be obtained by the .primitive_collections() method applied to the fan. The set $\mathcal{L}$ can be obtained by elementary set operations in SageMath.

For step (2), there is an exact sequence obtained from (7.2)

$$0 \longrightarrow \mathbf{Q}^{\dim \Sigma} \xrightarrow{M} \mathbf{Q}^{\Sigma(1)} \xrightarrow{M^\perp} \operatorname{Cl}(X_\Sigma)_{\mathbf{Q}} \longrightarrow 0$$

whose dual is

$$0 \longrightarrow \mathbf{\Gamma}(G)_{\mathbf{Q}} \xrightarrow{(M^\perp)^t} \mathbf{Q}^{\Sigma(1)} \xrightarrow{M^t} \mathbf{Q}^{\dim \Sigma} \longrightarrow 0\ . \tag{8.2}$$

In the computer program M is represented by a matrix whose rows are the rays of Σ. To compute a rational basis of W_S for an $S \in \mathcal{L}$, we first augment the matrix M by inserting column vectors e_ρ for $\rho \in S$. Call the augmented matrix M_S. Suppose A is a matrix whose column vectors form a basis of W_S in $\mathbf{Q}^{\Sigma(1)}$. Namely, A is a matrix representation of the inclusion $W_S \hookrightarrow \mathbf{Q}^{\Sigma(1)}$. Then by Remark 8.3.1 and sequence (8.2),

$$(M_S)^t \cdot A = 0.$$

In SageMath's terminology, appearing at the right hand side of $(M_S)^t$ in the equation above, A is the *right kernel* of $(M_S)^t$. The matrix A can be obtained by the right_kernel() method applied to $(M_S)^t$.

For step (3), let

$$B = \begin{bmatrix} M_S & \mid & A \end{bmatrix}$$

be the augmented matrix. Note that the column space of B is $\mathbf{Q}^{\Sigma(1)}$: The column space of M_S (resp. A) is $W_S^{\perp}$ (resp. W_S). Note that a vector $v \in \mathbf{R}^{\Sigma(1)}$ satisfies $v = \mathrm{Proj}_{W_S} \chi_D^*$ if and only if

- $v \in W_S$, and
- $\langle \chi_D, w \rangle = (v, w)$ for all $w \in W_S$.

If b_j is the j-th column vector of B, then solving for $\mathrm{Proj}_{W_S} \chi_D^*$ is just solving the system of linear equations

$$\mathrm{Proj}_{W_S} \chi_D^* \cdot b_j = \begin{cases} 0, \text{ if } b_j \in M_S \\ \langle \chi_D, b_j \rangle, \text{ if } b_j \in A \end{cases}$$

where $\mathrm{Proj}_{W_S} \chi_D^*$ is written as a row vector in $\mathbf{Q}^{\Sigma(1)}$. This is no difficult task using sageMath. Note that B is full rank so the solution is unique. In the program $D \in \mathbf{R}^{\Sigma(1)}$ is represented by a tuple of variables (a_ρ) for $\rho \in \Sigma(1)$. Hence each $-\mathrm{Proj}_{W_S} \chi_D^*$ is a vector whose components are $\mathbf{Q}$-linear combination of the a_ρ's.

The list of potential one parameter subgroups has the following data structure. Each entry in the list is a list

$$[v, l, ||v||^2]$$

where v is a column vector in $\mathbf{Q}^{\Sigma(1)}$ and l is a list of sets in $\mathcal{L}$ such that $v = -\mathrm{Proj}_{W_S} \chi_D^*$ for all $S \in l$.

Now take Σ_{ex} to demonstrate the ideas. For Σ_{ex}, the primitive collections are $[0, 1]$ and $[2, 3]$. Hence,

$$\mathcal{L} = [[], [2], [3], [2, 3], [0], [1], [0, 1]]$$

where the empty list $[]$ corresponds to $\emptyset$. For brevity, for a subset $S \in \mathcal{L}$, say $S = [0, 1]$, instead of writing $W_{[0,1]}$ (resp. $M_{[0,1]}$), we write W_{01} (resp. M_{01}). The matrix

$$M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

Take the subset $[0, 1] \in \mathcal{L}$ for example, we have

$$M_{01} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Note that the sum of the first and the fourth column of M_{01} yields the third column. This implies M_{01}, M_0, M_1 have the same column space and therefore

$$M_{01}^{\perp}, M_0^{\perp}, M_1^{\perp} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}.$$

Therefore we expect $\mathrm{Proj}_{W_{01}} \chi^* = \mathrm{Proj}_{W_0} \chi_D^* = \mathrm{Proj}_{W_1} \chi_D^*$ for every $D \in \mathbf{R}^{\Sigma(1)}$. Now

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Then $\mathrm{Proj}_{W_{01}} \chi_D^*$, written as a row vector $[v]$ in $\mathbf{Q}^4$, is the solution to the system of linear equations

$$\begin{bmatrix} v \end{bmatrix} \cdot B = \begin{bmatrix} 0 & 0 & 0 & a2 + a3 \end{bmatrix}.$$

It is computed that

$$[v]^t = \begin{bmatrix} 0 \\ 0 \\ -1/2*a2 - 1/2*a3 \\ -1/2*a2 - 1/2*a3 \end{bmatrix}.$$

Here is the list of the output of potential one parameter subgroups for $X_{\Sigma_{ex}}$.

$$\begin{bmatrix} 0 \\ 0 \\ -1/2*a2 - 1/2*a3 \\ -1/2*a2 - 1/2*a3 \end{bmatrix}, \quad [[0],[1],[0,1]], \quad 1/2*a2^2 + a2*a3 + 1/2*a3^2$$

$$\begin{bmatrix} -1/2*a0 - 1/2*a1 \\ -1/2*a0 - 1/2*a1 \\ 0 \\ 0 \end{bmatrix}, \quad [[2],[3],[2,3]], \quad 1/2*a0^2 + a0*a1 + 1/2*a1^2$$

$$\begin{bmatrix} -1/2*a0 - 1/2*a1 \\ -1/2*a0 - 1/2*a1 \\ -1/2*a2 - 1/2*a3 \\ -1/2*a2 - 1/2*a3 \end{bmatrix}, \quad [[]], \quad \begin{array}{l} 1/2*a0^2 + a0*a1 + 1/2*a1^2 \\ +1/2*a2^2 + a2*a3 \\ +1/2*a3^2 \end{array}$$

Looking at the first entry of the list, we get that $\mathrm{Proj}_{W_0} \chi_D^* = \mathrm{Proj}_{W_1} \chi_D^* = \mathrm{Proj}_{W_{01}} \chi_D^*$ for all $D \in \mathbf{R}^{\Sigma(1)}$ as was predicted. Since condition (8.1) is satisfied for the cone $\mathrm{Cone}(\rho_1, \rho_3)$, we can specialize the results to $\mathrm{Pic}(X_{\Sigma_{\mathrm{ex}}})_{\mathbf{R}}$ by setting $a_1 = a_3 = 0$. Here is the list of $-\mathrm{Proj}_{W_S} \chi_D^*$ for $D \in \mathrm{Pic}(X_{\Sigma_{\mathrm{ex}}})_{\mathbf{R}} \simeq \mathbf{R}\cdot D_0 \oplus \mathbf{R}\cdot D_2$.

$$\begin{bmatrix} 0 \\ 0 \\ -1/2*a2 \\ -1/2*a2 \end{bmatrix}, \quad [[0],[1],[0,1]], \quad 1/2*a2^2$$

$$\begin{bmatrix} -1/2*a0 \\ -1/2*a0 \\ 0 \\ 0 \end{bmatrix}, \quad [[2],[3],[2,3]], \quad 1/2*a0^2 \tag{8.3}$$

$$\begin{bmatrix} -1/2*a0 \\ -1/2*a0 \\ -1/2*a2 \\ -1/2*a2 \end{bmatrix}, \quad [[]], \quad 1/2*a0^2 + 1/2*a2^2$$

As was discussed earlier, $D = a_0 D_0 + a_2 D_2$ is ample if and only if $a_0, a_2 > 0$. This implies $-\mathrm{Proj}_{W_0} \chi_D^*$ (and therefore $-\mathrm{Proj}_{W_1} \chi_D^*, -\mathrm{Proj}_{W_{01}} \chi_D^*$) is parallel to $(0, 0, -1 - 1)$ whenever D is ample. Similarly, $-\mathrm{Proj}_{W_2} \chi_D^*$ (and therefore $-\mathrm{Proj}_{W_3} \chi_D^*, -\mathrm{Proj}_{W_{23}} \chi_D^*$) is parallel to $(-1, -1, 0, 0)$ whenever D is ample.

8.4. **Enumerating critical subsets.**

8.4.1. *Type one critical subsets.* Recall a type one critical subset is the collection of $D \in \mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$ such that $\mathrm{Proj}_{W_{S_1}} \chi_D^* = \mathrm{Proj}_{W_{S_2}} \chi_D^*$ where W_{S_2} is a codimension one subspace of W_{S_1}. The main reasoning behind type one critical subset computation is the following elementary observation:

Proposition 8.4.1. *Let V be a finite dimensional real vector space and $V^\vee$ be its dual. Let $\sigma \subset V$ be a full dimensional polyhedral cone and $f \in V^\vee$. Then the following statements are equivalent:*

(1) The hyperplane $H_f = \{v \in V \mid f(v) = 0\}$ intersects the interior of σ,
(2) H_f is not a supporting hyperplane of σ, and
(3) neither f nor $-f$ is in the dual cone $\sigma^\vee$.

Enumerating type one critical subsets breaks down into the following steps:

(1) Find pairs S_1, S_2 of sets in $\mathcal{L}$ where W_{S_2} is a codimension one subspace of W_{S_1}.
(2) For each such pairs, find a $\tilde{\rho} \in S_2$ but not in S_1 so that $W_{S_1 \cup \{\tilde{\rho}\}} = W_{S_2}$.

Let $f_{\tilde{\rho}} : \mathrm{Pic}(X_\Sigma)_{\mathbf{R}} \to \mathbf{R}$ be defined by

$$D \mapsto \langle \chi_{D_{\tilde{\rho}}}, \mathrm{Proj}_{W_{S_1}} \chi_D^* \rangle.$$

Then if $\ker f_{\tilde{\rho}}$ intersects $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$, it defines a type one critical subset. Hence we

(3) Apply Proposition 8.4.1 to the vector space $\mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$, the cone $\mathrm{Nef}(X_\Sigma)$, and the function $f_{\tilde{\rho}}$.

Note that the result does not depend on the choice of $\tilde{\rho}$ from step (2). For step (1), in the program each W_S is the row space of $M_S^\perp$. Whether or not $W_{S_2} \subset W_{S_1}$ can be checked by the .is_subspace() method in SageMath. The dimension of vector spaces can also be counted in SageMath as well. The more interesting task is to compute the equation of type one critical subsets. This amounts to step (2).

For step (2), if S_1, S_2 is a pair obtained in step (1), the program loops through $\rho \in S_2$ until $\dim W_{S_1 \cup \rho} = \dim_{S_1} - 1$ where S_1 is replaced in each iteration. Then we take the ρ at the end of the iteration. Call it $\tilde{\rho}$ here. The function $f_{\tilde{\rho}}$ is simply described by the $\tilde{\rho}$-th component of $\mathrm{Proj}_{W_{S_1}} \chi_D^*$ for $D \in \mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$.

Next, we determine if the critical subset intersects the interior of the nef cone. Taking $V = \mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$ and σ to be the nef cone as in Proposition 8.4.1, the .dual() method is available in SageMath to return the dual of $\mathrm{Nef}(X_\Sigma)$. The .contains() method can then be used to determine if $\pm f_{\tilde{\rho}}$ is in $\mathrm{Nef}(X_\Sigma)^\vee$.

Let us see what is going on with Σ_{ex}. Take the pair of subsets ([], [0]). It can be checked that W_0 is a codimension one subspace of $\mathbf{\Gamma}(G)_{\mathbf{R}}$. The condition that $\chi_D^* = \mathrm{Proj}_{W_0} \chi_D^*$ amounts to setting the first component of χ_D^* to 0. Referring back to list (8.3), this amounts to setting $-\frac{1}{2}a_0 = 0$, which is impossible in the ample cone. In fact, there are no type one critical subsets for the fan Σ_{ex}.

8.4.2. *Type two critical subsets.* Recall a type two critical subset is of the form

$$\{D \in \mathrm{Amp}(X_\Sigma)_{\mathbf{R}} \| \mathrm{Proj}_{W_{S_1}} \chi_D^* \| = \| \mathrm{Proj}_{W_{S_2}} \chi_D^* \| \text{ for } W_{S_1} \not\subset W_{S_2}, W_{S_2} \not\subset W_{S_1}\}.$$

To detect the containment, we would again use the .is_subspace() method and the equations are immediate as we already recorded the norm of each potential one parameter subgroups. The non-trivial task is to determine if the equation has solutions inside the ample cone. Due to computational difficulties, we currently do not test if a quadratic homogeneous polynomial has a solution in the ample cone or not. Instead, we list equations $||\operatorname{Proj}_{W_{S_1}} \chi_D^*|| = ||\operatorname{Proj}_{W_{S_2}} \chi_D^*||$ for any pair W_{S_1}, W_{S_2} without any containment. Doing this does not affect visualization of the decomposition of ample cone. SageMath can plot graphs in a certain region so anything outside the ample cone will not be plotted.

In terms of data structure that stores critical subsets, it is a list where each entry is a list

$$[f, l].$$

Here l is a list of tuples (l_1, l_2) of lists of sets in $\mathcal{L}$ and f is the polynomial such that for type one (resp. type two) critical subsets, $f = 0$ corresponds to the condition

$$\operatorname{Proj}_{W_{S_1}} \chi_D^* = \operatorname{Proj}_{W_{S_2}} \chi_D^* \text{ (resp. } ||\operatorname{Proj}_{W_{S_1}} \chi_D^*|| = ||\operatorname{Proj}_{W_{S_2}} \chi_D^*||) \\ \text{for all } S_1 \in l_1 \text{ and } s_2 \in l_2.$$

Take Σ_{ex}. The only pair of subspaces without containment is W_0 and W_2. The output is

$$-1/2{*}a0^2{-}a0{*}a1{-}1/2{*}a1^2{+}1/2{*}a2^2{+}a2{*}a3{+}1/2{*}a3^2, ([[0],[1],[0,1]],[[2],[3],[2,3]]).$$

Setting $a_1 = a_3 = 0$, we specialize the result to $\operatorname{Pic}(X_{\Sigma_{\text{ex}}})_{\mathbf{R}}$. The equation becomes $-1/2 * a0^2 + 1/2 * a2^2 = 0$. Since $a_0, a_2 > 0$ in the ample cone, it is the line $-a_0 + a_2 = 0$. We will see that this wall swaps the ordering of the strata.

We end this section with a strategy attempted for enumerating type two critical subsets in the case of simplicial projective toric varieties. It boils down to an optimization problem. To see this, we introduce the cross section of a strongly convex polyhedral cone σ. We say an affine hyperplane H is a *cross section for* σ if the polyhedron $P = H \cap \sigma$ satisfies

$$\sigma = \operatorname{Cone}(P).$$

Namely, P is a polytope and contains all the rays of σ. By Corollary 8.1.4,

$$\operatorname{Pic}(X_\Sigma)_{\mathbf{R}} \simeq \operatorname{Cl}(X_\Sigma)_{\mathbf{R}} \simeq \mathbf{\Gamma}(G)_{\mathbf{R}} \simeq \mathbf{\Gamma}(T)_{\mathbf{R}}$$

where T is the unique maximal torus of G. Now if m is the codimension of W_{S_1} in $W_{S_1} + W_{S_2}$ and n is that of W_{S_2} in $W_{S_1} + W_{W_2}$, then by Lemma 2.2.1, the type two critical subset is linearly equivalent to the zero locus of quadratic polynomial of the form $x_1^2 + \cdots + x_m^2 - y_1^2 - \cdots - y_n^2$. Moreover, the non-containment assumption implies both m and n are at least 1. We are now ready to state the

Proposition 8.4.2. *Let*

- *Q be the polynomial $x_1^2 + \cdots + x_m^2 - y_1^2 - \cdots - y_n^2$ with n, m at least one;*
- *σ be a strongly convex, full dimensional cone in $\mathbf{R}^{n+m+s}$ for some $s \geq 0$;*
- *Z be the vanishing locus of Q in $\mathbf{R}^{m+n+s}$;*
- *H be the cross section of σ;*
- *P be the polytope $H \cap \sigma$;*
- *m, M be the minimum and maximum of Q on P respectively.*

Then the following statements are equivalent:

(1) Z intersects the relative interior of P,
(2) Z intersects the interior of σ;
(3) $m \cdot M < 0$.

Proof. We let P^o be the relative interior of P and σ^o be the interior of σ. It is clear that (1) = (2). Now assuming (2). Let $(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_s)$ be a point in $Z \cap \sigma^o$. Choose $\epsilon > 0$ such that

$$(A_\epsilon, B_\epsilon) = \begin{cases} \Big((\alpha_1+\epsilon, \alpha_2, \ldots, \beta_1, \beta_2 \ldots \gamma_s), (\alpha_1, \ldots, \beta_1+\epsilon, \ldots, \gamma_s)\Big) & \text{if } \alpha_1, \beta_1 \geq 0 \\ \Big((\alpha_1+\epsilon, \cdots, \gamma_s), (\alpha_1, \cdots, \beta_1-\epsilon, \cdots, \gamma_s)\Big) & \text{if } \alpha_1 \geq 0, \beta_1 < 0 \\ \Big((\alpha_1-\epsilon, \cdots, \gamma_s), (\alpha_1, \cdots, \beta_1+\epsilon, \cdots, \gamma_s)\Big) & \text{if } \alpha_1 < 0, \beta_1 \geq 0 \\ \Big((\alpha_1-\epsilon, \cdots, \gamma_s), (\alpha_1, \cdots, \beta_1-\epsilon, \cdots, \gamma_s)\Big) & \text{if } \alpha_1, \beta_1 < 0. \end{cases}$$

and $A_\epsilon, B_\epsilon \in \sigma^o$. Then $Q(A_\epsilon) > 0$ and $Q(B_\epsilon) < 0$. The rays of A_ϵ and B_ϵ intersect P at two points a, b where $Q(a) > 0$ and $Q(b) < 0$. Hence $m < 0$ and $M > 0$ on P. Now assume (3), we will prove (1). Let p_1, p_2 be two points in P such that $Q(p_1) = m$ and $Q(p_2) = M$. Then Q achieves zero on the line segment $\overline{p_1p_2} \subset P$. If one of p_1, p_2 is in the interior of P we are done. Now suppose $p, q \in \partial P$ and (1) fails to hold. In this case we must have either $Q(P^o) < 0$ or $Q(P^o) > 0$ since P is convex. If $Q(P^o) < 0$, pick any interior point $q \in P$. Then Q achieves zero at some point interior to the line segment $\overline{p_2q}$. This is a contradiction. Similarly we cannot have $Q(P^o) > 0$. This shows (3) $\Rightarrow$ (1) and the proposition is proved. □

SageMath is able to optimize a quadratic polynomial on a polytope, only that it does optimization by numerical approximation instead of solving for absolute answers. Hence there are numerical errors so small that in some cases some type two critical subsets that are only tangent to the nef cone are counted as well. This is one of the reason why we do not enumerate type two critical subsets, but a larger set of equations that includes all equations of type two critical subsets.

8.5. **Computing stratifications with respect to ample divisors.** Given an ample divisor D on X_Σ, the program computes the stratification of $Z(\Sigma)$ induced by χ_D. The program also determines if the stratifications induced by two ample divisors are the same. For each $S \in \mathcal{L}$, recall

$$L_S = D(\prod_{\rho \in S} x_\rho) \cap V(x_\rho | \rho \notin S) \subset Z(\Sigma).$$

Computing the stratification induced by χ_D boils down to two steps:

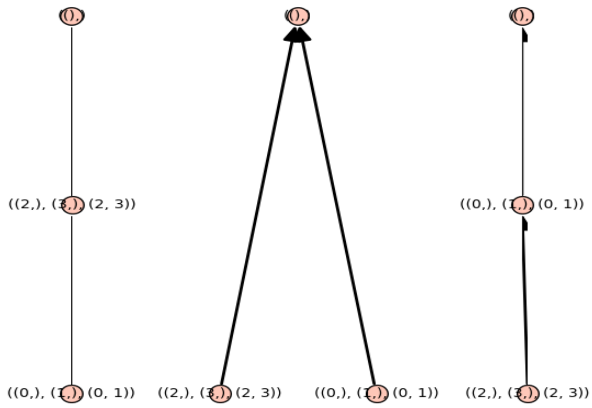
(1) Compute the one parameter subgroup that is χ_D-adapted to L_S for each $S \in \mathcal{L}$.
(2) Group $\{L_S\}_{S \in \mathcal{L}}$ together by the one parameter subgroups that are χ_D-adapted to them.

Step (1) amounts to minimizing the function $\frac{\langle \chi_D, - \rangle}{||-||}$ on the cone σ_S. By Corollary 2.1.8, this can be achieved by first collecting one parameter subgroups of the form

$$-\operatorname{Proj}_{W_{S'}} \chi_D^* \text{ for } S' \subset S \text{ and } -\operatorname{Proj}_{W_{S'}} \chi_D^* \in \sigma_S,$$

then find the longest one. This is a combination of basic set operations on the list of potential one parameter subgroups obtained earlier. Step (2) is also just a sequence of set operations. The stratification is then stored as a poset in SageMath where each node is a tuple of sets in $\mathcal{L}$, corresponding to a stratum. The strict partial ordering is defined by the norms of strata's indexing one parameter subgroups. With this data structure, to determine if the stratifications induced by two ample divisors are equivalent is to determine if two posets are isomorphic. SageMath's built in function allows us to do it.

Finally, SageMath can plot a poset. Take Σ_{ex} for example, the stratification induced by the divisors $(a_0, a_2) = (2,1)$, $(1,1)$, $(1,2)$ respectively looks like



Each node in the diagram is a tuple. The empty tuple $((),)$ corresponds to the strata $L_\emptyset$, which is the origin. The tuple $((0,),(1,),(0,1))$ corresponds to the strata $L_0 \cup L_1 \cup L_{01}$ and similarly for others. For some reason sageMath is not very consistent in drawing arrows or just an edge. In any case, higher order strata are placed higher in the diagram. In Section 8.4, we calculated that $a_0 = a_2$ defines a type two critical subset. Note that the divisor $(a_0, a_2) = (1,1)$ is on the type two critical subset. Crossing this type two critical subset swaps the orderings of the strata and on it the poset breaks up into two chains.

8.6. **Visualizing the ample cone decomposition.** The program can plot the ample cone and its critical subsets when the dimension of the ample cone is less than 3. We only discuss the case when the ample cone is of dimension 3. The ideas is to obtain a two dimensional slice of the picture. This boils down to the following steps:

(1) Obtain a cross section of the ample cone.
(2) Restrict all defining equations of critical subsets to the affine plane containing the cross section.
(3) Plot solutions to the equations in the cross section.

For step (1), here is the main tool to create a cross section of a polyhedral cone.

Proposition 8.6.1. *Let σ be a full dimensional, strongly convex polyhedral cone and $\sigma^\vee$ be its dual. Then*

(1) The sum of ray generators of σ is in the interior of σ.
(2) Let $m \in \sigma^\vee$. Then $H_m \cap \sigma = \{0\}$ if and only if m is in the interior of $\sigma^\vee$.

Our ample cone $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$ is strongly convex and full dimensional in $\mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$. So $\mathrm{Amp}(X_\Sigma)^\vee_{\mathbf{R}}$ is also strongly convex and full dimensional. Hence $\mathrm{Amp}(X_\Sigma)^\vee_{\mathbf{R}}$ has a well defined set of ray generators. SageMath can compute the dual of a polyhedral cone and enumerate its list of ray generators. If f is the sum of ray generators of $\mathrm{Amp}(X_\Sigma)^\vee_{\mathbf{R}}$, we see from Proposition 8.6.1 that $H_f \cap \mathrm{Amp}(X_\Sigma)_{\mathbf{R}} = \{0\}$. In particular, f does not vanish on the ray generators of $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$. Let u_ρ be any ray generator of $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$. Then if $u_{\rho'}$ is any other ray generator, we have

$$f(\frac{f(u_\rho)}{f(u_{\rho'})}u_{\rho'}) - f(u_\rho) = 0.$$

This means the affine plane $H \subset \mathrm{Pic}(X_\Sigma)_{\mathbf{R}}$ defined by the vanishing locus of the function

$$v \mapsto f(v) - f(u_\rho)$$

contains all rays of $\mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$. Namely, $H \cap \mathrm{Amp}(X_\Sigma)_{\mathbf{R}}$ is a cross section.

For step (2), the equation for H allows us to solve one of the variable in terms of the other two. Hence we can rewrite the equations of critical subsets in two variables. For step (3), we use implicit_plot() method to plot points implicitly defined by polynomial equations on the cross section. Readers may look at Example 8.7.5 for the decomposition of a three dimensional ample cone.

8.7. **Examples and counter examples.** Following the notation in SageMath, all fans will be presented as a list of rays followed by ambient indicies of cones in the fan. The highlights of this section are examples where

(1) there are critical subsets that are both type one and type two,
(2) a semi-chamber does not have to be convex,
(3) a type two critical subset may not have lattice points on it,
(4) a semi-chamber does not have to be connected,
(5) a proper part of a critical subset can be vacuous,
(6) there are more semi-chambers than IIT classes, and
(7) λ_C in Corollary 7.4.2 does not have to be primitive collection relation (Definition 8.7.4).

Example 8.7.1. The projective space $\mathbf{P}^n_{\mathbf{C}}$ is given by the fan in $\mathbf{R}^n$ with rays given by $\rho_0 = e_1, \cdots, \rho_{n-1} = e_n, \rho_n = -e_1 - \cdots - e_n$. The maximal cones consists of $\sigma_{0\cdots\hat{i}\cdots n}$ for $i = 1, \cdots, n$. The primitive collection in this case is the whole collection of rays $\Sigma(1)$. Therefore $Z(\Sigma)$ is the origin and also the only strata for any ample divisor. A torus invariant divisor $\sum_{i=1}^n a_i D_i$ is ample if and only if $\sum_{i=0}^n a_i > 0$. The Picard group of $\mathbf{P}^n_{\mathbf{C}}$ is one dimensional and the ample cone is a ray. We do not have variation of stratifications.

Example 8.7.2 (Blow-up of $\mathbf{P}^2$ at a point)**.** We will demonstrate some type two critical subsets that do not have lattice points on them. In this case there is no ample divisor in the wall but the stratification on two sides of the wall are different. The fan here consists of the rays

$$\begin{aligned}\rho_0 =& \mathrm{Cone}\ (e_1)\\ \rho_1 =& \mathrm{Cone}\ (-e_1 - e_2)\\ \rho_2 =& \mathrm{Cone}\ (e_2)\\ \rho_3 =& \mathrm{Cone}\ (-e_2).\end{aligned}$$

This is a smooth fan. It is computed that a divisor $D = \sum_{i=0}^3 a_i D_i$ is ample if and only if $a_1 + a_3 > 0$ and $a_0 + a_2 - a_3 > 0$. Since D_2, D_3 is a basis for the picard group, the ample cone can be described more concisely as

$$a_2 D_2 + a_3 D_3 \text{ is ample if and only if } a_2 > a_3 > 0.$$

The primitive collections consists of $\{0, 2\}, \{1, 3\}$. So the strata will be a grouping of

$$L_\emptyset, L_0, L_2, L_{02}, L_1, L_3, L_{13}.$$

The list of potential one parameter subgroups is

$$(1)\ \ -\chi_D^* = \begin{bmatrix} -2/5*a2+1/5*a3 \\ -1/5*a2-2/5*a3 \\ -2/5*a2+1/5*a3 \\ 1/5*a2-3/5*a3 \end{bmatrix},$$

$$(2)\ \ -\operatorname{Proj}_{W_0}\chi_D^* = -\operatorname{Proj}_{W_2}\chi_D^* = -\operatorname{Proj}_{W_{02}}\chi_D^* = \begin{bmatrix} 0 \\ -1/2*a3 \\ 0 \\ -1/2*a3 \end{bmatrix},$$

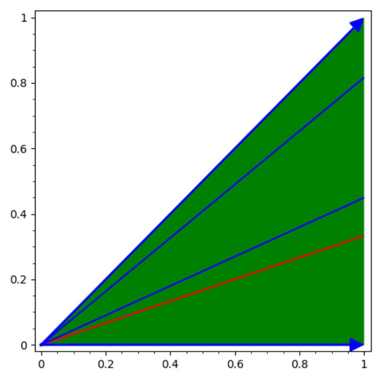
$$(3)\ \ -\operatorname{Proj}_{W_1}\chi_D^* = \begin{bmatrix} -1/3*a2+1/3*a3] \\ 0 \\ -1/3*a2+1/3*a3 \\ 1/3*a2-1/3*a3 \end{bmatrix},$$

$$(4)\ \ -\operatorname{Proj}_{W_3}\chi_D^* = \begin{bmatrix} -1/3*a2 \\ -1/3*a2 \\ -1/3*a2 \\ 0 \end{bmatrix}$$

Since $a_2 > a_3 > 0$, we see that $-\operatorname{Proj}_{W_3}\chi_D^* \parallel (-1,-1,-1,0)$ and $-\operatorname{Proj}_{W_0} \parallel (0,-1,0,-1)$. The type two critical subset $||\operatorname{Proj}_{W_3}\chi_D^*|| = ||\operatorname{Proj}_{W_0}\chi_D^*||$ is defined by

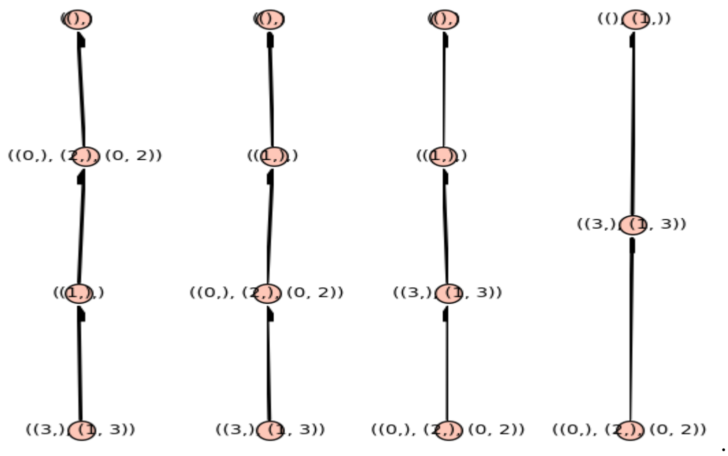
$$\frac{a_2}{\sqrt{3}} = \frac{a_3}{\sqrt{2}}.$$

This is evidently not rational. Overall there are two non-rational type two critical subsets and one type one critical subset in the ample cone. Here is a the picture.



.

The red line is a type one critical subset and the blue lines are type two critical subsets. Two blue rays are the extremal rays of the nef cone. The stratification listed from top chamber to bottom chamber:



We see that wall crossing under type two walls swaps orderings of some strata and wall crossing under type one wall collapses or divides some strata.

Example 8.7.3 (Blow-up of Hirzebruch surface at a point)**.** This is a rich example where items (1),(2),(4),(5),(6),(7) listed in the beginning of Section 8.7 can be found. We first need a thorough understanding of the one parameter subgroups that are χ_D-adapted to L_S for each $S \in \mathcal{L}$ first. Let $\Sigma \subset \mathbf{R}^2$ be the fan whose rays are given by

$$\begin{aligned}\rho_0 =& \text{Cone } (e_1)\\ \rho_1 =& \text{Cone } (e_2)\\ \rho_2 =& \text{Cone } (-e_2)\\ \rho_3 =& \text{Cone } (-e_1+e_2)\\ \rho_4 =& \text{Cone } (-e_1+2e_2).\end{aligned}$$

Let the ambient indicies of the maximal cones be given by $(0,1)$, $(0,2)$, $(2,3)$, $(3,4)$, $(1,4)$. The primitive collection consists of

(1) $\{\rho_0, \rho_3\}$
(2) $\{\rho_0, \rho_4\}$
(3) $\{\rho_1, \rho_2\}$
(4) $\{\rho_1, \rho_3\}$
(5) $\{\rho_2, \rho_4\}$.

Moreover,

$$\begin{aligned}\mathcal{L} = \quad & [[], [1], [2], [1,2], [4], [1,4], [2,4], [1,2,4], [3], [1,3], [2,3], [1,2,3], [0], [0,3],\\ & [0,4], [3,4], [0,3,4], [0,2], [0,2,4], [0,1], [0,1,3]]\end{aligned}$$

This is a smooth fan and by Corollary 8.1.4 applied to the maximal cone $\text{Cone}(\rho_0, \rho_1)$, $\text{Pic}(X_\Sigma)$ is free with basis D_2, D_3, D_4. The computer program computes that a divisor $\sum_{i=2}^4 a_i D_i$ is ample if and only if

$$a_2 + a_4 > a_3 > a_4 > 0.$$

For a point $D = \sum_{i=2}^4 a_i D_i \in \text{Amp}(X_\Sigma)_\mathbf{R}$, the potential one parameter subgroups in $\mathbf{\Gamma}(G)_\mathbf{R} \subset \mathbf{R}^{\Sigma(1)}$ are

$$(1)\ \ -\chi_D^* = \begin{bmatrix} -1/4*a2-1/3*a3-1/12*a4 \\ -1/4*a2+1/4*a4 \\ -3/4*a2-1/4*a4 \\ -2/3*a3+1/3*a4 \\ -1/4*a2+1/3*a3-5/12*a4 \end{bmatrix},$$

$$(2)\ \ -\mathrm{Proj}_{W_0}\,\chi_D^* = \begin{bmatrix} 0 \\ -2/5*a2-1/5*a3+1/5*a4 \\ -3/5*a2+1/5*a3-1/5*a4 \\ 1/5*a2-2/5*a3+2/5*a4 \\ -1/5*a2+2/5*a3-2/5*a4 \end{bmatrix},$$

$$(3)\ \ -\mathrm{Proj}_{W_1}\,\chi_D^* = \begin{bmatrix} -1/3*a2-1/3*a3 \\ 0 \\ -2/3*a2-1/3*a4 \\ -2/3*a3+1/3*a4 \\ -1/3*a2+1/3*a3-1/3*a4 \end{bmatrix},$$

$$(4)\ \ -\mathrm{Proj}_{W_2}\,\chi_D = \begin{bmatrix} -1/3*a3 \\ 1/3*a4 \\ 0 \\ -2/3*a3+1/3*a4 \\ 1/3*a3-1/3*a4 \end{bmatrix},$$

$$(5)\ \ -\mathrm{Proj}_{W_3}\,\chi_D^* = \begin{bmatrix} -1/4*a2-1/4*a4 \\ -1/4*a2+1/4*a4 \\ -3/4*a2-1/4*a4 \\ 0 \\ -1/4*a2-1/4*a4 \end{bmatrix},$$

$$(6)\ \ -\mathrm{Proj}_{W_4}\,\chi_D^* = \begin{bmatrix} -1/5*a2-2/5*a3 \\ -2/5*a2+1/5*a3 \\ -3/5*a2-1/5*a3 \\ -1/5*a2-2/5*a3 \\ 0 \end{bmatrix}.$$

$$(7)\ \ -\mathrm{Proj}_{W_{14}}\,\chi_D^* = \begin{bmatrix} -1/3*a2-1/3*a3 \\ 0 \\ -1/3*a2-1/3*a3 \\ -1/3*a2-1/3*a3 \\ 0 \end{bmatrix} \parallel (-1,0,-1,-1,0),$$

$$(8)\ \ -\mathrm{Proj}_{W_{01}}\,\chi_D^* = \begin{bmatrix} 0 \\ 0 \\ -1/3*a2+1/3*a3-1/3*a4 \\ 1/3*a2-1/3*a3+1/3*a4 \\ -1/3*a2+1/3*a3-1/3*a4 \end{bmatrix} \parallel (0,0,-1,1,-1),$$

$$(9)\ \ -\mathrm{Proj}_{W_{12}}\,\chi_D^* = \begin{bmatrix} [-1/3*a3+1/6*a4 \\ 0 \\ 0 \\ -2/3*a3+1/3*a4 \\ 1/3*a3-1/6*a4 \end{bmatrix} \parallel (-1,0,0,-2,1),$$

$$(10)\quad -\operatorname{Proj}_{W_{13}}\chi_D^* = \begin{bmatrix} -1/3*a2-1/6*a4 \\ 0 \\ -2/3*a2-1/3*a4 \\ 0 \\ [-1/3*a2-1/6*a4 \end{bmatrix} \parallel (-1,0,-2,0,-1),$$

$$(11)\quad -\operatorname{Proj}_{W_{23}}\chi_D^* = \begin{bmatrix} -1/6*a4 \\ 1/3*a \\ 0 \\ 0 \\ -1/6*a4 \end{bmatrix} \parallel (-1,2,0,0,-1),$$

$$(12)\quad -\operatorname{Proj}_{W_{03}}\chi_D^* = -\operatorname{Proj}_{W_{04}}\chi_D^* = -\operatorname{Proj}_{W_{34}} = -\operatorname{Proj}_{W_{034}}\chi_D^* = \begin{bmatrix} 0 \\ -1/2*a2 \\ -1/2*a2 \\ 0 \\ 0 \end{bmatrix} \parallel$$

$(0,-1,-1,0,0)$,

$$(13)\quad -\operatorname{Proj}_{W_{24}}\chi_D^* = \begin{bmatrix} -1/3*a3 \\ 1/3*a3 \\ 0 \\ -1/3*a3 \\ 0 \end{bmatrix} \parallel (-1,1,0,-1,0),$$

$$(14)\quad -\operatorname{Proj}_{W_{02}}\chi_D^* = \begin{bmatrix} 0 \\ -1/3*a3+1/3*a4 \\ 0 \\ -1/3*a3+1/3*a4 \\ 1/3*a3-1/3*a4 \end{bmatrix} \parallel (0,-1,0,-1,1).$$

Although there are 21 subsets in $\mathcal{L}$, we only listed $-\operatorname{Proj}_{W_S}\chi_D^*$ for 17 of them. This is due to the fact that some $W_S=\{0\}$.

For each $S\in\mathcal{L}$, we may describe the one parameter subgroup that is χ_D-adapted to L_S as a piecewise function of $D\in\operatorname{Amp}(X_\Sigma)_{\mathbf{R}}$. For example, take $S=[0]\in\mathcal{L}$. Then for $-\chi_D^*$ to be inside σ_S, we must have $-1/4*a2-1/3*a3-1/12*a4\geq 0$. This is clearly impossible inside the ample cone. Hence one parameter subgroups on the ray $\mathbf{R}_{>0}\cdot(-\chi_D^*)$ are never χ_D-adapted to L_0. Instead, one parameter subgroups on the ray of the projection $\mathbf{R}_{>0}\cdot(-\operatorname{Proj}_{W_0}\chi_D^*)$ are χ_D-adapted to L_0. Let us look at one more example where $S=[1,4]$. It can be checked that $-\chi_D^*\notin\sigma_{14}$ and $-\operatorname{Proj}_{W_1}\chi_D^*\notin\sigma_{14}$. But $\operatorname{Proj}_{W_4}\chi_D^*\in\sigma_{14}$ if $-2/5*a2+1/5*a3\geq 0$. This is possible in the ample cone so we conclude that the one parameter subgroups that are χ_D-adapted to L_{14} are on the ray of $-\operatorname{Proj}_{W_4}\chi_D^*$ if $-2/5*a2+1/5*a3\geq 0$ or else on the ray of $-\operatorname{Proj}_{W_{14}}\chi_D^*$. Below is a complete description of the one parameter subgroup that is χ_D-adapted to L_S for all $S\in\mathcal{L}$ and for all $D\in\operatorname{Amp}(X_\Sigma)$. For convenience, instead of saying one parameter subgroups on the ray of a vector v are χ_D-adapted to some L_S, we will just say L_S is maximally destabilized by v. For each ample divisor D,

(1) $L_\emptyset$, the origin
is maximally destabilized by $-\chi_D^*$,
(2) L_0
is maximally destabilized by $-\operatorname{Proj}_{W_0}\chi_D^*$,

(3) L_1
is maximally destabilized by $-\chi_D^*$ if $-1/4 * a2 + 1/4 * a4 \geq 0$, or $-\operatorname{Proj}_{W_1} \chi_D^*$ otherwise.
(4) $L_{124} \cup L_{24} \cup L_{12} \cup L_2$
is maximally destabilized by $-\operatorname{Proj}_{W_2} \chi_D^*$
(5) L_3
is maximally destabilized by $-\operatorname{Proj}_{W_3} \chi_D^*$,
(6) L_4
is maximally destabilized by $-\chi_D^*$ if $-1/4 * a2 + 1/3 * a3 - 5/12 * a4 \geq 0$, or $-\operatorname{Proj}_{W_4} \chi_D^*$ otherwise.
(7) $L_{034} \cup L_{34}$
is maximally destabilized by $-\operatorname{Proj}_{W_{34}} \chi_D^* \parallel (0, -1, -1, 0, 0)$.
(8) $L_{123} \cup L_{23}$
is maximally destabilized by $-\operatorname{Proj}_{W_{23}} \chi_D^* \parallel (-1, 2, 0, 0, -1)$.
(9) $L_{024} \cup L_{02}$
is maximally destabilized by $-\operatorname{Proj}_{W_{02}} \chi_D^* \parallel (0, -1, 0, -1, 1)$.
(10) L_{013}
is maximally destabilized by $\operatorname{Proj}_{W_{01}} \chi_D^* \parallel (0, 0, -1, 1, -1)$.
(11) L_{14}
is maximally destabilized by $-\operatorname{Proj}_{W_4} \chi_D^*$ if $-2/5 * a2 + 1/5 * a3 \geq 0$, or $-\operatorname{Proj}_{14} \chi_D^* \parallel (-1, 0, -1, -1, 0)$ otherwise.
(12) L_{03}
is maximally destabilized by $-\operatorname{Proj}_{W_0} \chi_D^*$ if $1/5*a2-2/5*a3+2/5*a4 \geq 0$, or $-\operatorname{Proj}_{W_{03}} \parallel (0, -1, -1, 0, 0)$ otherwise.
(13) L_{04}
is maximally destabilized by $-\operatorname{Proj}_{W_0} \chi_D^*$ if $-1/5*a2+2/5*a3-2/5*a4 \geq 0$, or $-\operatorname{Proj}_{04} \chi_D^* \parallel (0, -1, -1, 0, 0)$ otherwise.
(14) L_{13}
is maximally destabilized by $-\operatorname{Proj}_{W_3} \chi_D^*$ if $-1/4 * a2 + 1/4 * a4 \geq 0$, or $-\operatorname{Proj}_{W_{13}} \parallel (-1, 0, -2, 0, -1)$ otherwise.
(15) L_{01}
is maximally destabilized by $-\operatorname{Proj}_{W_{01}} \parallel (0, 0, -1, 1, -1)$.

Using the computer program, we have a list of equations of type one critical subsets (We refer the reader back to Section 8.4 for relevant data structures):

$$\begin{array}{l} 2/5 * a2 - 1/5 * a3, [([[4]], [[1, 4]])] \\ -1/5 * a2 + 2/5 * a3 - 2/5 * a4, [([[0]], [[0, 3], [0, 4], [3, 4], [0, 3, 4]])] \\ 1/4 * a2 - 1/3 * a3 + 5/12 * a4, [([[]], [[4]])] \\ 1/4 * a2 - 1/4 * a4, [([[3]], [[1, 3]]), ([[]], [[1]])] \end{array} . \tag{8.4}$$

Let us look at some wall crossing behavior. First we examine the type one critical subset

$$W_{0,03} := \{D \in \operatorname{Amp}(X_\Sigma)_{\mathbf{R}} \mid \operatorname{Proj}_{W_0} \chi_D^* = \operatorname{Proj}_{W_{03}} \chi_D^*\}.$$

This corresponds to the hyperplane

$$-1/5 * a2 + 2/5 * a3 - 2/5 * a4 = 0.$$

Let

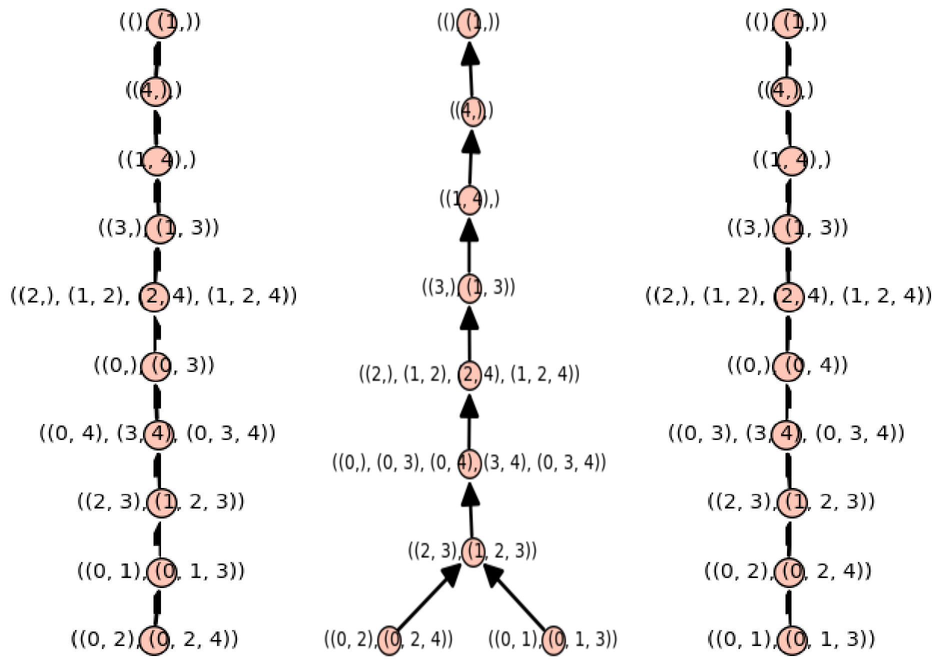
$$W_{0,03}^+ = \{(a2, a3, a4) \mid -1/5 * a2 + 2/5 * a3 - 2/5 * a4 > 0\},$$

$$W_{0,03}^- = \{(a2, a3, a4) \mid -1/5 * a2 + 2/5 * a3 - 2/5 * a4 < 0\}.$$

When $D \in W^{+}_{0,03}$, L_{03} is maximally destabilized by $-\operatorname{Proj}_{W_0} \chi^*_D$ and $L_{04} \cup L_{034} \cup L_{34}$ is maximally destabilized by $-\operatorname{Proj}_{W_{03}} \chi^*_D = -\operatorname{Proj}_{W_{04}} \chi^*_D = -\operatorname{Proj}_{W_{34}} \chi^*_D = -\operatorname{Proj}_{W_{034}} \chi^*_D \parallel (0,-1,-1,0,0)$. When $D \in W_{0,03}$, $L_{03} \cup L_{04} \cup L_{034} \cup L_{34}$ all come together in the strata indexed by $(0,-1,-1,0,0)$. When $D \in W^{-}_{0,03}$, L_{04} is maximally destabilized by $-\operatorname{Proj}_{W_0} \chi^*_D$, separated from $L_{03} \cup L_{034} \cup L_{34}$, whose maximally destabilizing one parameter subgroup is $(0,-1,-1,0,0)$. For concreteness we consider the three ample divisors

(1) $D = 45D_2 + 65D_3 + 55D_4$,
(2) $D' = 450D_2 + 775D_3 + 550D_4$,
(3) $D^* = 45D_2 + 80D_3 + 55D_4$.

Then it can be checked that $D \in W^{-}_{0,03}$, $D' \in W_{0,03}$ and $D^* \in W^{+}_{0,03}$. Here are the stratifications induced by D, D' and D^* respectively.



Notice how L_{03} and L_{04} move in the diagrams. As to why on $W_{0,03}$ the chain breaks up into two, we investigate the type two critical subset

$$Z_{01,02} := \{D \in \operatorname{Amp}(X_\Sigma)_{\mathbf{R}} \mid ||\operatorname{Proj}_{W_{01}} \chi^*_D|| = ||\operatorname{Proj}_{W_{02}} \chi^*_D||\}.$$

Using the computer program, we see that $Z_{01,02}$ corresponds to the condition

$$-1/3 * a2^2 + 2/3 * a2 * a3 - 2/3 * a2 * a4 = 0.$$

Since $a_2 > 0$, this condition is equivalent to

$$-a_2 + 2a_3 - a_4 = 0,$$

which is exactly the defining equation for $W_{0,03}$. Notice how L_{01}, L_{02} move in the diagrams. This is an example where a critical subset can be of both type.

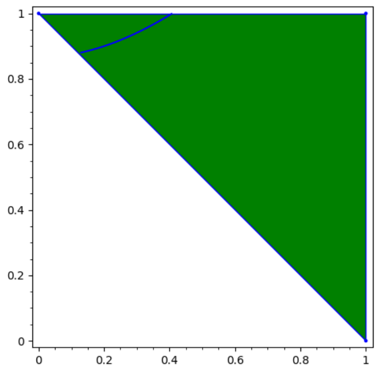
Second, we note that crossing some parts of a critical subset may not affect stratification. This will also provide examples that the semi-chamber decomposition is finer than the partition of IIT-equivalence classes of characters. We provide two examples here. The first one is easier. The type two critical subset

$$Z_{3,24} = \{D \in \mathrm{Amp}(X_\Sigma)_{\mathbf{R}} \mid ||\,\mathrm{Proj}_{W_3}\chi_D^*|| = ||\,\mathrm{Proj}_{W_{24}}\chi_D^*||\}$$

is vacuous as there are no points in $Z(\Sigma)$ whose maximally destabilizing one parameter subgroup is $-\mathrm{Proj}_{W_{24}}\chi_D^*$ throughout the ample cone. $Z_{3,24}$ is defined by the quadratic polynomial

$$3/4 * a2^2 - 1/3 * a3^2 + 1/2 * a2 * a4 + 1/4 * a4^2.$$

Note that $\dim W_3 = 2$ and $\dim W_{24} = 1$ so by Lemma 2.2.1, $Z_{3,24}$ is a quadric cone. Here is a visualization of $Z_{3,24}$ in a 2D slice of the ample cone.



.

The slice is the intersection of the ample cone with the hyperplane $a2+a4 = 1$. To verify that there are more semi-chambers than IIT-equivalence classes due to $Z_{3,24}$, take the two ample divisors $D = 30D_2+92D_3+70D_4$ and $D' = 30D_2+99D_3+70D_4$. It can be checked that

- $D \in Z_{3,24}^+$ and $D' \in Z_{3,24}^-$, but
- D and D' induces the same stratification.

The second one is more interesting where a proper subset of a critical subset is vacuous. For this, let us take the type two critical subset

$$Z_{1,4} := \{D \in \mathrm{Amp}(X_\Sigma) \mid ||\,\mathrm{Proj}_{W_1}\chi_D^*|| = ||\,\mathrm{Proj}_{W_4}\chi_D^*||\}.$$

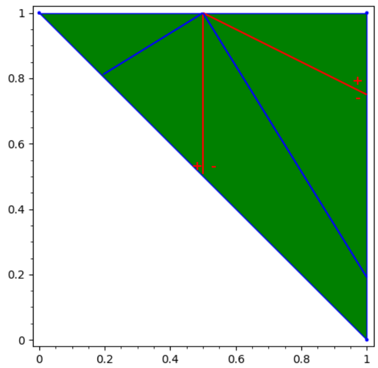
This corresponds to the hypersurface

$$-1/15*a2^2+2/5*a2*a3-4/15*a3^2-2/3*a2*a4+2/3*a3*a4-1/3*a4^2 = 0.$$

Note that since $\dim W_1 = \dim W_4$, Lemma 2.2.1 implies that $Z_{1,4}$ is a union of two hyperplanes. This critical subset $Z_{1,4}$ should swap the pairs of strata whose maximally destabilizing one parameter subgroups are $-\mathrm{Proj}_{W_1}\chi_D^*$ and $-\mathrm{Proj}_{W_4}\chi_D^*$. A look at the complete list of the maximally destabilizing one parameter subgroups for every $S \in \mathcal{L}$ provided earlier indicates that L_1, L_4, L_{14} are the only three locally closed subsets of $Z(\Sigma)$ whose maximally destabilizing one parameter subgroups are these two. According to the list, when $-a2 + a4 > 0$ and $-3 * a2 + 4 * a3 - 5 * a4 < 0$, the maximally destabilizing one parameter subgroup for L_1 is $-\chi_D^*$ and for L_4 is $-\mathrm{Proj}_{W_4}\chi_D^*$. In this case $Z_{1,4}$ is vacuous as $-\chi_D^*$ is longer than $-\mathrm{Proj}_{W_4}\chi_D^*$. Here is a visualization of the three critical subsets

(1) $-a2 + a4 = 0$,
(2) $-3 * a2 + 4 * a3 - 5 * a4 = 0$,
(3) $Z_{1,4}$.

in the 2D-slice of the ample cone.



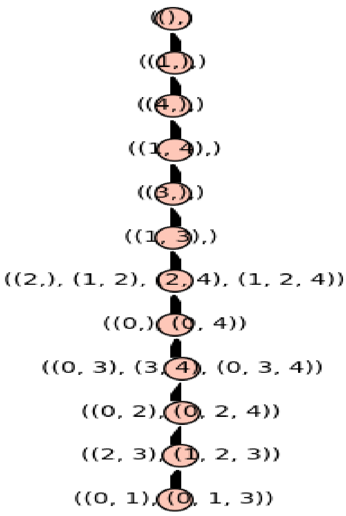
(8.5)

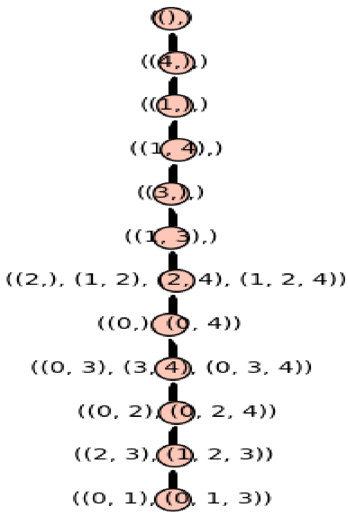
The two red lines are for type one critical subsets and the union of blue lines is $Z_{1,4}$. The vertical red line is the critical subset $-a2 + a4 = 0$ and the other is $-3 * a2 + 4 * a3 - 5 * a4 = 0$ with signs inserted to the graph. We see that $Z_{1,4}$ in the left upper corner of the graph is vacuous while the other part of $Z_{1,4}$ does swap L_1 and L_4. To carefully verify these, we first take the two ample divisors $D = 430D_2 + 960D_3 + 570D_4$ and $D' = 430D_2 + 955D_3 + 570D_4$. It can be verified by the computer that

- both D and D' satisfy $-a2 + a4 > 0$ and $-3 * a2 + 4 * a3 - 5 * a4 < 0$
- $D \in Z^+_{1,4}$ and that $D' \in Z^-_{1,4}$, but
- D and D' induces the same stratification.

Now take the two ample divisors $D = 55D_2 + 90D_3 + 45D_4$ and $D' = 65D_2 + 90D_3 + 35D_4$. It can be checked that

both D and D' satisfy $-a2 + a4 < 0$ and $-3 * a2 + 4 * a3 - 5 * a4 < 0$
$D \in Z^-_{1,4}$ and $D' \in Z^+_{1,4}$,
D and D' swaps L_1 and L_4. Here are the stratifications induced by D and D' respectively





.

Note that the two stratifications only differ by the ordering between L_1 and L_4.

Finally, recall in Corollary 7.4.2 we proved that for every ample divisor $D \in \mathrm{Amp}(X_\Sigma)$ and for every primitive collection $C \subset \Sigma(1)$, there exists a unique one parameter subgroup $\lambda_C \in \Lambda^{\chi_D}$ such that

$$\overline{S_{\lambda_C}} = V(x_\rho | \rho \in C).$$

We will show that $-\lambda_C$ need not be the primitive relation, which we now define. Suppose Σ is a complete simplicial fan and $C \subset \Sigma(1)$ is a primitive collection. Then the sum $u = \sum_{\rho \in C} u_\rho$ is in the relative interior of a unique cone $\sigma \in \Sigma$. Since Σ is simplicial, there exists unique $q_\rho \in \mathbf{Q}_{\geq 0}$ for each $\rho \in \sigma(1)$ such that $u = \sum_{\rho \in \sigma(1)} q_\rho u_\rho$.

Definition 8.7.4. The tuple $a \in \mathbf{Q}^{\Sigma(1)}$ where

$$a_\rho = \begin{cases} 1, \text{ if } \rho \in C \cap \backslash \sigma(1) \\ 1 - b_\rho, \text{ if } \rho \in C \cap \sigma(1) \\ b_\rho, \text{ if } \rho \in \sigma(1) \backslash C \\ 0, \text{ otherwise.} \end{cases}$$

is called the *primitive relation* of C.

In Proposition 7.2.6 we proved that the antipode of the primitive relation of C destabilizes all points in $V(x_\rho | \rho \in C)$. In this example the fan is smooth so the primitive relation is in $\mathbf{Z}^{\Sigma(1)}$ and in particular in $\mathbf{\Gamma}(G)$. Take $C = \{0, 3\}$. Since

$$\rho_0 + \rho_3 = \rho_1,$$

the primitive relation of C is

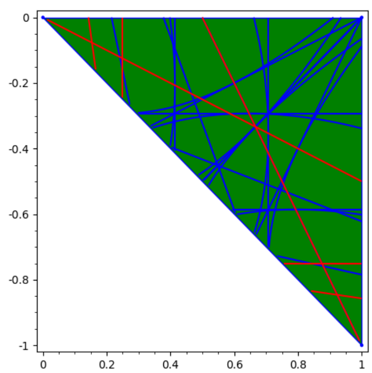
$$(1, -1, 0, 1, 0, 0) \in \mathbf{\Gamma}(G)_{\mathbf{R}}.$$

Using the divisor $D = 45D_2 + 65D_3 + 55D_4$ given earlier, we see that the stratum

$$L_2 \cup L_{12} \cup L_{24} \cup L_{124}$$

has closure equal to $V(x_0, x_3)$. This stratum is indexed by the one parameter subgroup $(-13, 11, 0, -15, 2)$ which is not equal to the antipode of the primitive relation of C.

Example 8.7.5. We end this section with a picture of a 2D slice of the wall and chamber decomposition of the ample cone for the toric variety obtained by blowing up $\mathbf{P}^3$ at two torus invariant points.



The red lines correspond to type one critical subsets and the blue ones to type two critical subsets. It is also clear from the picture that some semi-chambers are not convex.

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