

**Understanding Quadratic Functions and Solving Quadratic Equations:
An Analysis of Student Thinking and Reasoning**

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Abstract

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Understanding quadratic functions is critical to student success in high school mathematics and beyond, yet very little is known about what students understand about these functions. There is agreement in the field that quadratics are one of the most conceptually challenging subjects in the secondary mathematics curriculum. However, research on student learning in this area has focused on procedural aspects of solving equations, with very little known about student understanding of the behavior of quadratic functions. This study sought to learn what high school students who have completed an Algebra 2 or Precalculus class understand about quadratics. Specifically, what connections, if any, do they make between representations of quadratic functions? How do students approach solving quadratic equations, and how do they interpret the solutions? Lastly, what cognitive affordances support them in their learning and understanding of quadratic functions, and what cognitive obstacles do they encounter? This qualitative study employed cognitive interviews of 27 students in grades nine through eleven. The data included video and audio recordings as well as student work,

captured on a smart pen pencast. The data was analyzed in four phases: (1) focusing on one student at a time, (2) focusing on individual problems, (3) focusing across students, and then (4) revisiting individual problems across students using a conceptual framework grounded in big ideas and essential understandings of quadratics and a children's mathematical learning perspective. I found that students have a strong sense of the symmetry of the parent function, but are not consistently able to explain the cause of that symmetry. As students solved equations and graphed functions, they transitioned between equations set equal to constant values, expressions, and equations defining functions. At times this was a productive strategy, but for some students it reflected confusion about what they were solving. Lastly, I found that students apply their understandings from work with linear functions to solving and graphing quadratic equations. This study provides an initial framework for how students think about quadratic functions which may enable mathematics educators to better interpret how students' prior learning influences their understanding of big ideas within the study of quadratic functions.

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Understanding Quadratic Functions and Solving Quadratic Equations: An Analysis of
Student Thinking and Reasoning

CHAPTER 1: INTRODUCTION

Quadratic functions play a central role in secondary mathematics. They are one of the first families of non-linear functions that students encounter, and a strong understanding of quadratic functions is fundamental to success in much of the mathematics to come. Developing an understanding of quadratics is critical to students' learning trajectories in mathematics as they progress to working with higher-degree polynomials and rational functions, which feature heavily in higher-level math classes in high school and college. By the time students complete their Algebra 2 class, the Common Core State Standards for Mathematics (CCSSM, 2010) specify that they are expected to be able to solve quadratic equations using multiple methods; use their understanding of quadratic functions to create and analyze graphs; and apply these skills, knowledge and understanding to help them solve problems arising from a variety of contexts.

Quadratic functions are second-degree polynomial functions of the form $ax^2 + bx + c$ in which a , b , and c are constants and $a \neq 0$. Any quadratic function can be represented by an algebraic expression or graph. If f denotes a quadratic function, with x being the independent variable, the function can be written in the form $f(x) = ax^2 + bx + c$. In this case, the function f is defined as the function given by the expression $ax^2 + bx + c$, which maps each value d of x in the domain to a value $f(d)$ in

the range. Secondary teachers and students often write equations that define or represent quadratic functions in the form $y = ax^2 + bx + c$, where y is being defined as the quadratic function $ax^2 + bx + c$. In most high school math classrooms students interact with quadratic functions in which a , b , and c are integers. Teachers and students also work with quadratic equations that result from setting a quadratic expression equal to a constant (often zero), such as $3x^2 - 4x + 2 = 0$. Several big ideas emerge in the high school algebra study of quadratic functions. These ideas are tied together across algebraic, graphic and tabular representations and are expanded on below.

While understanding quadratic functions is critical for student success in mathematics, there appears to be agreement in the field that for many secondary students, understanding quadratic functions and solving quadratic equations is one of the most conceptually challenging subjects in the curriculum (Vaiyavutjamai, Ellerton, & Clements, 2005; Kotsopoulos, 2007; Didis, 2011). A review of the literature of student learning of quadratic functions and student solving of quadratic equations reveals that the existing research has primarily focused on procedural aspects of solving quadratic equations, with a small amount of research on how students understand variables and the graphs of quadratic functions. Very little is known about students' understanding of the behavior of quadratics and how the graphs and equations of quadratic functions are related.

This study sought to learn what high school students understand and are able to do after studying quadratic functions in Algebra 2, including what they bring to the topic from previous mathematical experiences. The study provides an initial framework for how students think about quadratic functions which may enable mathematics educators to

better interpret how students' prior learning influences their understanding of big ideas within the study of quadratic functions. Continued research in this area will support curriculum development and instructional decision-making.

The following research questions guided this study:

- What do high school students who have completed an Algebra 2 or Pre-Calculus class understand about quadratic functions?
- What connections, if any, do students make between equations and graphs of quadratic functions?
- How do students approach solving quadratic equations, and how do they interpret the solutions?
- What cognitive affordances support students in their learning and understanding of quadratic functions, and what cognitive obstacles do they encounter?

In this dissertation I will report on my study of student understanding of quadratic functions and solving quadratic equations based on data from 27 cognitive interviews with high school students. I asked the students to solve equations, describe graphs of equations, and to consider graphs of parabolas and tell me what they knew about the function. In these interviews I asked students to tell me how they knew what they told me and why they made the algebraic and graphical moves and decisions that they did. I recorded the interviews using audio- and video-recording, and additionally, I used a “smartpen” to create a “pencast” which allowed me to replay students' work and listen to what they said as they did mathematics on paper. I analyzed the student work and

transcripts to gain a deeper understanding of how students think and reason about quadratic functions and equations.

I argue that students have a deep understanding of the symmetry of the squaring function, and this symmetry can be an affordance to support them in solving equations. Students transition between equations, expressions and equations defining functions as they solve equations and graph functions. This can reflect an understanding of the relationships between the objects, and can be a productive strategy for understanding quadratics, but it may also reflect a lack of understanding of the three objects, what they are, and how they are connected. Additionally I found that students build their understandings of quadratic functions on their understandings of linear functions. There are affordances in this as linear techniques are a building block of understanding quadratics functions. However, linear thinking can also create obstacles when students incorporate these ideas without understanding as they take the linear idea of undoing and apply it to solving quadratics. Following the description of these foundational understandings, I offer an emergent framework of student learning of quadratics. The framework characterizes how these understandings evolve as students develop conceptual understanding of quadratic functions and equations.

In the next chapters I will describe framing ideas and informing literature, including a description of the mathematics students learn within the study of quadratics functions, a conceptual framework for student learning and a review of the literature on student learning of quadratic equations. In Chapter 3, I describe the study design and methods for data collection and analysis, and in Chapter 4 I share the findings of this study. Last, in Chapter 5 I offer a discussion of the findings in which I describe a

potential framework for student learning of quadratics as well as implications for research and instruction.

CHAPTER 2: FRAMING IDEAS AND INFORMING LITERATURE

This study is framed by the mathematical content that students learn as they study quadratic functions, which is described here using the framework of big ideas and essential understandings. The second set of framing ideas is a conceptual framework for understanding how students learn, in which I take a children's mathematical thinking perspective. Additionally this study was informed by a review of the existing literature on how students understand quadratic functions and equations.

Big Ideas and Essential Understandings of Quadratic Functions

The National Council of Teachers of Mathematics (NCTM) has developed a series of books that explore the big ideas and essential understandings in a range of mathematics topics. Big ideas are statements of concepts that are central to the mathematical topic. These big ideas link together essential understandings, which are smaller, more concrete ideas. For this exploration of quadratic functions, I have used *Developing Essential Understanding of Functions* (Cooney, Beckman, & Lloyd, 2010) as a starting point in identifying the big ideas and essential understandings in the study of quadratic functions.

I propose that the overarching idea that acts as an umbrella for the big ideas and essential understandings about quadratic functions is that quadratics are functions that can be used to model particular kinds of phenomena. All quadratic functions share common characteristics with the parent function $f(x) = x^2$, such as symmetry about a vertical line passing through the vertex of the corresponding parabola. This symmetry, which can be seen in the graph and table of values of the function, results from the fact that for every value in the domain of the function, the number squared is equal to that number's opposite squared (i.e. $n^2 = (-n)^2$). Quadratics can arise from situations with an

underlying multiplicative relationship such as the area of rectangles (Lappan, et al., 2009), and their equations can be written as the product of two linear binomials. Due to this multiplicative nature of quadratic functions, they have different patterns of change than linear and other functions. These patterns in symmetry and rate of change can be seen in the equations, graphs and tables of quadratic functions and in the connections among those representations.

The big ideas are further described below, along with the implications for learners. I propose that they can be delineated into five ideas. The first is the overarching idea that quadratics are functions that share key characteristics with the squaring function $y = x^2$ and can be used to model real-world situations. The remaining four interconnected big ideas are: (a) quadratics are expressed by second-degree polynomials, which can present in several different forms, each of which helps identify key features of the quadratic function; (b) equations of quadratic functions set equal to constant values may have one, two or no real solutions, and these solutions reveal information about the graph of the parabola; (c) graphs of quadratic functions are parabolas as a result of the squaring function; and (d) tables of values of quadratic functions can be used to identify patterns in the behavior of the functions. Each of these big ideas is comprised of several essential understandings, which overlap and provide different perspectives on key features of quadratic functions. As students develop a deep understanding of quadratic functions, they become fluent in each of these big ideas and flexible in their ability to see and leverage the connections among them.

Quadratics are functions. Quadratics are functions that share key characteristics with the squaring function, $y = x^2$, and can be used to model real-world phenomena.

Being a function means that quadratics are a single-valued mapping from one set, the domain, to another set, the range (Cooney, Beckmann, & Lloyd, 2010). Students have learned about linear and exponential functions in their prior math studies, and quadratics are the first non-linear polynomial function that they encounter. As students continue their math education, they will encounter additional polynomial functions, including cubic and quartic functions, and they will apply many of the concepts they learn about quadratics to those higher-degree polynomials to help them identify critical points such as local minima and maxima and zeros.

Many of the specific characteristics that quadratic functions have as a result of squaring can be seen in the parent function, $y = x^2$ (shown in Figure 1). The parent function is the simplest instance of the quadratic function $y = ax^2 + bx + c$, in which a equals 1 and b and c equal 0. In the parent function, for each value of x input into the equation, the opposite value of x gives the same result. For example, $3^2 = 9$ and $(-3)^2 = 9$. As a result, the function is symmetric about $x = 0$. This symmetry can be seen in the graph, as the parabola is symmetric across the y -axis; students also learn to see the symmetry in the table of values. The only situation in which the parent function will not have two x values that result in the same value of y is when $+\sqrt{y} = -\sqrt{y}$, which happens when $y = x = 0$. This point, $(0, 0)$, is the vertex of the parent function. Every quadratic function has a vertex that occurs at the function's maximum or minimum, meaning that the value of y is either the least or greatest in the range of the function. The vertex can be observed on the graph of the function and can be observed in the table of values when the vertex occurs at a point with integer coordinates. Because quadratic functions have

maximum or minimum values, their range is limited to all real numbers less or equal to the maximum or greater than or equal to the minimum.

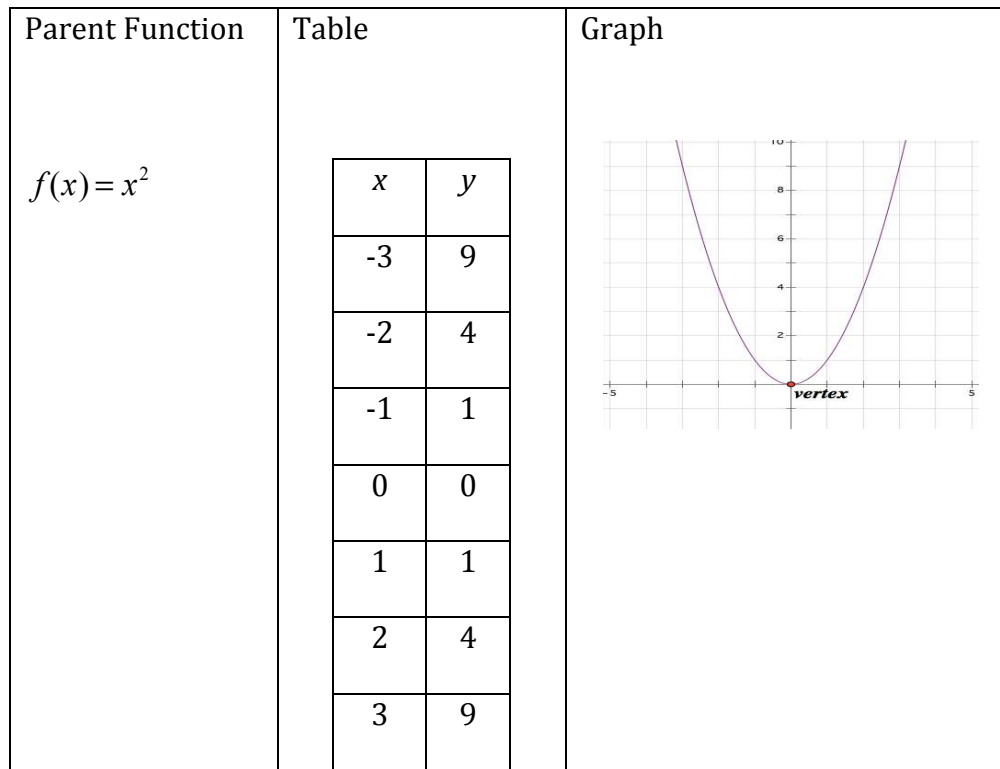


Figure 1. Table and Graph for the Parent Function

In addition to being non-linear, quadratic functions are not one-to-one. All but one value of y for any given quadratic function corresponds to two values of x . Because of the nature of the shape of a parabola, it will cross any given horizontal line in one, two or no points, depending on where that horizontal line is drawn. Students learn that as non-one-to-one functions, quadratics do not pass the “horizontal line test,” meaning that it is possible to draw at least one horizontal line on the graph that intersects the function more than once. They can also extend this thinking to understand why a quadratic function set equal to a constant will have one, two or no real solutions by thinking about whether and where the graph of the quadratic function crosses the x -axis, which is related to where the function’s maximum or minimum is relative to the x -axis.

Real-world phenomena that result from squaring relationships can be modeled by quadratic functions. Students encounter these relationships as they explore quadratics, as scaffolds in learning how to multiply binomials, and they use what they have learned about quadratics to solve application problems. For example, the area of a rectangle with fixed perimeter P and sides l and w can be modeled with functions such as $A = l\left(\frac{P}{2} - l\right)$ or $A = w\left(\frac{P}{2} - w\right)$, which are quadratic functions expressed as the product of two linear pairs (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2009). Exploring quadratic functions as models for area enables students to experience a real-world phenomenon that quadratics model and to explore how the quantities of length and area vary relative to each other. As the length of a rectangle with fixed perimeter increases, the area of the rectangle also increases until it reaches a maximum, and then it decreases. Previously, students have not encountered a function that increases and then decreases. Area as a model for quadratics also provides a scaffold to support students as they learn to multiply binomials. This can develop naturally from multiplication algorithms they have explored, such as partial products. Students also learn that quadratics model real-world situations that involve projectile motion and the effects of gravity on falling objects.

Quadratic functions are second-degree polynomials. Quadratic functions are expressed as second-degree polynomials, meaning that they always have two as the highest power of the variable. They can present in several different algebraic forms, and changing the form of the expression of a function does not change the function, the graph or the values in the table (Cooney, Beckmann, & Lloyd, 2010). As students learn to transform from one form to the other, they learn that each algebraic form of a quadratic

gives access and/or insight into different characteristics of the graph and table of the function.

The value of the coefficient a . All three forms of the expressions of quadratic functions (shown below) contain the non-zero constant a . This constant a is the leading coefficient of the polynomial in standard form, and it is the constant factored out of the factored and vertex forms. In all three forms, the sign of a indicates whether the corresponding parabola opens upward or downward, and the absolute value of a indicates how and whether the parabola is dilated. Early in their experiences with quadratic functions, students primarily encounter expressions in which the value of a equals 1 and is often not written in the expression. They come to recognize that when there is no written value of a , it equals 1.

Expressions in standard form: $ax^2 + bx + c$. Expressions in this form “look” like recognizable polynomials to students. The parameter c is the value of the y -intercept of the graph of the parabola, and the line $x = \frac{-b}{2a}$ is the line of symmetry.

Expressions in vertex form (translated form): $a(x - h)^2 + k$. This form consists of the square of a binomial which is then multiplied by the constant a and summed with the constant k . The vertex of the parabola is at (h, k) .

Expressions in factored form: $a(x - r_1)(x - r_2)$. This form is the product of two binomials, which are linear factors, and a constant a . The zeros (or roots) of the quadratic function are at $x = r_1$ and $x = r_2$, and students learn that transforming an expression into factored form is a method for finding the zeros. They also learn that the line of symmetry and vertex of the corresponding parabola lie halfway between the two roots.

Relationships. There are relationships between the forms of the expressions, and each of the forms can be manipulated into the other forms by multiplying binomials, factoring, or completing the square. When a real-world situation is modeled with a function, different expression forms may arise from the situation. For example, situations involving area may lend themselves to generating an expression in factored form, which a student may then manipulate into another form to further investigate the function. The following methods of algebraically manipulating the expressions correspond to methods of solving the related quadratic equations, which are explained in more detail in the next section.

- Transforming from standard to factored form involves factoring.
- Transforming from standard to vertex form involves completing the square.
- Transforming from vertex or factored form requires using order of operations to multiply monomials and constants and to add any constants in order to obtain standard form. Then the expression can be transformed to one of the other forms.

Expressions of quadratic functions can be set equal to constant values and solved. A quadratic function can be set equal to a constant value and solved for the variable. If the expression of a quadratic function is set equal to a real number, d , the resulting equation, $ax^2 + bx + c = d$, can be solved for x . Solving reveals the solution(s) to the system of equations consisting of the original quadratic function, $f(x)$, and the linear function $y = d$. In other words, the solution(s) indicate at what values of x the graph of the quadratic function and the line $y = d$ intersect. These solutions are the same as the solutions of the equation $f(x) - d = 0$, which is also a quadratic equation. When the value of d is zero, solving the quadratic equation yields the zeros (or roots) of the

function. As an example, consider the function $f(x) = x^2 - 4x + 5$ and the line $y = 2$ (shown in Figure 2). Discovering where $f(x)$ intersects the line $y = 2$ is the same as finding the zeros of $f(x) - 2 = 0$, which can be found algebraically by solving the equation $x^2 - 4x + 3 = 0$.

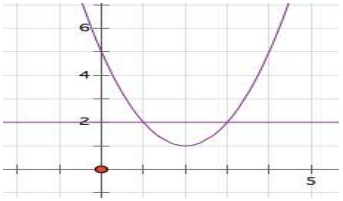
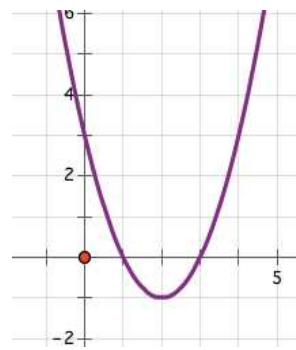
<p>Intersection of:</p> $f(x) = x^2 - 4x + 5$ $y = 2$  <p>Intersections at $x = 3$ and $x = 1$</p>	<p>Graph of:</p> $f(x) - 2 = 0$  <p>Roots at $x = 3$ and $x = 1$</p>	<p>Algebraic Solution:</p> $f(x) = 2$ $x^2 - 4x + 5 = 2$ $x^2 - 4x + 3 = 0$ $(x - 3)(x - 1) = 0$ $x = 3 \text{ or } x = 1$
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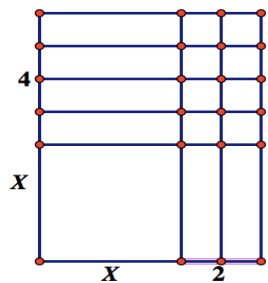
Figure 2. Solving Quadratic Functions

Quadratic equations in standard form ($ax^2 + bx + c = 0$) can be solved though by methods that are parallel to the methods for moving from one form of a quadratic equation to another, as outlined above.

Factoring involves using guess and check or a systematic process of making lists of factor pairs of the parameters b and c to factor a quadratic expression in standard form into the product of two linear pairs (binomials) and sometimes a constant. Students can use an area model to help them think about how to find the two linear factors of the

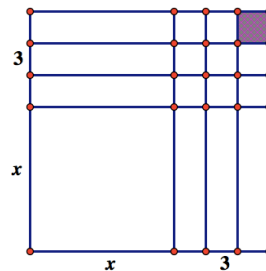
quadratic. The zero-product property can then be used to set the binomials equal to zero so they can be solved.

Completing the square involves understanding the structure of perfect square quadratic expressions and using that structure to algebraically manipulate an expression in standard form into an expression that involves a perfect square. When the initial expression can be represented with positive area, this process can be modeled with a geometric representation in which students work to transform an area from a rectangle to a square with a small adjustment. For example, $y = x^2 + 6x + 8$ can be rewritten as $y = (x + 3)^2 - 1$ as shown in Figure 3. This can be modeled by building a rectangle using algebra tiles, which models its area as the product of the lengths of the sides of the rectangle $(x + 2)(x + 4)$ or as the sum of the areas of the smaller rectangles, x^2 , $2x$, $4x$, and 8 which gives the sum $x^2 + 2x + 4x + 8$ or $x^2 + 6x + 8$. Then the pieces can be moved around to form a square with sides $x + 3$ units long. The area of the new square will be one greater than the area of the original rectangle, which can be shown by the missing (red) square in the figure.



$$(x+4)(x+2)$$

$$x^2 + 6x + 8$$



$$(x+3)^2 - 1$$

$$(x^2 + 6x + 9) - 1$$

$$x^2 + 6x + 8$$

Figure 3: Completing the Square

Using the quadratic formula involves being able to use the parameters a ($a \neq 0$), b and c in an algebraic expression, substitute in the appropriate values and then solve using order of operations. For high school students this can be a complicated process as it requires them to use addition, subtraction, multiplication, division, squaring, and to take the square root of a quantity while accurately applying the correct order of operations.

The quadratic formula also gives information about the graph of the parabola that can be

read from the equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. Here it is worth noting

that because a is not equal to zero, the denominator of these fractions is never zero. The

line of symmetry of the parabola is accessible from the formula $x = \frac{-b}{2a}$. The value of

$\frac{\sqrt{b^2 - 4ac}}{2a}$ dictates the distance between the line of symmetry and each root; the

discriminant, $b^2 - 4ac$, indicates whether there are real roots and, if so, how many. If

$b^2 - 4ac$ is greater than zero, there are two real roots; if it equals zero, there is one; and

if it is less than zero, there are no real roots. When the discriminant is less than zero, there are two complex roots.

Graphs of quadratic functions are parabolas. The graphs of quadratic functions are parabolas, and any graph that is a parabola opening upwards or downwards is the graph of a quadratic function¹. Students learn how to use symmetry and their understandings about the maximum or minimum values and intercepts to help them generate and understand graphs of quadratic functions. One essential understanding about parabolas is that they are always symmetric about the line of symmetry, which passes vertically through the vertex. This understanding supports students in moving from an equation to the graph. Understanding the symmetry of a parabola also enables students to identify key features of a graph so they can generate the equation of the function from its graph. As noted above, the vertex of the parabola occurs at the maximum or minimum of the function. Students develop an understanding that the vertex is a critical point in understanding the behavior of the parabola; learn how to locate it; and learn to investigate the behavior of the graph near the vertex. The equation of the line of symmetry and the vertex can be found from the algebraic representations, described above in the section on the algebraic forms of quadratic equations.

Students also learn that the zeros or roots of a quadratic are key features of the function and that the values of the solutions are instrumental in understanding the function's graph. Students learn that the graph of a quadratic function $f(x)$ intersects the

¹ This study focuses on parabolas opening up or downwards and not on relations such as $x = y^2$ or graphs in which the line of symmetry is not vertical.

x -axis in one or two places if the equation $f(x) = 0$ has real solutions. If $f(x) = 0$ has no real solutions, the graph does not intersect the x -axis. In this case, the function has complex solutions. When students have a deep understanding of the quadratic formula, they may use the value of the discriminant ($b^2 - 4ac$) to determine the number of real solutions.

Graphs of quadratic functions always have a y -intercept. This is because $x = 0$ is in the domain of all quadratic functions, so the graph of any quadratic function $f(x)$ will intersect the y -axis at $(0, f(0))$. Knowing both that a quadratic function has a y -intercept and how to find or identify it enables students to find this important point in the graph of a parabola.

Additionally, students learn that the coefficient of the x^2 term (a in $f(x) = ax^2$) indicates whether the parabola will open up or downwards. In the parent function $y = x^2$, the value of a is 1, and the graph of the function is a parabola opening upwards, with a minimum at the vertex, $(0, 0)$. In the function $f(x) = ax^2$, if the value of $a > 0$, the parabola will also open upwards, since each positive result from the parent function is being multiplied by a positive value of a . If $a < 0$, the graph will be a parabola opening downwards with a maximum at the vertex, since each positive result from the parent function is being multiplied by a negative number. This has the effect of reflecting the parabola across the x -axis. The sign of a has this impact for all algebraic representations of quadratic functions. Knowing whether the graph opens up or downwards enables a student to determine whether it has a minimum or maximum and to know how the y -intercept fits in with the rest of the parabola; it also may help the student understand more about a potential situation the quadratic function may be modeling.

In addition to moving from the equation of a quadratic function to its graph, students learn that they can use these critical features of the graph of a parabola (intercepts, vertex, line of symmetry, direction of opening) to generate its equation. If students identify the vertex of a graph, they may be more inclined to generate the vertex form of the equation, whereas if they know the intercepts, they may tend to generate the factored form. They also may use reasoning and additional points on the parabola to find the value of a for the equation of the function the parabola describes.

Tables of quadratic functions reveal patterns. Tables of values of quadratic functions can be used to investigate function behavior and to reveal patterns. Students learn that it can be beneficial to use a table of values to explore a function's behavior around critical points such as the x - and y - intercepts, the line of symmetry, and the vertex. As they explore the table of values, they will find that for all but one value of x , there is another value of x that has the same y value, and there is only one value of x with a unique value of y , which corresponds to the vertex of the parabola. In the table of values of a quadratic function, students may be able to identify the maximum or minimum value of the function, and observe the symmetry of the function around the maximum or minimum. The table also may reveal the zeros of the function if students input values of x that result in $f(x) = 0$. Additionally tables of values can support students in noticing that while the first differences are not equal as they are in linear functions, the second differences in quadratic functions are equal.

Conceptual Understanding of Quadratics

In the preceding section, I have described the big ideas and essential understandings that students learn in their study of quadratic functions. A student who

has developed an understanding of quadratics understands these big ideas and essential understandings as well as the relationships between them. Figure 4 is a depiction of what this understanding might look like. Students have understanding of graphs, expressions and equations, as well as tables. Additionally, they understand how to transition between the various forms, and they know that each form highlights information about key features of the function”

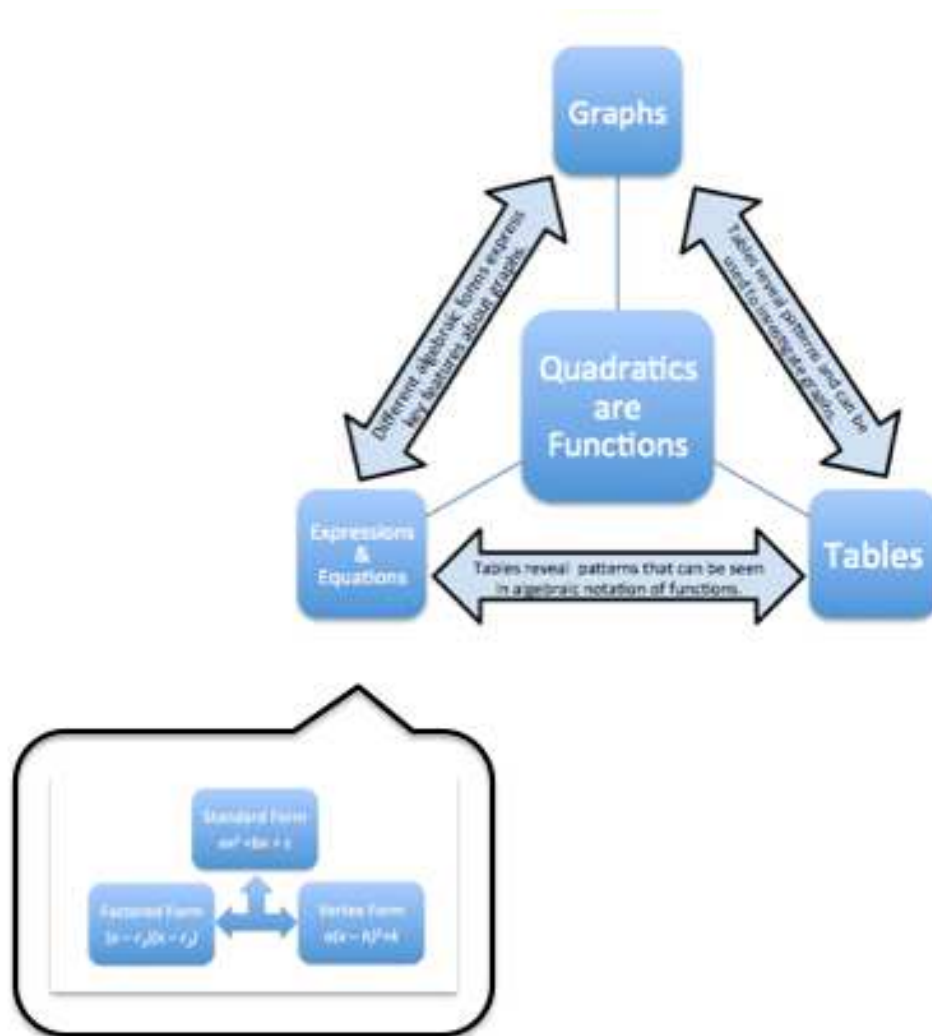


Figure 4: Conceptual Understanding of Quadratics

If one zooms in on expressions and equations (shown in Figure 5), one can see that students develop understanding of how to manipulate each form of a quadratic

algebraically to get other forms. Additionally students learn that the different forms highlight different information about a given function and learn how to decide what forms are advantageous to solve a particular problem.

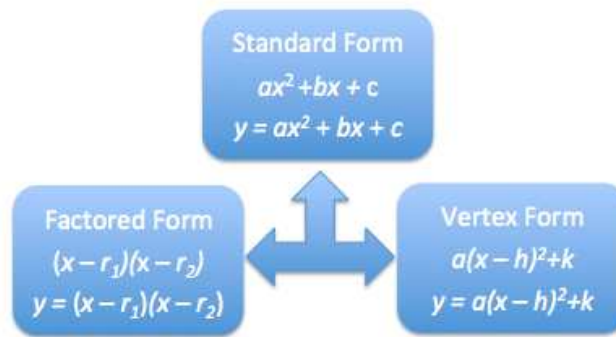


Figure 5: Relationships between expressions and equations

I have described the big ideas and essential understandings that students learn in their study of quadratic functions and what conceptual understanding of quadratics entails. I now consider how students learn mathematics more generally. In the next section, I describe how students learn taking a children's mathematical thinking perspective and examine a framework for understanding the development of conceptual understanding before going on to offer a review of the literature.

Conceptualizing Student Learning

This study examines students' learning about quadratic functions and equations from a children's mathematical thinking perspective (Carpenter & Moser, 1984, Bishop et al., 2014). That means that this study attempts to provide a detailed account of students' thinking about how they understand quadratic equations and functions rather than focusing exclusively on how students perform on procedures or on what they cannot do. This study is further grounded in constructivist learning theory, which takes the

position that learners create new knowledge and understanding based on what their previous knowledge and beliefs. As students develop conceptual understanding of quadratic equations and functions, their prior knowledge plays a critical role (Bransford, 2000). This prior knowledge may include cognitive affordances, which can lead to successful problem solving and learning, or cognitive obstacles, which can impede problem solving and/or learning (Bishop et al., 2014).

A children's mathematical thinking perspective. Since the early 1980s, a body of research has emerged that focuses on children's mathematical thinking. Carpenter and Moser (1984) studied elementary students' addition and subtraction concepts, and as a result they were able to provide a detailed account of children's solution strategies and how those strategies change over time. Further work found that though teachers had a great deal of intuitive knowledge about children's mathematical thinking, it was fragmented and, consequently, generally did not play an important role in most teachers' decision-making (Carpenter et al. 1988). Carpenter and colleagues further found that a teacher's knowledge of his or her students' thinking was related to student achievement. Furthermore, learning to understand the development of children's mathematical thinking could lead to fundamental changes in teachers' beliefs and practices, and these changes could be reflected in students' learning. Students of teachers who knew more about their students' thinking had higher levels of achievement in problem solving than students of teachers who had less knowledge of their students' thinking (Carpenter et al., 1988).

Initial studies of children's mathematical thinking were focused on addition and subtraction and other whole number operations. Subsequent studies have focused on other areas of mathematical content, including base ten concepts, algebraic reasoning

(Carpenter, Franke, & Levi, 2003), fractions and decimals (Empson, 1999, 2001) and integer reasoning (Bishop et al., 2014). These “research-based models of students’ mathematical thinking are used to help teachers make instructional decisions that support the development of more sophisticated conceptual understanding on the basis of students’ current understanding and strategy” (Bishop et al., p. 25).

Conceptualizing conceptual understanding. In this study, I am conceptualizing “forming a concept” as being synonymous with developing conceptual understanding, and in this section I explain how I am thinking about student conceptual understanding and how students develop conceptual understanding. Vygotsky’s theory offers a framework for the stages students progress through as they form a concept (Vygotsky, 1987). Skemp (1976) offers another way of thinking about conceptual understanding. He categorized being able to memorize and apply rules as *instrumental* understanding, and knowing what to do and why as *relational* understanding. Relational understanding and the idea of a concept is similar to Hiebert and Carpenter’s (1992) assertion that “a mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections” (p. 67). Hiebert and Carpenter further clarified that *understanding* refers to the way information is “represented and structured” in the mind (1992, p. 67). As students develop conceptual understanding, they link smaller pieces of knowledge, which may consist of ideas, words, graphs, mental pictures, mathematical signs, and associated properties and processes to each other and to other pieces of knowledge or already developed concepts. The links between the pieces of knowledge are as important as the pieces of knowledge they connect.

Novices have concept images that are made up of unorganized collections of pieces of knowledge, which diSessa (1993) calls *knowledge elements*. These elements may be in conflict with each other and are activated or “cued” and connected in various ways depending on the context. Knowledge elements are “considered to include but not be limited to ... facts, experiences, intuitive conceptions ... and (ideally) some mental models and concepts at various stages of development and sophistication” (Clark, 2006, p. 471). Change within a learner’s concept image occurs as the learner gradually adds, reorganizes, and refines knowledge elements and connections between them to create a more complex conceptual structure (diSessa, 1993; Özdemir & Clark, 2007).

Stages of Concept Formation – How do students come to understand?

Vygotsky regarded things as *concepts* if they consist of ideas and parts of ideas that are linked together and to other ideas by logical connections that form part of a “socially-accepted system of hierarchical knowledge” (Berger, 2005, p. 158). He suggested that concept formation proceeds through different pre-conceptual stages. Initially, in the *syncretic heap stage*, a student groups together ideas or objects that are objectively unrelated. Students may group ideas together simply because they are grouped together on a page in a textbook, or because they were discussed on the same day in class, but this grouping takes place according to “chance, circumstance or subjective impressions in the child’s mind” (Berger, 2005, p. 157).

Vygotsky describes *complexive* thinking as a stage (called the *complex stage*) in which the child begins to “unite homogeneous objects in a common group, to combine them in accordance with the objective connections that he finds in the things themselves” (Vygotsky, 1987, p. 136). The task of the student is to decide if, how and/or why certain

objects or ideas go together and what overarching idea unites them. As the student matures, he or she begins to rely more on the characteristics of the objects themselves. In the *complex stage*, students begin to link ideas together by associations or common attributes between the items. In this stage, the learner begins to notice or abstract different attributes of the ideas or objects and starts to organize ideas that share particular properties into groups, creating a basis for more sophisticated generalizations that will come later. In this stage, the learner does not use standard mathematical logic, but relies on “non-logical or experimental association.” Berger (2005) points out that this type of complex thinking may manifest in what she calls “bizarre or idiosyncratic usage” of mathematical objects, concepts or signs (p. 157). She gives the example of students associating the properties of a “new” mathematical sign with an “old” mathematical sign with which students are more familiar. In the case of quadratics, this might be exemplified by how students conceive of the constant multiplier b in the equation $y = bx^2$. Students may tend to regard that b as being the slope in the same way that it was in linear equations, in which, students learned that in $y = bx$, the number multiplying the variable x was always the slope. They might therefore use that understanding to predict how quickly the graph of $y = bx^2$ will grow as x changes.

Students who are guided by complex thinking may also attend to one particular aspect of a mathematical expression and not see the whole. For example, when graphing a line such as $y = 2x + 1$, a student might focus solely on the y -intercept and graph a line with the correct y -intercept, but not attend to the slope. Or a student might graph a line with the correct slope that passes through the origin, but not attend to the y -intercept. In

these cases, the student may understand the concepts of slope and y -intercept, but not know how to understand the graph of the line holistically.

In the complex stage, students use the signs and symbols of math and communicating about them. This gives them the opportunity to talk with others, including their teachers and peers. Through those social interactions and through reflection, students will eventually come to use and understand the signs in ways that are congruent with the “official” school mathematics. The child’s thinking evolves as he or she starts creating chained complexes in which ideas are linked together.

As a last step before the student forms a concept, he or she creates a *pseudoconcept*. Vygotsky envisioned the pseudoconcept as a bridge to the formation of concepts (1987, p. 142). Pseudoconcepts resemble true concepts in their use, but the thinking the student is doing is still complexive in nature. The student can use the mathematics without being able to understand what he or she is doing or explain it. For example, a student might be able to factor an expression in standard form such as $ax^2 + bx + c = 0$ and solve for x without understanding that he or she is finding the roots of the equation, which are also the x -intercepts of the graph. Berger (2005) makes the case that “the pseudoconcept can be used to explain how the student is able to use mathematical signs (in algorithms, definitions, theorems, problem-solving, and so on) in effective ways that are commensurate with that of the mathematical community even though the student may not fully ‘understand’ the mathematical object” (p. 159).

At this point in the development of understanding, the student can talk through the math with teachers and peers. Through these continuing conversations and interventions, and through being engaged in meaningful problem solving situations the student forms

concepts. Concept formation has a productive rather than reproductive character; “the concept arises and is formed in a complex operation that is directed towards the resolution of some task” (Ach, ca. 1921, as described by Vygotsky, 1987, p.124). The student has to create the concept in their understanding rather than replicate or repeat what they have been told. In order for students to form a concept, they need to be engaged in a task that can only be solved by forming a concept. In the case of quadratic equations, students might be able to form a concept of the forms of the equations if they have the opportunity to explore how each of the equations relates to the graph and to really grapple with what the equations have in common and how they are different.

Understanding and the Role of Prior Knowledge. One of the main tenets of constructivist learning is that everything a person learns is built upon what the learner already knows and understands. “People continually try to understand and think about the new in terms of what they already know” (Glaser, 1984, p. 100). Learners “come to formal education with a range of prior knowledge, skills, beliefs, and concepts that significantly influence what they notice about the environment and how they organize and interpret it” (Bransford, 2000, p. 10). Bransford goes on to suggest that the logical extension is that “teachers need to pay attention to the incomplete understandings, the false beliefs, and the naïve renditions of concepts that learners bring with them to a given subject” (2000, p. 10).

Cognitive Affordances and Obstacles. Bishop et al. (2014) suggest that students’ prior knowledge can be classified into two categories: knowledge that supports each student in being successful (affordances) and knowledge that may hinder student learning (obstacles). They trace the idea of cognitive obstacles to the French philosopher

Gaston Bachelard and his theory of epistemological obstacles in the development of scientific thinking (Bishop et al., 2014, p. 26). Bishop et al. analyzed how and why specific problems involving integers were or were not solved, looking for commonalities and patterns that enabled them to identify these affordances and obstacles. They suggest that cognitive affordances are ways of reasoning and prior knowledge that lead to successful problem solving. When cognitive affordances enable students to be successful connections in their understanding will be strengthened (Hammer, 1996; Smith, diSessa, & Roschelle, 1993). In contrast, cognitive obstacles are instances of “knowledge that is useful in solving a certain type of problem, but when applied to a new problem or context is inadequate or leads to a contradiction” (Bishop, et al., 2014, p. 26).

Generally, more work has been done with respect to students’ understanding of quadratics to understand the nature of cognitive obstacles rather than affordances. Additionally, in that work, cognitive obstacles are often thought of as misconceptions. Thinking of cognitive obstacles as misconceptions gives them the onus of being wrong or fallacies rather than as being ideas that apply to or were an extension of previous knowledge. Cory and Garofalo (2011) suggest that prior knowledge that may be considered an obstacle can also come in the form of *alternate conceptions*. These are ideas that may be unproductive in solving the mathematics, or they may be productive, but not “privileged by the mathematical community” (Cory & Garofalo, 2011, p. 68). In the beginning of their experience with quadratic equations, students might believe that solving a problem with an x^2 in it must be about area. This makes sense given the students’ prior experiences with squaring but may function as a cognitive obstacle if it leads to the generalization that the answers to problems with x^2 in them must be positive.

Students' Understandings Of Quadratics: A Review Of The Literature

This study takes a children's learning perspective as it focuses on how students learn the big ideas and essential understandings of quadratic functions. I have explored a framework for how students acquire conceptual understanding, including understanding students' prior knowledge as potential cognitive affordances and obstacles. I now provide a review of the research literature that focuses on student learning of quadratic functions, solving quadratic equations, and understanding their graphs with a specific focus on cognitive affordances and obstacles students encounter.

A review of the literature reveals that there has been a small amount of research on difficulties students encounter with quadratics as functions and on how students approach graphs of quadratic functions. Most research on students' understanding of quadratic functions has focused on students solving quadratic equations. There appears to be a dearth of research on how students understand tables of quadratic functions or on their understanding of how the squaring function behaves as evidenced in graphs and tables. Across the literature, the focus has been on students' difficulties and struggles, some of which may be cognitive obstacles, with quadratic functions. This review of the literature is organized around a subset of the big ideas and essential understandings described above related to the following topics: quadratics as functions, generating graphs and equations of quadratic functions, and solving quadratic equations.

Quadratics are functions. A modest amount of research has found that students encounter some obstacles in their study of quadratics that relate to the non-one-to-one nature of quadratic functions. Specifically, students may have difficulties with how

variables behave and may not completely understand that an equation can have more than one solution.

Variables. A difficulty for some students is that they may have a misunderstanding of the variable x in a quadratic in factored form (Vaiyavutjami & Clemments, 2006). For example, in the factored equation $(x - 3)(x - 5) = 0$, some students think that the first x stands for one value and the second stands for a different value. The authors analyzed the work of 231 students in government schools near Chiang Mai, Thailand, and 34 interview transcripts before and after a set of eleven lessons on quadratic equations. In the interviews, students said things such as “The solutions are 3 and 5 because $(3 - 3)(5 - 5) = 0 * 0$ which is 0.” In this case, the solution that $x = 3$ or 5 is correct, but the student’s explanation reveals that this student thinks that x can be 3 and 5 simultaneously. In fact, x can equal 3 or x can equal 5, but it cannot equal both at the same time because $(x - 3)(x - 5)$ is a function and x can only have one value at a time. Didis, Bas and Erbas (2011) confirmed this result using an open ended test given to one hundred and thirteen high school students in Antalya, Turkey. By analyzing the students’ written explanations, Didis et al., (2011) found that students can reach the correct answer that $x = 3$ or $x = 5$ without an understanding of how the zero-product property guarantees that one of the factors will be equal to zero.

More generally, quadratics are the first family of functions students encounter that may have one, two or no real roots, and they may tax students’ understandings of how the variable x behaves across the domain. When working with quadratics, students continue to develop the understanding that a function will have different values as x takes on different values across the domain, and that the value of a function can equal zero for

more than one value of x . This ties to understandings of the big idea of function as well as to ideas of variables representing values that “vary” rather than standing for an answer to a problem.

Focusing exclusively on the positive solution. Thorpe (1989) suggests that students do not understand that the solutions to equations in the form $x^2 = a$ have two solutions, positive and negative a . He asked students to solve problems such as $x^2 = 100$ and found that students most often gave the solution $x = 10$, forgetting the solution $x = -10$. Thorpe suggested students may expect equations to have only one solution, and he further suggested that the meaning of the plus or minus symbol (\pm) in the quadratic formula might not be fully understood.

Generating graphs and equations of quadratic functions. Research on how students understand graphs of quadratic functions reveals that students prefer generating graphs from equations of functions over generating an equation of a function from its graph. This research is primarily from the work of one researcher, Orit Zaslavsky (1997). Zaslavsky collected data from over 800 10th- and 11th-grade students from eight Israeli high schools in three stages. He observed classrooms and examined student notebooks. He then used his observations to construct a set of problems that might reveal students’ misconceptions regarding solving quadratic equations. Zaslavsky used students’ responses to the problems to develop a series of tasks to explore how students understand graphs. When given a set of tasks in which they were asked to determine which of four parabolas corresponded to an equation and which of four equations corresponded to a parabola, students appear to have worked in the “same direction,” consistently starting with the algebraic equations to see which matched the graphs (Zaslavsky, 2009, p. 34).

When working with graphs, students made assumptions about parabolas based on the graphs they saw and did not use their understandings of quadratic functions to help them interpret the functions. Zaslavsky (1997) characterized the students' actions as assuming that the part of the graph of a quadratic equation (parabola) they can see portrays the entire behavior of the function. For example, in considering a graph such as the one shown in Figure 4, students might assume that if the y-intercept is not shown, there is no y-intercept. Students might further assume that a parabola has a vertical asymptote.

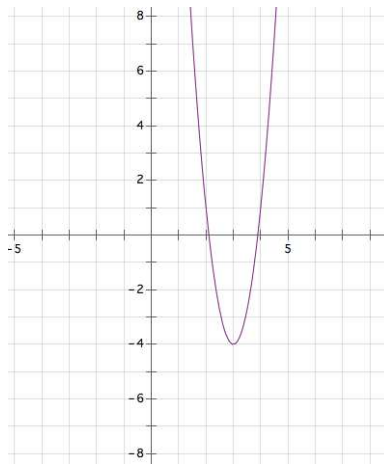


Figure 6: Graph of a Parabola Without a Visible y-intercept

Zaslavsky (1997) also found that students were “not able to use the implicit information related to line of symmetry” unless specifically directed to (p. 34).

When students graph equations of quadratic functions, they can encounter obstacles from their previous experiences with graphs of lines. The symbols for the parameters of linear and quadratic equations are often the same. Linear functions are often represented as $y = ax + b$ and quadratics as $y = ax^2 + bx + c$. Zaslavsky (1997) explains that students' experiences with linear functions in the form of straight lines go way back to preschool, so they tend to rely excessively on linear principles. This may lead some students to try to apply their understandings from linear functions to quadratics

when graphing. For example, as described earlier, when working to create graphs of parabolas, students might try to find the “slope” of a parabola using the value of b .

Solving quadratic equations. There seems to be agreement in the field that when solving equations, students tend to use procedures without understanding and that students have difficulties with aspects of solving quadratic equations such as factoring, applying the zero-product property, and solving equations that are not in general form (Sönnerhed, 2009; Didis, 2011). In an analysis of textbooks in Sweden, Sönnerhed (2009) found that the curricular materials provided students with the opportunity to learn to use factoring to solve quadratic equations quickly without paying attention to their structure and conceptual meaning. When students learn to solve quadratic equations, they are taught to memorize and enact rules and procedures with little understanding of the meaning of the quadratic equations or what the solutions they find might mean.

Sönnerhed’s finding fits well with Skemp’s (1976) categorization of being able to memorize and apply rules as *instrumental* understanding, and knowing what to do and why as *relational* understanding. When students only memorize procedures, they may develop instrumental understanding while their relational understanding lags behind. In their analysis of student work on quadratic equations, Didis et al. (2011) found that students incorrectly tried to transfer rules from one form of an equation to another.

Solving by factoring. One common method for solving quadratic equations involves factoring to transform an equation from standard to factored form and then applying the zero-product property to complete the solution. Didis, et al. (2011) found that students prefer factoring as a solution method when the quadratic is obviously

factorable, and that students can solve such equations quickly using factoring without paying attention to the structure and conceptual meaning.

Factoring can be problematic for students. This is a claim most secondary teachers would agree with, and researchers such as Kotsopoulos (2007) and Bossé and Nandakumar (2005) make the claim based on their experience in secondary and college classrooms. Some students have difficulties with their multiplication facts, which makes it difficult for them to quickly find factors for expressions in the form $ax^2 + bx + c$ (Kotsopoulos, 2007). These difficulties increase when the parameter a does not equal one (for example in expressions such as $6x^2 + 3x + 2$ and become even more challenging when a and/or c have multiple factors, leading to many possible factor pairs in expressions such as $20x^2 + 63x + 36$. Bossé and Nandakumar (2005) suggest that techniques such as using the quadratic formula or completing the square should be utilized more quickly and often, saving students the frustration of trying to factor expressions that are challenging. It is worth noting that the research literature on factoring quadratics attends to factoring when a , b , and c are integers resulting in expressions that can be factored into binomials with integer coefficients.

Once students have factored an expression and work to solve it using the zero-product property (if the product of two numbers is zero, one of the numbers must be zero), they run into additional obstacles. When working to solve an equation such as $x(x - 2) = 0$, students sometimes “cancel” the x from both sides (divide by x) leaving $x - 2 = 0$. They do not see that by doing so, they lose track of the root $x = 0$ (Didis, 2011; Bossé & Nandakumar, 2005; Kotsopoulos, 2007). Didis interprets this as students knowing the zero-product property but not being able to apply it appropriately when the

structure of the equation is changed. This could also be an example of students imposing linear structure on a quadratic as they work to solve the equation using techniques that have worked to solve linear equations.

Imposing linear structure. Because the symbols for the parameters of linear and quadratic equations are often the same, research suggests that some students try to apply their understandings from linear equations to quadratics when solving quadratic equations. In a study of 80 students in Brazil who had studied quadratic equations, de Lima and Tall (2010) asked the students to make concept maps, solve equations, and complete questionnaires that were comprised of problems involving quadratic equations. The researchers found that students take “rules” that they have developed from solving linear equations and either erroneously apply them to quadratics or use them to try to “linearize” quadratic equations. Working to isolate the variable by adding or subtracting terms from both sides is an example of the misuse of these rules, as is dividing both sides by x in the expression $x(x - 2) = 0$ in the example above.

Forms of quadratic equations. Research has shown that students consistently have difficulties working with and solving quadratic equations that are presented in a different form than what the student is used to. For example, a student who can solve an equation in standard form in which $a = 1$ and b and c are non-zero, such as $x^2 + 4x + 3 = 0$, might struggle to solve equations such as $x^2 - 2x = 0$, in which the constant term is 0 and therefore not visible in the equation, and $x^2 - x = 12$, in which the constant is on the other side of the equal sign (Didis, 2011). Didis (2011) suggests that a quadratic equation in which the parameters b or c equal 0 (examples: $y = ax^2 + c$ or $y = ax^2 + bx$) does not seem like a quadratic to students. Students may assume that if a parameter does not

appear in the equation, then it doesn't have a value, when in fact it has the value zero or one. For example, students might say that $y = ax^2 + bx$ does not have a y -intercept because they believe c doesn't exist. In this example, c does exist and has a value of zero, and the parabola would have the y -intercept at the point $(0, 0)$ (Zavlasky, 1997).

In addition, research suggests that students have strong preferences for standard form $y = ax^2 + bx + c$, rather than vertex form, $y = a(x - h)^2 + k$, or factored form, $y = (x - a)(x - b)$. Based on interview data, Vaiyavutjani, Ellerton and Clemments (2006) found that students in Thailand, Brunei and the US reported that they had been taught to reduce all quadratics to standard form and then factor or use the quadratic formula. In some instances, this results in students doing unnecessary procedures as they transform the equation out of a form that would have given them information into the standard form and then back into a form they can use to solve the given problem.

This review of the research reveals difficulties that students have with solving quadratic equations using the factoring method. It suggests potential cognitive obstacles with non-standard equations; with finding the positive and negative solutions to quadratic equations in the form $x^2 = c$ where c is a constant; and with variables. The literature also suggests that students tend to impose linear structures when working with quadratic functions and that students struggle to make the connections from graphs of quadratic functions to their expressions.

While the research literature described above provides an understanding of some of the difficulties students encounter with quadratic functions and equations, the field is in need of research that focuses on in-depth knowledge of what students understand about quadratic functions and how the squaring function behaves. How do students relate the

behavior of the squaring function to graphs and tables, and what connections do they make between those representations? We know that students encounter obstacles in solving equations, but we do not know the details of what lies behind those obstacles and how students think about them. Research also indicates that students have a strong preference for the general form of the quadratic equation of a function. We do not know, however, how students approach other equation forms. Nor do we know what connections students make among the three equation forms or between each of the forms and the graphs and tables of a quadratic function, which lies at the heart of being able to think about quadratic functions with flexibility and understanding.

In the pedagogy of teaching of quadratic functions, mathematics educators have not yet developed a research-based framework identifying and describing students' understanding or ways of reasoning. While individual teachers may have intuitive ideas of the obstacles and affordances their students might encounter or leverage, we do not have a framework that can be used to make instructional and curriculum development decisions. Bishop et al.'s work with students in grades six through ten points to the potential benefits of encouraging students to grapple with big, complex mathematical ideas and to use their questions and points of confusion as learning opportunities (Bishop et al., 2013). In order to support teachers in making these instructional decisions in the realm of quadratic equations, it is important to identify cognitive affordances and obstacles and to work to understand how learners form concepts for quadratic equations and functions.

CHAPTER 3: RESEARCH METHODS AND STUDY DESIGN

This study utilized qualitative methods to investigate students' mathematical thinking and understanding of quadratic functions through cognitive interviews (Ginsburg, 1997).

Specifically, I sought to learn what connections, if any, students make between equations and graphs of quadratic functions; how students approach solving quadratic equations and interpret the solutions; and what cognitive affordances support students in their learning and understanding of quadratics and what cognitive obstacles they encounter. I interviewed 27 high school students in their last month of an Algebra 2 or Precalculus course. Students were presented with a series of problems and asked about their thinking as they solved the problems. Additionally, students were asked what they knew and understood about quadratic functions and equations generally.

Setting

This study took place in a high school in the northwestern United States. Students in this high school take a series of math courses starting with Algebra 1 in 7th, 8th or 9th grade and then continuing with Geometry, Algebra 2, and possibly Precalculus and Calculus. Most students at this school complete at least Algebra 2. The selection of classrooms and schools was a purposeful convenience sample (Merriam, 2009). I relied on my connections with the district and teachers. I was not studying classroom instruction, and most interactions with participants took place outside of the classroom in interview settings.

Participants

Student participants were purposefully chosen with the help of their classroom teachers to represent different levels of mathematical understanding. Teachers were asked to identify students in ninth through twelfth grade who would be willing to engage in mathematics interviews and whom they experienced as more and less successful from their perspective. Students were invited to participate, and those who wished to participate and returned a signed permission slip were included in the study. Twenty-seven students agreed to participate in audio-recorded interviews, and 26 students also agreed to be video-recorded.

Of the 27 students who participated in the study, 20 were in Algebra 2, and 7 were in Precalculus. Seven ninth graders, 8 tenth graders, and 12 eleventh graders participated. No twelfth graders elected to participate. The students were distributed across 3 teachers, with 7 students from Mr. Whipple’s classes, 16 from Mr. Murphy’s classes, and 4 from Ms. Skeath’s classes². The distribution of students across the courses, grade levels and teachers is shown in Table 1.

Table 1: Participant Grade Level and Teacher by Course

Course	Grade Level			Teacher		
	9	10	11	Mr. W.	Mr. M.	Ms. S.
Algebra 2	7	6	7	7	13	0
Precalculus	0	2	5	0	3	4
Total	7	8	12	7	16	4

² Names of all teachers and students are pseudonyms.

Data Collection

Data was collected in semi-structured interviews that were video- and audio-recorded. As will be described below, in addition to audio- and video-recordings, the interviews were recorded with a “smartpen” pencast that links student writing to the audio.

Cognitive Interviews

I interviewed each student independently for approximately 40 minutes in conference rooms at the high school over the course of two three-day visits to the school in May and June of 2014. I gave each student the series of math problems, to solve one at a time. For each question, I stated the question verbally and wrote the algebraic representation in the notebook or showed the student a graph. The questions are described in the interview protocol in Appendix A. For each problem, I asked students to describe their thinking or how they solved the problem. When students had questions or problems that they were uncertain about, I offered to have them ask me questions in the last few minutes of the interview. At the conclusion of the interview, if time allowed, I followed up with a few more biographical questions to better understand the student’s attitudes towards and beliefs about mathematics and math instruction. Using semi-structured interviews enabled me to have a set of well-structured interview questions while allowing me to use those questions flexibly and respond to what students did and said (Merriam, 2009).

Interview protocol. I designed an interview protocol to elicit students’ understandings regarding the big ideas and essential understandings of quadratics. I included questions: (1) designed to elicit students’ ideas of what quadratics are in general

and how (and whether) they are related to functions; (2) that were specifically about the parent function $y = x^2$; (3) that asked students to solve quadratic equations to learn how they understand solving when equations are presented in different forms, when they have irrational or non-real complex solutions, or when one of the parameters (a, b, or c) is set equal to zero; (4) that asked students to describe and graph functions from their equations in standard, factored and vertex form; and (5) that asked students to examine graphs of quadratic functions, including graphs that have real solutions and complex solutions as well as functions for which the y-intercept is not visible on the graph, and explain what they know about the function. The interview protocol can be found in Appendix A, and the questions are listed along with the big ideas and essential understandings addressed in Appendix B. The interview protocol did not specifically include questions about tables of values and the patterns associated with the tables, but when students mentioned tables or it seemed appropriate, I asked follow-up questions to determine what patterns they were seeing or what connections they were making.

Refining and revising the protocol. The interview protocol was refined and revised with input from the teachers at the interview site, the University of Washington Mathematics Education Research Group, three pilot interviews and a mathematics educator in another state.

I met with teachers at the high school individually to learn how the students in this school learned about quadratic functions and to ask the teachers to help refine the interview protocol. In my interviews with teachers, I asked each teacher to work through the problems in the interview protocol with me (see Appendix A). For each problem, I asked teachers to describe how students might solve the problem and to anticipate

different solution strategies, including alternative correct strategies (different from the teacher's own strategy), incorrect strategies, and mistakes. I asked the teachers to consider the incorrect strategies and mistakes they anticipated students making and to identify what a student who uses that strategy or makes that error understands about quadratics. These conversations helped me identify potential cognitive affordances and obstacles. I used information from the teacher interviews to help refine the protocol and design follow up questions in anticipation of strategies, correct and incorrect, that the teachers thought their students might use.

I conducted three pilot interviews and received feedback on the protocol from an additional high school mathematics teacher in a different school setting as well as from the University of Washington Mathematics Education Research Group. I adjusted some of the questions and the order in which they were asked based on the student responses and feedback.

Question order. For all but four of the students, the questions on the protocol were asked in an order similar to the order listed on the protocol. In the first set of June interviews, I varied the order of the questions for a few students. For these four students, I began by asking students to look at graphs and tell me what they knew about the associated function. I did this to see whether students would have more to say about the graphs if these questions were asked earlier in the interview process. After starting with the questions about graphs for those four students, I decided that changing did not affect the interview, and I resumed the initial question order.

In some instances, I varied the order based on how the student responded. As an example, some students struggled with question 3b, $x^2 + 4x + 3 = 0$, and used linear

techniques. For those students, I often skipped question 4a, $y = 2x^2 + 5x - 12$, in favor of asking about graphing the equations that are presented in factored or vertex form.

When working on the questions that involved solving equations, students often made the connection between solving and graphing and wanted to graph the associated function.

When that happened, we followed the student's lead and I used that conversation to learn how the student thought about graphing.

Recording the interviews. I video-recorded 26 of the interviews, which allowed me to attend to student gestures in my data analysis. This was important in understanding what students were communicating, as they often used their hands to describe the symmetry of a parabola or the direction in which it opened as well as how dilations affected functions.

All of the interviews were also recorded using a Livescribe smartpen. I asked students to do all of their work using the smartpen in a notebook I provided that had digital graph paper. In addition to creating a paper version of student work, the smartpen created an audio-recording of the interview that is linked to what the students wrote and drew in the notebook. The student work can be viewed in the web-based Livescribe Player, where it is possible to replay the pencast of student work. Clicking on any mark in the student work opens Livescribe at the place in the audio-recording where that mark was made. This enabled me to see the student's work emerge while listening to what he or she said. The screen capture depicted in Figure 5 shows a page of student work online. The work that is green has already been completed at this point in the interview, and the work that is gray is what the student does after this point. I was able to watch the work being created by the student and replay it as many times as necessary. This enabled me to

determine the exact order in which students created graphs or solved equations, and it allowed me to link their words to their writing. At times, the smartpen audio also provided a higher-sensitivity recording than the video-recording.

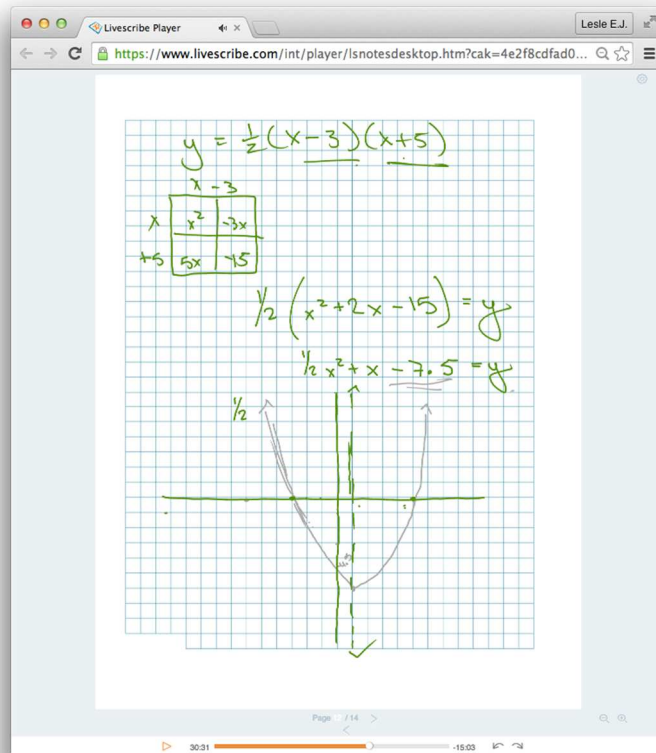


Figure 7: Pencast of Student Work in Livescribe Player

Data Analysis

The data collection process provided me with three sources of data to analyze for all but one student. These included the student work in a paper notebook as well as in Evernote and Livescribe, the linked audio-recording and the video of the interview. Data analysis was ongoing and began at the onset of my first interview. I engaged in preliminary analyses as I interviewed students, which informed my decisions about potential question order as well as which follow-up questions to ask each individual. As I

elaborate on below, to aid in the formal analysis of my data, I devised an organizational system to track student demographics and their success on the problems, and to link that information to information from the interviews. I transcribed the student interviews from the videos, using the pencasts as a resource, and developed coding systems. I then coded student transcripts and looked across students and problems to identify themes and patterns.

Phases of data analysis. My data analysis went through four phases. Initially, I focused on individual students as I transcribed each interview. Then I went through a second phase of analysis in which I focused on the problems and how students approached them. In my third phase of analysis, I focused across students as I used my coding system, which is described below, to code each student's transcript. In the fourth phase of analysis, I went back to re-examine the problems with insights from the previous passes through the data. From this repeated interaction with my data, I identified themes in students' understanding of quadratics, the connections they made between equations and graphs, how they approached solving equations and understood solutions, and the cognitive affordances and obstacles they encountered. As I examined students' thinking I attempted to differentiate levels of concept development. Specifically, I made memos for instances that might typify the syncretic heap stage, complexive thinking, and pseudoconceptual thinking. As I progressed, I made claims, and linked them to the existing literature and my conceptual framework, and I looked for counterexamples and instances of student thinking that disconfirmed my claims.

Organization. During my data collection process, I kept records in a spreadsheet, including information about each student's current math course (Algebra 2 or

Precalculus) and grade level, the time and date of the interview, initial notes about the interview, and the names of the files of each student's video-recording and pencast. I updated this spreadsheet each day at the conclusion of my interviews.

Code Development. I started with an initial set of codes that resulted from my review of the literature, pilot interviews with students and my interviews with the teachers. As data analysis progressed, I added codes in response to what students said and did, and ultimately, I developed two sets of codes to address my research questions.

The first set of codes stemmed from three sources; my conceptual framework, the review of the literature, and initial data analysis resulting from my first two passes through the data and can be found in Appendix E.

Big idea codes. This group of codes was developed to identify areas of the transcript in which students discussed big ideas about quadratic functions. I organized these into three sub-categories. One category was specifically about quadratics and related ideas such as the zero-product property or the square root of -1. The second category was used to identify conversations about algebraic representations, and the third was about graphical representations.

Solving techniques and graphing approaches. I created a set of codes which specifically described moves that students made as they solved problems. This category was further divided into solving techniques and graphing approaches. I also included a code that indicated the student was able to solve with support as well as a code that indicated the student was not able to solve a given problem. These codes enabled me to track how students approached solving and graphing quadratics.

Codes for Connections. My research questions included understanding the connections students make between equations and graphs and understanding the affordances and obstacles they encounter in solving quadratic equations. I created codes to identify connections that students made between equations, graphs, tables and factoring as well as other ideas.

Affordances and Obstacles. I created codes to identify the affordances and obstacles they encountered. The codes for Affordances and Obstacles evolved as I coded to include a subset that I called Interesting Errors so that I could tag things that students did that were not clearly affordances or obstacles, but that seemed to be worthy of closer inspection.

Justification Codes. This category includes codes I used to identify conversations in which students explained their reasoning. I sought to determine the quality of explanation students provided, gauging whether the student appealed to authority in their explanation, gave a specific example or a generic example, or made a deductive argument.

The second set of codes, which is described here, was designed to assess student success on the individual problems.

Correctness and support. To investigate how correctly students solved the problems in the protocol, I decided to focus on students' first attempt at the problem. This included their work and thinking from the start of the problem until they stopped or asked a clarifying question or I intervened. To develop a scoring system, I first solved the equations and anticipated what students might do to solve each of the problems. The anticipated solutions are shown in Appendix D. I then created categories for students'

initial responses to the problems. The categories included solving successfully; using valid methods but getting an incorrect solution due to a computational error; solving partially, meaning the student got off to a good start using a valid strategy, but got stuck on some conceptual idea involving quadratics; using invalid methods, such as using linear techniques to solve; and not knowing how to solve. I thought about this using a 4-point scale, with 4 corresponding to a successful solution. I chose to give a score of 0.5 when students engaged with a problem but said they did not know how to solve it, and a 0 for problems when students immediately said that they did not know what to do.

I applied a similar coding scheme to student explanations of the symmetry of the squaring function, noting whether they explained it (Explains), it was a weak explanation or was implied in their work (Weak), they didn't know (DK) or the question did not arise (NA). For the purposes of my scoring system, I also entered this on a 4-point scale, with 4 being assigned to Explains 2 to Weak, and 0 to Didn't Know

The resulting scoring system is shown in Table 2. I entered the student scores for each problem into a spreadsheet and found the average score per problem as well as each student's average score. This enabled me to look for trends across problems as well as across students and make connections between those trends and the codes described above for big ideas, approaches and techniques, connections, affordances and obstacles and justifications.

Table 2: Scoring Values for Students' Initial Responses to Problems

Student Response Code	Description	Score
Solved successfully	Student reached a correct solution	4
Valid methods, incorrect solution	Student started correctly but may have made sign errors, factoring mistakes, or computational errors.	3
Solved partially	Student started correctly but did not completely solve. An example is factoring, but not solving.	2
Invalid methods	Student may have tried to solve using linear methods and may have made order of operations errors.	1
Did not know or answer	Student thought about the problem and said that he or she did not know how to solve.	.5
Did not attempt	Student did not spend time considering the problem.	0

Focusing on individual students. One of the initial steps in my data analysis involved watching and listening to each interview, transcribing what was said, and watching the student's work unfold on the associated pencast. I also documented student strategies and noted where and how the students ran into impasses and what enabled them to work through those impasses. In this phase, I inspected each student's work closely, and I often replayed the pencasts of student work to understand the exact order in which the student constructed a graph or equation, which enabled me to see what a student said as he or she made different mathematical decisions and connections. I made notes from my observations on my data collection form, which can be found in Appendix C. The form is organized by the big ideas and lists all of the problems. The form is organized into three columns. The first identifies the big idea and question. In the second I made

notes of what the student wrote and did; here, I included screen shots of student work. In the third column, I included the associated transcript selection.

Focusing on problems. In my second pass through the data, I focused on students' solutions to each problem. In this stage, using student work, the interview transcript and my notes from the first pass through the data, I examined student thinking for each problem and recorded to what extent the student solved the problem correctly and with what level of support. I recorded these notes in a spreadsheet. For the initial questions about what quadratics are, I recorded the one-word answers or phrases that students gave.

Focusing across students. In my third phase of data analysis, I used the web-based data-analysis program Dedoose. I uploaded each student's transcript and engaged in more focused coding of the data using the codes I had developed (Emerson, Fretz, & Shaw, 1995). As I re-read each transcript, I used the sets of codes I had developed and coded for how students approached each problem, the strategies they used, the connections they made, and the explanations they gave. When appropriate, I added codes in response to student strategies and answers. In this phase, I attended to three data sources to ensure I understood as best I could what each student did and knew about quadratics. I worked with the transcript of the interview in Dedoose while also looking at my notes I had previously made in the data collection form. I also repeatedly consulted student work, focusing on what the student did and replaying pencasts. As I coded, I made memos about how students seemed to be approaching particular problems. This enabled me to make connections across the students for specific problems. I also wrote memos about themes I began to see emerging, such as the challenges of a missing

coefficient, students' strong sense of symmetry and how students explained the zero-product property.

Revisiting problems across students. In my fourth stage of data analysis, I examined the data and revisited my initial analysis of how students approached each problem. In this stage, I sought to understand how students understood quadratic functions and solving quadratic equations and the connections they made. I also attended to difficulties and confusion that students encountered. For each problem, I worked to categorize student responses to see what they had in common. For example, in solving $x^2 - 2x = 0$, several students gravitated towards trying to solve it using undoing, while others used factoring. I noticed some commonalities in student approaches, but I also noticed students who used unusual tactics or reasoning in their explanations.

In this stage of data analysis, I returned to the spreadsheet I had created which listed the students and their demographic information. I recorded each student's average score per problem as well as their one-word answers to questions about the shape of the parent function. The completed spreadsheet enabled me to sort the data by the student's average score per problem in descending order and used conditional formatting to help me visualize trends within and between any of the demographic categories. Having the data in this table enabled me to identify trends across problems and students.

Developing claims. As I sought to learn what students understand about quadratic equations and functions, I repeatedly cycled among the transcripts of student interviews, the student work and pencasts, the data that I had entered in Dedoose, and my notes and memos. I used my codes in Dedoose to look for co-occurrences of codes and of codes and descriptors. This became a process of zooming in on one code, idea or theme,

such as the zero-product property or what is and is not a quadratic, and then zooming out to get a larger perspective on how the idea or theme played out across students and problems.

I found that for each problem, student solutions could be clustered into similar approaches and types of reasoning. I looked for themes in the student responses and wrote memos for each problem, describing student approaches and noting trends in how students solved each problem and the difficulties that they encountered. I then examined what students said and did in those approaches and developed claims about how the students understand quadratic equations and functions and make connections. I also looked at how student solutions could be clustered across problems to notice whether there were similarities across problems. Several themes arose. One theme centered on students' strong understanding of the symmetry of parabolas. Another theme surfaced as I noted that students transition between quadratic equations, expressions and equations of functions while solving equations or graphing functions, at times not understanding the differences between them. Lastly, it became evident that students applied ideas from their experiences with linear functions as they solved quadratic equations and graphed linear functions. I describe these themes and my findings in the next chapter.

CHAPTER 4: FINDINGS

This study focuses on student understanding of quadratic functions in an effort to enable mathematics educators to better interpret how students' prior learning influences their developing understanding of quadratics. I have organized my findings into four sections. The first section gives an overview of how students did across the problems in the interview. I give a description of demographic trends among the students and for each of the problems I describe links to the existing literature. The second section has to do with how students think about quadratics, the parent function, and symmetry of quadratics. There, I argue that students have a very clear understanding of the shape of parabolas and their symmetry, and some are able to explain the symmetry using examples or through a general understanding that when you square a negative number you always get the same result as you would when you square its opposite. However, students struggle to give a mathematically precise and rigorous explanation for symmetry. The third section addresses the finding that as students solve equations or graph functions, they transition between quadratic equations, expressions and equations of functions, with and without understanding the differences between them or the limitations or affordances each has to offer. This can result in interesting consequences for understanding and interpreting the results of their solution methods. The fourth section focuses on the impact of students' linear thinking on solving quadratic equations and graphing quadratic functions. I claim that students apply techniques related to isolating the variable and balancing equations as they try to solve quadratic equations when they do not know how to solve them using

factoring or the quadratic formula. Students also apply their understandings from linear graphs to graphing quadratic functions.

Student Performance Across Solving Problems

The set of problems that are described here addresses the big idea that quadratic functions can be set equal to constant values and solved for the variable. I presented students with quadratic equations and asked them to solve. For each of the problems, I coded to what extent students were able to solve the problem on their first attempt and assigned a numerical value as shown in Table 3.

Table 3: Student Response Code Point Values

Student Response Code	Point Value
Solved successfully	4
Valid methods, incorrect solution	3
Solved partially	2
Invalid methods	1
Did not know or answer	0.5
Did not attempt	0

Scoring the problems on a 4-point scale allowed me to find an average score for each problem, as well as an average score for each student. These are shown in Table 4 on the next page.

Table 4: Student Performance on Solving Equations on a 4-Point Scale. See Appendix B for scoring details

Pseudonym	Graphs $y = x^2$	Explains symmetry of parent function	Solves $x^2 = 100$	Solves $x^2 + 4x + 3 = 0$	Solves $x^2 - 2x = 0$	Solves $x^2 - x = 12$	Solves $x^2 + x - 1 = 0$	Solves $x^2 + 1 = 0$	Student's average score per problem
Joanne	4	4	4	3	4	4	3	3	3.63
Stella	4	4	2	4	4	4	4	2	3.43
Chelsea	4	4	2	4	4	4	4	2	3.43
Clifton	4	4	4	4	4	3	2	2	3.38
Eva	4	4	2	4	4	4	2	2	3.33
Brad	4	4	2	4	4	4	2	2	3.25
Kimberly	4	4	2	4	4	4	2	2	3.25
Carson	4	4	2	4	2	4	4	2	3.25
Adam	4	4	4	4	4	4	2	2	3.20
Maria	4	3	2	4	2	4	2	4	3.13
Sarah	4	4	2	4	0.5	4	4	4	3.08
Sacha	4	3	2	3	3	4	3	2	3.00
Kelly	4	4	2	4	1	4	3	2	3.00
Bryce	4	4	2	4	2	4	2	2	3.00
Rudy	4	4	4	2	4	1	4	4	3.00
Claudia	4	4	2	4	2	4	2	2	2.86
Kerry	4	4	2	4	2	4	2	2	2.86
Dana	4	3	4	4	1	4	4	2	2.80
Tabitha	4	1	2	4	4	2	2	2	2.63
Deiliah	4	3	2	4	1	4	1	2	2.63
Tracy	4	4	2	1	2	4	4	2	2.50
Melody	4	3	2	2	1	4	4	2	2.40
Samantha	4	4	2	1	1	4	2	2	2.29
Annette	4	2	2	1	1	4	4	2	2.00
Margie	3	3	2	2	0.5	2	2	1	1.79
Michelle	3	3	2	1	0.5	4	4	2	1.70
Kenneth	4	1	2	0.5	1	1	4	2	1.64
Average score on each problem	3.92	3.41	2.31	3.02	2.29	3.42	2.40	2.09	

Demographic Differences

Although not the focus of my research, I did investigate whether the trends I found varied depending on demographic categories such as: student gender, which course students were taking, their year in school, whether they were accelerated in mathematics, their teacher, or the time of day their interview occurred. There appear to be no differences in how groups of students performed on the problems with the exception that students who were taking a math class two years above grade level appeared to be the most successful at solving the equations.

Acceleration. The group of students who were double accelerated seemed to be able to more consistently solve equations correctly. However some students who were double accelerated had difficulties with equation solving.

Six of the 10 students with the highest scores were two years accelerated in math. Joanne was a 10th grader who was double accelerated, and she had the highest average score across problems. She was very strong conceptually, and valued solving equations quickly. The errors she made were sign errors and not finishing problems. For example, when solving, she factored one side of the equation but neglected to solve the resulting equations and find the solutions.

There were two 9th grade students who were two years ahead who struggled. Samantha, who was one of the lower scoring 9th graders said that she loved Algebra, and then she went on, “I have an A in the class, but I feel I don't retain much. I've actually always struggled in math, but I tested in to Algebra as a 7th grader, and I've maintained a 4.0, but I just don't feel completely a 100%.” These students both shared a sense of

having been put in an advanced math class and never quite completely “getting it” after that.

One student, Clifton, was repeating Algebra 2 and was quite successful in solving equations. He had the fourth highest score across problems. Clifton made connections across problems, and he seemed to have benefited from a taking a second year of Algebra 2 which enabled him to make connections and have a deeper understanding of quadratics than many of his peers.

Precalculus. Students who had taken Algebra 2 and Precalculus did not perform differently from the students who were finishing Algebra 2. It is interesting that precalculus students did not perform better than students in Algebra 2. One might have expected more advanced math students to be more successful in solving equations. It may be that the precalculus students were further away from their specific study of quadratics. They may also have been trying to incorporate ideas about other families of functions into their understanding and been confused or distracted by the additional concepts. Specifically, several of the precalculus students mentioned functions they had learned about where they had to work with holes and asymptotes. These students were trying to make sense of quadratic functions in the bigger context of rational functions (functions in

the form $r(x) = \frac{p(x)}{q(x)}$) such as $y = \frac{x^2 + 4x + 3}{x + 3}$.

The Problems

In this section, I briefly describe what the research literature indicated for each of these findings, and how students in my study approached each of these groups of problems. I go into more depth about student understanding, the connections students make, and cognitive affordances and obstacles in the following sections.

Solving by factoring. To learn how students approach solving using factoring, I asked them to solve the equation $x^2 + 4x + 3 = 0$, and I observed them using factoring as a solution method across the problems. This fits with Didis et al.'s (2001) finding that students prefer factoring as a solution method when the quadratic is obviously factorable, and yet, factoring can be tricky for students, particularly when the leading coefficient does not equal 1 (Bossé & Nandakumar, 2005; Kotsopoulos, 2007).

Students were more successful in solving $x^2 + 4x + 3 = 0$ than any other equation in the interview, although only 15 of the 25 students who were asked it solved it correctly and completely. Of the 15 students who solved the equation, 1 used the quadratic formula and 14 used factoring. Additionally, 2 students factored correctly but did not complete the problem. One of them did not initially recognize that she needed to use the factors to solve the equation, and one student did not know what to do next.

In contrast, students had a more difficult time with the function $y = 2x^2 + 5x - 12$. Many of the students knew that they could solve the equation $0 = 2x^2 + 5x - 12$ to learn more about the graph, but only 5 students were able to factor the expression $2x^2 + 5x - 12$ successfully. Several students tried to factor but became discouraged, and a few expressed that they knew they could factor it to find the zeros, but they did not want to. None of the 12 students who were asked about this equation chose to complete the square or use the quadratic formula.

I also found that students often used an area model to help them factor, which was not discussed in the literature. Most of the students who were able to factor $2x^2 + 5x - 12$ used an area model, which they called the “box method,” to help them think it through. Tabitha’s work is shown in Figure 8. Some students used this method

throughout the interviews to factor as well as a tool to reason if they thought an expression was factorable into binomials with integer coefficients. Interestingly, the students who were most successful did not rely on the box method. They seemed to be able to think through how to factor while holding the numbers in their head without drawing the box.

$y = 2x^2 + 5x + 3$
 $(2x+3)(x+1)$
 $(2x+3)=0$
 $+3+3$
 $2x = -3$
 $\frac{2x}{2} = \frac{-3}{2}$
 $x = -\frac{3}{2}$
 $x+1=0$
 $x = -1$

Figure 8: Tabitha's Box Method for Factoring

The zero-product property. The zero-product property is a key component in solving by factoring. Once students have factored one side of an equation so they have an equivalent equation such as $(x+1)(x+3)=0$, they apply the reasoning that if two factors have a product of zero, then one or the other factor must be zero. This results in the two equations $x+1=0$ and $x+3=0$, which can each be solved for x . The literature credits a lack of understanding of the zero product property as the reason for why students solve equations such as $x(x-2)=0$ by dividing both sides by x , thus losing track of the solution $x=0$. (Didis et al., 2011, Nandakumar, 2005, Kotsopoulos, 2007). I found that students who were able to factor one side of a quadratic equation were usually able to apply the zero-product property and solve. However, most students' explanations lacked in completeness, and many provided explanations that appealed to authority regardless of how well they were able to apply the zero-product property.

To explore student understanding of the zero-product property, I had conversations with 16 of the students once they had moved from a step in which they had the product of the factors equal to zero such as $(x+1)(x+3)=0$ to a statement that $x+1=0$ or $x+3=0$. In these conversations I remarked on what they had done, and then asked why they were able to do that. Of the 16 students, 6 were able to explain through example or generalization, 6 appealed to authority saying it was a rule or they were told to do it that way, and 4 were not able to explain or said they did not know. Students' ability to explain or not was not related to how they performed in solving the equations. The 7 students with the highest average score per problem included 3 students who were able to explain (Joanne, Kimberly and Clifton), 2 students who appealed to authority (Stella and Chelsea) as well as 2 students who were not able to explain why the zero-product property worked (Eva and Brad).

The explanations that students gave were informal. Delilah's explanation revealed that her thinking about this is emerging. She said, "well, they're like two separate things. They are two different things that are multiplying together that equal zero, and so you can separate them." Kimberly knows that one factor or the other must be zero, but her explanation does not include the reasoning that because the two factors are being multiplied, when one factor is zero, the product is also zero. She said, "because no matter what the other one is, in order for it to equal zero, one of them has to be zero." Samantha thought this question through, and gave a more complete explanation when she said, "if this is equal to 0 and this is equal to zero, then they have to multiply together to equal zero... which would make that true." Her explanation may indicate that she thinks the factors have to be simultaneously zero. Throughout the explanations, it was apparent that

the students were trying to think through why they were able to make that mathematical move, and that this was a new question for them.

Many students appealed to authority giving me answers such as “because I was told,” or “it’s a rule” or, they didn’t know. Kerry’s understanding of using the zero-product property involved what she called “zeroing out.” When I asked her what she meant by zeroing out, she said:

Kerry: Zero out, make the whole .. make x so that it balances out whatever else is in the parentheses... to equal zero.

Leslie: and why do we care if this stuff in the parentheses equals zero?

K: because... it already equals 0 right here (above) and ... I don't know, actually. (pause) I mean, we were never taught... we were just kind of told, "do it"

Kerry had developed a way of thinking about this that was useful for her and enabled her to be successful, but she did not know the reason behind her method.

Tracy was able to successfully use the zero-product property to solve $x^2 + 4x + 3 = 0$ but struggled with $x^2 - 2x = 0$. When she had completed both problems, I asked her about the two different approaches she had taken and why the second equation had been difficult for her. She had this insight about why she was not able to apply the zero-product property in both cases:

I think it's because teachers teach us ... some teachers teach us stuff that instead of teaching you a logical way of how why something happens, they just teach you how to do problems like that... and so, when you run into something that you haven't seen very often... you just try to apply what you've learned instead of actually thinking about it logically... so then you're trying to do steps on something, and the steps are correct.. most of the time, so you're just applying steps and not really thinking about what you're looking at.. and I think that that's a problem, cause then you're just memorizing steps instead of actually learning what the problem is. ...

Tracy's insight captures what students meant when they said that they were given the rule without really understanding the reasoning behind it.

Forms of Quadratic Equations. The literature suggests that students have difficulties in solving quadratic equations when the value of b or c is zero such as equations of the form $ax^2 + c = 0$ or $ax^2 + bx = 0$ (Didis et al., 2011, Zavlasky, 1997). Didis et al. suggested that equations of this form do not appear to be quadratic to students. To explore students' understanding of this, I asked students to solve three equations that had a missing parameter or were presented in an unusual form.

When $c = 0$. It appears that when the constant c is 0, some students do not recognize that the resulting equation is a quadratic, and consequently find the equation challenging to solve. I asked students to solve $x^2 - 2x = 0$ and found that this was one of the most challenging problems for students. However, interestingly, one-third of the students were able to factor it with no difficulty using either the distributive property or an area model. The remainder of the students found it extremely challenging.

More generally, it appears that when students understand that this is a quadratic equation, they use strategies to solve it successfully. Five students who solved it successfully were able to recognize that they could "factor out" an x using the distributive property, while three students used a strategy of adding a 0 to the end, which Samantha called an "understood plus zero" and then factored it by trial and error or using an area model.

Three students, who had solved the previous problems successfully, were able to factor correctly and wrote $x(x - 2) = 0$, but they then encountered difficulties regarding what to do next. Tracy was one of these students. She factored using an area model, and

then decided to divide both sides of the resulting equation by x (See Figure 9). Then she said, “But that’s as simplified as it can get because it’s not factorable.” I asked if she could solve it from there. Tracy said, “Yeah, divide by do you have to divide by x first because it's outside the parentheses? or can you do the stuff that's inside the parentheses? First... I feel like you have to do some outside stuff... we'll just see what happens...

(divides both sides by x , then adds 2... gets $x = \frac{0}{x} + 2$)... I feel like that's just a little crazy.”

The image shows handwritten work on a grid background. At the top left, the equation $x^2 - 2x = 0$ is written. To its right is a small table with columns labeled x and $x-2$, and rows labeled x and $x^2 - x$. The table contains the values 1, 2, 1, and 0 respectively. Below the table, the equation $x(x-2) = 0$ is written. A horizontal line is drawn under the x in the first term, and the word "X" is written below it. This leads to the equation $x-2 = \frac{0}{x}$. Finally, the equation $x = \frac{0}{x} + 2$ is written at the bottom.

Figure 9: Tracy's Work

It appears that when the equation looked different, both in the factored and un-factored form, it made solving this equation feel like completely new territory to these students. This was a problem for many of the students, and eight of them attempted to use linear methods such as un-doing to solve the equation, which is described in detail in a later section.

Problems with zero. A difficulty that I had not anticipated students encountering was working with zero. I have not encountered literature which relates student understanding of zero to solving quadratic equations. Zero creates interesting stumbling blocks for students, and several students had moments in which they tried to remember

what the rules are about multiplying or dividing by zero. Tracy encountered this in her work when she got the result $x = \frac{0}{x} + 2$ and said, “but you can't divide by zero... so... uh... maybe... I also don't think you can just take the 2 out of the parentheses... so I think that's wrong... so, ... maybe... It just looks so weird.” Sacha also ran into difficulties with zero. He decided to use the quadratic formula to solve the equation $x^2 - 2x = 0$, and when he got the result $\frac{\sqrt{4-4}}{2}$, he was not sure what $\frac{0}{2}$ would be. At that point, he abandoned this approach and tried using other methods.

Other students had momentary issues with zero as they tried to factor. Clifton rewrote the problem as $x^2 + 2x + 0$, neglecting the $= 0$, and was stuck. When I asked him what he was thinking, he said, “I don't know. Cause nothing multiplies into zero, except.... ohhhhh maybe... maybe.” At that point, he factored into $(x + 2)(x + 0)$ and then multiplied it out again to check his work. I had not anticipated students having difficulties working with zero, but believe that if I had thought back to my own experiences as a student and teacher I would have predicted these struggles.

When $b = 0$. When the value of the parameter b is zero, quadratic equations can present as $ax^2 + c = 0$ or $x^2 = c$. Students commonly encounter these early in their math careers while solving equations that arise in the context of area problems such as $x^2 = 100$. I asked 26 students to solve $x^2 = 100$, and 23 students to solve the equation $x^2 + 1 = 0$. My findings were consistent the literature, which indicates that students tend to only find one solution and neglect the second root (-10 and $-i$ in these problems) when solving problems such as these.

Only 3 of 26 students initially solved the equation $x^2 = 100$ correctly giving both the positive and negative solutions, and most students confidently stated the positive root. All of the students who gave the answer +10 solved the equation by taking the square root of both sides and never mentioned the plus or minus sign. This is described in more detail below.

Later in the interview, I asked 23 students to solve $x^2 + 1 = 0$. This was more problematic for some students because the solutions are not real. However, for those students who considered complex solutions, 8 gave the solution $+i$, and only 3 of them correctly gave both plus and minus i as solutions. I found this interesting because solving this equation followed a conversation about solving $x^2 = 100$. In those conversations, most students came to realize that -10 was also a solution to the equation. (These are reported on in a section below.) That discussion did not seem to transfer to solving the equation $x^2 = -1$, which they solved by taking the square root of both sides.

When I asked students why they tend to forget the negative solution when solving $x^2 = 100$, they explained that negatives were not emphasized in their earlier grades. Stella explained that, “I don't think that it's necessarily that it's hard. It's just that when you first learn square roots, in like elementary school, teachers don't always teach about the negatives, and so their [students'] first instinct is to just stick with positive.” Chelsea also attributed forgetting the negative solution to not having enough experience. “They don't really enforce it much... usually they're in the positives... and sometimes there's like certain math where you only get the positive and you don't get the negative.” Kenneth also explained, “I don't think we really think about the negatives or the positives... if we see a positive on one side, we think it has to be positive on the other... it doesn't really

matter if it's squared or not, cause usually, you don't see a negative over here, so you think it's not going to be negative.” It appears that students feel they have not had enough experience with negative solutions.

There was one student whose response affirmed Thorpe’s (1989) assertion that the reason that students neglect the plus or minus sign is due to their not understanding it.

Kerry used the quadratic formula to solve $x^2 + 1 = 0$. Kerry initially wrote the plus or

minus sign in her work. She wrote her solution as $\frac{0 \pm \sqrt{-4}}{2}$ which she rewrote as

$\frac{-0 \pm \sqrt{4}i}{2}$. She then crossed out the $\frac{-0}{2}$ and wrote $\frac{2i}{2}$, $x = 1i$. I asked her what

happened to the plus or minus sign, and she said that it did not matter, and that it means that you “just don't get any more.” Kerry seemed to think that the plus or minus sign was important to the number that precedes it rather than the one that follows it.

Using the quadratic formula. Most of the equations that I asked students to solve did not require the quadratic formula, and most of the students preferred to factor whenever possible. This confirmed Didis et al.’s (2011) finding that students prefer factoring when a quadratic is obviously factorable. Many of the students I interviewed had the quadratic formula written in the upper right hand corner of the white board of their classroom. I found that when they were trying to remember the formula, they would glance up at that corner of the interview room as they worked to remember it. Very few students were able to correctly solve $x^2 + x - 1 = 0$ using the quadratic formula. However many of them were able to determine that the equation was not factorable, and thought that there was some formula they could use.

Some students regarded using the quadratic formula as a completely different kind of math which would have implications for the zeros of the graph. Clifton said, “it means you have to use a completely different math in order to get the graph itself.. or you can just use a graphing calculator and make it easier on yourself.” Clifton anticipated that the graph would cross the x -axis, but when I asked about the roots, he predicted, “they'd most likely be fractions.” Brad suggested that if an equation was not factorable, the graph would not cross the x -axis.

How Students Think About Quadratics, The Parent Function And Symmetry Of Quadratics

In this section, I describe findings that focus on how students think about quadratics, the parent function and the symmetry of quadratics. In my review of the research literature, I did not encounter any reference to students' understanding of the symmetry of quadratics or how that symmetry and the parent function can act as affordances for student learning. I found that all of the students who knew about the graph of the parent function had a strong sense of its symmetry, and they used symmetry to create the graph, with varying degrees of precision. There are a range of ways in which students understand and explain the symmetry of the squaring function. However the symmetry of the parent function helps students think about solutions to quadratic equations set equal to zero and equations such as $x^2 = 100$. Though students have a strong sense of the symmetry, they tend to struggle to explain why the function is symmetric in a generalized way that goes beyond explaining the behavior of a few points on the function.

Student ideas about what it means to be quadratic. The literature does not specifically address what students think quadratics are or how they conceive of quadratic equations and functions. I found that students know quadratics can be represented as graphs or equations, and many of them know that quadratics involve squaring. Students link the idea of quadratics to other related ideas such as exponents, polynomials, and the quadratic formula. However, students have incomplete ideas about whether certain equations and functions are quadratics. These findings emerged when I opened the interview with the question, “What are quadratics?” I gained insights into how students understand what quadratics are when I asked them to solve equations that were not in standard form or which in which b or c were zero.

What are quadratics? When asked what quadratics are, 14 students volunteered that they are parabolas or U-shaped graphs. Eight students said that their equations have an x squared term or exponent of 2, and 4 students mentioned that quadratics are equations involving exponents. Only 3 students mentioned that quadratics are functions. Three students mentioned the link to factoring, 3 mentioned polynomials, and 6 students said that quadratics are a formula associating quadratics with the quadratic formula.

The Parent Function and Symmetry of Quadratics

In my review of the literature, I did not find any reference to students’ sense of the symmetry of parabolas. Through the interviews, I learned that students have a well-ingrained understanding of the shape and location and symmetry of the graph of parent function, $y = x^2$. Many of the students knew and used the pattern of the squaring function in their graphing while some used the pattern of first differences. Most students were able to graph the parent function correctly. Their graphs ranged from a very rough sketch, to a

more exact graph that passed through the origin and the point (1, 1), to a precise graph with 3 to 5 points plotted which 12 students drew. Examples of student graphs are shown in Figures 10 through 13. The precision of student graphing was not related to students' grade or course level.

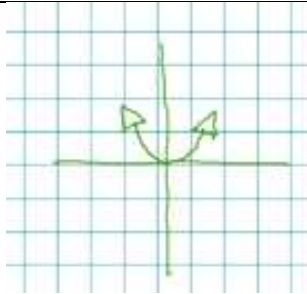


Figure 10: Tabitha's Sketch of the Parent Function

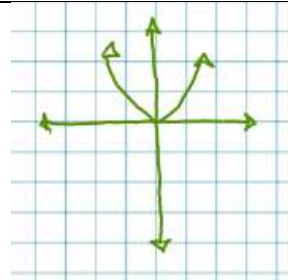


Figure 11: Stella's Graph: A sketch that passes through (1, 1) but is not very precise

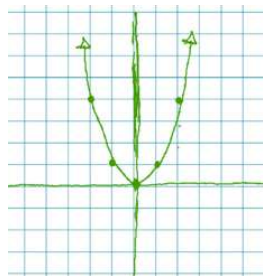


Figure 12: Chelsea's graph: The Parent Function with points plotted using the squaring pattern to graph the right hand side and then reflect over the y -axis

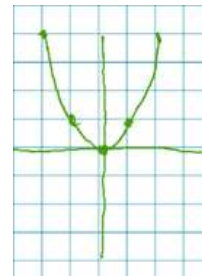


Figure 13: Carson's Graph: Parent function graphed using first differences

Most of the students who did an accurate graph (rather than a quick sketch) used the pattern inherent in the squaring function that when you move 1 to the right of the origin, you go up 1, and when you move 2 to the right of the origin, you go up 4. They generally graphed the right side of the parabola first and then reflected it across the y -

axis, an example of which is shown in Chelsea's graph in 12. A few students, including Carson (see Figure 13), used the pattern of first differences reasoning that from the origin, you go out 1, up 1. And from the point (1, 1), you then go out 1 more and up 3 to (2, 4) and then out 1 more and up 5 to (3, 9).

Explaining symmetry. When asked why parabolas are symmetric, most of the students were able to explain either using specific points or using more general properties of the squaring function. Many students explained in their own terms, which were not necessarily mathematically precise, but revealed a general understanding that when you square a positive number and square its opposite, you get the same value. Some students gave explanations that used a few specific points, and a few students gave explanations that confused squaring and taking the square root, or revealed that they did not know why the graph was symmetric.

Stronger explanations. Stronger explanations of symmetry were those that included some form of generalization. For example, Tracy explained, "The parent function means that it starts at zero, zero and reflects evenly over both sides because if you enter points in for x , and then entered the same negative point for x , you'd get the same y value out... each y value could have two different x values, and then that would make it symmetrical." Another example came from Joanne, who said, "Because, y equals x squared. There's negative... if the x is negative, it's still the same y value." Similarly, Kelly gave a generalized explanation, "When you enter numbers into the equation, the negatives of the x become positive, and so that's why the mirror shape kind of comes around and... it's , it looks identical to the other side." These students understand that $x^2 = (-x)^2$, but do not use precise language to describe it, using the words "negative x "

rather than the opposite of x . They typify many students in their deep understanding that a negative number squared will have the same value as its opposite squared and that a negative number squared will be positive.

Explanations relying on specific points. Several students grounded their explanation in one or two pairs of points. Delilah explained symmetry saying, “there would always be like mirrors of each other” while gesturing back and forth with her hand. I followed up and asked what she meant by mirrors of each other, and she explained using one point, “like for this one, you'd have like -1 would equal 1 , and then 1 would also equal 1 so it mirrors each other along the y axis.” Her language, that “ -1 would equal 1 ” is her way of explaining that if you input -1 into the parent function for x and square it, the resulting y will be 1 . In her explanation, she relied on one pair of points to explain the symmetry.

Similar to Delilah, four other students used one pair of points. Dana called this “reciprocating,” and Melody explained “because that's how parabolas are. They can't not mirror themselves, because that wouldn't be a parabola.” These students also have a well-established, deep sense of the symmetry of the parent function, but their explanation is localized to a few points rather than being generalized to all points.

Explanations that conflate ideas. A few students knew the graph of the parent function was symmetric and used that understanding to draw the graph, but their explanations revealed confusion about the interaction between the squaring function and its inverse. Maria and Kenneth both brought up the inverse of the squaring function. Maria explained symmetry as she graphed the parent function. She knew the pattern of the integer values of the points on the graph, but she confused squaring and taking the

square root. When I asked her what made the graph the same on both sides of the y -axis, she said, “Because it’s square rooting.” I asked her to clarify, and she said:

Like... why is it mirrored? Um.. it's like the opposite of b , so it all like... (uses hand to show a reflecting motion) I don't know, it all.. it's just, it's like, ... it's just how we were taught it... it's mirrored over, because the square root of 1 is 1... so that's right there... and the square root of 4 is 2... so... when you square root that, you get the square root of y is x .

The opposite of b in Maria’s explanation may refer to $-b/2a$ in the quadratic formula which she remembered and used later in the interview. But her focus on the square root in her explanation of why a parabola is symmetrical suggests she is mixing up several ideas that are not yet fully formed.

Kenneth, an 11th grader in precalculus, is another example of a student who has a strong sense of the symmetry of the parent function, but he is not sure why it is symmetric. He said he thought the graph is symmetric because of the equation, which he explained as, “I think it would even out.” But then he named the squaring function and its inverse, $x = y^2$, as both being parabolas. After a conversation about whether they were functions, he was not as strongly convinced about the symmetry of the squaring function. When I asked him how he would explain the symmetry of the parent function to a younger student, he said, “I can’t really think of a good reason for it.”

Affordances of Considering the Parent Function for Solving Quadratic Equations

Considering the graph of the parent function can act as a cognitive affordance for students as they solve a quadratic equation of the form $x^2 = c$, where c is a positive constant. Students who thought about the graph of the parent function as they solved the

equation $x^2 = 100$ were able to solve correctly and name both solutions. Asking students to consider the graph when they had only found the positive solution in their initial response overwhelmingly helped them see that the equation had two solutions. As described above, the literature indicates that students tend to neglect the negative root when solving problems such as $x^2 = 100$ (Thorpe, 1989). I did not find any discussion in the literature of affordances of considering a graphical representation to students.

As mentioned earlier, following the conversation about the parent function and symmetry, I asked 26 students to solve $x^2 = 100$. Three of those students initially solved the equation correctly giving both the positive and negative solutions. I then asked the students about the relationship between solving the equation and the parent function. Specifically I asked if the solution they had just found for the equation $x^2 = 100$ was related to the graph of the parent function that they had just graphed.

Each of those 3 students who solved the equation successfully was able to relate the solutions to the graph of the parent function. They all thought of the relationship between the parent function and the solutions to $x^2 = 100$ as being the solution to the system of equations $y=x^2$ and $y = 100$. They made that clear by sketching the graph and the line and identifying the points of intersection. Clifton took the square root of both sides and wrote the solutions as be “ ± 10 .” When I asked him if there was a connection to the graph of the parent function, he said, “yes... if you go left or right 10, then you will reach ..., you'd have to go up 100 spaces.” Furthermore, one of the students, Dana, used this understanding to remind himself of the negative solution without my prompting. He talked himself through solving this equation:

Dana: Ok.. so, take the square root of both.. so it would be 10. so it would only have 1 x... .. I see now I was wrong.

Leslie: I'm curious.. were you?

Dana: .. actually wouldn't you mirror that over, so wouldn't it equal both (sketches U with -10 and 10 on it.)

This was especially interesting in that Dana tended to struggle overall with the questions on quadratics. He did not use factoring to solve any of the equations, and the only other problem that he was successful on involved using the quadratic formula, which he accessed from an earlier conversation. However, his sense of the symmetry of the parent function enabled him to access his knowledge of the squaring function to find both solutions to .

I asked 20 of the 23 students who initially gave only the positive solution to consider the link between solving the equation $x^2 = 100$ and the graph of the parent function. Of those 20 students, 65% were able to make the connection to the graph and then named $x = -10$ as a solution. Most students who saw the connection considered the intersection of the line $y = 100$ and the equation $y = x^2$. Once they graphed the point (10, 100) on their graph, they had an “aha” moment and recognized that (-10, 100) would also be a solution. Being encouraged to consider the graph of the parent function and think about how the solutions of the equation related to the graph helped students recognize that the equation had a second solution.

Obstacles Related to Symmetry

Students in this study demonstrated a very strong sense of the symmetry of parabolas, which was useful to them as they graphed quadratic functions. However, their sense of symmetry also presented some interesting obstacles to graphing in conjunction with their understandings about the y -intercept of parabolas. At times, students used symmetry in conjunction with their belief that the vertex of a parabola will be on the y -

intercept to graph a parabola incorrectly. At other times, they abandoned symmetry when they found the y -intercept, assumed it will be the vertex of the parabola, but included other points they found to complete the graph. I did not encounter any discussion of this in the literature.

Assuming the y -intercept is the vertex. Several students found the y -intercept and then used symmetry to create the graph. Melody was convinced that the y -intercept was the vertex when she tried to use symmetry to graph the function . She used a table of values, and when she found the point $(0, -14)$ she decided that must be the y -intercept, which she called the “meeting point.” She graphed the rest of the points and sketched the graph shown in Figure 14. Melody decided that she did not know how this graph would turn out, and speculated that it should be reflected over the y -axis. Her initial intuition was that the y -intercept would be at $(0, 4)$ and the parabola would open down, but when she put 0 into the equation for x and got -14, her strong sense of symmetry led her to believe that what she had sketched in quadrants I and IV would be reflected over the y -intercept to create the parabola. When she found the next point at $(3, 4)$ she decided that this function not behaving like a regular parabola and decided that that this one would “just keep going.”

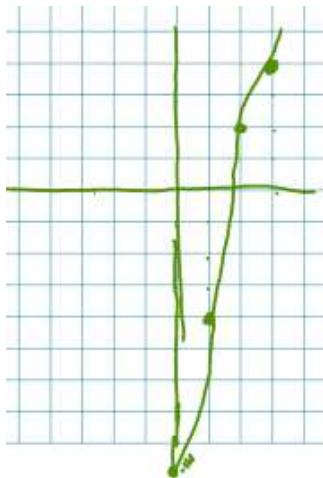


Figure 14: Melody's Graph

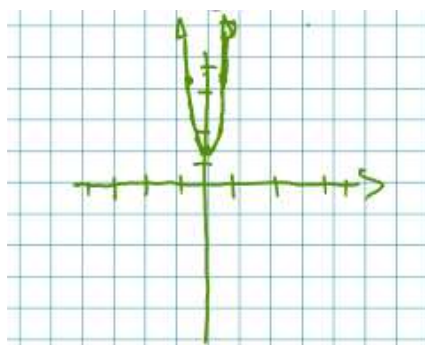


Figure 15: Bryce's Parabola

Bryce also used symmetry and the y -intercept to graph. When he graphed $y = x^2 + 4x + 3$, he knew the y -intercept would be at $(0, 3)$ and he said “you start there,” assuming that would be the vertex. Then he used ideas about slope, which he called “rise over run” to graph the parabola up 4 over 1. (Students’ use of linear concepts is discussed in a later section.) He graphed the parabola shown in Figure 15.

Abandoning symmetry when the y -intercept is assumed to be the vertex.

When students believe that the y -intercept is the vertex and they have found the x -intercepts, they sometimes graph the function abandoning symmetry, meaning that the vertex is not half way between the x -intercepts. When Tabitha graphed the function $y = 2x^2 + 5x - 12$, she factored to find the x -intercepts, and knew the y -intercept was -12 . She graphed these three points and then sketched in the parabola with the vertex on the y -intercept shown in Figure 16. Similarly, Claudia assumed the vertex of $y = x^2 + 4x + 3$ would be at the y -intercept. She graphed the parabola passing through the two x -intercepts with vertex at $(0, 3)$, abandoning symmetry, shown in Figure 17.

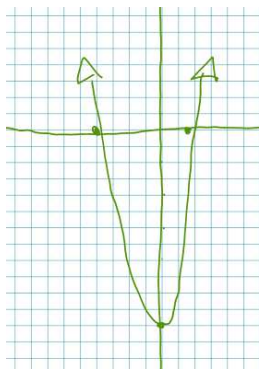


Figure 16: Tabitha's Parabola

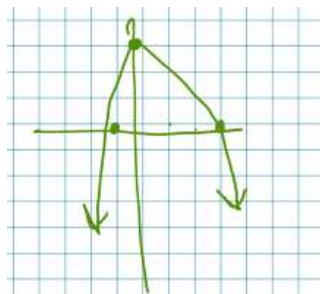


Figure 17: Claudia's Parabola

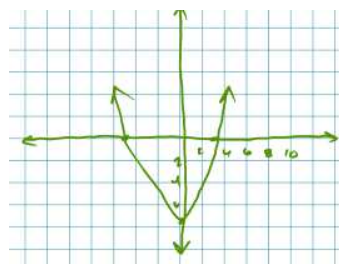


Figure 18: Maria's Initial
Parabola

Similarly, Maria put the vertex on the y -axis when she graphed $y = \frac{1}{2}(x - 3)(x + 5)$. She multiplied the factors and wrote $\frac{1}{2}x^2 + 1x - 7.5$ and said that the y -intercept would be at -7.5 . Then she identified a , b , and c and used the quadratic formula to find the x -intercepts and sketched the graph shown in Figure 18.

Symmetry as a scaffold. When students consider the symmetry of a parabola, it can support them in finding the vertex. I asked Maria if she thought the vertex would be at -7.5 . She thought for a moment about the symmetry of the function and said, “Yeah. Or at $1/2$... that would be.. it would be actually in the middle... it would be right here. (Draws line of symmetry and uses it to graph the point $(0.5, -7.5)$) So it would be right here, I think.. let's see... if I put in... so it's on this line, it's probably below here.. so down 2. 4. 6. 8... I don't know, it would be around here.” Asking her about the vertex seemed to cause her to revisit the function thinking about its symmetry, which led her to realize that the vertex had to be on the line of symmetry, “in the middle.” In this case, her sense of the symmetry became an affordance that supported her in correctly identifying where the vertex should be. Her completed sketch is shown in Figure 19.

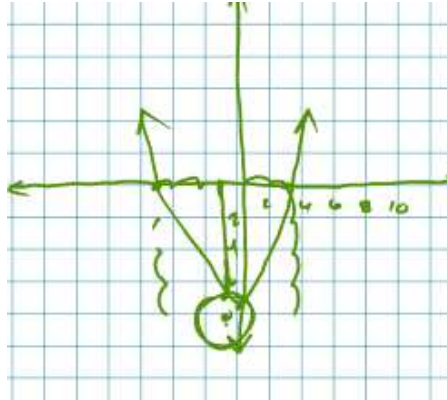


Figure 19: Maria's Completed Parabola

“Solving What?” What students think they are solving or solving for when they work with quadratic equations and functions

When working to solve equations or graph functions, students move between three objects: quadratic equations, equations defining quadratic functions and trinomial expressions. The first, $0 = x^2 + 4x + 3$ is an equation that can be solved for x . If one were to graph the solutions to this equation, the graph would consist of two points on a number line. The second object, $y = x^2 + 4x + 3$ defines a function. In high school mathematics classes, it is commonly referred to as the equation of a function. It cannot be solved, but one of the variables, x or y , can be set equal to a constant and the remaining equation can be solved. If y is set equal to 0, the result is the first equation, $0 = x^2 + 4x + 3$, and the solutions for x are the roots or zeros of the function and are the points where the graph of the function intersects the x -axis. The last object $x^2 + 4x + 3$ is an expression that can be factored but not solved.

Students transition between $0 = x^2 + 4x + 3$, $y = x^2 + 4x + 3$, and $x^2 + 4x + 3$ at times transitioning intentionally to better grapple with a task, and at times accidentally.

This is not a topic I encountered in the literature on student understanding of quadratics but arose as I analyzed student work. I found that students make connections between the objects, for example realizing that they would need to solve $0 = x^2 + 4x + 3$ in order to find the x -intercepts of the graph of $y = x^2 + 4x + 3$. However, at times students were unclear about exactly what they were solving for or which of these objects they were working with, which caused them to make erroneous connections.

Being able to flexibly navigate between the three objects can demonstrate understanding of the connections and is sometimes a productive strategy for solving the task at hand. However, moving between the objects while solving a problem can also lead to difficulties when students do not attend to the important features inherent in each object. Students make errors in switching between the three objects. At times this results in the student losing track of the question they were working to solve. Moving between the objects can also lead students to mix procedures and ideas from the different representations together. This results in confusion for students about what they are solving and/or what they are solving for.

Affordances of Moving Between Objects. When asked to solve quadratic equations in the form $ax^2 + bx + c = 0$, a productive strategy students sometimes used was to think about the graph of the related quadratic function, $y = ax^2 + bx + c$. They tended to do this for equations that were not solvable via factoring. Maria, Joanne, and Tracy each used the strategy of graphing the related equation to tackle solving equations that they found challenging to solve.

Maria used this strategy to try to get approximate solutions for the equation $0 = x^2 + x - 1$. She tried to factor the expression, $x^2 + x - 1$, and reasoned that it could not

factor into $(x+1)(x-1)$ because “that cancels out,” by which she meant the middle term in the product of $(x+1)(x-1)$ would be zero. Then, Maria said, “Like, I need to find the roots? Or I could graph it.” She sketched a graph of the function by graphing the y -intercept at $(0, -1)$ and finding the point $(1, 1)$. Then she reflected the point $(1, 1)$ across the y -axis, incorrectly assuming the y -axis was the line of symmetry. She used her graph to estimate that the solutions would be at about “plus or minus $\frac{1}{2}$.” Maria’s graph is shown in Figure 20. Maria’s strategy would have been productive if her graph had been accurate, and this strategy is productive when students use graphing technology to find solutions. In the interviews, we did not use graphing calculators, but in the classroom, graphing calculators are a frequently used tool. If Maria had access to a graphing calculator, using the graph to find the zeros of the function could have been a successful strategy. Her strategy reflects an understanding that she can solve the quadratic equation, $0 = x^2 + x - 1$ by considering the graph of $y = x^2 + x - 1$ and finding its x -intercepts.

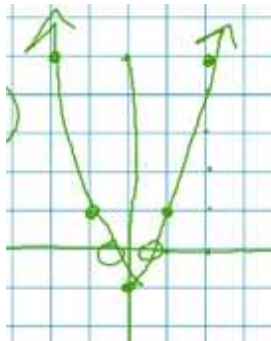


Figure 20: Maria's Graph

Joanne used a graph successfully (see Figure 21) to help her reason about the solutions to the equation $0 = x^2 + 1$. She began solving this problem by saying that she didn't think it could be factored, and then she said, “If it was a minus 1, then you would

be able to, but this one is factored as far as it can be, and this one is, oh... $y = x^2$ is just there, and this is one higher and so, problems.”

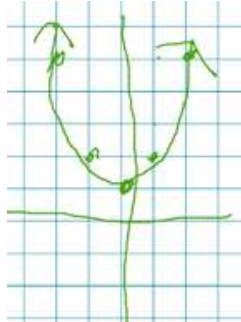


Figure 21: Joanne's Graph

Joanne went on to explain that this is just the parent graph, “only one higher.” When I asked if it had any solutions, she said, “No. It doesn’t cross the axis.” I referred to another problem where she had been talking about imaginary solutions, and she went on to use the quadratic equation to find the solutions to this equation, finding both positive and negative i . Though Joanne had strongly expressed that she dislikes graphs, she used them as a tool for understanding the behavior of a function, which allowed her to reason about the solutions of equations.

Tracy also made the connection to using a graph to find the solutions to an equation when she did not trust the solution she found algebraically. When she was solving the equation $0 = x^2 + 4x + 3$, Tracy initially did not remember factoring and tried unsuccessfully to complete the square. She was suspicious of her results because “the numbers came out kind of weird.” After thinking it through, Tracy was able to factor the expression, writing $(x + 3)(x + 1)$. I asked what that was, and she explained that those were the zeros of “the equation, the graph. So like at $x = -1$ and $x = -3$, the graph will be zero.” She sketched the axes and drew the two points. I pressed a bit more and asked

what she was graphing. Tracy said, “I’m graphing that [$y = x^2 + 4x + 3$]. But it’s easier.. but you have to, unless you put it all in your graphing calculator to find out what the zeros are, you have to do like, one of those things.” By one of those things, Tracy means solving by factoring, or another method she remembers from working with rational equations.

Difficulties Resulting from Moving between Objects. The difficulties that students encountered in moving between objects seem to fall into two categories. One was more about procedural errors, which might fall into a category teachers might call “careless” mistakes. The second category is characterized by a more complex conceptual piece. At times students appeared to arrive at an answer and then confuse their understanding of their answer by considering it as an answer to a different question. This occurred when students switched from one object to the other in mid-problem and lost track of what they were trying to figure out. This resulted in students giving an answer that shows understanding of some aspects of quadratic equations but not of the original question or how the ideas of factoring expressions, solving equations, and finding x -intercepts of graphs go together.

Procedural errors. One of the moves students made between objects was that when solving an equations, students often transitioned to operating on an algebraic expression. In the process of solving, it was not uncommon for students to stop writing the “0 =,” when solving an equation such as $0 = x^2 + 4x + 3$, which switched the object of their solving from the equation to the expression. When students do this, they were apt to state the answer as being the factors $(x + 3)(x + 1)$ and believe that they had solved the

problem. Joanne is one of the students who did this consistently. When I asked her if she solved for x , she said no, and said “that’s the step I almost always forget.”

Kerry routinely did not write the “=0” when solving an equation, and when she solved by factoring, she usually showed the original equation, then the expression she factored followed by the factored form as shown in Figure 22. After she wrote the factored expression she then wrote the opposite of the numbers in the factors. Kerry did not know how why this method worked. However she explained why she could write a “+0” at the end of the equation, “because it’s plus zero, and since it’s addition or subtraction, it doesn’t really impact it at all.” It is not clear whether she leaves off the zero to save writing, or if she does it because it does not matter or impact the equation or solution.

$$\begin{array}{l} x^2 - 2x + 0 \\ x^2 - 2x + 0 \\ (x-2)(x+0) \\ +2 \quad 0 \end{array}$$

Figure 22: Kerry's Work

In Kerry’s case, neglecting to write the zero did not impact the correctness of the solutions she finds. However this short cut seems to reflect a lack of understanding of the underlying concepts and process that allows her to find them. As described earlier, Kerry was not able to explain why the zero-product property works, and she has developed a

language she uses for the process where she refers to “zeroing-out” the factors. She seems to have adopted the procedure without understanding why it works.

Conflating ideas. Some students solved equations and then used other ideas in ways that were really interesting. Brad solved the equation $x^2 = 100$ and said it had one solution. Then he said, “that would be the zero, the zero would be at 10.” He reasoned that since 10 is a zero, he could think of $(x - 10)$ as a factor. “You could use $x - 10$, and you could put it back over here. I think that would be squared, because there’s... not, it will be a parabola that only intersects the x -axis once.”

Brad then squared $x - 10$ and said “it” would be $x^2 - 20x + 100$. I asked him if $x^2 = 100$ and $x^2 - 20x + 100$ were the same, and he said, “I am going to say yes, I’m not sure.” When I asked about the graph, he sketched a graph with root at $(10, 0)$ but also drew a line at $y = 10$ and indicated the two intersection points (see Figure 23).

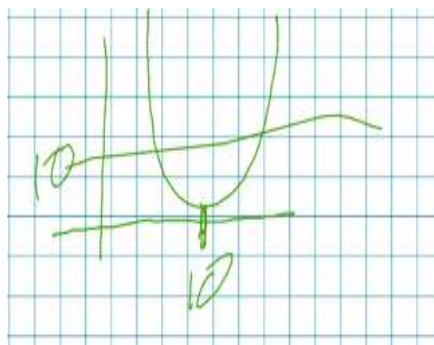


Figure 23: Brad's Graph

Brad made a really interesting transition between the equation he solved and the function he graphed. He found a solution $x = 10$, and thought of it as a zero of the function, meaning that $x - 10$ is a factor. He then decided that if a quadratic had only one solution at $x = 10$, the equation would be the factor squared, $(x - 10)^2$, and the graph would have its vertex on the x -axis at $x = 10$. In doing this he moved from

solving an equation, to finding a factor of part of the equation of a function, and then transitioned to thinking about the graph of the function. Brad understood that having one root means the parabola has a vertex on the x -axis. He made connections between the objects that are valid, but did not recognize that $x^2 = 100$ and $x^2 - 20x + 100$ are not the same.

The Impact Of Students' Linear Thinking On Solving Quadratic Equations And Graphing Quadratic Functions

Throughout the course of the interviews students frequently resorted to using concepts and skills that they had learned to solve linear equations and graph linear functions. The literature describes students solving equations such as $x(x - 2) = 0$ by dividing both sides of the equation by x , resulting in losing track of the solution $x = 0$. This is interpreted in the literature as students not applying the zero-product property properly (Didis et al., 2011, Nandakumar, 2005, Kotsopoulos, 2007). I found that students used linear strategies to solve equations when they could not remember how to solve by factoring or other methods and when they thought they could not factor with integer coefficients. My findings lead me to believe that when students use linear strategies, it may be more than not applying the zero-product property. Students are solving quadratic equations using the strategies they have been learning in preceding math classes for solving linear equations with a strong knowledge of keeping equations in balance by doing the same thing to both sides and using the strategy of undoing.

Additionally, students applied their understandings of linear functions as they thought about graphing quadratic functions. They thought of the parameter a as slope, and were able to identify the y -intercept from their previous work with linear equations. I

did not find any discussion in the literature regarding students applying linear strategies to *graphing* quadratic equations.

Applying Linear Techniques When Solving Quadratic Equations

The linear methods students applied to solving equations involve doing the same thing to both sides of an equation in an attempt to isolate the variable. For example, if one were trying to solve an equation such as $2x + 6 = 0$, a productive strategy would be to subtract the 6 from both sides of the equation and then divide both sides by 2 as shown:

$$2x + 6 = 0$$

$$2x = -6$$

$$x = -3$$

This solution process is sometimes taught or thought of as undoing. You try to undo the last thing done to the variable first, using your knowledge of order of operations. In this case, in the original equation, x is first multiplied by 2 and then 6 is added, so to solve the 6 is subtracted first, and then x is divided by 2.

Undoing can be used to solve quadratic equations when they are of the form $ax^2 + c = 0$, (i.e. $b = 0$). In this case, the c is subtracted from both sides, then both sides are divided by a . Lastly, you take the square root of both sides, taking care to note the positive and negative square root.

$$ax^2 + c = 0$$

$$ax^2 = -c$$

$$x^2 = \frac{-c}{a}$$

$$x = \pm \sqrt{\frac{-c}{a}}$$

Undoing is a component of the strategy of solving by completing the square. However, in completing the square, there is an additional step involved in which you have to think about what to add to both sides to create a perfect square trinomial on one side of the equation. In a problem such as $x^2 + 4x + 3 = 0$, that process is shown here:

$$x^2 + 4x + 3 = 0$$

$$x^2 + 4x = -3$$

$$x^2 + 4x + 4 = -3 + 4$$

$$(x + 2)^2 = 1$$

$$(x + 2) = \pm\sqrt{1}$$

$$x = -2 \pm 1$$

$$x = -1 \text{ or } x = -3$$

In this process, the first step of subtracting 3 from both sides is the same as solving a linear equation.

When linear methods work. Undoing is a productive strategy for solving $y = x^2 + 1$, and 11 of the 23 students who were asked this problem tried to solve it by undoing. Students seem to view this as a “real” algebraic procedure in contrast to factoring. Interestingly, none of the students who solved $0 = x^2 + 1$ found both solutions, plus and minus i . Nine students found only the solution $+i$, one student talked about i but didn’t identify it as a solution, and one said the solution was $x = \sqrt{-1}$. Students who solved by undoing in this problem included students who had previously solved problems using factoring or the quadratic formula as well as students who tried to apply linear strategies incorrectly.

Sacha referred to solving by undoing as solving “algebraically.” He reasoned that he was not able to factor it, and said “If you were to do this algebraically?” I replied that he could solve it however he would like to, to which he replied, “I get stuck on one

method because that's the way schools do it. If you do this algebraically, you get $x^2 = -1$ which means if you square root this, you get the square root of -1, which is i ... which is an imaginary number.” He sees this method as algebraic while he seems to think factoring is not.

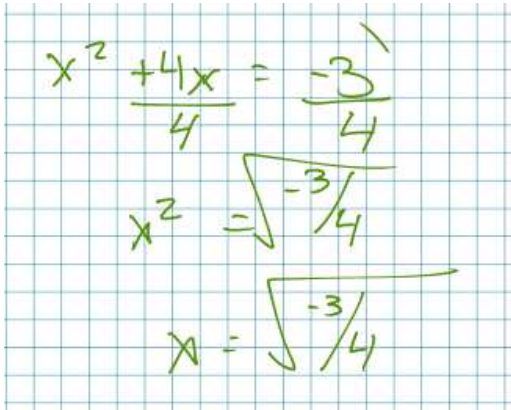
Tracy also seemed to think of undoing as a more normal type of algebraic operation than factoring. When she solved $y = x^2 + 1$, she said, “Yeah, so...because it's just a normal thing... um.. x squared ... I can just subtract 1, so it's just $x^2 = -1$. But then you take the square root of both sides, which gives you something interesting problems because you can't take the square root of -1, so ... that gives you ... i ?!.. that's i .”

When linear methods are used to no avail. Students encounter difficulties with undoing as a method. When students understand order of operations they sometimes try to apply linear methods, but realize that the equivalent equations they produce do not “look right,” whereas when students do not have a firm understanding of order of operations they may arrive at erroneous solutions.

Four students – Annette, Michelle, Samantha, and Tracy – tried to use linear techniques to solve the equation $0 = x^2 + 4x + 3$, subtracting 3 from both sides and then taking the square root of one or both sides and applying order of operations incorrectly to arrive at a solution. All four students were fairly sure that their answers did not look right, and Michelle specifically noted that this was not “balancing equations.”

I asked Samantha to solve $x^2 + 4x + 3 = 0$. Samantha used linear strategies too, and knew her answer did not look right. She incorrectly applied order of operations in her strategy. In this attempt, she divided both sides by 4, but only applied it to the $4x$,

which made the $4x$ disappear, shown in Figure 24. Samantha knew this didn't look "right" so she tried again.



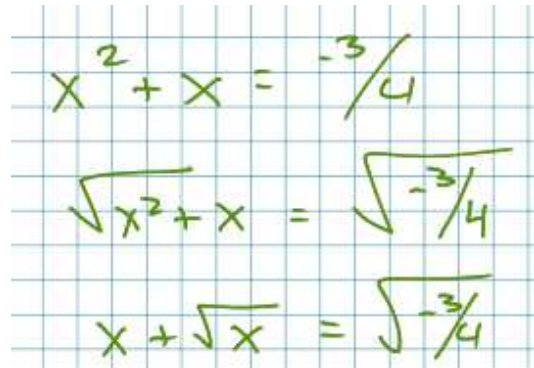
$$x^2 + 4x = \frac{-3}{4}$$

$$\frac{x^2}{4} = \frac{-3}{4}$$

$$x^2 = \sqrt{\frac{-3}{4}}$$

$$x = \sqrt{\frac{-3}{4}}$$

Figure 24: Samantha's First Attempt



$$x^2 + x = \frac{-3}{4}$$

$$\sqrt{x^2 + x} = \sqrt{\frac{-3}{4}}$$

$$x + \sqrt{x} = \sqrt{\frac{-3}{4}}$$

Figure 25: Samantha's Second Attempt

In her second attempt, shown in Figure 25, she divided by 4, which removed the 4 in front of the x , and then when she took the square root of $x^2 + x$ she got $x + \sqrt{x}$. Similar to Annette, Samantha took the square root of a sum by taking the square root of each of the addends individually.

Students also used linear techniques to solve $x^2 - 2x = 0$. Six of the 27 students tried to use linear methods to solve $x^2 - 2x = 0$. This equation was challenging to two-thirds of the students, and is discussed in an earlier section. These six students attempted to solve the equation using the linear methods they knew. Their methods involved adding the $2x$ to both sides. Some of them took the square root of both sides, arriving at

incomplete solutions such as $\sqrt{\frac{x^2}{2}} = x$ or $x = \sqrt{2x}$. These statements follow logically

from the original equation, and would have been equivalent equations if the students had included the positive and negative square root, but these students were not able to reason

from their solutions to determine that $x = 0$ or 2 . Other students made some errors in their algebraic manipulations. Annette took the square root of both sides of $\sqrt{x^2} = \sqrt{2x}$ and got $x = x$. She then told me that means the equation is true for all values of x .

Samantha said she knew it was a quadratic equation because there is an “understood plus zero.” She added $2x$ to both sides to get $x^2 = 2x$ and divided it by 2 to get $\frac{x^2}{2} = x$. She said “this is so odd” and tried again. Then she says, “this is... ok... I’m just stuck on where... I just can’t remember how to get the variable just on one side... when you don’t have that one... just definite number with no other variable... I guess you could try finding the square root.. but then you have x ’s on both sides... I don’t know... This is just so odd...” Samantha is firmly grounded in the solution method of getting the variable just on one side from working with linear equations, and cannot figure how to apply these techniques to quadratic equations.

$$x^2 - 2x = 0$$

$$\frac{x^2}{2} = \frac{2x}{2}$$

$$\sqrt{\frac{x^2}{2}} = x$$

$$\frac{x}{\sqrt{2}} = x$$

Figure 26: Samantha's First Attempt on Question 3c

$$x^2 - 2x = 0$$

$$\sqrt{x^2} = \sqrt{2x}$$

$$x = \sqrt{2x}$$

Figure 27: Samantha's Second Attempt on Question 3c

Linear methods when factoring does not work. Some students who were able to solve equations successfully via factoring resorted to linear methods when they were not able to factor expressions easily. For example, Delilah reverted back to ideas about isolating the variable, or as Delilah said “solve it solve it” when the quadratic strategies

she knew were not successful. Delilah solved the equations $x^2 + 4x + 3 = 0$ and $x^2 - x = 12$ successfully by factoring. However, when she encountered problems she didn't know how to solve via factoring, she resorted to linear techniques. When solving $x^2 - 2x = 0$, Delilah said, "I think this one you actually like have to *solve it solve it*, instead of like solve it, like move everything over so you find the value of x instead of factoring it." By "solve it solve it," she meant use linear techniques to isolate the variable as opposed to the methods of solving quadratic equations she had been using. For this problem, Delilah added $2x$ to both sides and then divided both sides by x and found that $x = 2$.

Solving the equation $0 = x^2 + x - 1$ also presented a challenge to Delilah. She tried to factor the equation and then reasoned that it was not factorable. Her next strategy was to use linear techniques to solve, shown in Figure 28. Delilah added 1 to both sides of the equations and took the square root of both sides.

The image shows handwritten mathematical work on a grid background. The work consists of several lines of equations and expressions:

- Line 1: $0 = x^2 + x - 1$
- Line 2: $(x)(x)$
- Line 3: $x^2 + x - 1 = 0$
- Line 4: $\sqrt{x^2 + x - 1}$
- Line 5: $x + \sqrt{x} = 1$

Figure 28: Delilah's Linear Techniques When She Can Not Factor

Near misses, or linear techniques mixed up with completing the square? One difficulty for students who may be prone to using linear techniques is that the process of completing the square begins with a step that resembles linear techniques. As mentioned

above, completing the square is a strategy that students learn for solving equations that are challenging to solve via factoring. When solving $x^2 + 4x + 3 = 0$, as shown in Figure 29, Brad started off trying to complete the square, and then briefly reverted to linear strategies, when he said, “then you subtract one from both sides, and then you take away 2.”

$$x^2 + 4x + 3 = 0$$

$$x^2 + 4x + 2^2 = -3 + 2^2$$

$$(x+2)^2 = -3$$

$$x+2 = -3$$

Figure 29: Brad's solution

Dana also started out trying linear strategies. He wrote out $x^2 = -4x$ and was about to write the 3. Then he crossed it out and said, “Actually no, you don't do that... so you do...” and then he factored it. As students learn to complete the square, they may need additional support in understanding how this is the same and different from applying linear techniques.

Linear Thinking When Graphing Quadratic Functions

Though the research literature referred to students applying linear techniques to solving quadratic equations, I did not encounter any reference to students applying linear approaches to graphing quadratic parabolas. When graphing linear functions, students learn about the slope-intercept form of a line, which is often represented as $y = mx + b$ or less often $y = ax + b$. In both representations, students learn that b gives them the value of

the y -intercept, and the coefficient of the leading term, x , is the slope. This has some similarities to equations of quadratic functions in standard form, $y = ax^2 + bx + c$. The constant term, which is c here, gives the value of the y -intercept, and the value of a determines the dilation of the parabola and indicates if it has been reflected across the x -axis.

Using ideas from slope. Many students think of the parameter a in the standard equation of a line as the slope of the parabola. They call it slope and talk about the “rise over run.” For example, when Joanne graphed the function $y = 2x^2 + 5x - 12$, I asked her to tell me what she knew about the graph of the function, and she said:

Joanne: well... humh.. the factor that it's a uh.. the 2 there probably means that it has a.. the slope? is that the right word for a quadratic? The thing that's probably like slope. It's larger.. it's dilated, and since it's a 2.. well, rather than be larger (gestures upwards) it would either be dilated larger or smaller vertically, I think.

Leslie: ok.. do you remember which one it is?

Joanne: I could figure it out, probably... smaller..

Leslie: ok.. smaller, meaning... (gestures) if this is the parent graph, what would it look like?

Joanne: holds up her hands.. like this.

Leslie: ok.. so, start with the parent graph.

Joanne: (she shows it scrunching in..) It would be like this and then like that

Joanne started gesturing with her arms up like the parent function and then brought them closer together and up to show how the 2 would impact the shape of the graph. She indicated that the graph would get skinnier, but not necessarily taller.

When Chelsea was graphing the function given by $y = \frac{1}{2}(x - 3)(x + 5)$, she multiplied it out to get $y = \frac{1}{2}x^2 + x - \frac{15}{2}$. When I asked her what she knew about the graph, she said, that it would maybe have a slope of $\frac{1}{2}$. I asked what she meant, and she

said, “It’s how far the x and y go over. Oh goodness, is it y over x or x over y ? It has to be one of them, up, up.. so it goes up up by y over x .” I asked again, and she said, Oh, that’s a vertical.... contraction, maybe?” she indicated with her hands that it would make the graph more “condensed” (flatter) reasoning, “because it’s vertical, it’s going to go, not inward, but the values are going to be downward, so they’d be bigger values, so it will widen up a little bit.” Chelsea has a sense of how the $\frac{1}{2}$ will impact the graph and is incorporating new vocabulary ideas such *vertical contraction* and *condensed*. However, she does not leverage the idea that the $\frac{1}{2}$ multiplies the quantity $(x - 3)(x - 5)$, which makes each resulting value of y one half as big.

The y -intercept. Students have a strong understanding that the constant term c in standard form is the y – intercept. However students did not necessarily understand why the constant term c gives the y value of the y -intercept. I asked 11 students about the graph of the function $y = 2x^2 + 5x - 12$. Of those 11 students, 8 attempted to graph the function, 6 of whom found the y -intercept.

I asked Joanne to tell me why she said she was “guessing” that -12 was the y -intercept. She said, “Well, on a linear graph, when it’s $y = mx + b$, it’s always the number that doesn’t have the x attached to it, which is that one (circling -12 on her paper).” I pressed when she said the y -intercept is where the “line hits the y -axis” and asked her why it hits the y -axis. She said, “because x is zero. When x is zero, the y value is 2 times 0 squared, plus 5 times 0 minus 12, and that’s $y = 0 + 0 - 12$.”

The vertex at the y -intercept. Several students successfully found the y -intercept and placed the vertex of their parabola on the y - intercept which is an extension of the linear graphing strategy of finding the y -intercept and graphing from there. Claudia did

this for both functions she graphed. Figure 7 shows Claudia's graph of $y = x^2 + 4x + 3$. Claudia began by finding the zeros of the function correctly by factoring at $x = -3$ and $x = -1$, but she graphed $x = -3$ incorrectly at positive 3. When I asked her what else she knew about the graph from the equation she initially said she didn't know anything else, but then identified the y -intercept at 3. I asked her why that would be, and she said, "cause I remember the formula $y = mx + b$, and that's what's throwing me off is the $4x$ with 4 being the slope. But I don't think that's right with this equation." She further explained, "I remember the $y = mx + b$, and I remember the b is the y -intercept, but I don't think 4's the slope." She then graphed the parabola as shown in Figure 30 opening downward with vertex at $(0, 3)$. The parabola is not symmetric (this is discussed in an earlier section), even though she earlier graphed the parent function correctly and described the symmetry in the table of values.

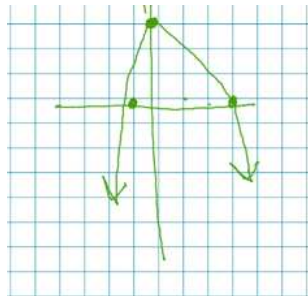


Figure 30: Claudia's Graph

Putting y – intercept and slope together. Some students used the y - intercept as a starting point and then used their ideas about slope and symmetry to graph the function without attending to other aspects of the function. In linear graphing, the y -intercept and slope are sufficient to graph a line, and these students seem to have adapted this to

graphing quadratic functions. Bryce knew the graph of $y = x^2 + 4x + 3$ would be a parabola. He began by thinking about the y -intercept and said the vertex would be on the y -intercept. He then used “rise over run” to get the shape, assuming the 4 in the equation is the slope, so he graphed a parabola with vertex at $(0, 3)$ that instead of going right 1, up 1, goes right 1, up 4 and then reflected it across the y -axis. Bryce’s graph is shown in Figure 31. He graphed a parabola with a vertical dilation of 4, but with the vertex placed incorrectly. This graph led Bryce to conclude that the equation had no roots because it did not cross the x -axis.

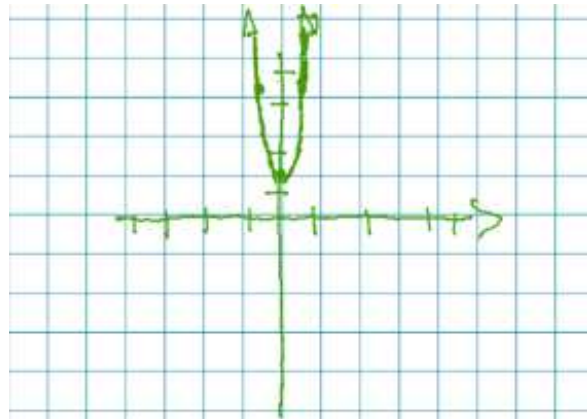


Figure 31: Bryce's Parabola

Samantha tried to make sense of the graph of $y = -2(x - 3)^2 + 4$, which is in vertex form, using her knowledge of the other forms. She used her previous knowledge of linear functions to determine that the +4 meant that the y -intercept would be at $(0, 4)$. She used her understanding of the factored form of equations of binomials squared, such as $y = (x - r_1)^2$ to reason that the x -intercept would be at -3. Samantha went on to ask if the slope would be -2. I asked her what she meant by the slope, and she said, “well in a basic line, if it was just going up like this (sketched a line with slope = 2)... it would be

up 2 over 1, and down 2 over 1.” When I reminded Samantha about the vertex form of the equation, which we had discussed earlier, she said that the -2 would make the parabola “point down” (she sketched it opening downwards) She said, “this negative, would mean its going to be flipped down... the 2 would make it .. I believe ... it's going to be a vertical stretch... I think it would grow, it would widen.” Samantha said that the parabola would get vertically stretched and would grow wider, naming both possibilities. Samantha is putting ideas from several different forms of quadratic equations together to create a graph, and like Chelsea, she does not have a clear sense of how the coefficient a contributes to the value of the y for each value of x .

In this chapter, I have described my findings. First, I characterized how students performed across the problems, examining both demographic differences and examining differences by problem types. I then shared my findings about how students think about quadratics, the parent function and symmetry of the parent function. I described how students transition between objects as they work to solve problems and how that can impact their solutions. Lastly, I described the impact of students’ linear thinking on both solving quadratic equations and on graphing quadratic functions.

In the next chapter, I offer a discussion of my findings.

CHAPTER 5: DISCUSSION

This study examined what high school students who have completed an Algebra 2 or Precalculus class understood about quadratic functions. This study has several major findings: (1) Students in this study relied heavily on understandings from their previous study of linear functions; (2) they had a strong sense of the symmetry of the parent function; and (3) they relied on the idea of taking the square root of both sides of an equation to solve quadratic equations of the form $x^2 = a$. At times, students in this study were unclear about what they are solving or solving for, and they found it challenging to give mathematically precise explanations for the symmetry of the parent function or the zero-product property. In this chapter, I discuss these findings. First I examine the implications of students' reliance on linear thinking when solving quadratic equations. Second, I discuss ways in which students move between equations, expressions, and equations defining quadratic functions. In the third section, I discuss the quality of student explanations.

One of the goals of this study was to begin to develop a research-based framework to identify and describe students' understanding or ways of reasoning in their learning of quadratics. Following the discussion of the major findings, I offer an emergent framework of student learning of quadratics in which I start with foundational understandings of quadratics and characterize how those evolve as students develop conceptual understanding of quadratic functions and equations. This chapter concludes

with a discussion of the contributions and limitations of the study and implications for future research and instruction.

Linear Thinking when Solving

Students reliably used linear techniques to solve equations when they did not know how to solve using other solution methods or those methods were not successful. Previous studies have suggested that students use linear strategies to solve equations such as $x^2 - 2x = 0$ because they do not correctly interpret the zero-product property (Didis et al., 2011; Bossé & Nandakumar, 2005; Kotsopoulos, 2007). I found that students use linear strategies on problems whenever they do not know how to proceed. Students may have used linear strategies because they were familiar with them and because some of the strategies for solving quadratic equations involve elements of linear strategies. For example, when students use completing the square to solve a quadratic equation in standard form, generally the first step is to subtract the constant term from both sides, which resembles steps students commonly use to solve linear equations.

Students begin solving linear equations in sixth grade (Common Core State Standard, 6.EE.B.1³), and by the time they encounter quadratic equations, they have spent at least three years using linear techniques. Making the transition from linear to quadratic strategies may be complicated for students because the linear strategy of “undoing” can be productive for solving quadratic equations when the parameter b equals zero. This is true in equations of the form $ax^2 + c = 0$ or $ax^2 = c$. In these equations, students use linear strategies to isolate the x^2 and then take the square root of both sides. However, to

³ CCSS.MATH.CONTENT.6.EE.B.7: Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p , q and x are all nonnegative rational numbers.

be successful they must also remember to take the positive and negative square root. For

example, the equation $ax^2 = c$ has the solution $x = \pm\sqrt{\frac{c}{a}}$.

In moving from linear to quadratic strategies, students learn the technique of completing the square after they learn to solve quadratic equations using factoring. The first step of completing the square, shown in Figure 32, in which students add the opposite of c to both sides, is the exact same as the first move students make when they use linear strategies (please see Figure 3 in Chapter 2 for the complete process of completing the square.) If students do not have a clear conceptual understanding of these different approaches to solve quadratic equations and how they are different from or build on linear strategies, they may confuse completing the square and undoing with linear techniques and revert to linear processes for solving. For example, after subtracting c from both sides of the equation, a student may attempt to solve by dividing both sides of the equation by one of the remaining coefficients (a or b), or try to take the square root of both sides.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ ax^2 + bx &= -c \end{aligned}$$

Figure 32: First step of completing the square

Students in this study encountered further difficulties with linear strategies when they had challenges remembering and/or imposing order of operations on their work and when they did not recognize that their solutions were not reasonable. When students do not have a firm grasp of order of operations, they do not necessarily recognize the operational errors they have made, which prevents them from recognizing that linear methods are not productive when the parameter b is not zero. Some of the students in this

study used linear strategies, correctly applied order of operations, and were able to determine that their work did not look right. Other students applied linear techniques without the correct order of operations. These students did not recognize their errors. These applications of linear strategies seem to reflect a lack of understanding of what quadratic equations and expressions are and how they behave.

An important element of learning and doing mathematics is trying to solve problems using the tools and strategies one knows. Correctly trying valid techniques and recognizing both fruitful avenues and dead ends is part of being a mathematically proficient student. In the absence of knowing a solution strategy for quadratics, it can be productive to experiment with adding something to both sides of an equation or taking the square root of both sides, or as some students did, graph the related function and find its zeros. As students take on quadratic equations, they have had experience with equations such as $4x = 3$ where it is appropriate to divide both sides by 4 in order to isolate x , or equations such as $x^2 = 100$, where they can take the square root of both sides (including the positive and negative square root) to get $x = \pm 10$. As students encounter equations such as $x^2 + 4x = 3$, their understanding of order of operations becomes critical. Students use their understanding that if they divide both sides of the equation by 4, they have to divide the entire quantity, $x^2 + 4x$ by 4, resulting in the equation

$\frac{1}{4}x^2 + x = \frac{3}{4}$. Alternatively, students may try to take the square root of both sides,

resulting in $\sqrt{x^2 + 4x} = \pm\sqrt{3}$. Using either approach, if students are able to apply the operations correctly to equations, they can try different approaches to solving, recognize those that are not successful, and persist in trying other techniques.

Solving What?

A second major finding of this study is that as students solved equations and graph functions, they have may started solving a quadratic equation, and then in the middle of their solution process changed the object of their attention to a trinomial expression or an equation defining a quadratic function. At times, students moved between these three objects intentionally as a part of their solution strategy. At other times, this was a result of a clerical error in which the student forgot to include a part of the original problem, such as “0 =”, in their work.

In a few instances, students moved from one of these objects to another as a result of a conceptual error. Brad did this when he solved the equation $x^2 = 100$, and then made the assumption that this would relate to the graph of the function $y = (x - 10)^2$. He used his understanding that the equation had one solution, $x = 10$, in conjunction with his understanding that quadratic equations that have one root intersect the x -axis in one place to conclude that the equation of the related graph would have a double root at $x = 10$ and therefore be $y = (x - 10)^2$. In these instances, the solution students reached working with one of the objects led them to leap to another object and then make assumptions about the related function.

These findings illuminate two issues. One concerns the precision of the words we employ to describe the objects being studied in high school algebra and the second relates to students' conceptions of the equal sign.

What are the objects of study? When students work with quadratic functions they use quadratic equations, such as $ax^2 + bx + c = 0$; equations defining quadratic functions, such as $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$; and quadratic expressions,

such as $ax^2 + bx + c$. Mathematics educators lack an agreed upon term for what to call this group. Entities? Objects? Algebraic representations? Through the process of carrying out this study, I have learned that my own language about these three has not always been sufficiently precise. I have tended to alternate between equations with a “ $0 =$ ” and equations with a “ $y =$ ” fairly fluidly in my own teaching practice, and I suspect that I am not alone. I believe that one of the reasons students float between these three objects is that teachers may not have had sufficient learning opportunities to reflect on and use precise mathematical language.

I settled on the word “object” as a result of Chazan’s (2000) observation that in algebra, the “objects of study” are not clear. He contrasts this with geometry, in which teachers and students know they are studying the properties of relationships between objects such as points, lines, planes, and polygons. Chazan puts forth that “If one one’s own understanding of the discipline allows one to describe its objects of study, perhaps one would then be able to appreciate how they manifest themselves in students’ experience. Thus, for me, a conceptual understanding of a discipline identifies its central objects of study” (p. 68). Chazan suggests that the study of algebra may more traditionally be organized as a study of actions taken, such as solving and graphing, rather than the study of particular mathematical objects.

If secondary mathematics educators are not able to articulate what the objects of study are, and are imprecise in the terms we use, this may impact the understanding that students develop. The students in my study were able to solve or attempt to solve equations and graph them, which is consistent with the study of algebra as a study of actions taken. However, I am not certain how many of the students have a conceptual

understanding of what the objects they are graphing and solving are, or of how they are connected and interrelated. A student who has a conceptual understanding of these three objects (expressions, equations, and equations defining functions) would be able to move between the three while maintaining an understanding of the properties of each. When solving an equation, such as $x^2 + 4x + 3 = 0$, the student would be able to factor the expression on the left hand side, make connections to the solutions of the equation, as well as make connections between the solutions to the equation and the zeros of the graph of the function defined by the equation $y = x^2 + 4x + 3$.

The equal sign. As students learn about functions and solve quadratic equations, their understanding of the meaning of the equal sign is being extended. Carpenter, Franke, and Levi (2003) describe the development of students' understanding of the equal sign. Many elementary students initially believe that the equal sign is a signal or command to carry out a calculation rather than a sign that “denotes the relation between two equal quantities” (Carpenter et al., 2003, p. 9). As students progress through middle grades and learn to solve equations, they do so by maintaining the balance between two equal quantities connected by an equal sign. Students add and subtract equal amounts from each side, and they learn to multiply and divide both sides by constants in order to maintain equality. This becomes more complicated as students learn to consider what happens if they multiply or divide both sides of an equation by a variable and to be wary of the possibility of dividing by zero.

When students begin to solve equations in which they take the square root of both sides of an equation, the meaning of the equal sign and what solving equations means is further expanded, because they are now solving equations of functions that are not one-

to-one. It would appear that the students in my study think about taking the square root of both sides of an equation, or “square-rooting it,” as an action done to both sides of the equation, without considering that the inverse of squaring generally results in two solutions. When students solve an equation such as $x^2 = 100$, they have to expand their understanding of “the” solution of an equation to mean that the variable can be one thing or another. In this case, $x = 10$ or $x = -10$. Two (or more) different solutions resulting from one equation is an expansion of student’s understanding of the equal sign and solving equations. I would suggest that students neglecting the negative square root in their solutions may be a result of not understanding that they are working with non one-to-one functions and that an equation may have more than one solution rather than as a result of not understanding the \pm as Thorpe (1989) suggested.

This way of thinking of “square-rooting” as a solution strategy yielding one solution may act as a cognitive obstacle resulting from students’ previous experiences with linear equations where solving techniques are grounded in doing the same thing to both sides of the equation resulting in at most one solution. This obstacle may be further compounded for students because their first experiences solving problems involving squares and square roots are generally presented as area problems and therefore only have positive solutions.

This expansion of understanding of the equal sign and solving equations continues as students learn factoring as a solution method for quadratics. Factoring is a solution method totally unrelated to previous ideas of balancing equations, but it depends on students remembering (and recording) that the expression they are factoring equals 0. For example, when students solve an equation such as $x^2 + 4x + 3 = 0$ by factoring, the

intermediate result is $(x + 1)(x + 3) = 0$. This equation should be read and understood as “the quantity $x + 1$ times the quantity $x + 3$ equals zero,” or it can be read as “the product of $x + 1$ and $x + 3$ equals zero.” When students factor, focus only on the expression, and do not keep the “= 0” attached, they lose sight of the relationship between the product of the factors and zero. Forgetting that the expression they are acting on is equal to zero may, in part, account for a lack of understanding of why the zero-product property can be used to solve quadratic equations.

Lastly, the equal sign has a meaning that many, if not most, secondary students and teachers do not use precisely. Namely, the object $y = x^2 + 4x + 3$ is a definition of the notation of y , which then allows the user to talk about the function y rather than repeatedly write out the function $x^2 + 4x + 3$. Students and teachers may be more familiar with this when using the notation $f(x) = x^2 + 4x + 3$, in which $f(x)$ is defined to be the function $x^2 + 4x + 3$. Understanding that one has defined a function as y or $f(x)$ gives students forms of notation that enable them to grapple with concepts yet to come, such as compositions of functions and derivatives.

Quality of Explanations

Students in this study found it difficult to give strong generalized explanations for why the squaring function is symmetric and why they were able to use the zero-product property to solve quadratic equations. Some students were not able to explain; others appealed to authority, saying that they were taught it or that the teacher told them. Many students were able to give explanations like Delilah’s for symmetry that were based on examples using one set of points, in which they pointed out that since 1 squared is 1 and -1 squared is 1, it would “mirror each other along the y -axis.” Very few were able to give

explanations that were generalized like Joanne did when she explained the symmetry of the parent function, “if the x is negative, it’s still the same y value.” Student difficulties in explaining the symmetry of the parent function and why the zero-product property works may stem from three potential sources. One potential source is that the reasons behind these mathematical concepts may not have been addressed in the students’ learning opportunities. Second, students struggle with the precision of language necessary for explanations. Third, students may not have learned how to construct viable arguments and explain their reasoning in the realm of algebra.

Learning opportunities. It is possible that many of the students in this study have not had the opportunity to explore why quadratics are symmetric or to develop an understanding of why the zero-product property enables them to solve quadratic equations. The students in this study received their algebra instruction from a curriculum that emphasizes procedural fluency, so it is possible that they have not had the opportunity to develop relational understanding of quadratic functions and their equations. Making connections between the instruction students received and their understanding of quadratics is beyond the scope of this study.

Attending to precision. Student explanations for symmetry of the parent function appeared to vary in the precision of their language, affecting the strength of their arguments. The Common Core State Standards for Mathematical Practice may offer a viewpoint for thinking about these student explanations. The sixth standard for mathematical practice is *Attend to Precision*. This standard states that students learn to “craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and

connecting their words clearly to their representations” (CCSS Mathematical Practice #6). The standard indicates that students learn to “craft careful explanations that communicate their reasoning” in elementary school, and that they “start by using everyday language to express their mathematical ideas.” It seems that in the relatively new domain of quadratic functions, the students in this study used language that they felt comfortable with, which was not necessarily mathematically precise or correct, such as “square-rooting,” Delilah’s phrasing that “-1 would equal 1, and then 1 would also equal 1,” and Chelsea’s reference to the vertex as the “as-tope,” by which she meant asymptote. The language these students used revealed understanding of how quadratics behaved and why, but the students did not yet have the more mathematically precise language that would enable them to explain symmetry or the zero-product property with a more generalized viable argument. It seems that the process of learning to be precise in explanations must be revisited as students learn new mathematics with new vocabulary.

Explaining reasoning. The explanations for symmetry and the zero-product property that students gave were generally grounded in one concrete example and did not involve generalization or the use of formal mathematical reasoning. The Common Core State Standards address students’ ability to construct arguments. The third Standard for Mathematical Practice states that “Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments” (CCSSM, 2010). Becoming proficient in constructing arguments requires that students have experiences that enable them to see mathematics as more than rules and procedures to solve exercises. This points to the importance of sociomathematical norms that promote student discussion and argumentation and help students learn that fully

understanding mathematics is more than knowing facts, rules, and procedures. A complete understanding of mathematics includes being able to engage in the process of mathematical thinking (Schoenfeld, 1994; Romberg, 1992). Mathematics is both an object to be understood and a process or means of understanding, and ideally students would be curious about why parabolas are symmetric and why we can set each factor of a quadratic equation equal to zero and solve. Learning to be persistent in asking why and pressing themselves to be able to explain should be goals for students and teachers can design tasks and opportunities to support students in developing this persistence.

Framework of Student Learning in Quadratics

In this study, I sought to learn what students understand about quadratics and about the connections students make between equations and graphs. My intention was to begin to develop a research-based framework of student learning in the area of quadratics. As I reflected on how these students understood quadratics in the light of Vygotsky's theory, I saw that there is not a clear, crisp progression of learning quadratics as a whole. I am coming to realize that developing conceptual understanding of quadratic functions entails developing conceptual understanding of several threads of ideas. These threads include at least the following: knowing the algebraic objects of study; understanding and becoming fluent in the actions one can take on those objects; understanding and being able to explain the reasoning behind those actions; using knowledge of the algebraic objects to graph quadratic functions; and knowing and being able to leverage the relationships connecting all of the above. I believe these threads of understanding develop in parallel, and the connections that students make between the threads support them in developing their understanding of the bigger concept of quadratic

functions. It appears that a student may have different levels of conceptual understanding of the various threads at any given time. The threads seem to have a common foundation on which all students seemed to base their understanding.

Foundational understanding. I propose that students have shared foundational understandings upon which they develop their conceptual understanding of quadratic functions and equations. These understandings include core ideas of the symmetry of the squaring function and using undoing to solve equations in the form $x^2 = a$. These core ideas about quadratics are situated within students’ understandings of graphing and solving linear functions and equations, as shown in Table 5.

Table 5: Foundational Understandings

FOUNDATIONAL UNDERSTANDINGS			
<p>Linear approaches to graphing: $y = mx + b$</p> <ul style="list-style-type: none"> • b is the y-intercept • m is the slope • Start at b and use slope $\left(\frac{\text{rise}}{\text{run}}\right)$ to graph 	<p>Symmetry of parent function $y = x^2$</p> <ul style="list-style-type: none"> • The graph of the parent function is symmetric about the y-axis 	<p>Use undoing to solve $x^2 = a$ ($a > 0$)</p> <ul style="list-style-type: none"> • To solve an equation such as $x^2 = 100$, “square-root” (take the square root of both sides)** <p>**This can be is a cognitive obstacle if students only consider the positive square root.</p>	<p>Linear techniques for solving:</p> <ul style="list-style-type: none"> • “Do the same thing” to both sides to isolate the variable • Use “undoing” techniques <p>**Potential Obstacle: Belief that equations have only one solution.</p>

The students in this study had a strong understanding of the shape and symmetry of parabolas and the parent function in particular. They learned procedures and have ideas from their study of linear equations that they apply to graphing and solving quadratics. Students also had a strong sense that quadratics are equations with squares and believed that to find a solution of $x^2 = 100$ they should take the square root of both sides of the

equation. The solution strategy of “square-rooting” both sides of an equation acts as a cognitive obstacle in students’ understanding of solving quadratic equations because they view it as a form of “un-doing” rather than adopting the new solution strategy that if a number squared equals a (where $a \geq 0$), then the number will either be the positive square root of a or the negative square root of a .⁴ It appears that these ideas about the shape and symmetry of quadratic functions, graphing and solving linear equations, and the idea of squaring and finding the square root to solve provide fundamental building blocks for students as their thinking and reasoning about quadratic functions, equations, and graphs develops. Furthermore, as this understanding develops, students gain both the precision of language and ability to understand explain their reasoning. As they become more precise, they develop deeper understanding and can better explain their thinking. Furthermore, as their understanding deepens, they can more fully appreciate the importance and nuance involved in being precise in their mathematical language. Students’ growth in their precision of language and ability to understand and explain their reasoning weave the threads in the framework together and support students in developing relational understanding of the big ideas of quadratics.

Development of these understandings. Table 6, depicts a potential framework for student learning of quadratics. If one begins with the foundational understandings described above, I argue that with careful scaffolded support, those grow, and as they grow, connections are made between them, new ideas are introduced and integrated,

⁴ In the event that $a < 0$, students would find that the solution would be plus or minus the square root of a or $\pm i\sqrt{a}$

students become more precise in their mathematical language and more able to seek explanations for and explain the mathematics. I nested the two ideas specific to quadratics in the second and third columns within the associated ideas from linear functions because students' understanding of quadratics is being constructed upon and within their understanding of linear functions. To the right of the four columns, there is a fifth column that describes the connections that are made and the new ideas that are developed and woven in to students' understanding. I described these as threads: knowing the algebraic objects of study; understanding and becoming fluent in the actions one can take on those objects; understanding and being able to explain the reasoning behind those actions; using knowledge of the algebraic objects to graph quadratic functions; and knowing and being able to leverage the relationships connecting all of the above.

The columns may give a false impression that these ideas develop in silos, but they are in fact interconnected by the connections students are making. I offer a brief description of how each of the four foundational understandings might grow below. These descriptions are informed by students' thinking in the interviews as well as by the conceptual framework of big ideas and essential understandings in quadratic functions and equations.

Symmetry. As students develop their ideas of symmetry, they come to understand that the symmetry of a quadratic function is guaranteed. This means that in addition to using symmetry to graph functions and find solutions to equations, students do not abandon symmetry if it appears to conflict with other information they have about the graph. As they have more experiences with quadratics, students come to realize that the

line of symmetry is halfway between the x -intercepts, and they can use that information to use the x -intercepts to find the line of symmetry, or the line of symmetry to help them find the x -intercepts. They realize that the line of symmetry passes through the vertex, and that the equation of the line of symmetry can be expressed as the equation $x = h$, where h is the x -coordinate of the vertex in vertex form. They further realize that the line of symmetry can be expressed with the equation $x = \frac{-b}{2a}$ where a and b are coefficients of the equation in standard form. The natural extension of this understanding is that the solutions as found by the quadratic formula, $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ are on either side of where the line of symmetry intersects the x axis, and the distance to the x -intercepts is the quantity $\frac{\sqrt{b^2 - 4ac}}{2a}$. Importantly, these are not rules and facts for students to memorize, but relationships to understand.

Linear approaches to graphing. As students move from graphing linear equations to quadratics, they can build on their prior understanding. In each type of function in standard form, the y -intercept is the constant term. But while the y -intercept can be an affordance when students consider quadratic equations in standard form, it can function as an obstacle for students when considering the vertex form. If students generalize and think that the number by itself (the c in standard form and the k in vertex form) is the y -intercept, this can lead to incorrect assumptions.

As students incorporate their new knowledge of quadratic equations, they come to understand that the value of the leading coefficient a is not the slope, though it has some similar impacts on the graph of parabolas that the m has on graphs of lines. It determines

the dilation, which students think of as “steepness,” and it determines if the parabola opens up or down, much like the m determines if a line slopes up or down. As students learn about quadratics, they incorporate these ideas into their growing ideas about symmetry and make links to solving.

Expanding undoing. Students have a strong understanding that to solve an equation such as $x^2 = a$, they should “square-root” both sides, resulting in one solution, $x = a$. This understanding, which probably emerges naturally from earlier experiences with the area of rectangles, offers a cognitive obstacle that students need to overcome. In fact, when they are solving this equation, students need to evolve to understanding that the solutions will be $x = +\sqrt{a}$ and $x = -\sqrt{a}$ because they are now working with functions that are not one-to-one. It probably is a further cognitive obstacle for students when this is taught as $x = \pm\sqrt{a}$ without fully developing what the \pm means.

Learning to solve equations such as $x^2 = a$ leads to solving equations by factoring, which is another strategy that does not rely on undoing. The equation $x^2 = a$ asks the question, “If x times x equals a , what might a be?” Students move on to learning that in equations such as $a \cdot b = 0$ where a and b are real numbers, either a or b must be zero, which leads to being able to solve equations such as $(x - 2)(x + 1) = 0$.

Table 6: Framework of Student Learning in Quadratics

FOUNDATIONAL UNDERSTANDINGS				CONCEPTIONS & NEW IDEAS
<p>Linear approaches to graphing:</p> <p>$y = mx + b$</p> <ul style="list-style-type: none"> b is the y-intercept m is the slope Start at b and use slope $\left(\frac{\text{rise}}{\text{run}}\right)$ to graph 	<p>Symmetry of parent function</p> <p>$y = x^2$</p> <ul style="list-style-type: none"> The graph of the parent function is symmetric about the y-axis 	<p>Use undoing to solve</p> <p>$x^2 = a$ ($a > 0$)</p> <ul style="list-style-type: none"> To solve an equation such as $x^2 = 100$, "square-root" (take the square root of both sides)** **This can be a cognitive obstacle if students only consider the positive square root. 	<p>Linear techniques for solving:</p> <ul style="list-style-type: none"> "Do the same thing" to both sides to isolate the variable Use "undoing" techniques <p>**Potential Obstacle: Belief that equations have only one solution.</p>	<p>Making Connections and Leveraging them</p> <p>Examples:</p> <ul style="list-style-type: none"> Considering the graph of a function supports students in solving the related equation. It can be helpful &/or productive to move between algebraic objects and between the forms of quadratic equations <p>New ideas that get integrated into student's existing understandings</p> <ul style="list-style-type: none"> There are three algebraic "objects" <ul style="list-style-type: none"> Expressions: $ax^2 + bx + c$ Equations: $ax^2 + bx + c = 0$ Function definitions: <ul style="list-style-type: none"> $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$ Precision of language <ul style="list-style-type: none"> Move from everyday language to more mathematically precise terms, symbols, and ideas. Development of the equal sign <ul style="list-style-type: none"> An equation can have two (or more) solutions (see expanding undoing) The equal sign can be used to define functions
DEVELOPMENT OF THESE UNDERSTANDINGS				
<p>Weave in & Build Upon</p> <p>Similarities /Differences between linear and quadratic equations in standard form:</p> <p>$y = mx + b$</p> <p>$y = ax^2 + bx + c$</p> <ul style="list-style-type: none"> b and c both give the y-intercept In linear functions, you can think of the y-intercept as a starting point, but in quadratics, the y-intercept is not necessarily the vertex. 	<p>Build Upon</p> <p>Symmetry is ALWAYS guaranteed.</p> <p>Symmetry can be used to graph.</p> <p>The line of symmetry is halfway between the x-intercepts – which can be used to find one, given the other.</p>	<p>Expand</p> <p>Expand solving by undoing to include functions that do not have a unique inverse</p> <ul style="list-style-type: none"> You can't "undo" squaring the same way you can undo other things. <p>$x^2 = a \Rightarrow$ $x = +\sqrt{a}$ or $x = -\sqrt{a}$</p>	<p>Weave in & Build Upon</p> <ul style="list-style-type: none"> You can use linear strategies from linear equations to add and subtract quantities from both sides of equations. You can use linear strategies to multiply or divide both sides of equations by constants (numbers.) 	

<ul style="list-style-type: none"> m and a give information about steepness and direction a indicates dilation, not slope <p>These similarities show up in other forms:</p> <ul style="list-style-type: none"> Factored form: $y = a(x - r_1)(x - r_2)$ 	<ul style="list-style-type: none"> Vertex form: $y = a(x - h)^2 + k$ <p>The a in factored form and vertex form is the same as the a in standard form. It indicates the dilation of the parabola.</p>	<p>The line of symmetry passes through the vertex.</p> <p>The line of symmetry is given by $x = \frac{-b}{2a}$ or $x = h$.</p>	<p>These combine with the quadratic formula in meaningful ways.</p> <p>In vertex form, the k being added on the end is NOT generally the y-intercept. (Special case when the vertex is on the y-axis.)</p>	<ul style="list-style-type: none"> Solving quadratics involves an expansion of the equal sign as there may be one, two or no answers. <p>Evolves with and into understanding solving by factoring.</p> <p>Initial experiences might include solving problems such as $xy = 0$ to give students a grounding in the zero-product property.</p> <p>Equations such as $ax^2 + bx + c = 0$ can be factored into $a(x - r_1)(x - r_2) = 0$. By the zero-product property, $x - r_1 = 0$ or $x - r_2 = 0$ so x equals r_1 or r_2. (r_1 and r_2 can be found by the quadratic formula, enabling students to factor all equations in standard form, not just those with integer coefficients.)</p>	<ul style="list-style-type: none"> When you multiply or divide both sides by variables, you have to be careful to not lose information or change the meaning. 	<ul style="list-style-type: none"> Understanding and Explaining Reasoning <ul style="list-style-type: none"> Seek out explanations and arguments for why quadratics behave as they do Work with explanations and arguments that use specific examples, (for example points or pairs of points) to more generalized formal arguments
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Linear techniques for solving. Students spend several years prior to learning about quadratics becoming proficient in solving linear equations by keeping equations in balance and doing the same thing to both sides. The technique of solving by balancing equations is used as they work to isolate the variable, which they sometimes refer to as solving by undoing. In addition to the description of the reframing of undoing described in the previous section, students continue to use and improve their linear techniques, including correctly using order of operations to move between forms of equations.

Connections. I found that when students considered the connection between the graph of the squaring function and the solutions to $x^2 = 100$, they were able to use that connection and their understanding of symmetry to solve the equation correctly. Students also used the connections between equations and graphs to find solutions to equations several times throughout the interviews. The algebraic and graphic representations of quadratic functions enhance each other, and support students in deepening their understanding of each.

Benchmarks, not stages. Carpenter et al., describe benchmarks for how students develop their conceptions of the equal sign, “Children do not necessarily pass through a sequence of distinct stages in developing their conceptions of the equal sign, and it should not be presumed that all children follow the same path to understanding how the equal sign is used” (2003, p. 19). Similarly, I argue that this framework offers potential benchmarks in the learning of quadratic functions, which may guide mathematics educators in supporting student learning.

Contributions

This framework of student learning in quadratic functions provides the field with an initial step towards understanding what students understand about quadratic functions and how that understanding might develop.

This study offers a contribution to the field of studying students' mathematical thinking in the area of methodology. I developed a framework for the big ideas and essential understandings of quadratic functions. I also offer an interview protocol, in which questions that the research literature highlights as problematic for students are organized using this framework of big ideas in quadratics functions.

Additionally, I believe that the methodology of using a smart pen as a tool in addition to video to capture student thinking as it unfolds is a contribution to research in student thinking in mathematics. Being able to replay student work and correlate what students and the interviewer say as they write was critical to my data analysis. It enabled me to more deeply and accurately analyze how students were constructing their understanding of quadratic equations and graphs and the connections they were making.

Limitations

This study was conducted with a small sample of 27 student volunteers at one high school in the northwestern United States in one window of time. As such, the study offers a view of how that group of students, all taught by a small group of teachers using the same curricular materials, understand quadratic functions and equations. However, this view into how these particular students understand quadratic functions and equations has allowed me to begin to investigate what students may know about quadratics, grasp

the connections they make between equations and graphs, and understand the affordances and obstacles they encounter as a result of their previous mathematics experiences.

Directions for Future Research

This study did not include any analysis of classroom instruction or make any connections between instruction and student understanding. It would be interesting to explore the connection between classroom instruction and student understanding to learn whether students who experience instruction that emphasizes explanations and justification would have more robust explanations for features of quadratic functions. The next step in better understanding the development of student understanding of quadratic functions could be to conduct a similar study with a larger, more diverse sample, including students from a variety of school settings who have experienced different curricular approaches.

An additional area for future research would be to explore how quadratics are and are not used to model real world phenomena with students and how students respond to having increased experience with authentic real world applications of quadratics. This was not a topic of my study, and would be an interesting further exploration.

Course taking and acceleration. One of my minor findings is that Precalculus students performed no better than students in Algebra 2 classes in this study. Why this was the case was beyond the scope of this study, but it seems worthy of further research. If an additional study were to take place including Algebra 2 and Precalculus students, it would be informative to include questions to dig more deeply into the obstacles as well as the affordances that students encounter in the two different courses.

I also found that while students who were two years accelerated were the strongest at solving equations, there were also students who were two years accelerated who struggled with the questions and ideas. This causes me to wonder how districts and schools make decisions about accelerating students in mathematics and to what extent the impacts of acceleration are being studied. This is not a simple thing to study, as many factors influence how students develop mathematically in their middle and high school years. However, we may be able to glean some information from further study, which can help inform students, parents, and teachers as they make decisions about accelerating students' mathematics course taking.

Implications for Instruction

The implications for instruction from this study address the goal of helping students develop conceptual understanding of quadratic functions by addressing the threads described above: knowing the algebraic objects, understanding and becoming fluent in the actions one can take on those objects, understanding and being able to explain the reasoning behind those actions, using knowledge of the algebraic objects to graph quadratic functions, and knowing and being able to leverage the relationships connecting these threads. Implications for instruction include being thoughtful in supporting students in making the shift from linear to quadratic thinking, in both equation solving and graphing; building on the foundational ideas students have about the symmetry of the squaring function, including instruction that supports students in being able to construct viable arguments; and addressing lingering confusions which students struggle from previous mathematics experiences.

Shift from linear to quadratic thinking. As students extend their mathematical knowledge to include quadratic functions, educators can support them by creating instructional experiences that help students see the similarities while also understanding the shift that is occurring. These could include using area models, not including taking the square root as a form of solving by undoing, and helping students make connections and see differences between linear and quadratic graphing.

Area models. Students in this study chose to use area models as a scaffold when they multiplied and factored quadratic expressions. I suggest that teachers can build on this comfort and develop the model more extensively. Instruction could use the area model earlier to explore length and perimeter to form a model for linear expressions and then, when developing the area model, draw students' attention to the difference between linear and quadratic as instantiated in length and area. An example is shown in Figure 33.

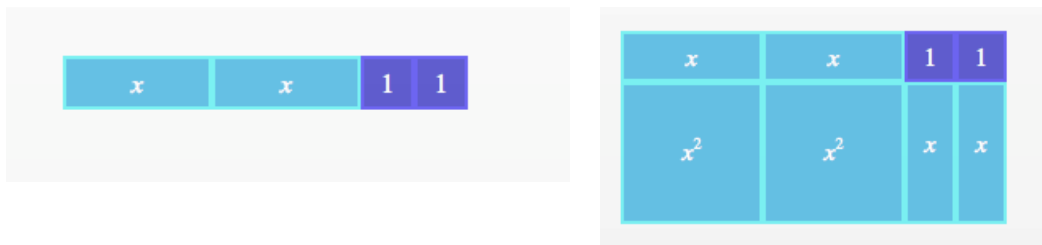


Figure 33: Length and Area Model

The rectangle on the left has a length of $2x + 2$ and a height of 1. The rectangle on the right has a length of $2x + 2$ and a height of $x + 1$. One can see that the area of the rectangle on the right, which is the product of the length and height, can be written as $(2x + 2)(x + 1)$ or $2x^2 + 4x + 2$.

Area models and algebra tiles are used to varying degrees in some algebra classes. Sometimes area models are presented as rectangle diagrams which can serve as graphic organizers (Murdock, Kamischke, & Kamischke, 2014, p. 527). In other mathematics curriculum, area models involve using blocks or tiles that students manipulate as well as various mats designed to help students develop an understanding of how expressions and equations behave (CPM, 2013). Additionally, there are web-based versions of algebra tiles that students can use and manipulate flexibly. The diagrams above were generated using an online tool from College Prep Mathematics and can be found at <http://www.cpm.org/technology/general/tiles/>.

Working with these manipulatives may support students in understanding how quadratic expressions behave differently from linear expressions. For example, when solving the equation $x^2 - 2x = 0$ students who have used an area model could build the rectangle to model the equation with one x^2 tile and two “negative” x tiles (modeled by red tiles that can be placed on top of the x^2). When students figure out how to build the rectangle, they see that it has height of x and length of $x - 2$ as shown in Figure 34, which is a way to use the fact that the area of a rectangle is equal to the product of its side to physically confirm that $x^2 - 2x = x(x - 2)$. To think about the solution to the equation $x^2 - 2x = 0$, students can use the rectangle to consider, for what value(s) of x would the area of this rectangle be zero? Since the area is the product of x and $x - 2$, they can reason that if $x = 0$, the area would be 0, and if $x = 2$ the area would be 0. Using area models will support students in understanding that when they are solving using factoring, they are changing a sum into the product of two binomials which are linear.

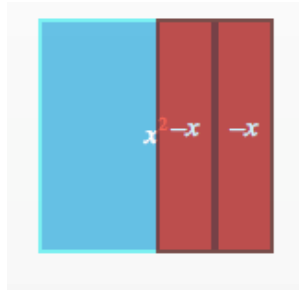
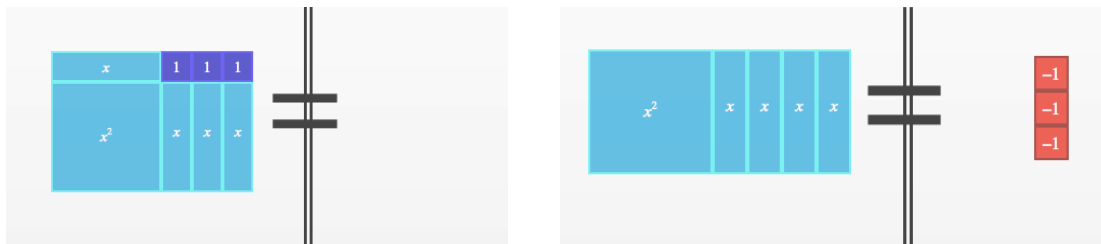


Figure 34: Area Model for $x^2 - 2x$

It should be noted that the area model is not a perfect model. In the picture shown in Figure 34, the two red x -tiles should cover the blue x^2 tile. When working with physical tiles, this may be a problematic limitation for students in that they cannot manipulate the tiles so that the situation is modeled correctly. An additional limitation is that it may be awkward for students to consider what values of x will make the area equal to zero.

Experience with these manipulatives may help students solidify their understanding of order of operations and how to multiply or divide an expression by a constant. For example, when students solve an equation such as $x^2 + 4x + 3 = 0$ using linear strategies, they can model their work using the tiles. If they first build the rectangle, the only rectangle they could build would be the rectangle shown on the left in Figure 35 (note: it could be rotated 90 degrees). The rectangle is shown on the left hand side of an equation mat, and since there are no tiles on the right hand side of the mat, the expression is equal to zero. It is likely that a student would think about area and factoring as a result of working with the rectangle and find that the expression can be factored into $(x + 1)(x + 3)$. However, if a student tried to use linear techniques, they might subtract 3 from both sides and get the equation $x^2 + 4x = 3$ as shown on the right side of Figure 35. That leaves the student with 1 x^2 tile and 4 x tiles. If they thought about dividing both

sides of the resulting equation by 4 at this point, they would see that while they can divide the $4x$ by 4, getting x , if they were to divide the x^2 by 4, they would get $\frac{1}{4}x^2$. If instead, they considered taking the square root, they would be able to see that the rectangle on the right is not a square. It would also make it possible for them to move the parts of the rectangle around to see that it is not possible to create a perfect square from the tiles, and specifically $x^2 + 4x$ is not the same as the quantity $x + 2$ squared.



$$x^2 + 4x + 3 = 0$$

$$x^2 + 4x = -3$$

Figure 35: Area Model Supports Order of Operations

This model would build on students' previous experiences of modeling division with manipulatives, and using it often will be useful to students as they learn to solve by completing the square.

Making the transition from undoing to understanding there may be two solutions. Students have a firm understanding from their linear experiences that to solve an equation you do the same thing to both sides to isolate x . They generalize this understanding to solving equations such as $x^2 = 100$ and think that they should take the square root of both sides without considering that they might anticipate two solutions. It may be productive to provide students with experiences that help them understand that they are solving the equation $x \square x = 100$ and, in fact, solving the puzzle, "I'm thinking of two numbers that multiply to give 100, and the two numbers are equal. What might the

two numbers be?” Teachers may want to be careful not to model taking the square root of both sides as an action that can be used to solve equations; instead, they could spend time reasoning with their students to help them understand that if $x^2 = 100$, then x could be positive 10, or x could be -10.

Graphing. Students in this study had clear ideas that the y -intercept is the constant term in the equation, and that the value of the leading coefficient somehow behaves like slope in quadratic functions. However, these ideas were instrumental in nature. Students used them; however, they were not always able to explain why these ideas worked, and sometimes students overused them. For example, students sometimes assumed that the constant term always indicates where the y -intercept is (in standard form and in vertex form). Having watched students graph equations without calculators, I had the sense that they needed more experiences playing with graphs. It seemed as if the students had not had many experiences exploring functions with paper and pencil. They did not generally use a table of values or have a sense of which points they could explore to create a graph.

A student might predict that the equation of $y = \frac{1}{2}(x - 3)(x + 5)$ would open up and be dilated, but when they were not sure what the graph looked like, they did not generally try putting values in for x and discover that the y values were multiplied by $\frac{1}{2}$. When students have graphed by hand, they experience creating a table of values, inputting x – values and then calculating the associated y – values. If they have experiences doing this that are carefully structured, they are likely to notice the impact of a on the y -value.

Students may need more time experimenting with graphing by hand, which moves more slowly than a calculator but enables them to see how the different parameters impact the graph. They may also benefit from explorations with graphing calculators that are

designed to target the behavior of the leading coefficients and constant terms of equations.

Building on foundational ideas of symmetry. Students' understanding that quadratic functions are symmetric is powerful. They rarely abandon this strong understanding. This seems to be something that can be used to help students better understand finding the solutions to quadratic equations, especially those that they find challenging, such as $x^2 = c$ and $x^2 - bx = 0$.

Constructing viable arguments. Supporting students in developing conceptual understanding of quadratics includes addressing their ability to explain how and why quadratics behave the way they do. This requires that we, as teachers, attend to precision in our language and assist students in attending to precision as well. We need to discuss with students what the objects are that we are studying and be clear with them why it matters that we attend to this. It will further support students to have sociomathematical norms in classrooms that value sense-making and explanation of thinking so that students learn that knowing mathematics is more than being able to apply procedures. This involves students having complex conversations in which they negotiate meaning and interpretation using mathematical tools and representations.

Anticipating and addressing lingering confusions. As students embark on their study of quadratic functions, concepts and ideas from previous math classes may present roadblocks. Some of the roadblocks students in my study encountered were about the behavior of zero, not remembering what -1 squared was, and how to use order of operations to solve equations. Student difficulties with these may reflect a lack of conceptual understanding of these concepts, or it may be a product of the stress of the

moment – being asked to perform math with a stranger. However, they are roadblocks that can be anticipated to some degree and that reveal themselves when students are asked to explain their thinking. It may be productive to anticipate these roadblocks in advance of students learning quadratics and provide students with experiences that will support them in remembering what they do know, or that will enable them to clarify confusions so that the roadblocks do not function as cognitive obstacles to their learning.

Alignment between language used in secondary and higher education mathematics. In the course of conducting this study, I encountered several areas in which the language I used as a long-time secondary math teacher and the language in secondary mathematics texts and resources was not in alignment with the language used by practicing mathematicians. I offer three examples.

In secondary math classes and texts and in this study, equations such as $y = 2x^2 + 4$ are commonly referred to as equations of functions. The function is in fact given by the expression $2x^2 + 4$, and what I have previously called an equation is actually a definition. A quadratic equation is an equation such as $0 = 2x^2 + 4$. The distinction between these three objects, as mentioned above, is not always made clear by texts and teachers. The Common Core State Standards also contain unclear language about functions when they state, “A function can be described in various ways... by an algebraic expression like $f(x) = a + bx$; or by a recursive rule” (CCSS, 2010, Function Introduction section, para. 4).

Another example is that when secondary math teachers and students say an expression is not factorable, we would be more correct to say that it is not factorable with integer coefficients. If an expression is not factorable with integers, the expression can be

set equal to 0 and the resulting equation can be solved using the quadratic formula. The resulting solutions, r_1 and r_2 can then be used to generate the factored form of the expression, $(x - r_1)(x - r_2)$. The last example arose in my discussion of the solutions of the equations $x^2 = 100$ and $x^2 = -1$. They each have two solutions, with similar notation, ± 10 and $\pm i$. However, the first is read and understood as positive and negative 10, while the second is plus or minus i , there being no positive or negative i .

A last example is that students refer to parabolas as “U” shaped, which might imply that the ends of the function are finite and they will become vertical lines at some point. Teachers use the letter U to describe parabolas to separate it from the V- shape of the absolute value function. While this may be a useful memory tool, it likely results in confusion about the behavior of parabolas.

These issues of precision of language or alignment with the practice of mathematics may result from attempting to limit the vocabulary and concepts being introduced to students at any given time. However, it may be wise to help students understand that as they are learning mathematics they are continually expanding the domain of numbers they are working with as well as expanding the number and types of families of functions and objects they are studying. Educators need to have opportunities to acquire and maintain content knowledge beyond the horizon of where they will travel with their students. This enables teachers to help students understand that their mathematical “playground” is constantly growing and expanding in many ways, and will help prevent teachers from inadvertently creating obstacles which students will need to overcome later.

Conclusion

This study focused on student thinking in the area of quadratic functions and equations. It was in many ways a rare privilege to have students take the time to solve problems and share their thinking with me. Creating the opportunity to talk with students and listening closely to their reasoning is an important foundation for improving teaching and learning, as it provides us with a better understanding of the conceptual work students do as they transition from linear to quadratic functions. Having this understanding enables us to support students with appropriate scaffolds and experiences. Knowing the obstacles students might encounter helps us know what questions to ask in the moment as well as supports us in creating meaningful instructional tasks that help students overcome those obstacles. Knowing what affordances students might bring with them from their previous math experiences also helps us build on those foundational understandings. This study is a first step in contributing to our knowledge of student thinking in quadratics.

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Appendices

This section includes four appendices and is provided as a source of additional information regarding my rationale for including specific problems and the process I used. Appendix A contains the protocol I used for the interviews including potential follow-up questions, prompts and larger versions of the graphs that I asked students to consider. Appendix B is a table giving the overview of my study design. In this table, I linked the interview questions to the big ideas and essential understandings discussed in Chapter 2 and to anticipated student responses. These anticipated responses included responses that I thought might indicate knowing, understanding and potential cognitive affordances along with the responses I anticipated indicating confusion, uncertainty, cognitive obstacles and use of procedures without understanding. Appendix C details the equations that I included in the interview including an explanation of why each was included, links to the research literature where appropriate and anticipated solutions strategies for each. Appendix C contains the form that I used in my data analysis. It consists of the questions in one column, a column in which I made notes of what I observed and inserted screen captures of student work as well as a column in which I copied the transcript of the interview.

APPENDIX A: INTERVIEW PROTOCOL

Preamble: Thank you so much for talking with me about math today. I am wondering about how students think about solving quadratic equations and would like to ask you some questions today about solving quadratic equations.

*Is it ok that we do some math problems today and talk about them?
May I videotape our conversation?*

Thanks!

Group 1

Introductory Questions

Big Idea: Quadratics are squaring functions

So, you've learned about quadratics in your math class, and I'm wondering if you could tell me a little about them.... What does it mean for a *thing* to be quadratic?

Follow-up questions:

- Can you tell me what their graphs are like? Could you sketch one?
- Are they functions? Tell me more? How do you tell?
- How are they the same or different from other functions? (linear, exponential?)
- What does it do to a function to have a squared term in it?
- What is important about them?
- If solving comes up, why do we solve for them? What does that mean?

I'm going to ask you to solve some problems to help me learn more. For each problem, I'm going to ask you how you think about it and how you got your solution.

Thank the student when they have a solution, and then ask them how they got the solution. Potential follow up questions:

- *How do they know?*
- *How did they think about the problem?*
- *What does it connect to....*
- *If it takes a longer time, what makes it more difficult?*
- *Specific prompts are with the questions.*

STOP: if in a series of problems if it becomes clear that the mathematics is too challenging or if the student wants to stop end the interview.

Group 2

Quadratic Functions

Focusing on the parent function

Build off of what the student said above...

I'm wondering about

- Can you tell me about it?
- Graph? What does the graph look like? What are the important things about this graph?
- Solving? What does solving mean?
- Follow-ups to the questions in Group 2... Does $y = x^2$ relate to solving $x^2 = 100$? How?
- Relate back to answers for Group 2 questions...

Group 3

Solving Quadratic Equations

Big Idea: Quadratic functions can be set equal to constant values and solved for the variable.

I have some quadratic equations here. Would you please solve this equation for x?

As students solve these, press for what they are thinking. What does it mean for an equation to have two solutions? --- Start with a) and move to the others... d) only if the students have learned the quadratic formula.

a) $x^2 = 100$

b) $x^2 + 4x + 3 = 0$

a. What does it mean to have solved this equation?

b. How does the problem (and solution) relate to the graph?

c. How does problem (and solution) relate to a table of values?

c) $x^2 - 2x = 0$

d) $x^2 - x = 12$

- e) $0 = x^2 + x - 1$
- f) $0 = x^2 + 1$
- g) How are these equations the same? How are they different?
- h) Which did you find most challenging and why?

Group 4

Graphs of quadratic functions are parabolas

Big Idea: Each algebraic form of a quadratic function gives access and/or insight into different characteristics of the graph and table of the function

I'm going to give you an equation. Please take a look at the equation and tell me everything you know about it.

Possible Follow up questions:

- What do you know about its graph?
- What would you do if you were going to graph it? What would you know?
- Please go ahead and graph it.
- If we take that equation and set $y = 0$, how does that relate to the graph you drew?

a) $y = 2x^2 + 5x - 12$

b) $y = \frac{1}{2}(x - 3)(x + 5)$

c) $y = -2(x - 3)^2 + 4$

d) Let's look at all three equations together. How are they the same? How are they different?

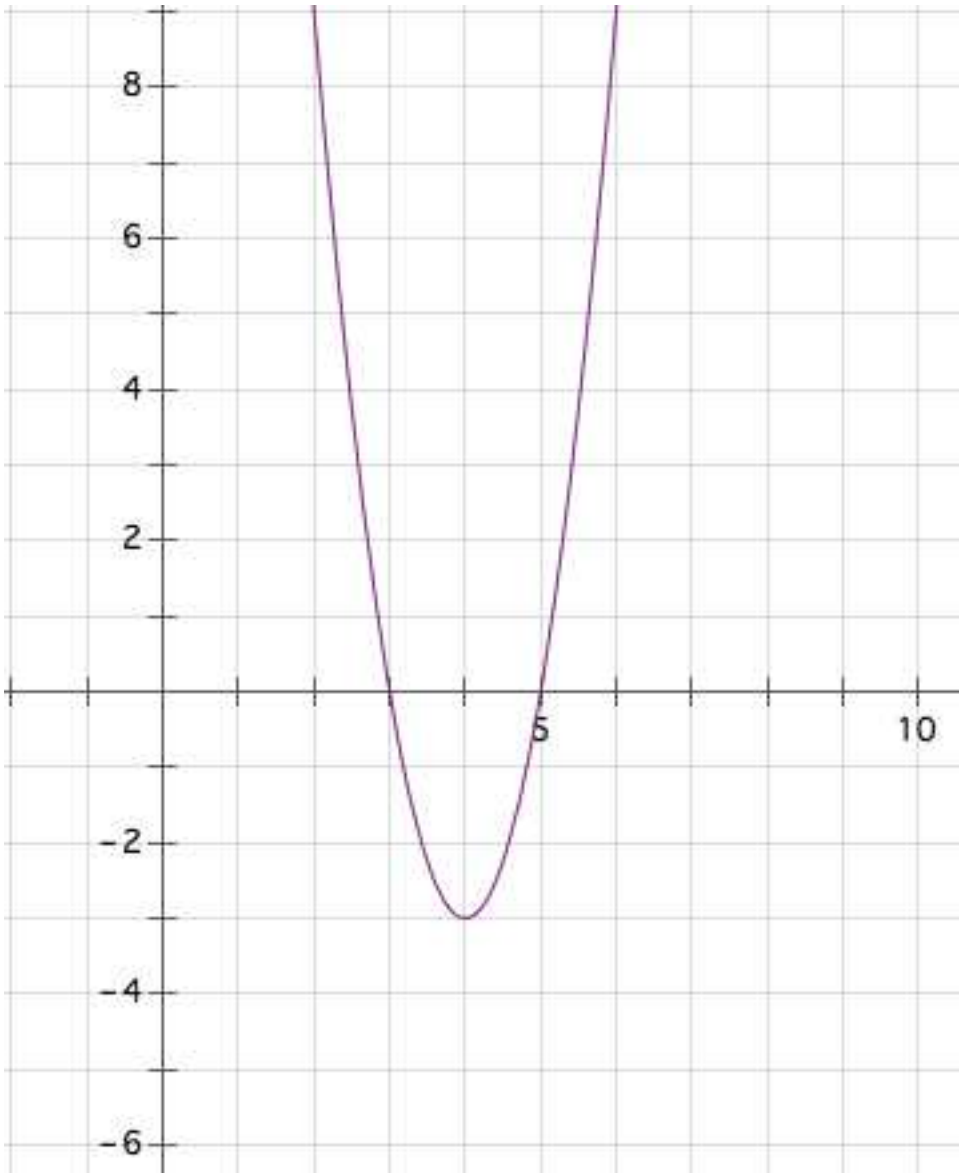
Follow up: Do they tell you the same things?

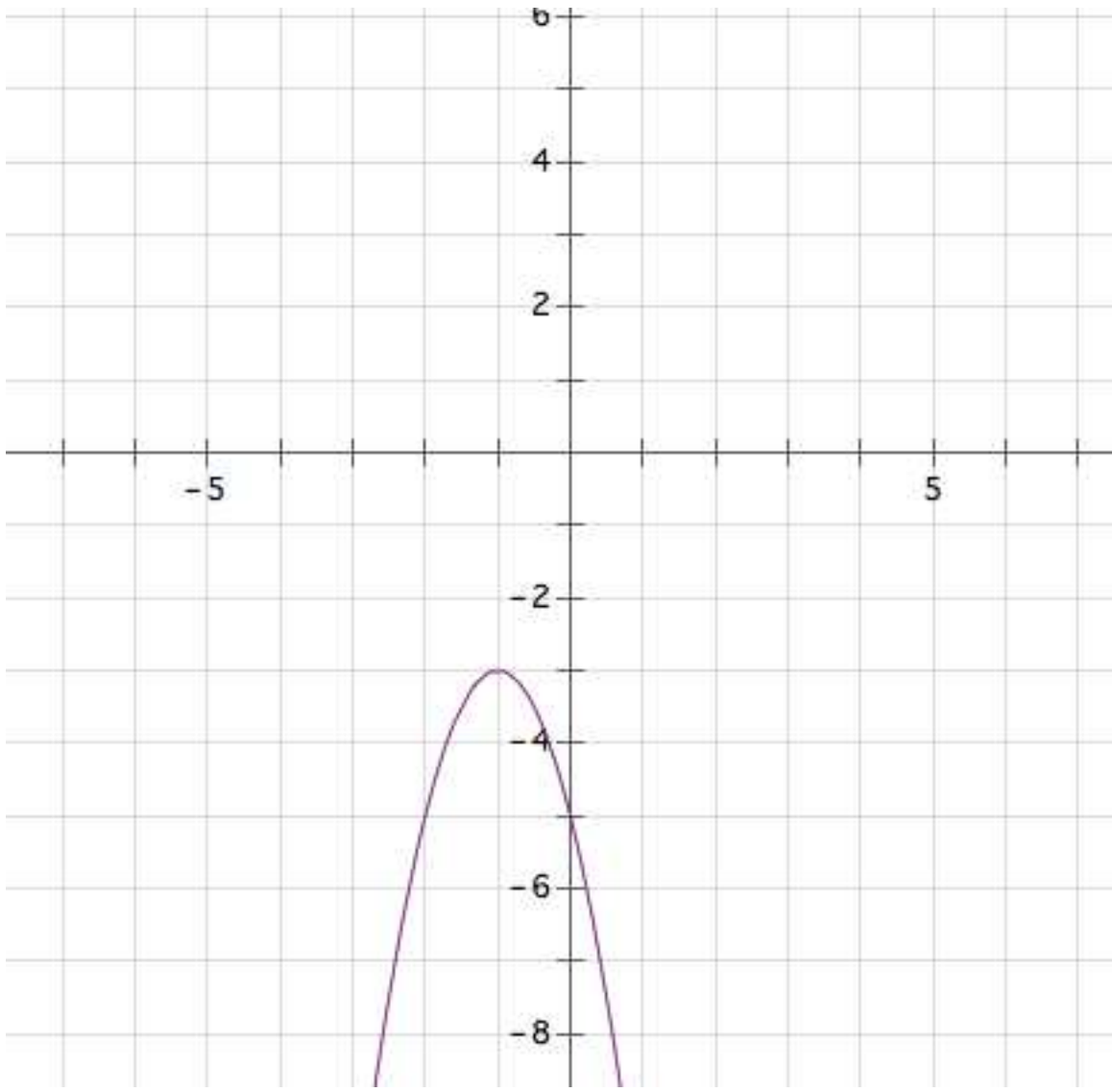
Group 5

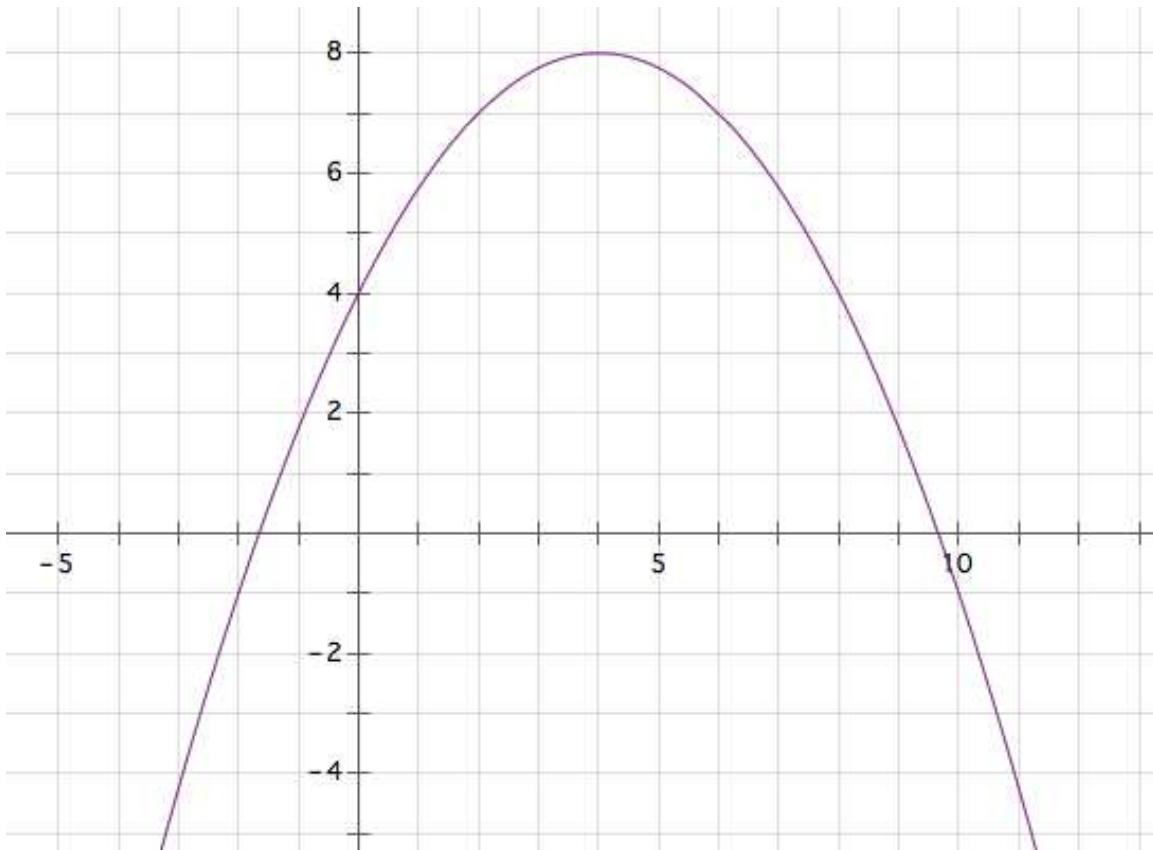
Any graph that is a parabola is the graph of a quadratic function.

Please look at this graph.

- What can you tell me about this graph?
- What is important about this graph?
- If you were to extend it, what would it look like? Please sketch that in.
- Ask about what happens as it approaches the y-intercept, x-intercept, solutions, maximum and minimum, roots
- What would be a possible equation for the figure?







Group 6) Math interest, affect and attitudes

Thank you so much for doing all of those problems with me! I really appreciate it. I'm wondering what you thought about this and think about math in general....

Follow up questions:

- Do you enjoy math? And math class?
- Do you like math?
- What does it take to be good at understanding math?
- Do you feel like you usually understand math?

APPENDIX B: STUDY DESIGN TABLE

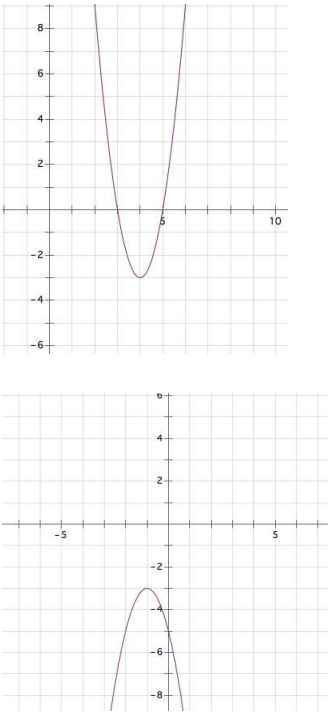
Table B. Interview questions organized by big idea with potential student responses.

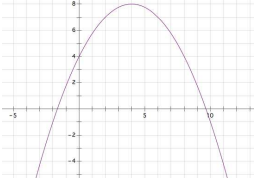
Big Idea &/or Essential Understanding	Question	Follow up questions	Student responses that indicate:	Student responses that indicate:
Quadratics are functions that can be used to model real world phenomena.	So, you've learned about quadratics in your math class, and I'm wondering if you could tell me a little about them.... What does it mean for a <i>thing</i> to be quadratic?	<ul style="list-style-type: none"> • Can you tell me what their graphs are like? Could you sketch one? • Are they functions? Tell me more? How do you tell? • How are they the same or different from other functions? (linear, exponential?) • What does it do to a function to have a squared term in it? • What is important about them? • If solving comes up, why do we solve for them? What does that mean? 	<ul style="list-style-type: none"> • Quadratics are squaring functions • Graphs are parabolas (can sketch) • Can be used for heights of objects thrown, area problems 	<ul style="list-style-type: none"> • Confusion/uncertainty • Cognitive obstacles • Procedures w/o Understanding • Not sure what a quadratic is

Big Idea &/or Essential Understanding	Question	Follow up questions	Student responses that indicate:	Student responses that indicate:
<p>The symmetry of the function is due to the squaring function.</p> <p>For most values of y, there are two corresponding values of x.</p> <p>The vertex occurs at the value of y that has only one corresponding value of x.</p>	<p>Build off of what the student said above...</p> <p>I'm wondering about $y = x^2$</p>	<ul style="list-style-type: none"> • Can you tell me about it? • Graph? What does the graph look like? What are the important things about this graph? • Solving? What does solving mean? • Follow-ups to the questions in Group 2... Does $y = x^2$ relate to solving $x^2 = 100$? How? • Relate back to answers to previous questions... 	<ul style="list-style-type: none"> • Sketches graph • Locates vertex • Can explain why the shape is symmetric • Knows it is a function and can explain why. • Knows that it is a function even though it is not one to one (fails horizontal line test) and can explain why. 	<ul style="list-style-type: none"> • Does not know the shape of the graph. • Does not know the vertex is at $(0, 0)$ • Knows the shape but does not know why it is symmetric • Does not know if it is a function or not – confuses vertical and horizontal line test

Big Idea &/or Essential Understanding	Question	Follow up questions	Student responses that indicate:	Student responses that indicate:
<p>Quadratic functions can be set equal to constant values and solved for the variable, which is often x.</p> <p>When set equal to zero, the solutions to quadratic equations correspond to the zeros (or roots or solutions) of the related function.</p>	<p>I have some quadratic equations here.</p> <p>Would you please solve this equation?</p> <p>Start with a) and move to the others... d) only if the students have learned the quadratic formula.</p> <p>3a) $x^2 = 100$ 3b) $x^2 + 4x + 3 = 0$ 3c) $x^2 - 2x = 0$ 3d) $x^2 - x = 12$ 3e) $y = x^2 + x - 1$ 3f) $y = x^2 + 1$</p> <p>How are these equations the same? How are they different?</p> <p>Which did you find most challenging and why?</p>	<p>As students solve these, press for what they are thinking. What does it mean for an equation to have two solutions? ---</p> <p>For each equation ask about:</p> <p>a. What does it mean to have solved this equation? b. How does the problem (and solution) relate to the graph? c. How does problem (and solution) relate to a table of values?</p> <p>If students graph them or create tables, ask about connections.</p>	<p>that indicate:</p> <ul style="list-style-type: none"> • Knowing • Understanding • Cognitive Affordances <ul style="list-style-type: none"> • Uses methods to solve flexibly • Remembers -10 in a) • Explains why a quadratic has two solutions. • Can explain connections between the equations 	<ul style="list-style-type: none"> • Confusion/uncertainty • Cognitive obstacles • Procedures w/o Understanding <ul style="list-style-type: none"> • Neglecting negative solution • Problems with variable –for example in 3b) $x = 3$ and 5 simultaneously • Factoring errors • Using linear methods to solve (note types of methods) • Uses rules, but not able to explain why • Misapplies procedures • Confused by parameters (b or c equal 0 or 1) • Declaring not possible

Big Idea &/or Essential Understanding	Question	Follow up questions	Student responses that indicate:	Student responses that indicate:
<p>Each algebraic form of a quadratic function gives access and/or insight into different characteristics of the graph and table of the function.</p> <p>There are relationships between the forms of the equations, and each of the forms can be manipulated into the other forms by multiplying monomials, factoring, or completing the square.</p>	<p>I'm going to give you an equation. Please take a look at the equation and tell me everything you know about it.</p> <p>4a) $y = 2x^2 + 5x - 12$ 4b) $y = \frac{1}{2}(x - 3)(x + 5)$ 4c) $y = -2(x - 3)^2 + 4$</p> <p>Let's look at all three equations together. How are they the same? How are they different?</p>	<p>I'm going to give you an equation. Please take a look at the equation and tell me everything you know about it.</p> <p>Possible follow up questions:</p> <ul style="list-style-type: none"> • What do you know about its graph? • What would you do if you were going to graph it? What would you know? • Please go ahead and graph it. • If we take that equation and set $y = 0$, how does that relate to the graph you drew? 	<ul style="list-style-type: none"> • Works from the equation given without needing to go to standard form • Looks for critical points such as intercepts and/or the vertex • Can confirm using points • Uses a table of values to track the behavior of the function when it would be useful 	<ul style="list-style-type: none"> • Confusion/uncertainty • Cognitive obstacles • Procedures w/o Understanding • Not making connections to graph • Preference for standard form • Does not use symmetry to help figure out the graph • Does not use a table of values to figure out the graph around critical points

Big Idea &/or Essential Understanding	Question	Follow up questions	Student responses that indicate:	Student responses that indicate:
<p>Any graph that is a parabola is the graph of a quadratic function.</p> <p>The line of symmetry passes through the vertex.</p> <p>The vertex corresponds to the maximum or minimum of the equation.</p> <p>If there are x-intercepts, they are the real solutions to the related quadratic equation.</p> <p>All quadratic functions have a y-intercept.</p>	<p>Please look at this graph. What can you tell me about this function?</p> 	<ul style="list-style-type: none"> • What is important about this graph? • If you were to extend it, what would it look like? Please sketch that in. • Ask about what happens as it approaches the y-intercept, x-intercept, solutions, maximum and minimum, roots • What would be a possible equation for the figure? 	<p>that indicate:</p> <ul style="list-style-type: none"> • Knowing • Understanding • Cognitive Affordances <ul style="list-style-type: none"> • Identifies critical points such as vertex, x and y-intercepts • Knows there will be a y-intercept. • Knows that if there are no x-intercepts the solutions to the associated equation will be complex (not real) • Makes conclusions about the function based on whether it opens up or downward. 	<p>indicate:</p> <ul style="list-style-type: none"> • Confusion/uncertainty • Cognitive obstacles • Procedures w/o Understanding <ul style="list-style-type: none"> • Does not know which points are critical • Not making connections to equations • Stating that graph a) does not have a y-intercept.

Big Idea &/or Essential Understanding	Question	Follow up questions	Student responses that indicate:	Student responses that indicate:
The critical features of the graph of a quadratic function can be used to generate an equation for the function. Different features lend themselves to different algebraic representations.			<ul style="list-style-type: none"> • Knowing • Understanding • Cognitive Affordances 	<ul style="list-style-type: none"> • Confusion/uncertainty • Cognitive obstacles • Procedures w/o Understanding
Math interest, affect and attitudes	Thank you so much for doing all of those problems with me! I really appreciate it. I'm wondering what you thought about this and think about math in general.	<ul style="list-style-type: none"> • Do you enjoy math? And math class? • Do you like math? • What does it take to be good at understanding math? • Do you feel like you usually understand math? 		

APPENDIX C: ANTICIPATED CORRECT SOLUTIONS METHODS

Question 3a: $x^2 = 100$

I asked students to solve the equation $x^2 = 100$. Thorpe (1989) based his assertion that students do not understand the plus or minus sign on his finding that students often find only the positive solution to problems of the type $x^2 = a$ where a is a positive number.

The correct solution to this equation is $x = \pm 10$, which can be reached by taking the square root of each side of the equation. When taking the square root of each side, students need to remember to consider both the negative and positive square root.

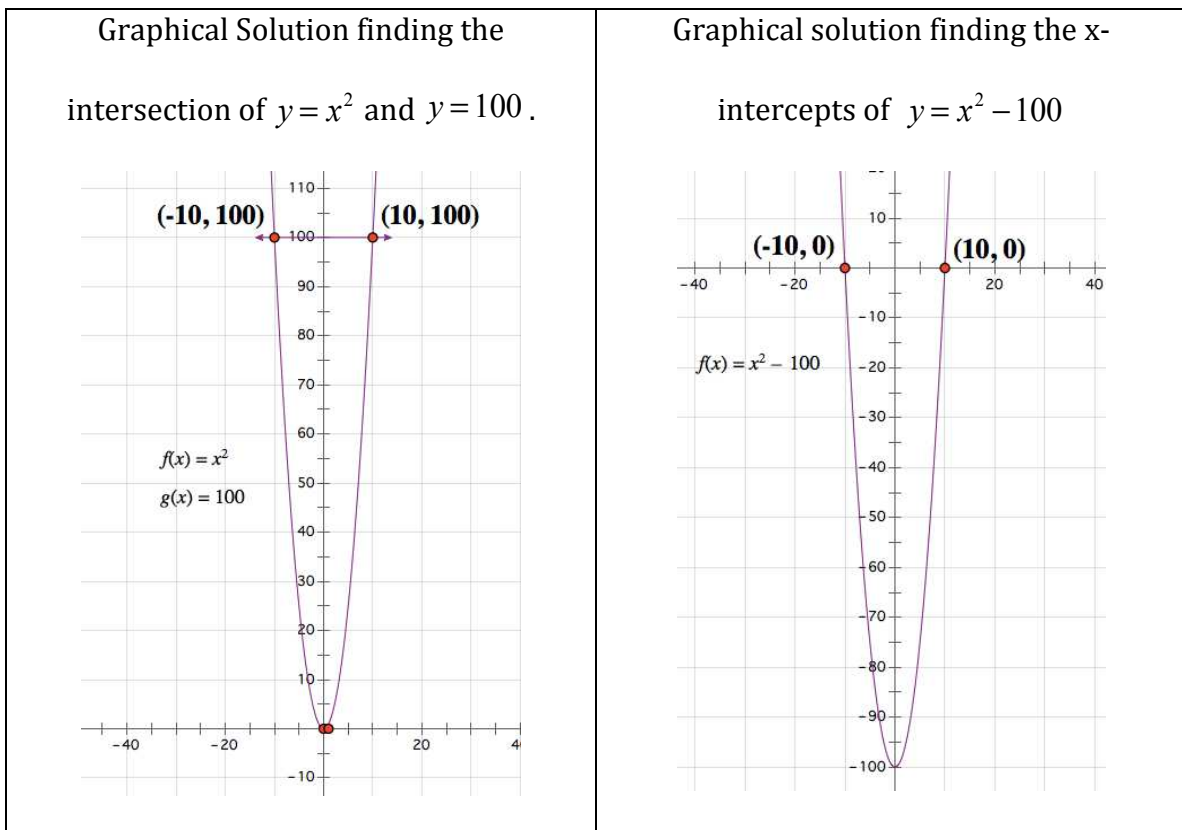
Algebraically, the steps would be:

$$x^2 = 100$$

$$x = \pm\sqrt{100}$$

$$x = \pm 10$$

Students might also solve this graphically in one of two ways (see Figure 1). They might consider the system of equations consisting of $y = x^2$ and $y = 100$, and find the intersection points at $(10, 100)$ and $(-10, 100)$, which gives the solutions $x = 10$ and $x = -10$. Alternatively, students might consider the quadratic function $y = x^2 - 100$,



which is the parent function translated down 100 units. If students consider this function, the solutions to $x^2 = 100$ are found by finding the zeros of the translated function which are at $(10, 0)$ and $(-10, 0)$.

Questions 3b through 4a: Solving Quadratic Equations

The literature indicates that students learn to use factoring to solve equations quickly without paying attention to their structure and conceptual meaning (Sönnerhed, 2009; Didis et al., 2011). Furthermore, students prefer factoring as a solution method when the

equation is easily to factor (Didis et al., 2011). Students use factoring as a solution method without understanding the zero product property (Didis et al., 2011; Bossé & Nandakumar, 2005; Kotsopoulos, 2007) I used the following questions in part to learn how students used and approached factoring and, how they understood factoring and the zero-product property. I also sought to learn how they understood what the solutions they reached meant, and what connections they made between the equations and the related functions.

Question 3b: $x^2 + 4x + 3 = 0$

I chose this equation because it is easily solvable via factoring. This question was asked in two ways. Sixteen students were presented with the equation $x^2 + 4x + 3 = 0$ and asked to solve it, and eight students were asked to tell me all they could about the graph of $y = x^2 + 4x + 3$.

Anticipated Correct Solutions

Solving: A complete solution to solving the equation $x^2 + 4x + 3 = 0$ could be found through factoring, completing the square or using the quadratic formula. Each solution method is shown in Table 2. To get a complete solution when factoring, students must use the product property which guarantees that if two quantities, a and b have a product of zero, then either $a = 0$ or $b = 0$. In this case, a student must recognize that since the factored form is the product $(x+1)(x+3)$, which is equal to 0, then either $x+1=0$ or $x+3=0$.

Table 2: Solution Methods for $x^2+4x+3=0$

Factoring	Completing the Square	Quadratic Formula
$x^2 + 4x + 3 = 0$ $(x+1)(x+3) = 0$ $x+1 = 0$ or $x+3 = 0$ $x = -1$ or $x = -3$	$x^2 + 4x + 3 = 0$ $x^2 + 4x = -3$ $x^2 + 4x + 4 = -3 + 4$ $(x+2)^2 = 1$ $(x+2) = \pm\sqrt{1}$ $x = -2 \pm 1$ $x = -1$ or $x = -3$	Identify $a = 1, b = 4, c = 3$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$ $x = \frac{-4 \pm \sqrt{16 - 12}}{2}$ $x = \frac{-4 \pm \sqrt{4}}{2}$ $x = \frac{-4 \pm 2}{2}$ $x = \frac{-2}{2} \text{ or } x = \frac{-6}{2}$ $x = -1 \text{ or } x = -3$

When completing the square, students generally start by subtracting 3 from both sides of the equation. They then reason about what they have to add to the quadratic expression on the right hand side in order for it to be a perfect square trinomial. In other words, they need to find a value for c by taking the value of b , dividing it by 2 and squaring the result. This results in the square of a factor, $(x - 2)$ in this case being equal to a number.

Students then solve this using an “undoing” process in which they take the square root of both sides and solve.

When using the quadratic formula, students remember the formula, which was written on the white board in one of the classrooms. They have to determine the values of a , b , and c from the equation they are solving, substitute them into the formula and solve. Solving the formula requires a strong grasp of order of operations, multiplying, adding,

and subtracting positive and negative numbers correctly and navigating the plus or minus sign.

Graphing. When asked to describe the graph of the function given by the equation $y = x^2 + 4x + 3$, there are several characteristics of the graph that can be identified in any order. The y -intercept of the function will be 3. This can be seen either because 3 is the value of c in the equation in standard form, or through reasoning that if the value of x is 0, then the value of y will be 3. Because the value of a is 1, the graph will have the same shape as the parent function and it will be opening upward. In order to find the intercepts, one solves the related equation $0 = x^2 + 4x + 3$ as described above, which means that the graph will intersect the x -axis at $x = -1$ and $x = -3$. Once the intercepts have been identified, the vertex can be found through reasoning that it will be halfway between the two intercepts, and therefore have an x value of -2. To find the y value, one can either substitute -2 in for x and solve for y , or one can reason that using the pattern of the squaring function, it is one x unit from the vertex to an intercept, so it will be $1^2 = 1$ unit up, and therefore the y -value of the vertex will be -1.

Question 3c: $x^2 - 2x = 0$

I asked students to solve the equation $x^2 - 2x = 0$. In problems of this type, the value of the parameter c in standard form is 0, and the equation is of the form $ax^2 + bx = 0$. The literature cites this problem as one where students tend to solve by imposing “linear structures.” This means that once the student factors, they divide both sides of the equation by x , causing them to overlook the solution $x = 0$. This specific problem was used in studies done by Didis et al. (2011), Bossé & Nandakumar (2005), and

Kotsopoulos (2007). For this question, I did not ask students to make the link to the graph, but sometimes asked them if the equation was a quadratic.

Anticipated Correct Solutions

Factoring. The equation $x^2 - 2x = 0$ can be solved using factoring, by completing the square or by using the quadratic formula. When factoring, this can be thought about in two different ways. One can use the distribute property and factor an x out of both terms resulting in $x(x - 2) = 0$ or one can add a 0 to the end of the trinomial and factor $x^2 - 2x + 0$ into $(x - 2)(x + 0)$. Both approaches result in the product $x(x - 2)$ equal to zero. Using the zero product property results in $x = 0$ or $x = 2$.

Completing the square. To solve this using completing the square, one would have to add a value to both sides of the equation so that the trinomial on the right hand side is a perfect square trinomial. In this case, adding 1 to both sides would give:

$$x^2 - 2x + 1 = 1$$

$$(x - 1)^2 = 1$$

Which can then be solved by undoing to give $x = 1 \pm 1$, which gives the solution $x = 0$ or $x = 2$.

Quadratic Formula. One can also apply the quadratic formula with $a = 1$, $b = -2$ and $c = 0$. This results in:

$$x = \frac{2 \pm \sqrt{4 - 0}}{2}$$

$$x = \frac{2 \pm 2}{2}$$

$$x = \frac{4}{2} \text{ or } x = \frac{0}{2}$$

$$x = 2 \text{ or } x = 0$$

Question 3d: $x^2 - x = 12$

I chose to include this question because the literature cites it as an example of question type that students struggle with because it is not presented in the form $ax^2 + bx + c = 0$ (Didis et al., 2011). I asked students to solve this equation but did not ask them to make the connection to the graph.

Anticipated Correct Solutions

This equation could be solved by subtracting 12 from both sides of the equation and then factoring, completing the square or using the quadratic formula. Factoring and the using the quadratic formula start by subtracting 12 from both sides and solving $x^2 - x - 12 = 0$.

Factoring	Completing the square	Quadratic formula
$x^2 - x - 12 = 0$ $(x - 4)(x + 3) = 0$ $x - 4 = 0$ or $x + 3 = 0$ $x = 4$ or $x = -3$	$x^2 - x = 12$ $x^2 - x + \frac{1}{4} = 12 + \frac{1}{4}$ $(x - \frac{1}{2})^2 = 12.25$ $x - \frac{1}{2} = \pm\sqrt{12.25}$ $x = \frac{1}{2} \pm 3.5$ $x = 4$ or $x = -3$	Identify $a = 1, b = -1, c = -12$ and substitute into the quadratic formula: $x = \frac{1 \pm \sqrt{1 - 4(-12)}}{2}$ $x = \frac{1 \pm \sqrt{49}}{2}$ $x = \frac{1 \pm 7}{2}$ $x = \frac{8}{2}$ or $x = \frac{-6}{2}$ $x = 4$ or $x = -3$

Alternatively, one could consider factoring to get $x(x - 1) = 12$ and reason, what numbers might make that true. For example if x equals 4, then $4(3) = 12$, and if x equals -3, then $-3(-4) = 12$.

Question 3e: $0 = x^2 + x - 1$

The literature reports that students are able to solve using factoring when equations are easy to factor, but little is known about how students approach non-factorable equations. I asked students to solve this equation to learn how students approach a non-factorable equation and how they understand the solutions.

Anticipated Correct Solutions

This question has two levels of correct answers. It is correct for a student to recognize that the equation can not be solved by factoring, and some students stop there saying that the solutions will not be integers. Students can use logic to try factoring, and then determine that they would have to try a different method to solve.

The equation can be solved by using the quadratic formula or completing the square (see Figure), and an approximate solution can be reached through graphing (see below).

Quadratic formula	Completing the square
Identify $a = 1$, $b = 1$ and $c = -1$ $x = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2}$ $x = \frac{-1 \pm \sqrt{5}}{2}$ $x \approx \frac{-1 \pm 2.24}{2}$ $x \approx 0.62 \text{ or } x \approx -1.62$	$1 = x^2 + x$ $1 + \frac{1}{4} = x^2 + x + \frac{1}{4}$ $\frac{5}{4} = (x + 1)^2$ $\pm \sqrt{\frac{5}{4}} = x + 1$ $-1 \pm \sqrt{\frac{5}{4}} = x$ $x \approx 0.62 \text{ or } x \approx -1.62$

Solving by graphing.

To solve by graphing, we graph the associated function $y = x^2 + x - 1$ and find where it crosses the x -axis. We know the y -intercept is -1 because $c = -1$. We also know this function is opening upwards and has the same shape as the parent function because $a = 1$. Then, we can create a table of values. I started with $x=0$, but since the function seems to be going up. Though we don't have the x intercepts, we can see that there is symmetry in this table, which can also be seen when the points are plotted (see figure XXX) which leads us to think the line of symmetry is at $x = -\frac{1}{2}$. We can then sketch the parabola and estimate the solutions. One is slightly less than -1.5 , and the other is slightly greater than 0.5 .

Table of values	Initial sketch	Sketch with estimated solutions												
<table border="1"> <thead> <tr> <th>x</th> <th>y</th> </tr> </thead> <tbody> <tr> <td>-2</td> <td>1</td> </tr> <tr> <td>-1</td> <td>-1</td> </tr> <tr> <td>0</td> <td>-1</td> </tr> <tr> <td>1</td> <td>1</td> </tr> <tr> <td>2</td> <td>5</td> </tr> </tbody> </table>	x	y	-2	1	-1	-1	0	-1	1	1	2	5		
x	y													
-2	1													
-1	-1													
0	-1													
1	1													
2	5													

Question 3f: $0 = x^2 + 1$

The literature did not specifically address problems in which the solutions were non-real. I chose to ask students to solve this equation because it because: 1) it was an additional opportunity to ask students to solve an equation involving finding the positive and negative solution of an equation in the form $x^2 = a$; 2) it can be solved using undoing;

and 3) it has imaginary solutions, and I was curious how students thought about imaginary numbers.

Anticipated Correct Solutions

The most straightforward method to solve this equation is to use “undoing” which students learn in Algebra 1. It is an extension of linear techniques and is used in completing the square. By adding -1 to both sides, the equation becomes $x^2 = -1$. Then one can take the square root of both sides, to get $x = \pm\sqrt{-1}$ which means $x = \pm i$.

Students could also solve this using the quadratic formula with $a = 1$, $b = 0$ and $c = 1$:

$$x = \frac{-0 \pm \sqrt{0^2 - 4}}{2}$$

$$x = \frac{\pm\sqrt{-4}}{2}$$

$$x = \frac{\pm 2i}{2}$$

$$x = \pm i$$

Question 4a $y = 2x^2 + 5x - 12$

The literature suggests that students have difficulty factoring when $a \neq 1$ (Bossé & Nadakumar, 2005). I asked students about this function because I wanted to see how they handled an equation in which the leading coefficient was not 1. Initially, I intended this to be the only question in which I asked students to graph a parabola presented in standard form. When it became clear that this equation was very challenging for them, I began to ask students to graph the parabola in question 3b as an alternative.

The question I asked all but one student for this problem was, “looking at this equation, what can you tell me about it’s graph.” I did not suggest solving the equation $2x^2 + 5x - 12 = 0$ unless the student was stuck. I asked one student how she would solve the equation $2x^2 + 5x - 12 = 0$.

Anticipated Correct Responses

When asked to describe the graph of this function or graph it, there are several characteristics of the graph that can be described in any order.

y-intercept. The y – intercept is at $(0, -12)$. Students might know this because the value of c in the standard equation is -12 , or because if they substitute 0 in for x , the resulting value of y is -12 .

Dilation. Because the value of a is 2 , we know that the graph opens upward, and that it is dilated by a factor of 2 . This means that if one considers the parent function, $y = x^2$, the graph of the function $y = 2x^2 + 5x - 12$ will have y values that are 2 times the y values of the parent function. Students might know this, or find it by plotting points. They might also use a table of values to find points on the function.

x – intercepts. The x -intercepts of the function can be found by solving the equation $2x^2 + 5x - 12 = 0$. This can be accomplished through factoring, completing the square, or by solving the using the quadratic formula. Students learn to factor equations where $a \neq 1$ through guess and check, by making lists of possible factor pairs, or by using an area model (sometimes referred to as the “box” method.” In this case, the right side of the equation factors giving $(2x - 3)(x + 4) = 0$, and the solutions are $x = \frac{3}{2}$ and $x = -4$.

Line of symmetry and vertex. Once the x -intercepts have been found, the line of symmetry can be identified as being half way between them at $x = -1.25$. The vertex can then be found by substituting -1.25 into the function for x and finding the y -value of -15.125 so the vertex is at $(-1.25, -15.125)$.

Alternatively, one can use an understanding of the quadratic formula to find the line of symmetry. As they learn about the quadratic formula, students learn that the line of symmetry is given by $x = \frac{-b}{2a}$.

The graph of the function $y = 2x^2 + 5x - 12$ is shown in figure XXX.

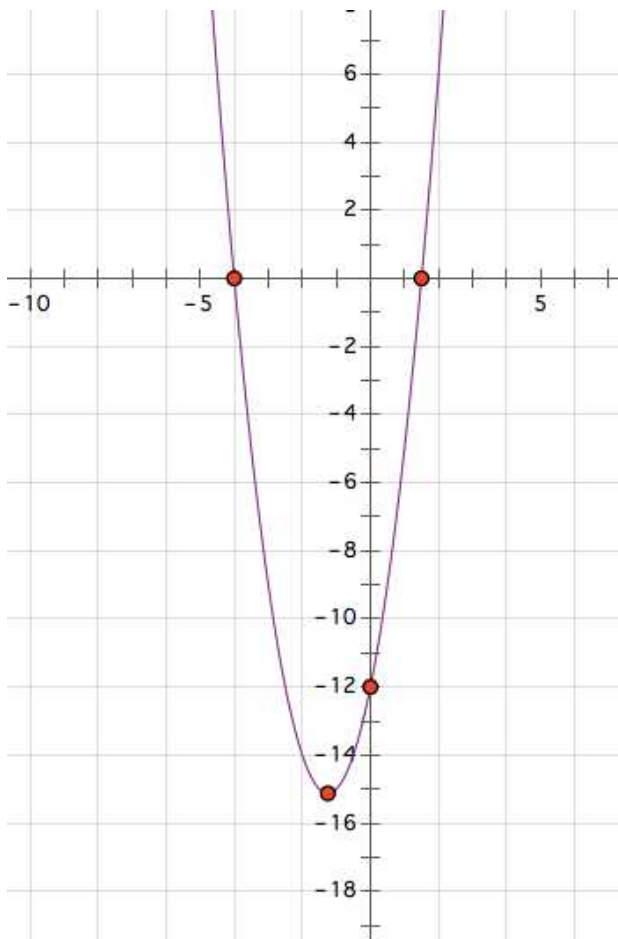


Figure 8: Graph of $y=2x^2+5x-12$ with x -intercepts, y -intercept and vertex

APPENDIX D: STUDENT RESPONSE FORM

Pseudonym:

Teacher: Pseudonym

Date of Interview

	Question (in <i>Italics</i>) Notes and observations about what the student says and does. Screen shots of student work	Transcript
Group 1 Big Idea: Quadratics are functions	<i>So, you've learned about quadratics in your math class, and I'm wondering if you could tell me a little about them.... What does it mean for a thing to be quadratic?</i>	

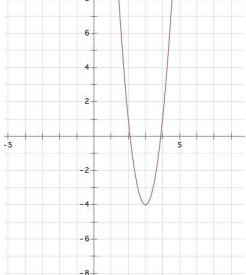
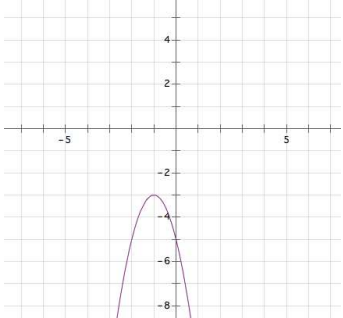
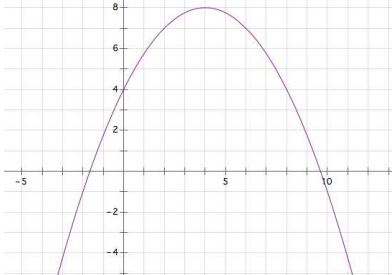
Group 2 Focusing on the parent function (this may happen as part of the first set of questions).	<i>I'm wondering about $y = x^2$ Can tell me about it?</i>	
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Group 3 Solving Quadratic Equations Big Idea: Quadratic functions can be set equal to constant values and solved for the variable.	<i>I have some quadratic equations here. Would you please solve this equation for x?</i>	
a) $x^2 = 100$		
b) $x^2 + 4x + 3 = 0$		

c) $x^2 - 2x = 0$		
d) $x^2 - x = 12$		
e) $0 = x^2 + x - 1$		
f) $0 = x^2 + 1$		
General conversation about solving		

<p>Group 4 Graphs of quadratic functions are parabolas Big Idea: Each algebraic form of a quadratic function gives access and/or insight into different characteristics of the graph and table of the function</p>	<p><i>I'm going to give you an equation. Please take a look at the equation and tell me everything you know about it.</i></p>	
a) $y = 2x^2 + 5x - 12$		
b) $y = -2(x - 3)^2 + 4$		
c) $y = \frac{1}{2}(x - 3)(x + 5)$		

<p>Group 5 Any graph that is a parabola is the graph of a quadratic function.</p>	<p><i>Please look at this graph.</i></p> <ul style="list-style-type: none"> • <i>What can you tell me about this graph?</i> • <i>What is important about this graph?</i> 	
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Student Questions of me	
Attitudes towards Math	

APPENDIX E: CODES

Codes for Conversations about Big Ideas: Quadratics and Related Ideas, Algebraic Representations, and Graph Characteristics

Quadratics and Related Ideas

- What are Quadratics?
- Importance of Quadratics
- Conversations about whether quadratics are functions.
- Conversations about the Zero Product Property
- Conversations about Imaginary numbers and the square root of -1

Algebraic Representation - conversations about specific algebraic representations

- Standard Form
- Factored / Intercept Form
- Vertex Form
- Polynomials

Graph Characteristics – conversations about specific characteristics of graphs

- Symmetry
- Vertex
- y-intercept
- x-intercepts
- Asymptotes
- Line of Symmetry
- Dilations
- Translations
- Domain and Range

Solving Techniques and Graphing Approaches

Solving Techniques

- Factoring
 - Connects to area model
- Completing the square

- Quadratic Formula
- Tries to use linear techniques
- Mentions graphing calculator
- Solves by undoing (appropriate linear techniques)
- Uses a table of values
- Tries but not able to solve
- Says not possible
- Supported by the graph
- Invalid method (linear and/or interesting approaches)

Graphing Approaches

- Knowledge of squaring function: uses knowledge of squaring function.
Example: “It goes over one, up one, over 2, up 4”
- Knowledge of first differences: uses understanding of change between points (first differences) to graph. Example: “It goes over 1, up 1; over 1, up 3; over 1, up 5”
- Vertex
 - Uses vertex as a starting point
 - Places vertex at y -intercept
 - Knows vertex is halfway between the x -intercepts but not sure of y value
 - Knows vertex is halfway between the x -intercepts
- Intercepts
 - Plots x and y -intercepts, tries values to find remaining points
 - Plots x and y -intercepts, tries values to find remaining points (may use table)
 - Plots x and y -intercepts, but no intuition of how to find other points
 - Plots x -intercepts first
 - Plots x -intercepts only
 - Plots x -intercepts, but no idea of how to find y -intercept or other points
- Y -intercept
 - Knows quadratics have one
 - Not sure if a quadratic will have one.
 - Knows the y -intercept occurs where $x=0$
 - Identifies as the c in the standard formula
 - Knows the x -intercepts are solutions to the equation
- Translations
 - Correctly identifies the translations from the translation form
 - Incorrectly identifies the horizontal translation
 - Incorrectly identifies the vertical translation
 - Correctly identifies the vertical translation
 - Switches vertical and horizontal translations
 - Correctly identifies the horizontal translation

- Uses incorrect info in the equation to get the translation
- Dilations
 - Knows whether it opens up or down
 - Articulates $a > 0$ means opens up
 - Articulates $a < 0$ means opens down
 - Thinks that if $a < 0$, the parabola will be reflected over the y-axis
 - Thinks that if $a < 0$, the parabola will always be reflected over the x-axis
 - Confusion re horizontal and vertical stretches
 - Knows that if absolute value of $a > 1$, the parabola will be vertically stretched (get taller)
 - Knows that if absolute value of $a < 1$, the parabola will be horizontally stretched (get wider)
 - Knows that $a = 1$ means the graph will have the same shape as parent function
- Uses symmetry to create graph
- Makes a table
- Moves the parent function (uses translations)
- Multiplies out factored or vertex form
- Doesn't know how to graph
- Assumes the y-axis is the line of symmetry

Codes for Connections

Connections

- Connects factoring to area model
- Connects Equation to Graph
 - Spontaneously
 - If asked
 - Cannot connect between function and graph
 - Graph not discussed
- Connects Graph to Equation
- Connects Graph to Table
- Connects Equation to Table
- Makes connections between ideas
- Connects (or not) result of factoring to solution
- Connects being unfactorable in the real numbers to having no real roots

Codes for Affordances and Obstacles

Affordances and Obstacles

- Affordances – mathematical ideas that students use that appear to support their

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- understanding
- Obstacles – mathematical ideas that are useful in some problems/contexts, but hindrances in others
 - Slope
 - Errors & Struggles
 - From Precalculus and more advanced math
 - Missing constants
 - Remembers equation or formulas incorrectly
 - Zero
 - Squaring -1
 - Fractions
 - Order of operations
 - Only finds positive root
 - Plots points incorrectly
 - Decimals and “weird” numbers
 - Other interesting errors

Codes for Justifications

Justifications

- Explains thinking – why something works, or how things are connected
 - Appeal to authority – says the teacher or book said so
 - Through example
 - Generic example
 - Deductive argument
- Says they can't really explain