An intuitive explanation for Proposition 3.1

In our 2002 paper “Separating equilibria in continuous signalling games”, we present a general method for finding the separating equilibria of a continuous signalling game as integral curves of a vector field determined by the game’s payoff structure. At the heart of this method is the result presented in Proposition 3.1 of that paper:

**Proposition 1** Define the vector field \( V \) as

\[
V(s', q') = \left( \frac{d}{ds} C(q', s) \right|_{s = s'}, \frac{d}{dq} H(q', R^*_q(q)) \right|_{q = q'}
\]

If a separating equilibrium exists for the game \( \Gamma \) with \( S(q_0) = s_0 \), the integral curve of \( V \) through \((q_0, s_0)\) will be an equilibrium signalling strategy \( S(q) \), provided that everywhere along this integral curve...[a second-order condition given in the paper]...is satisfied. The equilibrium receiver strategy is given by \( R(s) = R^*_q(S^{-1}(q)) \) where \( S^{-1}(q) \) is the inverse of \( S \).

(As a terminological refresher, \( q \) is quality, \( s \) is signal intensity, \( r \) is response. \( C(q, s) \) is the signal cost function, \( H(q, r) \) is the benefit function, \( s = S(q) \) is a signalling strategy and \( r = R(s) \) is a response strategy.)

In the paper we provide a rather complicated proof of this proposition. A shorter intuitive explanation would be useful. Here we provide such an explanation.

For the signalling strategy \( s = S(q) \) to be an equilibrium, the marginal cost to the signaller of changing a signal must be equal to the marginal benefit everywhere along this equilibrium strategy path.

\[
\frac{\partial C}{\partial s} = \frac{\partial H}{\partial r} \frac{dR}{ds}
\]

We aim to find a vector field such that an equilibrium signalling strategy \( S(q) \) is an integral curve of that vector field. That is, we want to know how the signal changes as the signaller’s quality changes: \( dS/dq \). Let’s multiply both sides of (1) by this:

\[
\frac{\partial C}{\partial s} \frac{dS}{dq} = \frac{\partial H}{\partial r} \frac{dR}{ds} \frac{dS}{dq}
\]

Note that the right hand side above is the marginal benefit to a change in the signaller’s perceived quality, because we are only looking at how change in quality affects change in benefit through change in signal.

Although the value of \( dR/ds \) by itself is unknown and the value of \( dS/dq \) is what we are looking for, we do know the product of these two partials, because we know that at equilibrium the composition of \( R \) and \( S \) yields the optimal response to each level of quality: \( R(S(q)) = R^*(q) \). Thus we can write:
\[ \frac{\partial C}{\partial s} \frac{dS}{dq} = \frac{\partial H}{\partial r} dq \]  

(3)

Finally, we can solve for \( dS/dq \) as desired:

\[ \frac{dS}{dq} = \frac{\partial H}{\partial r} dq \frac{\partial C}{\partial s} \]  

(4)

Then \( S(q) \) will be an integral curve of the vector field given by \( (\frac{\partial C}{\partial s}, \frac{\partial H}{\partial r} dq) \).

These terms are equal to the total derivatives given in Proposition 3.1.

Matina Donaldson
Carl Bergstrom
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