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Discussing Controversial Public Issues in Secondary Social Studies Classrooms: Learning from Skilled Teachers

by

Diana Hess

A dissertation submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

University of Washington

1998

Approved by

Chairperson of Supervisory Committee

Program Authorized to Offer Degree

College of Education

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Doctoral Dissertation

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Abstract

Discussing Controversial Public Issues in Secondary Social Studies Classrooms: Learning from Skilled Teachers

by Diana Hess

Chairperson of the Supervisory Committee: Professor Walter Parker College of Education

This study was about the discussion of controversial public issues (CPIs) in middle and high school social studies classes. Its purpose was to address the problem that while CPI discussions are valuable for students and for the broader society, few students are actually taught how to participate in them. I sought to better understand the instruction and conceptions undergirding the instruction of secondary social studies teachers who are skillfully teaching their students to participate more effectively in CPI discussions. To accomplish that, I studied three such teachers by interviewing them, examining discussion-related classroom artifacts, and observing videotapes and listening to audiotapes of CPI discussions in their classes. Data were analyzed in a four-step process using grounded theory methodology. During the analysis of the data, seven propositions emerged, along with six ways in which the propositions related to one another. The propositions were: (1) Teachers teach for, not just with, discussion; (2) Teachers work to make the discussions the students' forum; (3) Teachers select a discussion model and a facilitator style that is congruent with their reasons for using discussion and their definition of what constitutes effective discussion; (4) Decisions about whether and how to assess students' participation in CPI discussions are
influenced by an enduring tension between authenticity and accountability; (5) Teachers' personal views on CPI topics do not play a substantial, visible role in classroom discussion itself – however, teachers' views strongly influence the definition and choice of CPIs for discussion; (6) Teachers engage in CPI discussion teaching practices that are informed by their conceptions of democracy; and (7) Teachers are receiving support for their CPI discussion teaching from school administrators, the overall culture of the school, and their schools' missions. Implications of these propositions for teachers, teacher educators, and researchers interested in classroom discussion are also presented.
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Introduction

Finding support for the claim that students should learn to participate effectively in small and large group discussions is easy; finding evidence that students are currently provided opportunities to engage in such discussions is not. In fact, research indicates that students rarely participate in class discussions (Goddard, 1984; McNeil, 1986). As one high school student reported, "Talking is one of the things we are pretty deprived of at school" (Alvermann et al., 1995).

Even when students do participate in classroom discussions, it is rarely the case that discussion focuses on controversial public issues (Hahn, 1991; Shaver, Davis, & Helburn, 1980). A controversial public issue (CPI) is a matter of public concern about which there is (or was in the past) disagreement (Harris, 1996). Discussing such issues has powerful educational outcomes for students (Hahn, 1996; Harris, 1996) and for the maintenance and development of a democratic society (Barber, 1989; Parker, 1996). A problem exists that can be summarized simply: students rarely participate in class discussions of CPI, even though such discussions could be educative for them and helpful for the broader society.

Notwithstanding that few students participate in classroom discussion of CPI, there are teachers who do include such discussions in their curriculum and teach their students to participate effectively in such discussions (Miller & Singleton, 1997; Rossi, 1995). Here, I studied three such teachers in order to generate an initial theory about what constitutes effective teaching of CPI discussion skills. This initial theory helps to address the problem of little emphasis on CPI discussions by providing teachers, teacher
educators, and other researchers with a better understanding of the conceptions and practices of these skilled teachers.

This dissertation has five chapters. Chapter One explains the background of the problem. It focuses on how social studies educators have traditionally made the case for including CPI discussions in the secondary social studies curriculum and reviews what previous studies have shown about the effects of CPI discussions on a number of student and societal outcomes. Chapter One concludes with an explanation of why few secondary social studies teachers include CPI discussions in their curriculum. It also explains how studying teachers who do teach students to participate effectively in CPI discussions can enhance our understanding of what constitutes effective CPI discussion teaching practice.

Chapter Two identifies and explains the three research questions that animate the study: How do secondary social studies teachers who are skilled in the use of CPI discussions teach their students to participate effectively in such discussions? What role do instructional strategies, issues, materials, and assessments play in this teaching process? What accounts for these teachers' approaches to CPI discussions? In particular, I am interested in how the teachers' conceptions of democratic citizenship, the purposes of social studies education, what constitutes good discussion, and their rationales for CPI discussions inform and influence their CPI discussion teaching practice. The explanation of the research questions emphasizes their origins, meaning, and importance. Following this, I present the conceptual framework that guides the study.

In Chapter Three, I explain the grounded theory methodology that was used in the study. This chapter focuses on the criteria used to select the three
teachers, how data was collected and analyzed, and how the initial theory was
developed. Illustrations of the data analysis and theory generation process are
included, along with a discussion of methodological concerns.

The conceptions and practices of the three teachers with respect to CPI
discussions is presented in Chapter Four through portraits organized in
accordance with the research questions and conceptual framework of the
study. Each portrait explains the teacher's CPI classroom practice, the relative
roles of specific parts of his/her practice (e.g., issues selection, assessment),
and his/her conceptions of factors (e.g., purpose of social studies) that may
influence CPI discussion teaching practice.

Chapter Five describes and explains the initial theory about effective
CPI discussion teaching that emerged from the study. Drawing on similarities
and differences throughout the three portraits, the theory includes seven
propositions about the use of CPI discussions in secondary social studies.
Relationships among the propositions are identified and a revised conceptual
framework is presented. In this chapter, I also discuss how and why the initial
theory generated from this study is potentially important for secondary social
studies teachers, teacher educators, and other researchers of classroom
discussion. It concludes with the limitations of the study.
Chapter One

Background and Explanation of the Problem

In this chapter, I describe two rationales social studies educators have traditionally used to support the use of CPI discussions: (1) positive connections between a citizenry that knows how to discuss CPIs and a healthier democracy and (2) the influence participation in such discussions has on important student outcomes. I briefly review existing research on the effect of participation in CPI discussions and the extent to which CPI discussions are actually used in social studies classrooms. The chapter concludes with a discussion of how studying secondary social studies teachers who do teach their students to participate effectively in CPI discussions can help other teachers, teacher educators, and researchers.

Since 1916, when the National Education Association's Commission on the Re-organization of Secondary Education recommended the development of a course examining the problems of democracy, social studies reformers have repeatedly called for inclusion of CPIs in the social studies curriculum. In recent years, the enthusiasm for CPIs in the social studies has not waned. Both Social Education (1996) and The Social Studies (1989), two prominent social studies education journals, devoted full issues to issues-centered social studies curriculum. Additionally, the National Council for the Social Studies sponsors a special interest group on issues-centered social studies and published a lengthy Handbook on Teaching Social Issues (Evans & Saxe, 1996).

CPI Discussions as an Outcome for Democracy's Sake

Why have social studies educators been so interested in the teaching of CPI discussions? The connection between learning how to discuss divisive
public topics and preparing for democratic citizenship is the predominant and most compelling answer. Fred Newmann (1989) argues that the most important component of effective democratic citizenship education is teaching young people how to deliberate about the nature of the public good and how to achieve that. Walter Parker (1996) makes a similar claim:

Curricula need to emphasize civic discourse, particularly face-to-face discussion. Discussion of the public’s problems, the causes of which they are effects, and alternative courses of action need to become centerpieces of the curriculum--taught, modeled, studied, practiced, and assessed. This elevates democratic deliberation to the high point of the school curriculum. (p. 197)

By deliberation, Newmann and Parker mean a particular kind of classroom discourse, one designed to teach students how to determine both public ends and means. For example, secondary social studies students might deliberate about ends when they talk about the tension between individual rights and public safety embedded in questions regarding how much safety citizens are willing to give up to be less encumbered by governmental regulations. Deliberation about means could involve analyzing whether a city council should adopt a curfew for teenagers as a means of diminishing violence by and against youth. Thus, the ends-means connection is whether a curfew would enhance or diminish the likelihood of achieving a desired goal (various levels of public safety or individual rights). Deliberation about the public good, by definition, concerns topics that are controversial, or “at issue.” What makes these topics controversial is the disagreement that exists about how they should be resolved.
From this viewpoint, the rationale for deliberation of CPIs is that
democratic citizenship depends on it because healthy democracies have many
citizens who are engaged in high-quality public talk (Barber, 1984, 1989). As
Ruth Grant reminds us:

Decent politics, and democratic politics particularly, is conducted
through talk, and thus conversation has an impact beyond the
individual development of character. Conversation is a civics
education as well as a moral education because the capacity for
conversation is a crucial public capacity. (1966, p. 477)

The democracy rationale for CPI discussions posits discussion competence as
the primary outcome; that is, young people should be taught to discuss CPIs
because good citizens in a democracy need to be able to participate effectively
in such discussions (see Figure 1).

| CPI Discussions → CPI Discussion Competence → Healthier Democracy |

Figure 1. Relationship Between CPI Discussions and Democracy

CPI Discussions as a Method for Achieving Other Student Outcomes

The democracy rationale for CPI discussions is based on discussion
competence as the primary student outcome of participation in such
discussions. Another rationale for CPI discussions is that other student
outcomes, such as the development of certain values or enhanced
understanding of content, may be achieved through participation in CPI
discussions. I am calling this rationale a method rationale because it is based
on the belief that participation in CPI discussions will be a method (or
avenue) toward desired student outcomes. In other words, the discussions teach students something in addition to how to participate in discussions.

Within the method rationale are several claims about the positive student outcomes that could result from CPI discussions. The claims include the development of democratic values, increased willingness to engage in political life, enhanced content understanding, and improved critical thinking and interpersonal skills (Gall & Gall, 1990; Hahn, 1996; Harris, 1996).

![CPI Discussions Diagram]

Figure 2. Relationship Between CPI Discussions and Student Outcomes

While the first two outcomes (democratic values and participation in political life) may appear to be part of the democracy rationale just discussed, I am conceptualizing them in the method category because they support types of citizenship behavior in addition to participation in public issues discussion. For example, students who develop democratic values may be more prone to behave in support of those values. A person who believes in
equality, for example, may be more likely to treat others in an equal manner. Similarly, a person can engage in political life through activities other than participation in CPI discussions, such as political protest, volunteer service, jury duty, or voting. To summarize, CPI discussions work toward the democracy rationale when the end is more effective participation in such discussions, which some theorists suggest correlates positively to a healthy democracy. CPI discussions are supported as a method when aimed at democratic values or behaviors in addition to participation in CPI discussions.

The first claim in the method rationale is that CPI discussions influence the development of democratic values, such as toleration of dissent and support for equality. This claim presumes that schooling can influence students' values and that the dynamics of effective CPI discussions, in particular, help students form values supportive of democracy (Lockwood & Harris, 1985). Informed by the research of Lawrence Kohlberg (1981), the democratic values claim posits that the cognitive dissonance created by CPI discussions, as well as the likelihood that students will hear and be attracted to moral reasoning more sophisticated than their own, will combine to shape the development of democratic values.

CPI discussions are also recommended as a way to enhance students' willingness to participate in the political world. Derived from the research on the political socialization of youth (Hahn, 1996), this claim suggests a connection between participation in discussion of CPIs and an interest in political participation. Discussing CPIs is seen as a way to help students feel more politically efficacious, an attitude correlated positively to willingness to participate in political affairs.
Participation in CPI discussions is often advocated as a means of helping students better understand important content, just as writing is recommended as a method toward enhanced understanding of content. David Harris (1996) explicates this claim when he wrote, "The effort to produce coherent language in response to a question of public policy puts knowledge in a meaningful context, making it more likely to be understood and remembered" (p. 289). For example, in a CPI discussion about physician-assisted suicide, a teacher may hope that students will form a deeper understanding of social studies content, such as the meaning of liberty in the U.S. Constitution. The idea underlying this hope is that talking with others will shape (in fact, improve) one's understanding because ideas will be challenged, broadened, and refined by the dynamics of the group discussion.

Finally, CPI discussions are advocated because it is believed they improve both students' ability to think critically and their interpersonal skills. As applied to CPIs, a critical thinker is able to think rationally by supporting and justifying arguments and conclusions (Common, 1985). By critically examining positions, and evidence supporting positions, students are able to carefully analyze their reasoning and the reasoning of their classmates. This analysis is seen as an avenue toward improving the ability to think critically. In terms of interpersonal skills, one of the overall goals of CPI discussions is to improve students' ability to work well with others, even those with whom they disagree. For example, listening attentively and disagreeing respectfully are key interpersonal skills that research suggests may be enhanced by participation in such discussions.


Effects of CPI Discussions

There are many rationales that are used to support CPI discussions. What does the research show about the effects on society and on students from participation in such discussions? In 1991, Carole Hahn wrote an extensive review of the research on controversial issues in the social studies. The review characterized the empirical evidence as "meager . . . and coming from non representative samples," yet concluded that it "consistently supports the position that positive citizenship outcomes are associated with giving students opportunities to explore controversial issues in an open, supportive classroom atmosphere" (Hahn, 1991, p. 470). Hahn was clearly unsatisfied, however, by the amount and quality of research that had been done on CPI discussions in the social studies. Her review concludes with this charge to other researchers: "There is much yet to be learned about controversial issues in social studies, and given the long-standing commitment of the profession to teaching controversial issues, this topic warrants a top priority in our collective research agenda" (1991, p. 471).

Although there is not a robust body of research on the effects of CPI discussions, the research that has been done sheds light on many of the claims made by those who advocate such discussions. In particular, two questions are addressed by the research on CPI discussions. First, what influence does participation in CPI discussions have on the building of a healthier democracy? Second, what influence does participation in CPI discussions have on students' knowledge, skills, and attitudes?
Improving Democracy

The democracy rationale for CPI discussions is based on the belief that a citizenry that can and does discuss CPIs leads to a healthier democracy. Empirical evidence suggests there is merit to this claim.

It is difficult to find examples of exemplary public talk in the United States. Barber (1989) suggests this is so because public deliberation has been reduced to “an instrument of symbolic exchange between avaricious competitors who are seen as having only private, animal interests” (p. 355). There are, however, some particularly strong exemplars of public talk. Public deliberations that are part of the National Issues Forum (NIF) network are one such example. The NIF process is designed to bring together people from all walks of life to deliberate about public issues, such as youth violence, immigration, and affirmative action. Research on the effects of NIF deliberation suggests that a positive connection exists between CPI discussions and a healthier democracy. Specifically, public deliberation “establishes and enhances communication between groups” and “improves a community’s . . . ability to deal with its issues, concerns, and problems” (NIF literature, no date).

Recent focus group research on how people form relationships with public concerns (Kettering Foundation, 1993) provides additional evidence for the claim that CPI discussions influence the formation of a healthy democracy. Specifically, the researchers found that citizens want to participate in public talk, and that when they do so, they “enlarge, rather than narrow, the way they see and act on public concerns” (Kettering Foundation, 1993, p. 1). Conversation about public problems is positively linked to what people learn from other citizens and to solving important problems. The researchers
concluded that the importance of CPI discussions to citizens and to a healthy
democracy shows that "talk is not cheap to people, as the axiom goes; it is the
valued currency of their public life" (Kettering Foundation, 1993, p. 2). In a
society where talk is often criticized in comparison to other forms of action,
this research is an important reminder that public talk is a form of democratic
action that appears to strengthen democracy.

In addition to the relationship between CPI discussions and a healthier
democracy, researchers have also investigated what effect CPI discussions
have on the students who participate in them. Recall, advocates of CPI
discussions have theorized a positive relationship between CPI discussions
and five categories of student outcomes: development of core democratic
values, willingness to engage in political life, enhanced content
understanding, improved critical thinking, and improved interpersonal
skills. Empirical evidence suggests there is merit to each of the claims.

Development of Core Democratic Values

Advocates of the inclusion of controversial issues in the social studies
have traditionally theorized a connection between the discussion of such
issues and the development of core democratic values, such as support for
civil liberties and tolerance of dissent. Research has fairly consistently shown
that such a connection exists (Avery, Bird, Johnstone, Sullivan, &
Thalhammer, 1992; Baughman, 1975; Brody, 1994; Goldenson, 1978;
Grossman, 1975/76). For example, a study of a curriculum on free expression
that included CPI discussions showed that experimental group students
significantly increased their scores on a scale of Political Tolerance, and that
these scores persisted four weeks after the curriculum was completed (Avery
et al., 1992). Another study analyzed the influence of a nationally
disseminated curriculum on the Constitution and Bill of Rights that includes classroom CPI discussions. Participation in this curriculum, *We the People* (Center for Civic Education, 1987), correlated positively to students being more likely than a comparison group to support the rights of free speech, freedom of assembly, and due process for diverse groups (Brody, 1994).

**Willingness to Engage in Political Life**

Researchers have studied the influence of discussing CPIs on factors related to involvement in the political world, including political efficacy, political participation, and trust in the political system (Baughman, 1975; Blankenship, 1990; Ehman, 1969, 1970). The single most significant research finding focuses on the differences between CPI discussions in an open versus a closed classroom climate. An open classroom climate has three attributes: students examine multiple “sides” of an issue, students feel free to express their opinions, and teachers allow dissent. CPI discussions in an open climate have consistently been correlated positively to four effects on students:

1. an interest in the political world, 2. a sense that they and citizens like themselves can have some influence on political decisions in a democracy, 3. a belief that citizens have a duty to be actively engaged in politics, and 4. integration into--rather than alienation from--the school culture and the wider society. (Hahn, 1996, p. 32)

Research has also fairly consistently shown a correlation between discussion of controversial issues (even in an open climate) and increased distrust of government (Ehman, 1970; Long & Long, 1975; Zevin, 1983). These studies showed that discussion of controversial issues can enhance cynicism about politicians and government. Depending on one’s political vantage point, this finding could be interpreted as either a positive outcome (i.e., good
citizens need to be cynical, it will prevent them from being "conned" by political leaders) or a negative outcome (i.e., if young people are too cynical they will not support the political structure, and ultimately might be less likely to be involved citizens).

Enhancement of Content Mastery

Several studies of CPI discussions have included a focus on the acquisition of content knowledge (Cousins, 1963; Johnston, Anderman, Milne, Klenk, & Harris, 1994; Levin, Newmann, & Oliver, 1969; Oliver & Shaver, 1974/1966). Results of these studies are mixed; some show that CPI discussions have no effect on students' content mastery (Levin et al., 1969; Oliver & Shaver, 1974/1966), while others show that CPI discussions help students learn and retain important social studies content (Cousins, 1963; Johnston et al., 1994). For example, two studies of the Harvard Social Studies Project, one with junior high students and one with high school students, showed that CPI discussions caused students to learn as much content as students in non-CPI classes (Levin et al., 1969; Oliver & Shaver, 1974/1966). In other words, CPI discussions did not cause students to learn less content, which is a concern to some teachers.

The most recent study of CPI discussions (Johnston et al., 1994) showed that the discussions were correlated positively to the mastery and retention of social studies content. In this study, experimental group students viewed Channel One broadcasts, followed by CPI discussions of issues explained on the program, while control group students viewed the broadcasts but did not participate in CPI discussions. The experimental group students scored significantly higher on a test of current events knowledge. This finding lends
support to the claim that participation in CPI discussions can enhance students' mastery of social studies content.

**Improvement in Critical Thinking**

More than thirty years ago, the study of the Harvard Social Studies Project showed that participation in CPI discussions enhances students' analytic competence (Oliver & Shaver, 1974/1966, pp. 262-274). This finding has recently been replicated in the Channel One study described above (Johnston et al., 1994). Using a written test adapted from the Harvard Social Studies Project study, the Channel One researchers found that students who participated in CPI discussions after viewing Channel One were better able to analyze the functions of various statements in a written dialog. As an indicator of critical thinking about CPIs, this finding is important because it illustrates the connection between the development of higher-level thinking (which is what the written test measured) and participation in CPI discussions.

**Improvement in Interpersonal Skills**

There is little research about the influence of participation in CPI discussions on students' interpersonal skills, such as listening, sharing the floor, and challenging others' statements without attacking. Research on various cooperative learning strategies, however, is helpful to assessing the potential of CPI discussions to improve students' interpersonal skills. Cooperative learning strategies, such as Structured Academic Controversy (Johnson & Johnson, 1979), improve students' interpersonal skills because students are provided instruction and opportunities to practice civil ways of dealing with one another. Given that much of what occurs in high-functioning cooperative learning groups is discussion, it is logical to infer that
high-quality CPI discussions can be a pathway to helping students develop interpersonal skills.

**Summary of Research**

In summary, research on the discussion of controversial issues in secondary social studies classes lends support to the claims of controversial issues advocates that such study can have a positive influence on important societal and student outcomes. However, there has not been much research on CPI discussions, and some of the best quality research (e.g., Levin et al., 1969; Oliver & Shaver, 1974/1966) was done three decades ago.

It is important to note that research on CPI discussions is stronger when compared to what we know about students' learning in other forms of citizenship education. For example, traditional civics classes without a focus on CPI discussions had little or no effect on student outcomes that many civic educators value. Students in such classes, for example, were not apt to be more interested in political affairs or willing to participate in the political life of their communities (Langston & Jennings, 1968; Litt, 1963). In other words, the effects of CPI discussions on a variety of student outcomes look more positive when compared to what empirical evidence suggests results from classes without an emphasis on such discussions.

**Why Few Teachers Use CPI Discussions**

Evidence suggests that few teachers include CPI discussions in their secondary social studies classes (Goodlad, 1984; Hahn, 1991; McNeil, 1986). Yet, survey research indicates that social studies teachers support the inclusion of CPI discussions in their courses (Engle, 1993; Hahn, 1998). Observational evidence, however, suggests that what teachers say they support on surveys does not translate into what actually happens in classrooms. For example,
John Goodlad (1984) observed that little discussion of any sort occurs in high school classes, and Fred Newmann (1988) was unable to find many social studies teachers who included discussion, even though the observations were in schools nominated for thoughtful teaching, a type of teaching that includes the use of classroom discussion.

Little is known about why teachers do or do not include CPI discussions in their courses. However, other bodies of research about why teachers teach what they do suggest some possible explanations: (1) the influence of the apprenticeship of observation (Lortie, 1975), (2) lack of attention to CPI issues in teacher education programs, (3) the difficulty of teaching CPI discussions (Dillon, 1994), (4) the impact of standardized tests, (5) lack of knowledge about how to teach students to participate effectively in CPI discussions, and (6) teachers' concerns about maintaining control of the classroom (McNeil, 1986).

**Studying Effective CPI Discussion Teachers**

While research suggests that many secondary social studies teachers' instruction relies on recitation and lecture (Goodlad, 1984; Hahn, 1991; McNeil, 1986), there are teachers who do teach their students to participate effectively in CPI discussions (Miller & Singleton, 1997; Rossi, 1995). What can be learned from these teachers? Lee Schulman (1983) suggests that studying "good cases" has value because it allows us to learn about the possible, instead of just the probable:

The well-crafted case instantiates the possible, not only documenting that it can be done but also laying out at least one detailed example of how it was organized, developed, and pursued. For the practitioners concerned with process, the operational detail of case studies can be
more helpful than the more confidently generalizable virtue of quantitative analysis in many cases. (p. 495)

Examples of research on effective teachers, such as the study of exemplary secondary history teachers by Samuel Wineburg and Susan Wilson (1988) and the study of exemplary teachers of African-American children by Gloria Ladson-Billings (1994), show that an in-depth look at teachers who are doing what is difficult can contribute to the knowledge base about what constitutes effective teaching. In both of these studies, the researchers selected teachers who were atypical in order to gain a greater understanding of good teaching. Such studies are often referred to as “best practice” or “models of wisdom” studies.

In this study I sought to address the problem that few secondary social studies teachers teach their students to participate effectively in CPI discussions. I did so by studying “good cases” in order to create an initial theory of what constitutes good CPI discussion teaching practice. This, in turn, can be used to improve teachers’ practice by providing examples and a theory based on them. Such a “models of wisdom” study follows the common-sense model of learning concepts based on examples. Teachers have lacked examples of what the concept “good CPI discussion teaching” entails. Lacking exemplars, it is exceptionally difficult to learn how to teach something as sophisticated as participation in CPI discussions. Moreover, the initial theory connects the examples to one another and induces from them more general ideas about the characteristics of good CPI discussion teaching.

Chapter Summary

Two rationales are commonly cited to justify teaching secondary social studies students to participate in CPI discussions: (1) positive connections
between a citizenry that knows how to discuss CPIs and a healthier democracy and (2) the influence participation in such discussions has on important student outcomes. While research on the effects of participating in CPI discussions is limited, existing research lends support to claims that such study can have a positive influence on important societal and student outcomes. Still, few teachers actually use CPI discussions in their classrooms. The reasons for their failure to do so are unknown, but this "best practice" (i.e., "model of wisdom") study should help teachers, teacher educators, and researchers understand what constitutes good CPI discussion teaching practice.
Chapter Two

Research Questions and Conceptual Framework

As documented in the previous chapter, few secondary social studies teachers teach their students how to participate effectively in CPI discussions. Neglect of CPI discussions is a problem because it deprives students, and society writ large, of opportunities to obtain the potential benefits participation in such discussions offers. This study addresses three research questions that may help us better understand how this problem can be addressed. In this chapter, I explain these questions and the conceptual framework on which the study is based.

The research questions are: How do secondary social studies teachers who are skilled in the use of CPI discussions teach their students to participate effectively in such discussions? What role do instructional strategies, issues, materials, and assessments play in this teaching process? What accounts for these teachers’ approaches to CPI discussions? In particular, I am interested in how the teachers’ conceptions of democratic citizenship, the purposes of social studies education, what constitutes good discussion, and their rationales for CPI discussions inform and influence their CPI discussion teaching practice. What follows is an explanation of the origin and meaning of each of the three research questions.

First Research Question: CPI Instructional Practice

The first research question is: How do secondary social studies teachers who are skilled in the use of CPI discussions teach their students to participate effectively in such discussions? This question stems from my interest in better understanding how these teachers do something that is very difficult. James Dillon (1994) reminds us that discussion is unnatural: ‘Discussion is
difficult. Far from coming naturally, it has to be learned” (p. 105). Moreover, discussion of CPIs is arguably more difficult than other types of discussion because the very nature of the subject matter at hand often causes fight (i.e., rancorous debate) or flight (i.e., refusal to participate). Therefore, I assume that especially skillful discussion teachers do something (most likely, many things) to engage their students in CPI discussions. That is, they do not simply provide students with the opportunity to discuss CPIs, but also teach them how to discuss these issues.

The first research question is fundamental: learning what such teachers do to teach CPI discussion skills is necessary in order to understand the role of various components of their discussion-teaching practices and the reasons for the choices they make about curriculum and instruction, which are the topics of the two other research questions.

**Second Research Question: Roles**

The second research question (What role do instructional strategies, issues, materials, and assessments play in this teaching process?) stems from the belief that skilled CPI teachers most likely make decisions about several basic questions intrinsic to teaching discussion. James Dillon (1994) writes that the questions include: how to conduct a discussion, how to talk in discussion, what to discuss, and how to prepare for discussion. Barbara Miller and Laurel Singleton (1997) advocate the exploration of an additional question: how to assess a discussion. In the following subsections, I examine the four components of the second research question: instructional strategies (Dillon’s how to conduct a discussion and how to talk in discussion), issues for discussion, materials (the key aspect of Dillon’s preparing for discussion), and assessment.
**Instructional Strategy**

Two of Dillon's questions (i.e., how to conduct a discussion and how to talk in discussion) imply some type of instructional strategy used by the teacher. I am defining instructional strategy to include both particular models of discussion and the ways teachers help students learn the skills necessary to participate effectively in the models. This purposely broad definition reflects the variation I expected to find in how these teachers instruct students to participate effectively in CPI discussions. For example, the instructional strategy could be selecting and orienting students to a particular model of discussion, such as Structured Academic Controversy (Johnson & Johnson, 1988) or Public Issues Discussions (Singleton & Giese, 1996), or it could be providing examples (e.g., by viewing a videotaped exemplary discussion or doing a lesson on a specific skill, such as listening) designed to help students form a more general concept of what constitutes effective discussion. Skilled CPI discussion teachers may have many instructional strategies in their pedagogical quiver, and they may use different strategies for different purposes, with different students, at different times in the school year.

Within this broad definition of instructional strategy, I am also including decisions the teacher makes about whether, when, and how to disclose her/his personal views about the CPI being discussed. Framing these decisions under the heading, "teacher's role," Tom Kelly (1986) has conceptualized four positions that a teacher can take with respect to disclosing personal views on a CPI: exclusive neutrality, exclusive partiality, neutral impartiality, and committed impartiality (pp. 114-132). Of the four, Kelly prefers committed impartiality. Committed impartiality has two components. First, it involves the teacher's stating, rather than concealing, her/his beliefs
about CPIs. Second, the teacher should encourage truth-seeking by encouraging critical discourse about competing perspectives. This means opinions other than the teacher's perspective will also get a fair hearing.

While I will be attentive to other factors related to instructional strategy (such as what the teacher does to create a safe environment for discussion), I anticipate that the three factors mentioned above will be particularly important.

Issue Selection

Of course, discussions must be about something, a topic at hand. While this study focuses only on discussions of CPIs, there are still many questions about how the teachers select them. For example, to what extent do the teachers involve students in selecting CPIs to be discussed? What specific criteria do the teachers (and/or students) use to select the issues? How do the teachers (and/or students) define what constitutes a public issue?

Many civic educators urge an in-depth approach to a relatively small number of issues instead of focusing on many issues for a short amount of time (Harris, 1996; Miller & Singleton, 1997; Rossi, 1995). Given that time is the currency of teaching, selecting issues carefully is imperative. Sound criteria must be utilized when choosing the small number of issues which most deserve students' attention.

Various criteria have been proposed for selecting issues for CPI discussions. Richard Gross (1964, pp. 3-4) advised teachers to use the following questions to select issues:

1. Is this issue beyond the maturity and experience level of the pupils?
2. Is this issue of interest to the pupils?
3. Is this issue socially significant and timely for this course and grade level?

4. Is this issue one which the teacher feels he can handle successfully from a personal standpoint?

5. Is this issue one for which adequate study materials can be obtained?

6. Is this issue one for which there is adequate time to justify its presentation?

7. Is this issue one which will clash with community customs and attitudes?

Gross's final criterion about community norms has itself been a controversial public issue for many years. Almost ten years before Gross published his list of questions, Maurice Hunt and Lawrence Metcalf (1955/1996, p. 111) urged a clash with community customs and attitudes. Unlike Gross (1964), who warned the teacher against "cutting his own throat" (p. 3) by selecting issues that would upset the community, Hunt and Metcalf recommended an examination of "closed" or taboo issues for students' own psychological well-being and for the good of democratic society at large. Only by taking issues out of the closet, so to speak, could a legitimate and productive discussion of them occur.

Other selection criteria that have been advanced include selecting issues because of their overall importance to the society and body politic or because of the possible personal significance the issues may have to particular students (Singleton & Giese, 1996). In this study I am most interested in investigating three questions related to issues selection: What issues are selected? Who selects the issues? For what reasons are the issues selected?
Materials

Once issues are selected, materials about the issue (such as films, software, written text, etc.) are gathered and studied in preparation for discussion. Without such preparation, the discussion would not be a discussion. Instead, it would be what Tom Roby (1988) calls a “bull session.” Here, I will investigate how the teachers (and/or students) select materials used to prepare for discussion.

In particular, I am interested in both the depth and breadth of materials selected for discussion preparation. By depth, I mean how much students learn about an issue in preparation for the discussion. By breadth, I mean the range of materials used for discussion preparation, especially in terms of variance of perspectives on the issue. For example, if a teacher has selected affirmative action as a CPI for discussion, how many different positions on affirmative action do students study in preparation for the discussion? Do these different perspectives represent a narrow (i.e., liberal and conservative) range of views, or do they include a broader range of the political spectrum?

Finally, I will investigate whether the teacher selects the materials students use to prepare for discussion, whether students select the materials, or whether they are selected jointly by the teacher and students. This point is important because if students are finding materials about the issue they may be learning some ancillary skills to discussion, such as how to research public policy issues.

Assessment

The last component of the second research question deals with assessment. I am most interested in whether skilled CPI teachers make judgments (i.e., assessment) about how students are progressing toward the
goal of participating effectively in CPI discussions. While some experts in CPI discussions urge fairly formal assessment (Harris, 1996; Miller & Singleton, 1997), some students object to assessing discussion because of concern about whether they should be required to participate orally in class. Rahima Wade’s (1994) research on preservice teachers’ beliefs about oral participation in class shows that the “choice issue” (whether students should be given a choice about oral participation) is itself highly controversial. For example, 66% of preservice teachers agreed with this statement: “Participating in class discussions is a matter of personal choice. It is not essential that everyone contributes in this way” (p. 235). The Wade study suggests that teachers who do require (and assess) oral participation in class discussion may be operating at odds with their students’ desires.

My primary focus for investigating the teachers’ assessment of CPI discussions is whether they are formally assessing individual students’ participation or assessing the class as a whole. If they are assessing individual students’ participation in CPI discussions, then I would also like to know the dimensions on which they base their assessments. For example, do the teachers value some characteristics of CPI discussion participation more than others? How do the teachers determine what constitutes more or less skillful participation in CPI discussions? Have the teachers formalized their assessment of CPI discussion participation through codifying desired characteristics on a rubric? Finally, to what extent does participation in CPI discussions influence students’ grades in the course?

**Interactions Among Components of Teaching Practice**

Embedded in the second research question is an emphasis on understanding the roles of the various components of teachers’ discussion
teaching practice; that is, the function and position of the components in teaching students to participate more effectively in CPI discussions. By function and position, I mean possible relationships between components (position), such as a relationship between issues selection and instructional strategy. For example, teachers may vary their instructional strategies depending on the nature of the CPI at hand. Highly contentious issues (such as abortion or race-related controversies) may require a more structured strategy than less controversial issues.

**Third Research Question: Conceptions**

The third research question (What accounts for these teachers' approaches to CPI discussions?) is based on research illustrating that skillful practitioners have reasons for teaching the way they do. Samuel Wineburg and Suzanne Wilson (1988) suggest that teachers who are "wise practitioners" possess "rich and deep understandings of many things, understandings that manifest themselves in the ability to draw from a broad range of possibilities" (p. 58). I am defining these "rich and deep understandings" as conceptions, following Shavelson and Stern's (1981) thinking that conceptions include knowledge, beliefs, thoughts, and images. Together these form a mental picture--a conception--that then may inform classroom practice.

As previously mentioned, I am particularly interested in how the teachers' conceptions of four things (democratic citizenship, purposes of social studies education, what constitutes good discussion, and rationales for CPI discussion) explain and influence how they engage students in CPI discussions. I will address each in turn.
Democratic Citizenship

Preparing citizens for participation in democracy has traditionally been the mission of the public schools generally, and particularly the role of the social studies curriculum (Patrick & Hoge, 1991). This study seeks to understand how these teachers define democratic citizenship because their definitions may inform their CPI discussion teaching practice. Recall that the preparation for democratic citizenship rationale for CPI discussions contends that citizens in a democracy need to participate in decision making about public issues. The preeminence of the democratic citizenship rationale for CPI discussions suggests that how teachers conceptualize democratic citizenship will influence their CPI discussion teaching practice.

This study will investigate two questions about the teachers' conceptions of democratic citizenship. First, how do teachers conceptualize what a good citizen in a democracy should know, be able to do, and be inclined to do? Second, how do these teachers believe difference (diversity) should be dealt with in a democracy?

In an earlier study (Hess, 1997), I found that secondary social studies teachers have differing conceptions of what citizens need to know, be able to do, and be inclined to do in a democracy. One teacher believed that good citizens should be highly involved in developing answers to pressing public policy issues, while another believed that good citizens should vote wisely and participate in volunteer activities designed to alleviate human misery. Their conceptions roughly mirrored one difference found in the theoretical literature on what constitutes effective democratic citizenship: the distinction between "weak" and "strong" democracy.
Historian Paul Gagnon's (1996/1989) theory of democratic citizenship is revealed in the following:

We seek to develop at one and the same time a taste for teamwork and a taste for critical, thorny individualism, at once the readiness to serve and the readiness to resist, for no one can foretell which way the "good" citizen ought to turn in future crises. . . . Civic education asks all this, and that citizens inform themselves on the multiple problems and choices their elected servants confront. (p. 247)

The citizen has few roles in the democracy Gagnon describes. After "serving" (for instance, willingly paying taxes and going to war) and "resisting," the citizen has only one other role--to elect the people who will do the work of deliberating about the nature of the common good and how to achieve it. This vision of democracy is "weak" because citizens are not directly involved in creating public policy. In contrast, strong democracy, as political scientist Ben Barber (1989) defines it:

is not simply a system whereby people elect those who govern them, but a system in which every member of the community participates in self-governance. It entails not merely voting and overseeing representatives but ongoing engagement in the affairs of the civic community at the local and national levels. (p.355)

Teachers with "weak" and "strong" conceptualizations of democratic citizenship may make different choices about how to teach students to participate in CPI discussions. For example, a teacher who shares Barber's view may put greater emphasis on preparing students for public discussion of public issues, whereas a teacher who shares Gagnon's view might be more
likely to teach CPI discussions as a means of analyzing issues to prepare to vote in elections.

A second component of my investigation into teachers' conceptions of democratic citizenship is how they think about difference in a democracy. In particular, to what extent do these teachers think it is possible to have discussion across difference, especially about issues that are quite controversial? By difference, I mean the variety of ways that people define who they are relative to others, including such categories as race, gender, class, religion, and ways of looking at the world.

The theoretical literature illustrates that the possibility of discussion across difference is highly controversial. Some feminists, such as Leach (1992), suggest that power differences in contemporary American society are so firmly entrenched along lines of race, class, and gender that it is impossible to have a genuine public discussion among people who are different. Other theorists (Burbules & Rice, 1991) disagree, claiming that even though discussion across difference is exceptionally difficult, it is still possible as long as specific steps are taken to ensure that people who have historically been marginalized are allowed and encouraged to speak and be heard.

Understanding how these teachers think about difference in democracy may be important to illuminating their CPI discussion teaching practice. For example, a teacher may purposely select certain discussion models that mandate equal sharing of air time. Teachers' thinking about difference may also influence the selection of issues. For example, a teacher may select an issue that is of concern to students in the minority as a way of ensuring that minority concerns are emphasized in the classroom. Conversely, the teachers
who deny the importance of difference in democracy may make CPI
discussion teaching decisions based on that assumption.

Purpose of Social Studies Education

A second conception area I am interested in investigating is how these
teachers define the purposes of social studies education. The driving question
undergirding this area of inquiry is: What do these teachers want their
students to know and be able to do as a result of social studies? Given that CPI
discussions are typically only one part of a teacher's social studies curriculum,
here I seek to understand the larger whole (in terms of student outcomes) in
which CPI discussions are nested.

Numerous frameworks for delineating the purposes of social studies
have been proposed, including Barr, Barth, and Shermis' (1977) typology
which separates social studies into three historical traditions: social studies as
citizenship transmission, as reflective inquiry, and as social science. Research
since the development of this framework has suggested that teachers'
conceptions of the purposes of social studies are much more complicated and
variegated (White, 1982; Goodman & Adler, 1985). That is, the number of
categories into which teachers' conceptions of social studies can be placed are
more numerous than simply the three proposed by Barr, Barth, and Shermis
(1977). For example, one of the categories that the research by Goodman and
Adler (1985) added to the stew was education for social action. Given that
secondary social studies teachers have tremendous power as curricular gate-
keepers (Thornton, 1991), understanding how they conceptualize the
purposes of social studies may be a significant step toward the larger goal of
understanding why they use CPI discussions.
**Good Discussion**

Another area of teachers' conceptions that I will investigate is how the teachers define what constitutes good discussion. Discussion is a relatively generic term, often used to describe a range of classroom talk from recitation to more open-ended deliberation. This generality creates both a problem and an opportunity for researchers who seek to better understand the nature of classroom discussion. The problem is that social studies teachers disagree about what discussion is and about what constitutes good discussion. Recent research (Larson, 1997; Miller & Singleton, 1997) demonstrates that social studies teachers have multiple conceptions of discussion that they variously employ in the classroom. These multiple definitions of discussion can be a problem: If a concept means everything, then often it means nothing. Teachers have difficulty using discussion effectively if they have no firm concept of what constitutes discussion in the first place and, equally as significant, what makes a discussion a good discussion.

Teachers' multiple and conflicting definitions of discussion also provide an opportunity for researchers, however. Effective CPI discussion teachers may have formed an understanding of what makes a good discussion and are carefully organizing their instruction to ensure that students reach that target. In an earlier study (Hess, 1997), I found that two exemplary high school social studies teachers had strong and conflicting ideas about what constituted good CPI discussions. As an example, here I include what they said after viewing a videotape of the same discussion in a high school classroom:

**John**: There's a lot of important work going on in this classroom . . . but this is not discussion because most of the questions are not only
directed at the teacher’s questions, but they’re compartmentalized. They almost stand as separate entities. . . . My initial definition of discussion is two or more people talking together, maybe linking their comments, spring boarding from one comment to another, give and take, response. These people might have well been talking in an empty barn.

Jan: I think they’re curious to hear what other people are going to say, and it’s moving along. . . . These kids are, seem to understand that they can’t, you know, hold the floor forever, . . . they’re reasoning, they’re doing a great job . . . thinking it all the way through. They’re putting pieces together and they’re backing up what they’re saying. They seem to have a respect for each other. . . . Reasonable people can disagree here. These kids are bringing in their factual information. I just love the way they’re really, you know, interacting and following up, picking up on each other’s comments.

The author of a nationally disseminated assessment rubric for CPI discussions encourages teachers to assess students’ participation in CPI discussions more favorably if they use relevant background knowledge, engage others in the discussion, and constructively challenge the accuracy, clarity, relevance or logic of statements made (Harris, 1996). Other experts in CPI discussions are more interested in promoting an “open climate for discussion . . . where all points of view are valued, and all ideas merit critical examination” (Massialas & Cox, 1966). A discussion in which “all points of view are valued” may differ significantly from one in which students are encouraged to “challenge the accuracy . . . [etc.] of statements made” by other students. In the latter, accuracy and challenge receive center stage; in the
former, openness is the most significant attribute. For this study, I elicited the teachers' definitions of discussion and the specific attributes they believe a discussion must possess to be characterized as a good discussion.

**Rationales for Using CPI Discussion**

The last component of the third research question focuses on why secondary social studies teachers use CPI discussions. Here I am mainly interested in whether the teachers treat discussion as the outcome or as a method for achieving other outcomes. In the former, engagement in CPI discussions is designed to teach students how to participate effectively in CPI discussions. That is, this conception would identify students' ability to discuss as the primary outcome. In the latter, teachers have other reasons for using CPI discussions, such as enhanced content understanding or developing critical thinking skills.

Understanding why teachers use CPI discussions is important because rationale influences practice. For example, if teachers see enhanced discussion skills as the primary outcome, they may be more insistent that all students participate in discussions.

**Conceptual Framework**

The conceptual framework (see Figure 3) of the study is made up of four parts: CPI Instructional Practice, CPI Instructional Plans and Strategies, Teachers Conceptions, and Classroom, School, Community Contexts. Within each part of the framework are smaller categories, what I am calling sub-parts. I will be looking at the relationships between the sub-parts and the parts. For example, within the part of the framework on Teachers' Conceptions, I will seek to understand the teachers' conceptions about the four sub-parts (democratic citizenship, etc.), as well as how the sub-parts are interrelated,
such as a relationship between how a teacher conceptualizes democratic citizenship and the purposes he or she holds for social studies. Moreover, I will look at how elements within a particular part of the framework, such as CPI Instructional Practice, are informed and influenced by other parts of the framework, such as CPI Instructional Plans and Strategies.

Three of the four parts of the conceptual framework are directly correlated to the research questions that have already been explained. The part of the conceptual framework dealing with contexts has not been explained by the research questions because, while I think it will be important, it focuses on a more generic influence on teaching and learning than the other parts of the conceptual framework. By contexts, I mean the circumstances and settings in which the use of CPI discussions occurs, including the classroom, the school, and the community.

The classroom context includes the grouping and grade level of the students and the courses in which the teacher is including CPI discussions. This part of the conceptual framework, then, deals with ways that students and curriculum can influence how a teacher uses CPI discussions. For example, teachers may scaffold CPI discussion instruction based on how much experience they think their students have participating in such discussions. Students who have already taken courses that include a focus on CPI discussions might receive more advanced instruction on CPI discussion than students who are relatively new to this type of classroom discourse.

By school and community context, I am referring to a range of ways that school and community climate may influence how teachers use CPI discussions. For example, if administrators and other teachers value CPI discussions the teacher may be more likely to include them in the curriculum
they use in the classroom. Conversely, in some communities CPI discussions that focus on highly contentious issues are not valued, in fact, are often effectively banned. Thus, this part of the conceptual framework focuses on the need to understand the context outside of the classroom in which the teacher works.

Figure 3. Conceptual Framework

Chapter Summary

Three research questions guided this study: How do secondary social studies teachers who are skilled in the use of CPI discussions teach their students to participate effectively in such discussions? What role do
instructional strategies, issues, materials, and assessments play in this teaching process? What accounts for these teachers' approaches to CPI discussions? The foci of these questions form three of the four parts of the conceptual framework on which the study is based; the fourth component of the framework is contexts—the classroom and school/community circumstances and settings in which use of CPI discussions occurs.
Chapter Three
Methodology

Models of wisdom or "best practice" studies about especially effective teachers have typically used qualitative methodology. For example, Ladson-Billings (1994), in a study of teachers who were exceptionally effective at teaching African-American students, used a combination of classroom observations, teacher interviews, and teacher focus groups to collect the data. To develop a theory about the characteristics of these effective teachers Ladson-Billings used analytic induction to make meaning from the data. Her research had the features typically associated with qualitative research, including data collected in the natural setting, data that was inductively analyzed (often likened to putting together the parts of a puzzle), and a primary emphasis on what things meant (Bogdan & Biklen, 1992). Similarly, recent research into teachers' conceptions has relied on qualitative methodology. For example, Larson's (1995) study of teachers' conceptions of classroom discussion relied on interviews, observations, and inductive analysis to develop a theory explaining teachers' purposes for using discussion.

This study, an investigation of secondary social studies teachers who are skilled at teaching their students to participate effectively in CPI discussions, is a models of wisdom or "best practice" study that focuses, in part, on the teachers' conceptions. As such, I too used qualitative methodology. Within the large family of qualitative methods, I relied on grounded theory, a qualitative method that "uses a systematic set of procedures to develop an inductively derived grounded theory about a phenomenon" (Strauss & Corbin, 1990, p. 24).
Because the grounded theory approach is appropriate for research questions that involve the understanding of complex social phenomena (Strauss & Corbin, 1990, p. 250), it was appropriate for this study. Teaching students to participate effectively in class discussion is a complex and difficult enterprise (Dillon, 1994). Understanding the conceptions and practices of teachers who do this well, by definition, involves the understanding of a complex social phenomena. The grounded theory approach is also appropriate because little is known about how teachers instruct their students to participate more effectively in CPI discussions. In other words, no existing data-based theory about how to teach CPI discussions well exists. Building the first layer of a "close to the ground" or substantive theory, defined by Gehrke and Parker (1982) as "theory that is developed for a relatively specific area of inquiry in a given context" (p. 2), then has practical utility for people interested in better understanding how to teach students to discuss controversial public issues.

In this chapter, I describe the components of the grounded theory approach as I applied them through three phases of the study: sample selection, data gathering, and data analysis. The first section of the chapter explains how I selected the teachers who participated in the study. The chapter's second section focuses on the data collection methods used in the study. The third section turns to data coding and analysis.

Selection of Teachers

Instructing young people to participate more effectively in CPI discussions can be taught by school teachers in a variety of subject areas (Engle, 1993) or by other people, such as parents, youth group leaders, or peers. This study, however, investigated only secondary social studies teachers who
were skillfully teaching their students to participate in CPI discussions. Moreover, the teachers studied were chosen purposely, instead of at random. This section addresses two questions about their purposeful selection: Why did I narrow the sample to secondary social studies teachers? How did I find and select the secondary social studies teachers who participated in the study?

Narrowing the Sample

In the grounded theory approach, a sample of groups or individuals is chosen not because it statistically represents a given population, but because it has theoretical usefulness to "discover categories and their properties, and to suggest the interrelationships into a theory" (Glaser & Strauss, 1967, p. 62). When beginning the discovery, or generation, of substantive theory, experts recommend that differences in the sample be minimized (Glaser & Strauss, 1967).

Four reasons prompted me to select secondary social studies teachers who were skilled at teaching their students to participate in CPI discussions. First, given that the purpose of the study was to generate a theory of the thinking and teaching of skilled CPI discussion teachers, a sample consisting of these skillful teachers was necessary. Second, although CPI discussions do occur in classes other than social studies, evidence suggests that social studies teachers are more likely than teachers of other subjects to include such issues in the curriculum (Engel, 1993). Third, the importance of the democracy rationale (see Chapter One) suggests that the school subject that has historically been charged with citizenship education, social studies, was a good fit for this study.

The fourth and final reason supporting my decision to study secondary social studies teachers was that secondary social studies is my area of expertise,
so selecting a sample of teachers from that category enhanced my theoretical sensitivity. As a novice researcher, I was especially concerned with theoretical sensitivity, defined by Strauss and Corbin (1990) as the researcher’s “attribute of having insight, the ability to give meaning to data, the capacity to understand, and the capability to separate the pertinent from that which isn’t” (p. 42). Theoretical sensitivity can come from different sources, including two that pertained to my situation: familiarity with the literature and professional experience. As a teacher of discussion classes and professional development workshops for social studies teachers, and an experienced secondary social studies teacher, my professional experience would, I reasoned, help me more readily understand the nature of what happens in CPI discussions that occur in social studies classes.

**Identifying and Selecting the Teachers**

Social studies educators who work with teachers in many different school districts, such as professional development providers for inservice teachers and university professors, were asked to nominate teachers for a selection pool from which I would select my sample. I sought only teachers who lived near two major cities in the West because finances precluded traveling across the United States. The social studies experts were asked to nominate secondary social studies teachers who were skilled at teaching students to participate effectively in CPI discussions. I then contacted the teachers, explained the study, and asked if they would be interested in participating. Many of the nominated teachers expressed interest in the study but did not have time to participate. Those who indicated an interest and did have time to participate were asked to identify a social studies expert who had recently seen them lead CPI discussions, either in person or on videotape. By
contacting the social studies experts they identified I received verification that the teachers were, in fact, especially skilled at teaching CPI discussions. One of the recommended teachers did not make it through this verification stage.

Once teachers were nominated and their CPI discussion-teaching prowess verified, I selected three to participate in the study on the basis of three additional criteria: the teacher was teaching a class that included discussion of CPI during the 1997-98 school year, the teacher could accommodate the interviews into his/her schedule, and necessary permission to audio- or videotape class discussions could be readily obtained from the school's principal, students, and their parents.

**Data Collection**

Research on especially skillful teachers and research on the conceptions of teachers typically rely on multiple data types (Ladson-Billings, 1994; Larson, 1995; Wineberg & Wilson, 1988). Researchers commonly interview teachers, observe them in the classroom, and analyze documents (e.g., lesson plans, student materials, assessments). Multiple data sources are also recommended when developing grounded theory because "it yields more information on categories than any one mode of knowing" (Glaser & Strauss, 1967, p. 66).

In this study I collected three kinds of data that correlated to the conceptual framework (see Figure 4). The three kinds of data were: (1) audiotapes and notes from interviews of the three teachers, (2) field notes from observations of three CPI discussions in each teacher's classroom (and/or notes from listening to audiotapes/viewing videotapes that the teachers made of CPI discussions in their classrooms), and (3) written artifacts related to CPI discussions. I collected all of the data from the first teacher
before moving on to teacher two. Then I collected all of the data from that
teacher before moving on to teacher three.

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Figure 4. Types of Data Correlated to the Conceptual Framework

Key to Figure 4

Note that the x means that a particular part of the conceptual
framework was addressed by kind of data collected. For example, I learned
about how the teachers conceptualized democracy from the semi-structured
interviews.

Semi-structured interviews: Five semi-structured interviews were
conducted with each teacher, many involving tasks, such as concept
mapping, that the teachers performed and explained.

Observations: For the first teacher ("Joe"), I listened to audiotapes of nine
CPI discussions, each lasting more than an hour. For the second teacher
("Elizabeth"), I observed two CPI discussions in her classroom and listened
to three additional discussions that had been audiotaped. For the third teacher ("Ann"), I observed three CPI discussions in her classroom and viewed three additional discussions that had been videotaped. 

Post Observation/Listening/Viewing Interviews: The first teacher (Joe) and I listened to parts of two CPI discussions together; I then interviewed him about various elements of the discussions. The second teacher (Elizabeth) and I listened to parts of two audiotaped discussions together and I then interviewed her about them; I also interviewed her immediately after two CPI discussions I had observed. The third teacher (Ann) and I viewed portions of two videotapes together, immediately followed by an interview. Additionally, another interview occurred after I observed three CPI discussions in her classes.

Artifacts: I collected, for each teacher, written artifacts that pertained to CPI discussions, including a list of classroom rules for discussion, readings used to prepare students for the CPI discussions that were observed/listened to/viewed, and assessment instruments.

Interviews

Most of the data for this study came from two kinds of interviews with the teachers, semi-structured interviews and open-ended interviews. The first four interviews described below were semi-structured, meaning that I was using a detailed interview protocol for each one. Interviews five through eight were open-ended, meaning that I did not use a protocol. Below is a thumbnail sketch of each interview, describing the part of the conceptual framework the particular interview focused on and what kind of questions or tasks were part of the interview. Interview protocols for those interviews that involved specific pre-planned questions or tasks are located in the Appendices.

Interview One: Context, Purpose of Social Studies, Democratic Citizenship. In the first interview, each teacher responded to open-ended
questions about the school, community, and particular class (or classes) that included CPI discussions. They also described their backgrounds and gave short biographies of their teaching careers. Then they drew two concept maps, one illustrating what effective citizens in a democracy should know, be able to do, and be disposed to do, and another illustrating what they want their students to know, be able to do, and be disposed to do as a result of their social studies classes. After they had drawn each concept map, they explained it to me and responded to questions about its meaning. See Appendix B for the protocol for this interview.

Interview Two: Characteristics of Good Discussion. In an adaptation of the think-aloud method (Clark & Peterson, 1986), the teachers viewed two short teaching tapes that showcased class discussion and explained their reactions to the discussions. The first videotape was of a high school United States History class discussing a moral dilemma drawn from an excerpt of Richard Wright’s autobiography in *Reasoning with Democratic Values* (Lockwood & Harris, 1985). This class discussion is on the film *Minds on Social Studies* (National Council for the Social Studies, 1998). The second discussion was of six students in a high school American Government class discussing whether the electoral college should be abolished or reformed. This small group discussion is included in the film series *Preparing Citizens* (Social Science Education Consortium, 1997).

I selected these two videos mainly because of their differences. The first discussion is a whole class discussion with students seated in rows facing the teacher. The teacher takes an active role as the discussion leader, asking many questions and occasionally correcting students’ factual errors. The discussion focuses on an individual’s moral decision-making about whether or not to
steal. The second discussion is not led by the teacher. Instead, a small group of discussants are seated in an inner circle surrounded by their classmates and the teacher who are evaluating the discussion using a rubric (explained in Miller & Singleton, 1997, p. 69). The discussion focuses on the public policy issue of whether the United States Constitution should be amended to change the presidential election system.

**Interview Three: Rationales for CPI Discussions.** Using labels drawn from the rationales for CPI discussions in the literature (see Chapter One), the teachers ranked those rationales to reflect the reasons they use CPI discussions. This kind of ranking exercise is drawn from research that connects the ability of teachers to clearly articulate their teaching aims and purposes to activities that require some kind of discrimination or ranking (Feiman-Nemser & Floden, 1986). See Appendix B for the list of rationales the teachers ranked in this interview.

**Interview Four: Issues Selection.** This interview involved another selection and ranking exercise. The teacher organized various CPI topics in terms of the likelihood they would be included in his/her curriculum. I selected issues drawn from a broad range of categories of public policy concerns: domestic and international topics; money and morals topics; and contemporary and historical topics. I purposely included topics that might be particularly controversial in some communities, such as gay rights, abortion, and physician-assisted suicide. See Appendix B for the list of CPI topics used in this interview.

**Interview Five: Assessment.** The teachers responded to open-ended questions about how they assess CPI discussions. If the teacher had developed a scoring guide for CPI discussions, then much of this interview focused on
why and how that scoring guide was developed and how it worked in practice.

**Interviews Six, Seven, Eight: Enacted CPI Discussion Teaching/Post-Observation.** These interviews occurred immediately after each observation or immediately after each teacher and I had listened to/viewed an audiotape/videotape of a CPI discussion. The teacher explained and evaluated CPI discussions by answering open-ended questions. The teacher also explained and analyzed the materials used for at least one of the CPI discussions I observed (or listened to) and evaluated how the materials worked with students.

**Field Notes from Observations/Audiotapes/Videotapes**

In addition to collecting data from interviews, I also took field notes during classroom observations and listening and viewing notes while listening to audiotapes or viewing videotapes of CPI discussions. As explained in the key to Figure 4, the way in which I learned about enacted CPI discussions in the three teachers' classes varied. For the first teacher, Joe, I listened to nine CPI discussions that had been audiotaped and did not conduct any in-person observations because the class Joe said was most suitable for this study (*Important Supreme Court Cases*) ended before the field work began. For the second teacher, Elizabeth, I observed two CPI discussions in person and listened to two CPI discussions on audiotape. I did not videotape these discussions because the students had not been videotaped before and I was concerned that video taping would interfere with the normal course of the discussion. For the third teacher, Ann, I observed three CPI discussions in person and viewed three on videotape. Because Ann videotapes all of the CPI discussions in her classes, I had ready access to these videotapes and was not
concerned that the additional taping would change the dynamics of the discussions. I consider the limitations caused by this variance in how I learned about each teacher’s enacted CPI discussion teaching practice in the final section of this chapter.

When conducting in-person observations, I both audiotaped the discussions and took extensive field notes. The audiotapes were not transcribed as were the interviews, but I did listen to them carefully and took listening notes. The notes (both in person and from listening and viewing) focused on the quantity and content of teacher and student participation. For example, I tracked who was participating and what they were saying. When observing in person, I also noted what the teacher and students were doing (i.e., taking notes, having side conversations, looking at the speaker, etc.).

**Classroom Artifacts**

In addition to interviews and notes from observations (both in person and via audiotapes and videotapes), I collected a third type of data: materials used by the teacher and students to prepare for participation in CPI discussions, assessment instruments, and classroom rules that pertained to CPI discussions. Because the teachers’ approaches to CPI discussions varied, these materials also varied by teacher. For example, two of the teachers formally assessed and graded CPI discussions using rubrics, but the other teacher did not. Consequently, I have assessment rubrics for only two of the three teachers.

**Summary of Data Collection**

Three types of data were collected for this study: semi-structured and open-ended interviews, field notes from observing and/or listening/viewing tapes of CPI discussions, and classroom artifacts related to CPI discussions,
such as assessment rubrics and preparatory readings. All three teachers were interviewed at least six times. Several of the interviews were semi-structured, followed a standardized protocol involving various tasks (such as ranking activities and concept maps) completed by the teachers. For the reasons given, both the number of CPI discussions I observed in each teacher’s classes and the form (i.e., in-person observation vs. listening to an audiotape or viewing a videotape) of the observation varied. The classroom artifacts I collected also varied from teacher to teacher because of the different ways they approached CPI discussions.

Data Analysis

The grounded theory approach is distinguished from other types of qualitative research because data gathering and data analysis occur throughout the study, rather than all of the data being collected before the data analysis stage. This process allows tentative hypotheses to be checked against new data, thus assuring that verification is built into the method (Gehrke & Parker, 1982). Following this tenet of grounded theory methodology, I collected and analyzed all of the data from each teacher before moving on to the next.

In this section, I describe the four-stage process (see Figure 5) I used to analyze the data and generate the initial theory. In the first stage, I initially coded and analyzed the data by proceeding through four steps: (a) transcription and verification; (b) coding and the development of visual displays and conceptual memos; (c) portrait writing; and (d) analysis to integrate categories and their properties. After these four steps were completed for the first teacher (Joe), I moved on to the second teacher,
completing the four steps, but taking with me what I had learned from Joe. Then the entire process was repeated for the third teacher.

The second stage of the process was to compare and contrast the conceptions and practices of the three teachers by additional integration of the categories and their properties. That is, I looked for similarities and differences among the three teachers to better understand the categories, their properties, and the interconnections between categories.

In the third stage of the process, I delimited (by both limiting and demarcating) the categories and their properties, working to enhance both the parsimony and scope of the theory. This all led to the fourth and final stage, the writing of an initial theory explaining the conceptions and practices of secondary social studies teachers who are skilled at teaching their students to participate more effectively in CPI discussions.
Methodological Sequence Used to Generate the Initial Theory

Collect data from Teacher One (Interviews, Observations, Artifacts)

Stage One: Code Data and Do Initial Analysis

1. Transcribe and verify
2. Code, including inter-rater coding, develop visual displays, and conceptual memos
3. Write portrait
4. Perform initial integration of categories and properties

Collect data from Teacher Two

   Repeat Stage One with Teacher Two’s Data (see above)

Collect data from Teacher Three

   Repeat Stage One with Teacher Three’s Data (see above)

Stage Two: Integrate Categories and Their Properties Among Three Teachers’ Data

Stage Three: Delimit Categories and Their Properties

Stage Four: Write Initial Theory

Figure 5. Methodological Sequence
Stage One: Coding and Initial Analysis

After collecting the data from the first teacher, I began to code and analyze that data. It is important to note that this coding and analysis process was initially applied to the data from teacher one, before any of the data was collected for teachers two and three. Then, the process was followed, with modifications that are explained beginning on page 58, for the data from teachers two and three. The coding and initial analysis stage included the following steps, which are described below: transcription and verification; coding, creating visual displays, and writing conceptual memos; writing portraits; and the initial integration of categories and their properties.

Transcription and Verification Opportunity. The first step in the data analysis process was to transcribe the interviews for the first teacher (Joe) and check them for accuracy. After a professional transcribed the interviews, I edited the transcriptions to correct misspellings and other inaccuracies. This editing also served to remind me of what had transpired during the interviews. After this editing, the transcripts were printed with very wide right-hand margins to allow adequate space for manual coding. I offered Joe the opportunity to read the transcripts and correct them. He did not have time to read them and indicated that he trusted the accuracy of their contents. Later, the other two teachers also declined the opportunity to read their transcripts, giving the same reasons.

Coding. Working with the transcripts, field notes, and classroom artifacts from the first teacher, I began coding. I coded each definable "chunk" of the data. Some chunks were just fragments of sentences, others were complete paragraphs. As a general rule, I defined a chunk of data as an idea
that was (1) different from what had come before and (2) important enough to stand alone.

As recommended by Glaser and Strauss (1967), I coded each chunk in the data into as many categories as possible "as categories emerge or as data emerge that fit an existing category" (p. 105). The conceptual framework provided some of the existing categories, such as the large categories of rationales for CPI discussions and characteristics of effective discussions. As I explored the data, many categories not in the conceptual framework also emerged; to illustrate, "knowledge creation" emerged early in the coding of the data from the first teacher. In many cases, a chunk of data was coded into several categories. For example, when the first teacher explained that only "good discussion" would enhance students' interpersonal skills, that "chunk" of data was coded into two categories: interpersonal skills and the defining what constitutes good discussion.

Throughout the coding process, I followed what Glaser and Strauss term "the basic, defining rule for the constant comparative method" (1967, p. 106): always compare an incident (what I am calling a "chunk") being coded into a specific category with the incidents previously coded in that category. For example, one overall category in the conceptual framework was teachers' reasons for selecting which CPI issues students would discuss. Everything in the data related to reasons for selecting issues was coded in that overall category; then sub-categories began to emerge. For example, from the first teacher's data the category of "social justice" was identified. Each time I coded an incident for "social justice" I mentally compared it to the other incidents similarly coded and to other incidents that had been coded in the overall
category of criteria for issues selection. By doing this, I began to generate theoretical properties of each category.

**Inter-rater.** Early in the coding process, I asked another researcher familiar with my study to code 10% of the data from the first teacher. I did this as a check on whether the categories I saw emerging from the data could be seen by another researcher. By enlisting this help, I sought to "compare the analyst’s ideas [the other researcher’s ideas] with [my] ideas and knowledge of the data; this comparison generates additional theoretical ideas" (Glaser & Strauss, 1967, p. 108). This process is different from the typical one used for inter-rater reliability because the other researcher was constructing categories and their properties, instead of using the categories I had identified to code the data. The other researcher and I talked about the codes she induced from the data, comparing her codes to the ones I had previously identified. This process was enormously useful because it refined my coding scheme and broadened how I was interpreting the data.

Throughout the coding process, I created visual displays and wrote conceptual memos both to reduce and to better understand the overwhelming amount of data (Miles & Huberman, 1994). For example, I created a data retrieval chart that summarized and showcased the teachers’ conceptions of the rationales for using CPI discussions (see Figures 6, 8, 10). I also wrote conceptual memos explaining how I thought a particular category was emerging from the data and the properties of that category. For example, one memo explored how the first teacher used CPI discussions to help his students create new knowledge about the issue being discussed. Another memo focused on the data about the first teacher’s role in facilitating discussions that emerged from the audiotaped discussions. Throughout the
process of creating visual displays and writing conceptual memos, I continually referenced the data, as is typical of the constant comparative approach of joint coding and analysis.

Writing Portrait One. The next step in the data analysis process was to write the portrait (Lawrence-Lightfoot & Davis, 1997) of the first teacher. Using the visual displays, conceptual memos, and the coded data, I wrote an extensive portrait to describe the first teacher's CPI discussion thinking and teaching. The portrait is not the theory, although it does include many of the categories and their properties that were later used for theory-creation. To organize the portrait, I relied on the conceptual framework (see Figure 3). For example, the overall categories of the portrait (such as contexts and conceptions) mirror those in the conceptual framework. Within those categories, however, are glimmers of the emerging theory. For example, within the section of the portrait describing how the first teacher defines effective discussion are categories and their properties that emerged from the coding and initial analysis (such as classroom seating to promote equality).

The portraits contain many quotes from the interviews, observations, and classroom artifacts. These quotes, however, tend to be short and are used to lend authenticity to the portrait. In keeping the quotes short, I followed Wolcott's advice (1990) to not "let informants rattle on in the written account just as they may have done during the interviews" (pp. 66-67). I developed a citing system for the quotes so readers would know who made each statement, and in which context (i.e., interview, observation, etc.). Using pseudonyms for the teachers, quotes are first cited to a specific teacher (J.P. for Joe Parks, E.H. for Elizabeth Hunt, and A.T. for Ann Twain) or to a student (ST), then to its source (IV for interview, AT for audiotape, VT for videotape,
and ART for artifact), and then, if applicable, to the page number of the transcript. For example, “J.P. IV#1, p. 4” means that the quote came from Joe Parks, during interview one, and can be found on the fourth page of the transcript of that interview.

After extensive editing, each portrait was shared with the teacher it described. This was done to obtain “member feedback” (Miles & Huberman, 1994, p. 277). Inviting the teachers’ reactions to the portraits served two purposes. First, the feedback enhanced the overall reliability of the study because it ensured that factual errors would be corrected. Second, I shared the portraits and invited feedback to enhance the trust that had begun to form during the data collection process. In short, I did not want the teachers to feel this research was something being done to them, but instead with them. Notwithstanding that every effort was made to ensure that their participation in the study was confidential, the teachers’ principals, other teachers, and their students knew they were participants in the study. Thus, I felt an ethical obligation to provide the teachers with an opportunity to react to my description of their conceptions and practices.

All of the teachers were pleased with the portraits that I had written describing their CPI discussion conceptions and teaching practice. In particular, they said the portraits were accurate and complete. Two of the teachers, however, caught factual errors and corrected them. One was a major error caused by how I had misinterpreted the teacher’s description of how issues were selected for discussion in the curriculum. Another teacher pointed out that my use of the word “consequence” implied a negative judgment to her, which was not what I had intended. The teachers’ positive
reactions to their portraits were an important indicator to me that the process I had used to collect data and write the portraits had worked well.

Initial Integration of Categories and their Properties. After the first portrait was completed, I turned to the next stage in the data analysis process, which was to further refine the categories and their properties that had emerged from the first teacher's data and to begin looking for relationships across the categories. I revisited the coded data, visual displays, conceptual memos, and the portrait. As suggested by Glaser and Strauss (1967), "this process started out in a small way; memos . . . are short" (p. 108).

At the beginning of this stage, I wrote another series of conceptual memos and created additional visual displays. I frequently returned to the coded data to be reminded of all the incidents that had been coded a particular way. Categories grew larger (becoming more integrated) during this process. For example, I created a macro category labeled "Effective Democratic Citizen Behaviors (EDCB)" that encompassed all of the previous categories about what effective citizens in a democracy should know, be able to do, and be disposed to do. By collapsing these categories into a larger one, I began to see how the first teacher's conceptions of EDCB were linked to one another. Additionally, I began to see how the larger, more encompassing categories were connected. For example, I began to identify ways in which the first teacher's conceptions of EDCB were linked to how he defined the characteristics of effective discussions. At this point, I wrote additional conceptual memos that explained my tentative hypotheses about the CPI thinking and teaching of the first teacher. While still not a theory, the theory was beginning to emerge. It was time to move to the second teacher.
Repeating the Process for Data from Teachers Two and Three. Because grounded theory requires a constant comparison between data and analysis, I next moved on to teacher two, but took what I had learned from teacher one with me. I did this in two ways. First, the interview protocol for the second interview on rationales for CPI discussions changed as I moved on to teacher two, and again as I moved to teacher three. Recall, this interview involved a card-sorting task requiring the teachers to rank order reasons for using CPI discussions drawn from the literature. Teacher one added several additional rationales. For the interview with teacher two, I added to the sorting task the rationales created by teacher one. Teacher two also added rationales that were then brought to the interview with teacher three. Allowing the list of rationales to grow facilitated theory-generation by eliciting data that would allow for comparisons across the three teachers.

The second way that the data coding and analysis process I used for teacher one informed the process for teacher two (and later, how teacher two informed the process for teacher three) was in the initial coding. Instead of starting a fresh coding system with teachers two and three, I brought the codes from teacher one (and later from teacher two) with me. Thus, as I was coding the data from teachers two and three, I constantly referred to the incidents from the previous teacher(s) that had been coded in the same categories. For example, all three teachers talked about the importance of voting, and all of these data chunks were coded into a voting category. However, they talked about voting in different ways, and comparing those differences caused this category to be understood differently. Another example can be found in the second teacher’s definition of interpersonal skills. She talked about the importance of students learning how to “yield” the floor in CPI discussions.
Under the previously created category of "interpersonal skills," the property of "yielding" was added.

Except for those changes (i.e., rationale interview and coding categories), the rest of the process for coding and initially analyzing the data from teachers two and three followed the pattern established for teacher one (see Figure 5). After completing all of these steps for the three teachers, I was ready to move on to the second stage in the theory-generation process: integrating categories and their properties among the three teachers.

Stage Two: Integration of Categories and Properties Among the Three Teachers

In the second stage of integrating the categories and their properties I worked with the data, visual displays, and conceptual memos from all three of the teachers. Here I was using comparison and contrast to develop categories and properties that would apply to all three teachers. For example, the category that would eventually be titled "students' forum" emerged as I collapsed previously-coded incidents from the data of the three teachers into a general category that described ways in which the CPI discussions were controlled and influenced by the students. Several properties of this category began to emerge, such as the students' influence on issue selection, and the role of the teachers as facilitators who worked to encourage interaction among the students.

It was during this process that I realized the potential significance of what was not in the data. Based on CPI discussion literature, I had expected to find that the teachers would have to make decisions about whether and how to disclose their personal views on the CPI under discussion. As I compared and contrasted the data, I realized that I had never heard a student ask the
teacher to disclose his or her views on the issue. Moreover, none of the teachers volunteered their views. This realization illustrates the recursive nature of grounded theory, for it sent me back to the data to double-check whether I was right. I was, and the absence of teachers’ disclosure of personal views on CPIs then became a property within the larger category of “students’ forum.”

As a result of this stage in the data analysis process, I had created many categories based on the data from the three teachers. Some of the categories were more fleshed out than others, and some were closely related to one another, while others still seemed fairly distinct. Each, however, had specific properties. At this stage in the process, the emerging theory was cumbersome and awkward, signaling to me that it was time to move to the next step in the process, one of winnowing the categories and developing relationships among them.

Stage Three: Delimiting Categories and Their Properties

In grounded theory methodology, the emerging theory is “delimited,” that is, limited and demarcated, reducing the number of categories and interrelationships and constantly looking for larger and more abstract categories. This defining is done with two goals in mind: to create parsimony of variables and formulation and to enhance the scope of the theory to a wide range of situations (Glaser & Strauss, 1967, p. 111).

I progressed through this delimiting stage by systematically reducing the categories into seven propositions, and then working to develop ways in which the propositions influenced one another. As explained in Chapter Five, I sought to create propositions that described and explained the CPI discussion conceptions and practices of the three teachers. The propositions
were created by collapsing categories that were closely related to one another. For example, the proposition about how the teachers teach for and with discussion (Parker, 1996) was created by combining two categories that emerged during stage two. I combined the two categories into one larger and more abstract category to enhance the explanatory power of the emerging theory. That is, what seemed important about how the teachers taught with and for discussion was that they were doing both, not just focusing on one or the other. I viewed the propositions as building blocks to the emerging theory—not the theory itself.

Once I had developed the seven propositions, I began looking for ways that they influenced one another in order to develop a theory about skillful CPI discussion teaching. For example, the teachers' conceptions of democracy were informing many of their decisions about what kind of discussion to teach their students, so a key piece of the emerging theory was that teachers' conceptions of democracy were important to their CPI discussion teaching practice.

**Stage Four: Writing the Initial Theory**

The final stage in grounded theory methodology is to write the initial theory by collating the memos on each category, often returning to the data for illustrations. The written theory is considered a “theory-in-progress” because “researchers expect, even welcome, refinements and extensions of the theory” (Gehrke & Parker, 1982, p. 5). I wrote the theory in two stages. First, I

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1 Walter Parker's definition of the difference between using discussion to teach other outcomes and teaching for discussion as an outcome in its own right (the with/for distinction) informed how I labeled the first proposition. While retaining Parker's meaning, I reversed the order of the distinction to for/with to represent the primacy these teachers place on teaching for discussion.
described and explained the seven propositions (see Chapter Five). At this stage, a number of readers offered critical feedback that was used to revise the explanations to enhance their clarity. The next step was to write the theory about how the propositions related to and influenced one another. To accomplish this, I began with the conceptual framework that had informed the design of the study. I placed each proposition on the conceptual framework, re-arranging the arrows that illustrated relationships between the propositions. Using the revised conceptual framework as an outline, I then wrote a description of the emerging theory (see Chapter Five).

Chapter Summary

I used grounded theory methodology throughout three phases of the study: sample selection, data gathering, and data analysis. Three secondary social studies teachers who are especially skilled at teaching their students to participate effectively in CPI discussions were selected to participate in the study. I gathered data by interviewing the teachers, observing their CPI discussion teaching (either in person, or by listening to audiotapes/viewing videotapes), and collecting CPI-related classroom artifacts. Data analysis followed the recommended procedures of grounded theory methodology and progressed through four stages: coding and initial analysis, integrating categories and their properties among the three teachers’ data, delimiting categories and their properties, and writing the initial theory.
Chapter Four

Portraits of Three Teachers' Conceptions and Practices of CPI Discussions

At the heart of this study are the CPI discussion conceptions and practice of three teachers. Just as researchers of “wise practitioners” in other areas of teaching expertise have found (Ladson-Billings, 1994; Wineberg & Wilson, 1988), no single set of beliefs and practices defines effective CPI discussion teaching. The three teachers in this study think about and enact their CPI discussion teaching in different ways. Yet similarities can also be found among the three. This chapter illustrates these similarities and differences through portraits that describe the three teachers’ CPI discussion thinking and teaching; the portraits lay a foundation for understanding the theory of what constitutes effective CPI discussion teaching that will be presented in the next chapter.

The portraits are organized using the factors described in the conceptual framework for the study (see Chapter Two). Recall, the conceptual framework was developed based on the literature about how teachers think about and practice CPI discussions in their teaching. The factors in the conceptual framework are organized into four categories: contexts, conceptions, instructional plans and strategies, and CPI teaching practice. I use the categories in the conceptual framework to organize the portraits as one would use building blocks to develop a structure. Throughout the portraits I refer to the teachers and their schools by using pseudonyms.

The first section of each portrait begins with a brief biographical sketch and description of the contexts in which the teacher is working. This introduction focuses on how and why the person became a teacher, the school and community in which she/he works, and the students he/she teaches.
Following this introduction, each portrait explains the teacher’s conceptions of factors that may influence CPI discussion teaching, including beliefs about what students should know and be able to do as a result of her/his social studies teaching, and the knowledge, skills, and behaviors he/she thinks effective democratic citizens should possess and exhibit. Each portrait then turns to a specific focus on CPI discussion, explaining the teacher’s rationales for including such discussions in the curriculum, conceptions of what constitutes an effective discussion, and how he/she makes decisions about the important matter of what should be discussed.

Following the explanation of the teacher’s contexts and conceptions, each portrait then turns to a more specific focus on how the teacher enacts his/her CPI discussion teaching. In this third section of each portrait, I describe and explain the teacher’s instructional plans and strategies when using CPI discussions. I begin with how the teacher makes decisions about what should be discussed and what materials students should use to prepare for discussions.

The centerpiece of the third section of each portrait is a “snapshot” of a CPI discussion that illustrates what CPI discussions look like in each teacher’s classroom. These “snapshots” are narrative descriptions of CPI discussions that reveal the CPI instructional models used by each teacher, how students prepare to participate in a discussion, the role the teacher takes throughout the discussion, and whether and how students’ participation in discussions is assessed. The “snapshot” narratives alternate between describing what actually happened in a specific discussion and the teacher’s explanation of various instructional plans and strategies.
Portrait One: Joe Park at New Horizons High School

Joe Park has taught secondary social studies for 22 years, a career choice initially motivated by his interest in social studies content, especially African studies. As an undergraduate student in one of the nation's premier universities, Joe's preservice teacher education was "pretty phenomenal. . . . I learned how to teach from people who really believed that social studies teaching was a special and very, very important trust" (J.P. IV#1, pp. 2-3). Joe credits the superb preservice training he received as the reason he both likes teaching and feels successful as a teacher.

Since graduating with a B.A. in history and a certificate to teach secondary social studies in 1975, Joe has taught both middle and high school social studies at several schools, completed an M.A. degree in curriculum and instruction, and participated in a wide variety of professional development programs, in both the teacher and student roles. For example, early in his teaching career, Joe spent a year as a teacher associate for a social studies think tank, where he developed curriculum and led teacher inservice professional development programs. A few years later, he spent two years as a clinical professor in a university program, supervising student teachers and facilitating staff development programs for teachers in his school district. Since that time, Joe has continued to participate in a number of professional development activities, including a national project on developing authentic assessments in civic education, Educators for Social Responsibility (a national organization dedicated to peace education), and a locally-based study group in which teachers share their curriculum and students' work as a way to improve their teaching. As an adjunct instructor for his state's flagship university, Joe teaches a preservice class on secondary social studies methods.
Contexts

The contexts in which teachers work influence their practice (Meier, 1995; Newmann & Wehlage, 1995). Defining contexts broadly to include the community, the school, the students, and the curriculum, in this section I provide a brief orientation to the contexts in which Joe’s CPI discussion teaching practice is situated.

Community. The school in which Joe teaches is located in a university community outside of a major Western city in the United States. Joe describes his community as a middle to upper middle class college town, high expectations, high income, lots of scientists, privileged community, that is obsessive about the success of its own children. . . . For the majority of the parents in this community, or a very vocal segment of the community, traditional schooling worked for them, they perceive that it will work for their children, and they don’t want anything to get in the way of their children having the same privilege that they have. It is a community that prides itself on enlightened progressivism. (J.P. IV#1, p. 8)

Not all young people and parents in the community are drawn to schools that use traditional approaches. In fact, the school district has sponsored the development of several non-traditional schools, including the high school that Joe helped design and where he has taught for six years.

New Horizons High School. In 1992 Joe joined a design team working to develop a public high school of choice (similar to a magnet school) that opened as a "break the mold" high school in the fall of 1993. New Horizons High School was specifically designed to be "a place to experiment . . . to do
high school right. As we said, to take 25 years of research and try to implement it” (J.P. IV#1, p. 5).

Founding principles of the new school included using the community as a viable learning resource; valuing diversity, including not just race, gender, and ethnicity, but a vast spectrum of “other ways that kids are” (J.P. IV#1, p. 5); actively engaging students in their learning; teaching students to take responsibility for their learning; creating and fostering a climate of mutual respect; holding high expectations for all students; and personalizing education for students.

Many of the original plans for the school did not work; after much trial and error, the teachers “came into a very strong understanding of what it means to teach from your passions, and that authentic curriculum is different than active learning” (J.P. IV#1, p. 6). Using a shared governance structure that Joe characterizes as “real wonderful and cozy” (J.P. IV#1, p. 8), the school experienced few disputes among students, parents, and staff regarding its philosophy, in large part because students choose to attend New Horizon. Any high school student in the 540-square-mile school district is eligible to apply to attend the school.

During the 1997-98 school year, New Horizons High School had 350 students and 20 teachers. Joe, one of three social studies teachers in the school, chooses to work 70% time, teaching three courses per quarter.

Students. As a school that serves students in grades 9-12, New Horizons uses a multi-aged, mixed-ability, full-inclusion model. The school, by its very design, attracts students who want a non-traditional high school education. The promotional material for the school describes the kind of student well served by its unique approach:
Based on our experience, New Horizons High School works really well for students who: are willing to be partners with teachers, parents and other adults; are willing to be partners with other students; work hard when they are treated with respect and given autonomy; believe a school should be a community of learners; advocate for themselves and negotiate with others to solve problems; are ready for more responsibility for their own learning. (New Horizon Promotional Literature, 1997)

Curriculum. Undergirding the philosophy of the curriculum at New Horizons High School is student choice. There are no required classes, although students must successfully complete units in such areas as science, language arts, and social studies. All courses are one quarter in length. Students also create their own “Individual Student Paths,” which culminate in major projects completed for graduation in their senior year.

Joe has designed a number of nine-week elective courses for New Horizons students, including The Vietnam War, Protest and Reform, and a new course for the fall quarter of 1997 on landmark Supreme Court cases about the free speech and press clauses of the First Amendment. Class discussions in this Supreme Court course, along with Joe’s conceptions of a number of factors related to his use of CPI discussions, were the focus of this study.

Summary of Contexts. Joe has a rich and varied background in social studies teaching. As both a leader and learner in multiple professional development programs, Joe views his teaching as a work in progress. The community he both lives and teaches in is supportive of education, almost obsessive, and the unique school in which Joe does his daily work is based on
principles he helped to develop and implement. The students Joe teaches are
drawn to the school because of its non-traditional approach, a fact that causes
a higher than usual degree of agreement about the school’s philosophy and
curriculum. While these contexts inform and influence Joe’s teaching, it is
also important to understand his conceptions of a number of other factors
that may influence CPI discussion teaching.

Conceptions

Understanding the thinking that undergirds classroom practice
requires identification and analysis of teachers’ conceptions, defined as
teachers’ knowledge, beliefs, thoughts, and images (Shavelson & Stern, 1981)
that may inform classroom practice. In this section, I examine Joe’s
conceptions of a variety of factors that research suggests may be important to
CPI discussion teaching practice (see Chapter Two). These factors include the
purposes of social studies; the characteristics of effective democratic
citizenship; the rationales for CPI discussions; and the characteristics of
effective CPI discussions.

Conceptions of the Purpose of Social Studies. At the core of Joe’s
teaching is his belief that social studies is “interesting, engaging, and
important” (J.P. IV#1, p. 10). He tries to counteract what he considers the
typical outcome of social studies instruction, which is to “teach kids that
social studies is stupid, uninteresting, and has nothing to do with my life and
is to be avoided at all costs” (J.P. IV#1, p. 10). Drawing a concept map of what
he wants his students to know, be able to do, and be disposed to do as a result
of his social studies teaching, Joe identified four central outcomes: attitudes
about social studies, content and values, theory making, and citizen action.
Joe seeks to inculcate in students a positive attitude about social studies because he believes there is a strong connection between what students will learn and what they think is worth knowing and doing. Two ways Joe works to help his students develop this positive view are by providing them with alternative perspectives in his history courses and by linking what they are studying to present-day concerns. For example, Joe teaches a People’s History of the United States course using Howard Zinn’s book of the same title because “they’ve gotten plenty of the straight history in their lives, and they ought to hear the other side of that story” (J.P. IV#1, p. 11). While teaching a Civil War course, Joe engaged students in a discussion of whether it is ever right to assassinate a political leader, explicitly linking that question from the Civil War era to contemporary concerns about Saddam Hussein. By providing students with alternative views and connecting the past and present, Joe hopes to interest students in social studies and thereby enhance their ability to learn.

Joe does not believe there is a core of social studies content that all students should master. He thinks that the current standards movement, which attempts to identify core content, is misguided and unworkable, in large part because the standards sacrifice depth for breadth.

The selection of which “stuff” students will learn is driven by Joe’s conception of himself as a teacher of democracy in a democracy. “It’s my job to teach the concepts of citizenship and democracy. . . . In a democracy such as ours, it’s my job to teach the concept of justice” (J.P. IV#1, p.12). By teaching these concepts, Joe distinguishes between working toward students’ developing a general understanding of and appreciation for concepts such as justice, and specific views on controversial issues. Interchanging the words
"concepts" and "values," Joe says "those are values I will adhere to and be explicit about, though we'll argue about how they play out" (J.P. IV#1, p. 12). When discussing controversial issues that involve concepts such as democracy and justice, Joe will say "I don't have an answer, but it's my job to help you ask the questions" (J.P. IV#1, p. 12).

Joe's interest in helping students ask questions is driven by his view that one of the purposes of social studies is to involve students in the "theory-making business" (J.P. IV#1, p. 12). Drawing on constructivist learning theory, Joe believes that knowledge is created, not transmitted. Thus, when studying the First Amendment, Joe's students are encouraged to create their own theories that respond to two central questions: Why do we have the First Amendment? How absolute ought it be?

Educating young people for citizen action is another goal embedded in Joe's conception of the purposes of social studies. When asked what he would like to see one of his students doing ten years in the future, Joe responded, somewhat tongue in cheek, "burning the flag" (J.P. IV#1, p. 13). He then told a story about a recent conversation with the mother of a former student. Since graduating from college, the young woman has been a witness in Guatemala (accompanying indigenous people to protect them from government oppression) and is currently working as a political organizer. The mother attributed her daughter's citizen action to the influence of Joe and his wife, who is also a social studies teacher. Upon hearing this, both Joe and his wife "swelled up three sizes too big" (J.P. IV#1, p. 13). Joe cited numerous other examples of what he wants his students to learn about citizen action from social studies, all of them emphasizing personal agency and action.
Conceptions of Effective Democratic Citizenship. While drawing a concept map of effective citizenship, Joe included the following words and phrases: "engaged," "value this engagement," "ask questions," "know something (but not everything)," and "current events" (J.P. IV#1, ART). He used these words to describe the particular kind of democracy he values and to explain how the current democracy in the United States is lacking.

At the core of Joe’s conception of democratic citizenship is engagement stemming from personal agency. "Citizenship is about action . . . an effective citizen is someone who believes that he or she has agency of some sort" (J.P. IV#1, p. 16). Joe cites numerous examples of what political engagement might look like, such as standing up for the rights of others, running for office, and voting. He believes that each of these examples is indicative of a central belief held by effective citizens: that they have a voice.

One way that Joe thinks effective citizens should use their voice is by asking questions. Characterizing the ability to ask questions before acting as "the most important thing" that citizens should be able to do, Joe defines effective citizens as people who substitute "ready, ask a question, think about it" for the more typical tendency of citizens to engage in "ready, fire, aim" (J.P. IV#1, p. 14). The high premium Joe places on thoughtful questioning is rooted in his belief that most of the decisions citizens need to make are very difficult.

The questions Joe wants effective citizens to ask are built on content knowledge related to democracy, what he calls the fundamentals of democracy, defined as certain attitudes rooted in the Bill of Rights. For example, effective citizens should believe that "free speech is better than
restricted speech and the rights of the accused are more important than kangaroo courts” (J.P. IV#1, p. 15).

Knowledge of current events is another important facet of Joe’s conception of effective democratic citizenship because “citizenship is not about history, but it’s about the present and the future” (J.P. IV#1, p. 15). In addition to becoming critical consumers of the news, citizens should use that knowledge to engage in voting and partisan politics because “that’s the coin of the realm” (J.P. IV#1, p. 16).

**Conceptions of the Rationales for CPI Discussions.** Joe includes CPI discussions in virtually all of the classes he teaches. Why are CPI discussions so prominent in his social studies curriculum? To address this question, Joe engaged in a card-sorting task (see Chapter Three) designed to elicit his conceptions of the rationales for CPI discussions. Figure 6 illustrates Joe’s responses to a number of rationales typically cited in the literature on CPI discussions, along with additional rationales he created.
<table>
<thead>
<tr>
<th>Rationales for CPI Discussions</th>
<th>Do CPI discussions accomplish this goal?</th>
<th>Reasons why CPI discussions do/do not accomplish this goal:</th>
<th>Caveats and/or further explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Improve interpersonal skills</td>
<td>Yes, but only if the discussion is facilitated with ground rules and norms</td>
<td>Good CPI discussions teach interpersonal skills, bad discussions reinforce bad behaviors</td>
<td>Interpersonal skills are real, real important, but must be developed in the context of doing something</td>
</tr>
<tr>
<td>Understand important democratic values</td>
<td>Yes, but only if the discussion is good and appropriate</td>
<td>Because you bump own ideas against other people's ideas</td>
<td></td>
</tr>
<tr>
<td>Participate in political life of democracy (e.g., voting, serve on jury)</td>
<td>No</td>
<td>Discussion could increase cynicism</td>
<td>But, discussion is model for political participation defined as participating in discussion</td>
</tr>
<tr>
<td>Learn important social studies content</td>
<td>Yes, if focus is substantive and students are pushed</td>
<td>Depending on discussion model, social studies content is learned</td>
<td>Purpose of Seminar model is not based on content teaching, there's no official content, least important rationale</td>
</tr>
<tr>
<td>Learn how to think critically</td>
<td>Yes, assuming it is not in a critical thinking way</td>
<td>Because you bounce your ideas off others' ideas and interact with them in a thoughtful way</td>
<td>Discussion is one of the best ways to get students to think critically</td>
</tr>
<tr>
<td>Improve democracy</td>
<td>Yes, behaviors I expect in discussions are integral to how civic discourse should work</td>
<td>Because you have to listen and justify what you say, learn about democratic processes</td>
<td>This is the broad, overall purpose of CPI discussion</td>
</tr>
</tbody>
</table>

Figure 6. Rationales for CPI Discussions: Joe Park
(Note: Joe’s responses are from J.P. IV#2, pp. 1-8)
The primary reason Joe uses CPI discussions is that they provide the practice budding democrats need to "become part of the great conversations that take place in our society, and are taking place increasingly poorly" (J.P. IV#2, p. 8). Characterizing "great conversations" as exemplars of civic discourse, Joe explicitly links CPI discussions to preparation for effective citizenship in a democracy. He is not, however, using CPI discussions to prepare students to participate in the kind of civic discourse that currently occurs in the United States:

One of the things that drives me crazy is that what goes for political conversation in our society, on TV especially, and talk radio, is shouting matches. And has absolutely nothing to do with thoughtful dialogue and the complexities of issues . . . and that, I think, is antidemocratic. (J.P. IV#2, p. 4)

Joe teaches his students to participate in CPI discussions to counter this antidemocratic trend, in the hope of creating a better democracy. Thus, the rationales Joe uses to support CPI discussions focus more on developing specific kinds of thinking and process skills than specific social studies content.

Describing CPI discussions as "one of the best ways to get kids to think critically" (J.P. IV#2, p.4), Joe says this goal is realized if three factors are present: (1) students are pushed to justify their thinking, (2) they are presented with alternatives, and (3) there is interaction between the ideas of various discussants. CPI discussions promote more complex thinking (i.e., more critical thinking) if the discussion is structured to help students become more comfortable "accepting a lack of closure regarding issues and questions" (J.P. IV#2, p. 5).
Joe also includes CPI discussions as a way to help students develop process skills, such as the ability to both listen and talk well. Emphasizing the importance of practice to the development of these skills, Joe exclaims, "There has to be some sort of environment in society where kids practice doing that. They practice baseball batting, for God’s sakes, why can’t they practice talking?" (J.P. IV#2, p. 7).

Joe recognizes that CPI discussions don’t automatically achieve the outcomes he values. Throughout his discussion of rationales, Joe distinguished between outcomes achieved in good discussions and those achieved in ineffective discussions. This raises the question: What, in his view, is effective discussion?

**Conceptions of Effective Classroom Discussion.** Joe has developed a highly specific definition of the characteristics of effective classroom discussion. These characteristics emerged as we watched and Joe discussed two videotaped excerpts of high school students’ discussions in social studies. To Joe, effective discussions are based on the following factors: setting, climate, ownership, preparation, facilitation, and knowledge construction through interaction.

1. The setting promotes equality and interaction between students. Referring to the seating arrangement in the classroom as “geography,” Joe emphasizes the importance of this geography to the quality of discussion. “I actually believe you cannot, by definition, have a quality discussion when kids are looking at the back of people’s heads. And where their words are going in one direction” (J.P. IV#3, p.1). Arguing for seating students in a circle, or at least in semi-circles, Joe says the physical layout of the room should communicate the message “you ought to be talking” (J.P. IV#3, p. 6).
The circle is important because it is authentic to public discourse outside of school, whereas the artificial nature of rows in a classroom "is not the way society orients itself and organizes itself" (J.P. IV#3, p.1).

2. The climate encourages students to talk and listen. Many environments are too cold or intimidating for students to participate in discussions. Creating a climate in which the opposite is the case—students want to talk and listen to one another—is a key factor for Joe when evaluating whether a discussion has been effective.

3. Students own the discussion. When evaluating class discussion, Joe asks himself the question, "Whose discussion is this?" (J.P. IV#3, p. 2). He wants students to feel they have some control over what happens in the discussion. Joe does not believe that students need to select what is being discussed to have ownership; rather, students must know they have some control over and responsibility for what direction the discussion takes.

4. Students are prepared for the discussion. Effective discussions involve student preparation of common content. "I think the quality of the discussion really is about the degree to which kids are prepared to actually do some careful thinking about what's going on. Or at least share the common content in some way" (J.P. IV#3, p. 3).

5. If the teacher is facilitating the discussion, she/he does so actively to encourage knowledge construction based on probing and critical challenge. Joe does not believe that an effective discussion must have a facilitator; when a person takes that role, however, Joe favors relatively active facilitation. Through the use of prompts and challenges, teachers who are facilitating discussions can help students construct new knowledge.
6. Students interact with one another, extending ideas, questioning, and probing to create new meaning. Whereas some teachers believe that equality of participation is more important than what students say, Joe is concerned with interaction among students, and between students and the teacher, in order to both develop the critical thinking skills of individual students and to create new meaning derived from mixing ideas. Simply sharing ideas, without any transformation of those ideas, does not constitute effective discussion. Evaluating one of the videotaped discussions we viewed, Joe used a baking metaphor to express this belief:

   But, it's almost like . . . now, on the counter, she's [referring to the teacher on the tape] got yeast, and flour, and water, and honey, and mixing bowls, and all that kind of stuff. And the question is whether or not they're going to stay ingredients. You know, whether it's going to become bread that will get leavened up and become this really, you know, a whole bunch of fermentation of those ideas will go around.
   (J.P. IV#3, p.4)

Summary of Joe's Conceptions. How Joe conceptualizes what he wants students to learn from his social studies classes is remarkably congruent with what he thinks effective democratic citizens should know, be able to do, and be disposed to do. In both cases, Joe puts a premium on thoughtful participation, informed by an understanding of, and appreciation for, democratic values, such as justice. His conceptions of CPI discussions highlight an important link between teaching students how to participate effectively in discussions of important public issues and the necessity for thoughtful discussion of such issues in a functioning democracy. In defining the important characteristics of effective classroom discussion of CPIs, Joe is
most interested in interactions between participants that promote the development of new understandings and meanings.

Selecting Content for Discussion: Issues and Materials

Although some educators view discussion as primarily a content-neutral process, others (Miller & Singleton, 1997; Parker, 1995) emphasize the importance of carefully selecting the content to be discussed. Emphasis on content selection raises two questions: Who decides what content (i.e., issues) will be discussed? What criteria are used to select the content? Once an issue is selected, a third question is introduced: What materials will students interact with to prepare for CPI discussions?

Joe includes controversial public issues in many of the courses he teaches. Students are infrequently given a choice about the issue to be discussed because, “One of the responsibilities that an adult has in a classroom is to determine, amongst all the various contents and things, which ones are the most important to use.” (J.P. IV#6, p. 3). Given that Joe is the adult in the classroom who selects issues for discussion, what criteria does he use when making those decisions?

These criteria emerged as Joe explained how he selected the nine Supreme Court cases students discussed in the Important Supreme Court Cases class and as he sorted a list of controversial public topics into three piles: those he would include in his curriculum, those he wouldn’t, and those he was unsure about (see Chapter Three and Figure 7). Joe selects issues based on the extent to which they are about social justice, are relevant to students’ lives, focus on important content, and fit into his curriculum.
<table>
<thead>
<tr>
<th>CPI Topics</th>
<th>Joe’s decision</th>
<th>Joe’s reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abortion</td>
<td>rarely include</td>
<td>issue is too complicated and sophisticated</td>
</tr>
<tr>
<td>Affirmative Action</td>
<td>yes</td>
<td>connected to curriculum, human rights connection</td>
</tr>
<tr>
<td>Balanced Budget</td>
<td>no</td>
<td>phony issue, it’s about manipulation of numbers</td>
</tr>
<tr>
<td>Gay Rights</td>
<td>yes</td>
<td>teach as a civil rights issue, not a CPI</td>
</tr>
<tr>
<td>Immigration</td>
<td>yes</td>
<td>connected to curriculum, human rights connection</td>
</tr>
<tr>
<td>Legalizing Drugs</td>
<td>no/maybe</td>
<td>plays into students’ culture too much</td>
</tr>
<tr>
<td>Physician-assisted suicide</td>
<td>no</td>
<td>doesn’t fit into his curriculum</td>
</tr>
<tr>
<td>Trade policy</td>
<td>yes</td>
<td>connected to curriculum, human rights connection</td>
</tr>
<tr>
<td>Vouchers for private schools</td>
<td>yes</td>
<td>connected to curriculum</td>
</tr>
<tr>
<td>Welfare Reform</td>
<td>yes</td>
<td>connected to curriculum, human rights connection</td>
</tr>
</tbody>
</table>

Figure 7. Joe’s Decisions and Reasons About Which CPIs To Have Students Discuss

Social Justice. In selecting issues for discussion, Joe favors issues that call for the analysis of how the “words of our democracy play out against our . . . actions. You know what we say we believe versus what everybody is so proud of on the Statue of Liberty, versus what’s in Proposition 209” (J.P. IV#6, p. 9). Defining social justice largely as a quest for equal rights, Joe distinguishes between the goal of equal rights, which is not a controversial issue in his schema, and specific questions about the means to achieve that goal, which are controversial issues. For example, whether minorities should have equal rights is not a controversy, but whether affirmative action is an appropriate way to achieve that goal is a controversial issue.
Relevance to Students' Lives. A second selection criterion Joe employs is relevance to the lives of his students. In the Important Supreme Court Cases course, Joe included three cases that dealt with students' rights in schools. Experience with these cases had taught Joe that students found them particularly interesting because they were "related to their lives; the differences of being kids in school and what those rights are, versus being adults out, or even kids, out in the community and what those are" (J.P. IV#6, p. 2). Joe's interest in selecting issues that are relevant to his students is tempered, however, by concerns about "playing into their culture too much" (J.P. IV#6, p. 9). Explaining why he placed legalizing drugs in the "maybe" pile, Joe said "It's too high school... I've got limited amounts of time to work with kids. It's not the issue that I want to deal with them. It's sort of playing into their culture too much, in some ways" (J.P. IV#6, p. 9).

Focus on Important Content. Although Joe opposes the notion of a core curriculum, he does think that some issues involve content that is especially important, and selects issues with that criterion in mind. For example, in the Supreme Court course students studied a famous 1948 case, Terminello v. Chicago, that broadened the First Amendment rights of speakers whose words spark an angry crowd. Joe selected this case because it is a "pretty important foundational free speech case" (J.P. IV#6, p. 2). Joe refers to others cases as "classics" by way of explaining why they were selected: "I don't think a kid should walk out of school without knowing something about Tinker v. Des Moines School District. You know, in terms of civic education" (J.P. IV#6, p. 3). Joe's concern with externally imposed content standards influences how he explains what makes content important. Consider the following exchange, which followed Joe's explanation of why
the *Terminello* and *Tinker* cases were important for students to learn: "Hess: So, they're kind of core content? Joe: You wouldn't get me to say that. [Both laugh]" (J.P. IV#6, p. 3).

Another determinant of whether issues are important is whether they are currently matters of public debate. For example, Joe selected the flag burning case *Texas v. Johnson* because Congress is currently considering an amendment to the United States Constitution that would effectively overturn the Court's decision that flag burning was a protected form of free speech. Labeling this "a good hot case," and another on censoring the Internet (*Reno v. ACLU*) a "good recent case" that had "been in the press" (J.P. IV#6, p. 3), Joe communicated that issues can be judged important because they are currently matters of public concern.

**Curricular Fit.** Joe selects issues for his students to discuss with an eye toward curricular fit. All things being equal, Joe seeks issues that are directly related to the topics of the courses he teaches. Given that Joe's high school has no required social studies classes and that he has tremendous flexibility in creating courses to offer as electives, this selection criterion is probably less constraining for Joe than for teachers in other contexts.

**Selecting Materials.** In the Supreme Court cases course that was the primary focus of this study, Joe assigned the texts of entire Supreme Court cases as the materials for classroom discussion. Even though Joe believes more than half of the students in the class are reading below grade level, and Supreme Court decisions are written in complex language, he says, "If you give kids real stuff, and treat them as real learners, they will do amazing things" (J.P. IV#6, p. 4). By scaffolding instruction on the difficult materials, Joe seeks to provide all students with access to the "high stats [status]
curriculum that's given to AP [Advanced Placement] and IB [International Baccalaureate] students because [mockingly] 'they can handle it' . . . everybody can handle that if they're led through it and there's support for it" (J.P. IV#6, p. 5).

   Materials selection is informed by Joe's conception of democracy and the connection between democracy and schooling. He is most interested in providing difficult and authentic materials to students to enhance equality.
   
I really want to emphasize that, if we live in a democracy, not only the brightest three percent should be reading Supreme Court cases and understanding what the nature of the Supreme Court is, it seems inherently anti-democratic to me. And what do we do to those other kids? We give them worksheets on, there are a blank number of justices on the Supreme Court. (J.P. IV#6, p. 6)

   Summary of Joe's Approach to Selecting Content. Joe believes that the teacher is responsible for selecting which issues should be discussed in the classroom. He selects issues based on the extent to which they are about social justice, are relevant to students' lives, focus on important content, and fit into his curriculum. Joe selects materials for student use that are difficult and authentic.

Snapshot of a Seminar

   During the 1997-98 school year, Joe taught a nine-week course focusing on historically significant controversial public issues related to freedom of speech and press. Joe designed the course to rely on Supreme Court cases because "you would be hard pressed to find a more authentic text than a Supreme Court decision" (J.P. IV#4, p.3). Although all nine of the cases the students read are about the First Amendment's speech and press clauses, Joe
also hopes his students will gain a general understanding of content that extends beyond this amendment. On the first day of class he said to his students: “I want to grow old in a society that has many people understanding the way the Constitution and the Supreme Court works” (AT#1).

The 24 students enrolled in the course met three times per week, for a weekly total of four hours. During most weeks the grade 9-12 students read one First Amendment Supreme Court case, prepared to participate in a seminar discussion on the case by completing a pre-discussion assignment called a “ticket,” worked in small groups to review the facts of the case, participated in a seminar, and wrote an issues-analysis paper.

The seminar model of discussion that Joe used in the Supreme Court course is pervasive throughout New Horizons High School. Joe learned the model from the school’s principal (who participates in the seminars as a model participant and, on occasion, as the facilitator) and has been using it for several years. The model, labeled simply “seminars” at the school, is text-based large group discussion designed to help participants develop a deeper understanding of the issues, ideas, and values in the text (Gray, 1989). Joe favors the model because of its potential to enhance critical thinking and the generation of new ideas.

Preparing for the Seminar. This day’s seminar focuses on the Supreme Court’s decision in *New York Times Co. v. United States*, the famous “Pentagon Papers” case decided in 1971. This case focuses on the tension between freedom of the press and national security. Joe’s students read the 50 pages of the case and completed a ticket (a pre-seminar assignment) to participate in the seminar discussion. The ticket for this case required students to create and complete a data retrieval chart that identified the basic
arguments made by each of the justices in the nine separate opinions issued in the case. Joe's ticket assignments require students to read and interact with the text. Joe does not expect, however, that the ticket will cause students to understand the text: that is the purpose of the seminar discussion. The day before the seminar, the students work in small groups to figure out the basic facts of the case and how it moved through various courts to be heard by the United States Supreme Court.

As the students enter the classroom on seminar day, Joe checks whether their tickets are completed. Graded as a pass if completed and fail if not completed, the tickets also determine who may participate in the discussion. Students without a ticket are not allowed to sit in the circle, even if they say they have done the reading. Instead, they are assigned an observer role and must sit outside the circle and take notes on who is participating. Joe encourages students to complete the ticket by saying, "It's preferable to sit in the circle and choose not to participate, rather than be on the outside and not be able to participate" (J.P. IV#4, p. 6). Requiring students to complete the ticket is one way that Joe attempts to deal with the difficulties involved with talking across difference:

The only thing that we know we have in common in a seminar is the text that we share in common. We've been raised differently. We have studied different materials in this class. We may have had U.S. History classes, others have not had U.S. History classes. All sorts of things. But what we do know is that we all have the text in common. A good discussion, a good seminar, begins from the premise that we are talking about a shared text. (J.P. IV#4, p.7)
Setting and Maintaining Guidelines for the Seminar. As the two-hour class period begins, Joe, 19 students, and the school’s principal (acting as a model seminar participant) are seated in the seminar circle. One student who didn’t complete the ticket is creating a list of participants, with check marks for each time they talk. Before the seminar begins, Joe reminds the students to work hard and to “do the work of the seminar” (AT#9). By that Joe means adhering to the guidelines created by the students at the beginning of the course; these guidelines are now posted on butcher paper on the classroom wall. Some of the guidelines written on the poster are “listen, respond to ideas out there, make the agenda yours, and refer to the text” (ART#1). In a later interview, Joe explained what he meant by that exhortation to his students: “I think it was a phrase they all understood . . . that doing the work of the seminar is using the behaviors [required in seminars]; is working hard with the text; it’s living with ambiguity” (J.P. IV#5, p.1).

Focus Questions to Begin the Seminar. Joe begins the discussion with a focus question: “What was the most compelling argument in the case?” (AT IV#9). Joe has developed this focus question using specific criteria: it cannot be answered without using the text; it is open-ended in that there is no right or wrong answer; and it is a question about which he, as the seminar facilitator, has some genuine curiosity.

A student immediately responds to Joe’s focus question by changing the question. “Well, I can tell you the least compelling argument” (AT#9). The student then points the class to a part of Chief Justice Burger’s dissenting opinion that laments the short amount of time the Court had to spend on the case and says, “he is just whining here” (AT#9). Later, I asked Joe why he didn’t direct the student to stick with the question that was asked. Joe
responded, “That’s a no brainer. Just because I asked a question, doesn’t mean that I asked the right question. . . . Just because I was fishing for trout doesn’t mean that I’m going to ignore the bass that bites” (J.P. IV#5, p.3). Moreover, Joe believed the student’s response accomplished the primary purpose of his focus question, to open a door to the text in a way that will focus students on the reasoning of the justices.

**Referring to the Text.** None of the other students comment on Burger’s reasoning; after a short pause, several chime in and say that Justices Douglas and Black had particularly compelling reasons supporting their opinions. Joe asks the students to find where the Douglas opinion begins, and they turn to a specific page in the text. Joe immediately probes with a question to one of the students who liked the reasons of Justice Douglas, “Betty, what was your sense of what Douglas was arguing?” She responds by paraphrasing the position Douglas takes in his opinion. Joe follows up by labeling Douglas’s reasoning, “So he was a First Amendment absolutist?” Students agree and Joe follows up again, “Talk to us more about Douglas’s arguments.” Another student responds with an elaborated description of why he finds the arguments compelling. This type of interchange continues for several minutes. Students refer to the text and talk about the basic tenets of the two First Amendment absolutists. During the opening several minutes of this seminar, Joe asks quite a few questions, continually reminding students to find a specific part of the text they are talking about.

**Taking Minority Views Seriously.** During the seminar, it becomes apparent that most of the students support the opinions of the court majority, which held that publishing the Pentagon Papers was protected by the First Amendment. One student, however, takes the contrary position. Joe then
says to the class, "There's our lone conservative, this time. We actually don't have to support Logan, but let's . . . pretend to do so for a minute. Okay? Let's try to construct and give credence to the argument of the government in this case" (AT#9). For several minutes the seminar continues with students identifying parts of the dissenting opinions that represent the view of the government and explaining what they think those arguments mean. Later, Joe explained why he refocused the seminar on the arguments that did not have the support of most of the students.

I think a real important critical thinking skill is the ability to take a different position and to argue it with credence and credibility. I think it's an incredible skill for citizens, for enlightened citizens in a democracy, because it's rare that issues are completely black and white. It's important to give minority voices a really serious airing in a classroom. Because then people will give their true opinion. I think it's also real important to have kids take on different viewpoints as a way of better understanding their own viewpoints. So related, back to the earlier question, that's about doing the work of seminars. Doing the work of seminars is trying on ideas. (J.P. IV#5, p. 5)

Investigating the Meaning of Words. Several times in the seminar students do not understand the meaning of words in the text. The first time this occurs, Joe says, "Let's look it up; here's the dictionary" (AT#9). A student then looks up the word and reads the definition to the class. Later, when another word is not understood by several students, a student says, "I already looked this up last night; it means . . ." (AT#9). Throughout the seminar, it is apparent that students believe that the meanings of words matter; they are willing to stop the seminar to make sure they understand a word.
Moving to Moral Issues. The Pentagon Papers were stolen government documents, a fact that becomes the focus of conversation toward the end of the seminar. Joe asks the students, "So, what should the New York Times have done when Daniel Ellsberg came to them with boxes of stolen government documents? If Logan steals a TV and gives it to me, and I know that he stole the TV, have I done something wrong?" Several students exclaim, "Yes." Joe asks, "Is that the same thing as the New York Times did with the documents?" A student replies, "They didn't know." Another counters, "Oh yes, they knew." A third says, "But they thought the public had a right to know." Joe now takes them back to the text: "Doesn't one of the justices say something to the effect that there is this right to know right now and the New York Times feels a responsibility to provide that information? Who said that?" After a few seconds of looking, someone shouts, "page 749," and Joe then reads an excerpt from that page. Joe then says, "You guys, most of you believe, that what the Supreme Court did was right in this case." Several students say, "Yes." Joe continues, "Did the NYT do the right thing?" A student responds, "In my opinion, it's just a matter of your opinion, more important that the public know - they did what they needed to do and I agree with them." Another student says, "I agree, it's like this pull - they were publishing stolen documents which was basically not the right thing to do, but yet it was important to let the public know what the government was doing. I have a question, did anything happen to the New York Times as a result of this?" Joe answers, "The New York Times was fine, Daniel Ellsberg was tried for taking the Pentagon Papers--do you want to know now or later what happened to Daniel Ellsberg?" One student says, "Now, right now, Joe." Another jokingly adds, "We have a right to know."
This excerpt from the seminar illustrates a move frequently made by Joe to use the text to spark discussion of larger moral questions, in this instance, Is it ever right to steal? The text still reigns supreme, however, as can be seen in Joe's reference back to the text.

**Seminar Participation.** During the hour-long seminar, the student observer counts 150 different contributions, 104 (or 70%) made by seminar participants and 46 (or 30%) by Joe. Of the 19 students in the seminar circle, 13 verbally participated. However, 20 of the 104 statements were from the principal. Comparing this seminar to the eight others in this class, the overall participation numbers are fairly constant. Joe talks quite a bit, although most of his participation is in the form of questions to the students.

**Student Critiques of the Seminar.** While students are not required to participate orally in the seminar discussion, they are required to share their critique of the discussion during a debriefing period held immediately after the seminar ends. This particular seminar was the final one in the nine-week class. In a celebratory manner, Joe begins the debriefing session with the statement, "Give yourselves a round of applause, you guys got this thing" (AT#9). Following enthusiastic applause, one student exclaims, "I was terrified when I first saw it" (AT#9). Joe then says, "Regarding this seminar, I would like to know what your sense of this seminar was as it compared to others and on its own merits" (AT#9). A student volunteers:

I'll start... I just thought this was a really comfortable seminar, not a lot of people talked, but those people who did really knew what their ideas were about the case, and that helped me, a person who didn't understand it a whole lot, to get a better sense of it all. I enjoyed the
relaxed energy of it because it made it a lot more easy to get into.

(AT#9)

Although many students agree the seminar had a relaxed pace, views about the text differ. Some students liked the text, but a few others said it was confusing or worse. One student plainly states, "This text sucks" (AT#9). Another student critiques her own participation in the seminar:

I finally completed my goal, which was to not talk during the seminar. I kept wanting to talk because I think this case was very confusing, but the seminar cleared it up. But I thought it was pretty good, but it is kind of weird trying not to talk, I think I listen more when I am talking because I listen in order to respond. (AT#9)

Throughout the critique, Joe says very little. But, when his turn comes as the critique moves around the circle, he says: "... the coolest thing about this seminar was the opportunity to read this case because I have known about this case for a very long time. I found the ticket helped me a whole lot in terms of organizing nine separate opinions" (AT#9).

Joe described the purpose of the seminar critique as "feedback loop for all of us" (J.P. IV#5, p. 14) and a way to enhance the students' abilities to be reflective:

It's a whole metacognitive process that I think is incredibly powerful for kids. It also, again, goes to that notion of power and voice and the politics of the classroom that I value by holding a space for them to reflect on what's going on--which says . . . whether or not it's my classroom or their classroom. My conversation or their conversation. And this is very, very concrete evidence that it's our conversation. (J.P. IV#5, p. 15)
Joe's Views on Assessment of Seminar Participation. Joe distinguishes between informally assessing the quality of students' participation in seminar discussions and formally assessing, or grading, that participation. He supports the former and is resolute about not doing the later. While Joe does provide each student with oral and written feedback on their seminar discussion skills, he does not factor in their participation in seminars into their course grades. Joe believes the authenticity of the seminar would be harmed if students were being graded on their verbal participation. The unique nature of seminars as discussions aimed to collaboratively create meaning makes the grading of individual's verbal participation problematic for Joe:

Seminars are about public performance[,] . . . are explicitly about collaborative work. Seminars are about coming together in a public space to interact with people who are and are not like you. And that is freighted with different things, not less important, but different things than individual writing assignments [which Joe does grade]. Not the least of which is the public performance aspect of it. And the fact that we're trying to make meaning together. If I want to make meaning together, I want only the contribution of authentic ingredients. (J.P. IV#5, p. 4)

Another reason for Joe's refusal to grade seminars is his belief that students participate in seminars in various ways. "You see, I like it when kids speak in seminars. I've learned to get just as big a kick out of kids who say, 'I really, God I was there the whole time. I just didn't have anything to say because I was listening so intently and trying to figure stuff out.' So I value being in seminars in a variety of ways" (J.P. IV#5, p. 7). Joe believes it is impossible to
create an assessment rubric for seminar participation that would honor the various ways that students participate in the discussions.

**Summary of the Seminar.** The seminar discussion on *New York Times Co. v. United States* typifies how seminar discussions work in Joe’s classes. Before a seminar, students read a text, complete a ticket, and work in small groups to become more familiar with the basic facts of the case. On seminar day, guidelines for seminar behavior are reviewed, and Joe then begins the seminar with a focus question. Although there is often disagreement about what the text means and what its implications are, the conversation is rarely heated or adversarial. Instead, the pace is relaxed and the tone is highly civil. Frequent references to the meaning of specific words and to the text pepper the conversation, although both Joe and the students use non-text metaphors and analogies. Each seminar ends with a critique in which the verbal participation of all students is required. The critique focuses not on the issues or ideas in the text, but on students’ reactions to the text and their opinions about the quality of the seminar. While Joe informally assesses his students’ participation in the seminars, he believes that grading students’ verbal participation would decrease the authenticity of the seminars. After most seminars (but not this one because it was the end of the class), students are required to write a paper about the issues in the text.

The next chapter of the dissertation interprets the conceptions and practices of Joe and the other two teachers who participated in the study by presenting an initial theory that describes and explains seven propositions. Before this theory is presented, however, portraits of the two other teachers’ CPI discussion conceptions and practices are described and explained.
Portrait Two: Elizabeth Hunt at Summit Ridge Middle School

Elizabeth Hunt always knew she wanted to be a teacher: "I can remember even when I was in third grade . . . having an excellent teacher and thinking, 'Oh, what fun this would be'" (E.H. IV#1, p. 4). Elizabeth now teaches social studies to eighth-graders at Summit Ridge Middle School, located in a suburb of a major city in the West. Elizabeth has taught for seventeen years. She taught language arts for twelve years but switched to social studies five years ago for the challenge of teaching different content.

Elizabeth earned a B.A. in Middle School Education at a university that has one of the nation’s largest teacher certification programs. At the time Elizabeth attended the university, it was one of only two in the country that offered a special certification in middle school education. Within that program, Elizabeth specialized in social studies and language arts.

Since receiving her B.A., Elizabeth has taken numerous graduate-level courses in language arts and social studies. The past several summers she has attended institutes focusing on civic education. As part of the 1996 institute, she took a course about teaching with discussion that I taught.

Since the 1996 institute, Elizabeth has participated in a study group with other middle school social studies teachers. The study group gathered once a month to discuss how to improve social studies teaching and learning. Facilitated by social studies experts from a nationally recognized social studies think tank, the study group meetings initially focused on improving classroom discussion. For each study group meeting the teachers brought representations of their students' experiences in classroom discussions (such as videotapes) and using a "show and ask" format, solicited feedback from their peers. During the past year, the focus has broadened beyond classroom
discussion to encompass a range of issues related to teaching social studies. For example, several recent meetings centered on how to teach such important social studies concepts as federalism and executive privilege.

### Contexts

Similar to the CPI discussion conceptions and practice of Joe Park, Elizabeth Hunt’s work is informed by the various contexts in which it is situated. Her community, the school, her students, and the curriculum all shape how Elizabeth approaches teaching her students to participate more effectively in CPI discussions.

#### Community

The community in which Elizabeth lives and teaches is typical of the suburban sprawl found outside many major cities in the United States. Primarily residential, it is a “middle to upper middle class, predominantly white community” (E.H. IV#1, p. 5). Many adults in the community work in a nearby technology center, which is often labeled the “new downtown” (E.H. IV#1, p. 5) to distinguish it from the downtown of the major city it borders. Elizabeth’s community is a new one—still working to create its traditions and institutions. One institution that both reflects and creates the community is the schools. Many people, in fact, moved to this community because of the excellent reputation of the school district in which it resides. Parents in the community are supportive of education, and want to be involved in the schools. One way adults show their support is to vote for increased funding to build new schools. Summit Ridge Middle School is only six years old, and new middle and high schools are under construction.

#### Summit Ridge Middle School

Elizabeth’s school is huge, with more than 1600 students in grades 6-8. The wealth of the community is immediately apparent when touring the school. Its library, for example,
overflows with computers, expensive research materials, and new books. Students are placed in three separate "communities" organized around the classic middle school team concept. The "communities" operate like schools within a school, each with a separate administrative team and teaching staff. Students stay in the same community from sixth through eighth grade. Teachers work in teams to facilitate communication about individual students and to coordinate the curriculum. Additionally, subject matter teachers communicate across the teams to ensure some commonality in the curriculum.

**Students.** Summit Ridge Middle School is a neighborhood school and, as such, is filled with students who run the gamut in terms of interests, capabilities, and goals for the future. While the overall student population is not very diverse in terms of race (approximately 90% of the students are white), it is diverse in other ways. For example, as the school district's designated school for students with multiple physical disabilities, some students are dealing with serious physical difficulties. Other students have learning and behavioral challenges. For example, Elizabeth has students in her classes who read significantly below grade level, in some instances as low as the first or second grade levels. In this school that follows a special education inclusion model, some of the students in Elizabeth's classes have individualized education plans (IEPs) and receive supplemental services from special educators.

Most of the students in Elizabeth's classes, however, are not dealing with serious physical or learning challenges. Quite the contrary. They are reading at or above grade level and behave in age-appropriate ways. Moreover, the students seem to get along with one another. Walking
through the hallways, one hears few taunts or bursts of anger. Instead, the warm atmosphere seems to reflect the many posters on the hall walls encouraging students to respect one another, take learning seriously, and celebrate diversity.

**Curriculum.** The required course that all eighth-graders at Summit Ridge Middle School take is officially labeled “American Studies,” and focuses on United States history and civics. Using the state’s civics standards for direction, teachers from the five middle schools in the district recently developed a common curriculum for the eighth-grade course that mandates beginning with a unit on the American Revolution, then spending more than half of the school year on an in-depth study of the Constitution and civics. Within each school, the curriculum varies after the Constitution and civics unit is completed. At Elizabeth’s school, the eighth-grade teachers in the three communities fill out the curriculum with the study of political parties, Westward expansion, and the beginning of the Civil War. Thus, unlike Joe, who has total control over the content of the courses he teaches, Elizabeth teaches a common curriculum that she has developed with other teachers in her school and district.

**Summary of Contexts.** Elizabeth is an experienced middle school teacher who works hard to continually improve her practice. Through taking courses and participating in a unique study group, she demonstrates a commitment to reflective practice. The young and growing community Elizabeth teaches in values education, financing the building of new schools and purchasing expensive educational resources. Elizabeth teaches in a new school, a large middle school organized into three separate communities. As the eighth-grade social studies teacher in one of the school’s communities,
Elizabeth teaches a district-wide curriculum she helped to develop that focuses on 18th- and 19th-century United States history and civics. While these various contexts influence how Elizabeth teaches her students to participate effectively in CPI discussions, her conceptions of a number of factors also inform that practice.

Conceptions

Recall, in this study I am defining conceptions to include a teacher’s knowledge, beliefs, thoughts, and images that may inform classroom practice (Shavelson & Stern, 1981). In this section, I focus on Elizabeth’s conceptions of four factors: the purposes of social studies; the characteristics of effective democratic citizenship; the rationales for CPI discussions; and the characteristics of effective CPI discussions.

Conceptions of the Purposes of Social Studies. When drawing a concept map to illustrate her thinking about the purposes of social studies, Elizabeth created a wheel, with “participatory citizens” (E.H., ART) in the center. This visual display represented the primacy she attaches to the citizenship goal: “I’m hoping that I’m teaching students to be effective citizens” (E.H. IV#1, p. 18). Toward that goal, Elizabeth identified six outcomes of social studies: knowledge of the development, structure, and function of the United States government; understanding of how historic events and policies have shaped the United States; informed about domestic and foreign events and relations through current events; civic virtue; civic discourse; and preparation for future voting. In combination, Elizabeth conceptualizes these six outcomes as leading to a greater likelihood that her students will be effective citizens.

The first two are knowledge goals, one focusing on civics (structure and function of the government), the other on history. Elizabeth sees a
connection between understanding the development of constitutionalism in the United States and future citizenship participation: "I think if they have good background knowledge about how our government works they may be more likely to participate in it" (E.H. IV#1, p. 12). In particular, she wants her students to know how to effect change and that "they do count" (E.H. IV#1, p. 13).

Important historical events and policies are also crucial for students to learn, especially if "it's something that is still ongoing" (E.H. IV#1, p. 13). For example, Elizabeth teaches a unit on "Turning Points of Equality" that focuses on significant milestones related to equality in United States history. Defining equality broadly, Elizabeth explains the rationale for the unit: "I think when we talk about civil rights, a lot of people think we talk just about the black minority. But I think that there are other minorities in our country that are still striving for equality. ... I put a big emphasis on that because I want them to be aware ... that we're all struggling for civil rights" (E.H. IV#1, p. 13).

Elizabeth's emphasis on teaching about contemporary domestic and foreign relations issues through current events is aimed at helping her students develop a habit that she hopes they will carry into adult life: to "want to read the newspaper and watch CNN, other news programs--to be informed" (E.H. IV#1, p.11).

After arming her students with knowledge about government structure, and historic and contemporary events, Elizabeth wants them to demonstrate civic virtue, defined as "putting the good of the community ahead of their own individual needs" (E.H. IV#1, p. 12). Demonstrating civic virtue occurs through volunteerism; "It can be at a very simple, basic level, where you are involved in a neighborhood association in your community to
keep it a nice place or be involved in your church” (E.H. IV#1, p.12).

Regardless of venue, the purpose of exercising civic virtue is the same: to
“help our nation be a better place” (E.H. IV#1, p.12).

Two additional purposes that Elizabeth holds for teaching social studies are to teach students the skills to engage in civic discourse and to prepare
them for their future roles as voters, both of which she detailed when
describing what she thinks effective citizens in a democracy should know, be
able to do, and be disposed to do.

**Conceptions of Effective Democratic Citizenship.** Elizabeth’s thinking
about effective democratic citizenship is remarkably similar to how she
conceptualizes the purposes of social studies. Effective citizens must possess
background knowledge about how the government works and believe that a
common good should take precedence over their individual needs.
Moreover, such citizens should engage in certain activities, including public
discourse, voting, local political activity, and volunteerism in the civil sector.

Elizabeth cited many reasons for supporting the inclusion of public
discourse in her definition of effective democratic citizenship. Engaging in
public talk with others is a way to become informed and to communicate that
you are informed. It is a way to both learn about issues and influence the
opinions of other people. Public discourse also has social utility: “I think in
social situations, lots of people talk about current events, about issues that
matter to them, and it’s fun to be able to participate in those kinds of
discussions” (E.H. IV#1, p. 16). While many of the other behaviors embodied
in Elizabeth’s conception of effective democratic citizenship are done
occasionally, public discourse is a daily activity. After reading the newspaper
or listening to the news, effective citizens, Elizabeth believes, should talk about what they have learned:

When things come up on the news or in the newspaper that you’re really curious about, are really interested in, or that make you really angry, you talk about it . . . and when you have those discussions, I think it helps you to flesh out your opinions. And if you have to defend your opinion with things that you’ve read or things that you’ve heard, I think it helps you to understand the issue better yourself. (E.H. IV#1, p. 17)

Although Elizabeth urges daily public discourse, she also thinks that effective citizens should engage in activities that communicate their views to a broader audience, including policy-makers. Citing a recent example from her own life, Elizabeth explained how postcards were distributed at her church so the congregants could write messages to legislators urging a ban on partial birth abortions. Although the proposed abortion legislation was federal, Elizabeth believes citizens have more influence at the local level because the smaller numbers make it more likely that citizens’ concerns will be heard. Regardless of the level of government that one attempts to influence, Elizabeth places a primacy on voting. Responding to my question about the oft-stated view that voting does not matter, Elizabeth said, “Well, there have been elections in our nation’s history where the votes were very close. And I think that if everybody had that attitude, that one vote doesn’t matter, then it would make a huge difference. I think that voting does matter” (E.H. IV#1, p. 19).

Effective citizens, however, should not limit their activities to influencing political issues. Echoing political theorists who argue for the need
to re-establish a stronger civil society in the United States, Elizabeth said that people need to participate in their community in order to feel a connection to the community. By joining a local homeowner’s association, involvement in local church activities, or similar activities, effective citizens work to build a healthier civil society.

Conceptions of the Rationales for CPI Discussions. Throughout the school year, Elizabeth’s students participate in at least one full-blown CPI discussion each month. Given that preparation for the discussions can take a few class periods, and the discussion and debrief a few more, this is a major commitment of instructional time. Elizabeth has numerous rationales to support this emphasis on CPI discussions, including democratic citizenship goals, critical thinking, interpersonal skills, and better understanding of issues (see Figure 8).
<table>
<thead>
<tr>
<th>Rationales for CPI Discussions</th>
<th>Do CPI discussions accomplish this goal?</th>
<th>Reasons why CPI discussions do/do not accomplish this goal:</th>
<th>Caveats and/or further explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Improve interpersonal skills</td>
<td>Yes</td>
<td>Students get practice exercising key skills, such as listening and yielding</td>
<td>This is one of the most important reasons Elizabeth uses CPI discussions</td>
</tr>
<tr>
<td>Understand important democratic values</td>
<td>To some extent</td>
<td>If a CPI is value-laden, then students will learn about it through discussion</td>
<td>Some CPIs (but not all) emphasize conflicts between values</td>
</tr>
<tr>
<td>Participate in political life of democracy (e.g., voting, serve on jury)</td>
<td>Perhaps in the long term</td>
<td>Because students want to participate in discussion, they watch the news and read the paper</td>
<td>Broaden the definition of participation to include being informed</td>
</tr>
<tr>
<td>Learn important social studies content</td>
<td>Yes</td>
<td>Issues are chosen that link to important content</td>
<td>Especially helpful to students with learning difficulties</td>
</tr>
<tr>
<td>Learn how to think critically</td>
<td>Yes</td>
<td>Students are exposed to the ideas of others; this stretches their brains</td>
<td>Improving critical thinking skills is a major rationale for Elizabeth</td>
</tr>
<tr>
<td>Improve democracy</td>
<td>Yes</td>
<td>Talking helps democracy--exposes people to multiple perspectives</td>
<td>Discussion can cause action, also function as a democratic safety valve</td>
</tr>
</tbody>
</table>

Figure 8. Rationales for CPI Discussions: Elizabeth Hunt
(Note: Elizabeth’s responses are from E.H. IV#3, pp. 1-14.)

When asked to rank rationales for including CPI discussions in her curriculum (see Chapter Three), Elizabeth placed the connection between such discussions and a healthier democracy at the top of the list. Her primary explanation of this choice is rooted in the First Amendment: “The First
Amendment is what allows us to have, or helps us maintain a healthy democracy; that people are able to talk about it, are able to criticize the government, are able to get their feelings out” (E.H. IV#3, p. 8). By providing her students instruction and practice in how to participate in public discourse about CPIs, Elizabeth believes she is helping them learn how to exercise their First Amendment right to speak. While Elizabeth distinguishes speech from action, she does think that speech can cause action: “When people can talk about these things [CPIs], it can initiate action” (E.H. IV#3, p.8).

The other rationales Elizabeth holds for CPI discussions are more clearly rooted in the unique needs of middle school students. Because students of this age are beginning to think on a more abstract level, Elizabeth believes CPI discussions can improve their critical thinking skills. Asked to define critical thinking skills, Elizabeth responded: “Stretching their brain beyond where they would if it was just left to them to do themselves, looking at Bloom’s Taxonomy and reaching for the higher levels like synthesizing and evaluating. I think that public issues discussions really foster those skills” (E.H. IV#3, p.9). The key element in CPI discussions that enhances critical thinking skills is that students hear and are challenged by viewpoints different from their own.

Another rationale Elizabeth holds for CPI discussions, improving interpersonal skills, also stems from the needs of middle school-age students. Heading a long list of specific interpersonal skills that students need to develop is “simple manners,” defined, at the minimum, as calling other discussants by their names. The other skills are more complex and difficult to develop, including teaching students to yield the floor, not dominate the discussion, and listen carefully to one another.
Finally, Elizabeth includes CPI discussions in her curriculum because she believes they are a useful tool for helping students, especially students with learning difficulties, better understand social studies content. Even if students with learning difficulties only participate by listening, Elizabeth thinks the discussions help them learn more: “A lot of times they struggle with the content of what we’re studying. But the discussions really help them to understand it better” (E.H. IV#3, p. 12).

Conceptions of Effective Classroom Discussion. As Elizabeth and I viewed two videotaped excerpts of classroom discussions (see Chapter Three), it was immediately apparent that she, like Joe Park, had well-formed ideas about the characteristics of effective classroom discussion. These characteristics are normative beliefs about what Elizabeth thinks should be present in the following categories: seating, interpersonal skills, facilitator’s role, critical thinking, interaction, preparation, and the role of prior knowledge. Through brief descriptions of Elizabeth’s conceptions of each of these categories, an overall picture of what she values in class discussions will emerge.

1. Seating. Where students are seated in relation to one another and to the teacher is a cue to Elizabeth about the kind of discussion one can expect to hear. She recommends seating discussants in a circle. When asked about what a circle produces that rows would not, Elizabeth replied, “Eye contact is the biggest thing. And the feeling among the students that they’re speaking to each other and not to the teacher” (E.H. IV#2, p. 14).

2. Interpersonal Skills. Effective discussions are characterized by discussants demonstrating interpersonal skills. Elizabeth is interested in a highly civil atmosphere in which discussants listen carefully, yield the floor,
make eye contact with one another, exhibit patience, and make one another feel a part of the discussion. One way that students who are reluctant to participate orally can be encouraged is by other students consciously trying to draw them into the discussion, either with a broad invitation, "What do the rest of you think about . . .?" or a more directed overture, "John, what is your opinion about . . .?"

3. Facilitator’s Role. Elizabeth distinguishes between teachers who are discussion leaders and those who are discussion facilitators, and favors the latter role. Discussion leaders direct the flow, pace, and agenda of the discussion, whereas discussion facilitators open the discussion up to where students want it to go. If a discussion gets off track, a facilitator will intervene, but the students generally have more responsibility for the content of the discussion than in one with a discussion leader. Elizabeth favors discussions with facilitators because leaders so often preempt what would come up naturally if left to germinate.

4. Critical Thinking. Effective discussions are not merely sharing information, but places where critical thinking about the information being shared is exhibited. Elizabeth wants students to challenge one another, use analogical reasoning, direct questions to one another, and use statements like "Wait, Let me see if I get this right" (E.H. IV#2, p. 18) to seek clarity.

5. Interaction. Not surprisingly, given Elizabeth’s preference for the teacher’s role as discussion facilitator vs. discussion leader, she places a premium on interaction between students and is wary of teachers who ask too many questions. Instead, Elizabeth wants to hear students talking directly to one another.
6. Preparation. Students who have prepared for a class discussion (through reading or working through pre-discussion questions) are more likely to produce one that is of high quality. Thus, Elizabeth’s conception of effective discussion is not talk that occurs “off the cuff,” but talk that is informed by preparatory work by the discussants.

7. Role of Prior Knowledge. Elizabeth also views favorably evidence that the students are making connections between what they have learned in the past and the topic under discussion. Commenting on a student’s reference to Plessy v. Ferguson to substantiate a point about institutional racism during the discussion of Richard Wright’s autobiography, Elizabeth said, “They’re doing a great job of making connections between history and literature” (E.H. IV#2, p. 4).

Summary of Elizabeth’s Conceptions. Elizabeth’s conceptions of the purposes of social studies, the rationales for using CPI discussions, and the characteristics of effective discussion share common themes. Similar to Joe Park, she cites a classic reason—preparation for democratic citizenship—to support what she is teaching her students in their eighth-grade social studies class. The democratic citizenship theme also trumps the many rationales she has for teaching students to participate more effectively in CPI discussions. Another reason Elizabeth has for using CPI discussions is to teach interpersonal skills, which she links to the development of an atmosphere of overall civility. When defining what makes a discussion particularly effective, Elizabeth draws on her rationales for CPI discussions and for teaching social studies. She emphasizes students talking directly to one another, instead of through the teacher, and the use of interpersonal and critical thinking skills in discussion.
Selecting Content for Discussion: Issues and Materials

To this point, the portrait of Elizabeth has focused on her background, contexts, and conceptions of a variety of factors that may influence her conceptions and practice about CPI discussions. Now I turn to a description of how Elizabeth selects issues and materials for CPI discussions. This section addresses three questions: Who decides what content (i.e., issues) will be discussed? What criteria are used to select the content? What materials will students interact with to prepare for CPIs discussions?

Who Selects Issues. In most instances, Elizabeth decides which CPIs will be included in the curriculum of her eighth-grade social studies classes. If students are particularly interested in a current events issue, Elizabeth will sometimes include that in the course. For example, in February 1998, when it appeared that the United States might go to war against Iraq, Elizabeth responded to her students’ desire to discuss the issue by including it in the curriculum. Periodically Elizabeth teaches an elective mini-course, Public Issues Discussions, that has as its primary curricular outcome the ability to participate more effectively in CPI discussions. In that course, students select the issues to be discussed.

Criteria for Selecting Issues. Elizabeth uses four criteria to determine which issues should be included in the required social studies course: curricular connection, student interest, anticipated community reaction/student age, and teacher interest/knowledge. These criteria emerged as Elizabeth engaged in an issues-selection task. Recall, this task required Elizabeth to sort ten CPI topics into three piles to reflect what she would include in her curriculum, what she might include, and what she would not
include (see Chapter Three). Figure 9 illustrates Elizabeth’s decision-making on issues selection.

<table>
<thead>
<tr>
<th>CPI Topics</th>
<th>Elizabeth’s decision</th>
<th>Elizabeth’s reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abortion</td>
<td>maybe, focus on structural issues (i.e., which govt. should regulate?)</td>
<td>community disapproval; generate more heat than light</td>
</tr>
<tr>
<td>Affirmative Action</td>
<td>yes</td>
<td>connected to curriculum, high teacher knowledge</td>
</tr>
<tr>
<td>Balanced Budget</td>
<td>no</td>
<td>low teacher knowledge; not tied to curriculum</td>
</tr>
<tr>
<td>Gay Rights</td>
<td>maybe, focus on structural issues</td>
<td>community disapproval</td>
</tr>
<tr>
<td>Immigration</td>
<td>yes</td>
<td>connected to curriculum, ties to current events</td>
</tr>
<tr>
<td>Legalizing Drugs</td>
<td>no</td>
<td>students too young, community disapproval</td>
</tr>
<tr>
<td>Physician-assisted suicide</td>
<td>maybe, focus on structural issues, more inclined to include if on ballot</td>
<td>community disapproval</td>
</tr>
<tr>
<td>Trade policy</td>
<td>no</td>
<td>low teacher knowledge</td>
</tr>
<tr>
<td>Vouchers for private schools</td>
<td>no</td>
<td>too complex; not tied to curriculum</td>
</tr>
<tr>
<td>Welfare Reform</td>
<td>maybe</td>
<td>not connected to curriculum</td>
</tr>
</tbody>
</table>

Figure 9. Elizabeth’s Decisions and Reasons About Which CPIs To Have Students Discuss

1. Curricular Connection. The criterion that most influences which issues Elizabeth selects is the extent to which the issues are connected to the eighth-grade curriculum. This is not surprising, given that Elizabeth believes that CPI discussions can help her students better understand important social
studies content. For example, the curriculum includes a unit on the United States Civil War, which causes Elizabeth to favorably view the inclusion of affirmative action:

It's [affirmative action] a topic that I would do with students, and I would do it as an outgrowth of slavery, to talk about what are the effects of slavery in our society today. We study slavery during the Civil War and I think that's a good tie-in of a current event. (E.H. IV#4, p. 2)

The way Elizabeth employs this criterion can be illustrated by her thinking about whether she would include welfare reform as a CPI in the eighth-grade course. She indicated that she probably would not include welfare reform because it has no direct link to the curriculum. If her curriculum extended beyond the United States Civil War (to include the New Deal), then welfare reform would be more likely to be included because its historical antecedents can be found in the New Deal period.

2. Student Interest. Another criterion Elizabeth uses when selecting issues is her assessment of whether the issue would be interesting to students. Because the study of current events is such a mainstay of her classes, her students tend to be particularly interested in CPIs that are matters of current public discussion. Student interest as a CPI selection criterion often conflicts with other criteria Elizabeth uses to select issues. For example, Elizabeth knows that her students are interested in discussing many issues that she thinks community members don't want them talking about, either because they challenge community values or because the students are perceived as being too young.
3. Anticipated Community Reaction/Student Age. When selecting issues for class discussion, Elizabeth is enormously attentive to whether the community would support having students discuss the issue. As Figure 9 illustrates, gay rights, legalizing drugs, abortion, and physician-assisted suicide are all issues that Elizabeth would be hesitant to include in the curriculum because she fears community members would disapprove. Her fear of community disapproval is not just paranoia. A teacher in her school who taught a unit on the Salem Witch Trials was castigated by some community members who thought she was encouraging the practice of witchcraft. Elizabeth admits, however, that her fear of community disapproval is entangled with her concerns about whether some issues are inappropriate for her students because of their young age.

4. Teacher Knowledge. Elizabeth is more likely to select issues about which she possesses strong background knowledge. Given the numerous issues that could be selected for her curriculum, she is candid about how her own knowledge influences the selection process. For example, when describing why she probably would not include trade policy issues, Elizabeth said, "I guess I wouldn't include it because I don't know enough about it. It's something that I probably could research and figure out a way to connect [to the curriculum], but I haven't done that yet" (E.H. IV#4, p. 2).

Materials for CPI Discussions. Most of the materials Elizabeth's students use to prepare for CPI discussions are readings, although she occasionally shows a video for background information. Because her students are reading on such different grade levels, the materials Elizabeth selects are carefully targeted for various students. For example, in preparation for a discussion on gun control, Elizabeth found more than ten different articles,
some written on a very elementary level, others at a much higher level. Finding materials to meet the needs of students reading on varying grade levels is an extremely difficult task.

This year I have the lowest students that I've ever had. I have kids who are reading at the first, second, and third grade levels. And the materials, it's been really tough on me—just with the materials I provide them to prepare for a discussion . . . so, I've tried to differentiate for those low end, and for the high end kids too. But it's a phenomenal task to be able to find materials that are appropriate at every level. (E.H. IV#5, p.1)

The need to find articles about the same CPI written at various grade levels has an effect in addition to the considerable work for Elizabeth. Because different articles have different information and perspectives, her students come to the discussions with different knowledge about the issue under discussion.

**Summary of Issues Selection and Materials.** Although Elizabeth's students sometimes get the opportunity to select which CPIs will be discussed, Elizabeth typically makes those decisions. When doing so, she employs a number of criteria, including connection to the curriculum, student interest, anticipated community reaction/students' age, and her own background knowledge about the issue. Once she selects an issue, Elizabeth searches for background reading materials written at various grade levels to accommodate the diverse needs of her students. As an effect, students often bring different background knowledge to the CPI discussion.
Snapshot of a Public Issues Discussion

During the 1997-98 school year, Elizabeth's eighth-grade social studies students participated in nine CPI discussions. Interspersed throughout other instructional activities, the discussions usually focused on issues that were directly connected to the unit being studied. On occasion, however, a CPI was inserted because it was related to an important current event. The CPI discussion showcased in this snapshot was such an occasion. Shortly after two boys in Jonesboro, Arkansas, kills four of their classmates and a teacher with hunting rifles, Elizabeth prepared her students to participate in a discussion of the question, "Should the United States place more limits on guns?"

This snapshot of the gun control discussion describes the model Elizabeth uses for CPI discussions, how her students prepared to participate in the discussion, and the process used by the students to develop guidelines for the discussion. Then, I turn to the actual discussion and describe what students say in the discussion, Elizabeth's role as a facilitator, and how Elizabeth assesses their participation.

The Public Issues Model. As previously mentioned, Elizabeth was a student in a class I taught in the summer of 1996 that focused on various models of classroom discussion. One of the discussion models taught in the class was Public Issues, a model developed as part of the Harvard Social Studies Project in the 1960s (Oliver & Shaver, 1974/1966). The Public Issues model involves selecting issues that bring to the fore tensions between core democratic principles (such as liberty vs. property), and the use of three different types of sub-issues within the discussion: definitional, ethical, and factual. Although some teachers use the model in small-group discussions (Miller & Singleton, 1997), Elizabeth learned the model in a large group and
uses it in that manner. Elizabeth taught the Public Issues model to her students at the beginning of the school year by explaining the different types of sub-issues and working as a large group to practice identifying the sub-issues in an article they read together. Additionally, each of her classes developed a list of discussion guidelines that she then compiled into a master list that is posted in her classroom.

Preparing for the Discussion. Three days before the discussion occurred, Elizabeth distributed various articles on the gun control issue to her students. As previously mentioned, the articles were selected because of their varying difficulty. Some articles were chosen for students reading well below grade level, others were more challenging. Students were instructed to read the articles and create a chart that listed arguments in favor of and against placing more limits on guns, and to identify ethical, definitional, and factual issues undergirding the larger gun control issue. Almost all of the students completed this assignment, which is typical; without it, they cannot participate in the discussion. Students “want to be a part of it. . . . More kids do their work for discussions than on an average basis because they really enjoy them” (E.H. IV#5, pp. 12-13).

Classroom Arrangement. As Elizabeth’s 30 students enter the room, they take a seat in either an inner or outer circle. The two circles represent a change in how the classroom furniture is typically arranged for CPI discussions, which is one large circle. Only students in the inner circle are allowed to participate orally in the discussion. When students in the outer circle want to participate orally, they must tap the person in front of them on the shoulder and change seats. Because Elizabeth’s students have difficulty inviting one another into the discussion, she hopes that the physical
movement will cue them to yield the floor and invite others to participate. Moreover, she thinks the new seating arrangement will cut down on the airtime of students who often monopolize the discussions. She says to the class:

The people who have typically monopolized the discussion, this cuts their airtime almost in half. You are lined up with somebody who talks about as much as you do, and if you talk a lot, you probably have a tendency a lot of times to control the discussion. This going to give more airtime to people who don't talk as much. (E.H. AT#2)

Reminding Students of Assessment Rubric and Discussion Guidelines.

Elizabeth begins the discussion by reminding students about what is on the rubric that she is using to assess their participation in the discussion:

Please remember that I am still observing for all of the things that I have before on discussions: that you bring in your background knowledge, that you state some of the issues you have read about this week—the definitional, factual, ethical issues, that you build on or challenge someone else’s comments, that you question when you want more clarity, and inviting others in is a big one this time. (E.H. AT#2)

Next, Elizabeth directs her students’ attention to the board, where she has written the discussion guidelines previously developed by the class. The guidelines state: listen, participate, invite others in, be responsible, be open-minded, and respect (field notes). She then encourages her students to read through them as a reminder of the types of behaviors they should demonstrate in the discussion.

Posing the Discussion Question. Elizabeth then states the discussion topic, “The question that we pose is: Should the United States place more
controls on guns? First period realized when they got part way through the
discussion that they needed to clarify what kinds of guns they were talking
about. Do you think we should do that in advance?” (E.H. AT#2). A student
responds, “No, guns are guns.” Several others murmur agreement with his
view. Elizabeth then says, “But, one thing I wanted you to remember is that
when we say more controls, that some weapons have already been regulated.
Who knows about this?” (E.H. AT#2). Two students explain that assault and
fully automatic weapons are banned. Elizabeth then focuses the entire class
on the issue once again: “So, let’s go ahead and begin with the people in the
inner circle, Should the United States place more controls on guns?” (E.H.
AT#2).

The Discussion. For the next forty minutes, students have a wide-
ranging discussion on gun control that focuses on various proposals to
regulate guns, such as requiring stricter background checks, requiring that
people who have guns take training courses, requiring secure storage of guns,
and a total ban on all guns except those used by the police and the military.
Throughout the discussion of these proposals, students use various kinds of
evidence to support or challenge other students’ views, such as factual
information from their readings and personal experience.

Referencing What They Have Read. The first several statements in the
discussion include direct references to the articles the students have read. For
example, one student says, “In one of those articles I read it said it is better to
protect your life than your property” (ST, AT#2), thereby immediately
surfacing the value conflict between property vs. life that becomes a major
focus of the discussion. Although students occasionally talk about using guns
for hunting, most of them think people have guns to protect themselves and
their property. After several minutes, a student summarizes the conflict between values by saying, “Wait, Trina said something about wanting to protect your property, but someone else said that you should put your life above your property, because if you try to get your stuff back you could end up shooting yourself or the robber” (ST, AT#2).

The factual issue of whether people are safer if they have guns in their homes becomes the next point of dispute when a student says, “I read an article that said that gun owners are more likely to kill themselves than the attacker” (ST, AT#2). For the next few minutes, the students talk about whether that is accurate and, if so, what is the cause. For example, a student says, “To add on to what was said, many people who shoot themselves are not trained to use a gun, and I read an article that with training anyone can learn to use a handgun, but I think they should still be banned or regulated, but hunting rifles should not be” (ST, AT#2).

Using Personal Experiences. In addition to drawing on the articles they have read, the students also make liberal use of personal experience. When talking about whether people should be required to participate in training to use a gun, a student comments, “I know how to use a gun because my Dad taught me, we have a bunch of them in our house, but if a kid, like, if kids aren’t taught how to use them, then we’ll have what happened in Arkansas” (ST, AT#2). Another student responds to this by saying, “Kids do know how to use guns; the kids in Arkansas sure did” (ST, AT#2). Referencing personal experiences again, many students state that there are guns in their homes, even going so far as to explain where they are stored.

Challenging the Views of Other Students. Throughout the discussion, students challenge the views of their classmates. During one interchange
comparing gun crime in the United States to that of other nations, a student states that crime in Great Britain has increased on a percentage basis more than crime in the United States, even though Great Britain has stricter gun control laws. A student immediately corrects that statement by pointing out that, overall, gun crime is still much more prevalent in the United States. On occasion, the students use gentle sarcasm to respond to others’ views. For example, one student references an article that said that criminals interviewed feared an armed citizenry more than the police, so having guns causes crime rates to go down. A student immediately quips, “Well, aren’t we just one happy society” (ST, AT#2).

Seeking a Compromise. A few students take on the role of compromisers, attempting to synthesize the views of others and forge a middle ground. For example, one student says, “We need to compromise and figure out a solution; guns are good for protection but only in the right hands, but a lot of the problems in society today are young teens using guns in the heat of the moment—don’t use your best judgment so there needs to be some kind of control” (ST, AT#2).

Elizabeth’s Stance as a Facilitator. During the discussion, Elizabeth rarely intervenes; when she does, it is almost always to encourage equal participation. For example, about midway through the discussion, Elizabeth says, “Okay, I am going to interrupt for a minute because some of you guys are monopolizing again. Bob just opened his mouth to talk and you talked right over him, so Bob what did you want to say?” (E.H. AT#2). A few minutes later, she encourages students who have not yet talked to do so by saying, “I am going to ask that the people who have not yet been in the inner circle move to the inner circle” (E.H. AT#2). Later, she encourages careful listening
by saying, "He’s agreeing with you," and reinforces the goal of inviting others
to participation by pointing out, "I want you to notice that Deanna has just
moved into the circle" (E.H. AT#2). Only once during the discussion does
Elizabeth say anything about the content. In response to a student’s direct
question about the Brady Bill, Elizabeth directs the students’ attention to the
chalkboard, where the Brady Bill has been summarized, and briefly explains
its provisions.

**Closing the Discussion.** As the end of the period nears, Elizabeth points
out that time permits only a few more comments. Many students groan,
indicating they don’t want the discussion to end. With just a few minutes left
in the period, Elizabeth ends the discussion and quickly directs the students to
assess their participation in the discussion by filling in a copy of the
discussion rubric. Additionally, she asks them to write down anything they
had wanted to say but didn’t, to comment on the inner/outer circle seating,
and to identify a goal for the group for future discussions. The students then
hand in the rubric and the notes they wrote to prepare for the discussion.

**Debriefing and Assessing the Discussion.** Because the class periods are
so short (forty-eight minutes), students rarely have time to debrief the
discussion until the next day of class. The debriefings discussions are usually
10-15 minutes long and focus on what went well and what didn’t go so well.
Because Elizabeth has assessed each individual student’s participation in the
discussion, students also receive feedback on where they stand in relation to
developing the discussion skills that Elizabeth values. Elizabeth has
developed a new rubric for assessing discussions that draws on the work of
other discussion experts. She completes the rubric for each student after each
discussion, using notes taken on a specially designed class roster. While it
may seem difficult to both facilitate and assess the discussions, Elizabeth says that is not the case. Students receive points for their participation in the discussion and for the assignment they completed to prepare for the discussion. Students who don’t participate orally in the discussion but did complete the assignment receive some points, but they are penalized for not talking. Thus, Elizabeth is communicating the value she places on participating orally in discussion through her assessment and grading policies.

**Summary of Gun Control Discussion.** Although the quality of discussions varies from class to class, the gun control discussion that took place in Elizabeth’s third period social studies class typified her approach to Public Issues discussion. Students prepared for this discussion by reading several articles and completing an assignment. When the class period began, Elizabeth reminded the students of what she was assessing and of the discussion behaviors they had previously developed as a class. Elizabeth rarely intervened in the discussion; when she did, it was usually to encourage equal participation. Students used evidence from their readings and personal experiences to support and challenge one another’s views. The discussion ranged over a number of specific proposals to regulate guns with no expectation that students reach consensus. The discussion was debriefed during the next meeting of the class, when students also found out how they scored on the discussion rubric.

**Portrait Three: Ann Twain at Douglas Middle School**

Unlike Joe and Elizabeth, who started teaching immediately after being awarded their B.A. degrees, Ann Twain’s teaching career began in her early 30s, after she earned B.A. degrees in film and dance at a major university and
spent several years as a video producer in Washington, D.C. Working as a video producer exposed her to many impressive young people in the nation’s capital who were political activists. As a result, she began thinking about the “notion of kids as activists” (A.T. IV#1, p. 1) and realized she could help more young people become politically active if she taught social studies.

After Ann moved to the Pacific Northwest, she produced an educational film about service learning and the social studies and gave educational lectures to high school students. These experiences caused her to think, “This is something I really enjoy” (A.T. IV#1, p.3), and propelled her to enter a teacher education certification program in her new state’s flagship university. She enrolled in a special Middle School certification program, “which included a middle school seminar, twice weekly, for two hours, with four professors. I felt totally spoiled” (A.T. IV#1, p.4). After receiving certificates in both K-8 education and 4-12 language arts, Ann landed a position as a social studies teacher for seventh- and eighth-graders at an alternative school in one of the state’s largest school districts.

During her five years of teaching, she has participated in numerous professional development programs, including a national program on authentic assessment in civic education and a yearlong district-sponsored program on linking curriculum, instruction, and assessment to the district’s new curriculum standards. She regularly attends social studies conferences and has presented workshops for other teachers on assessing classroom discussion. Now, after five years of teaching, Ann is thinking about earning a graduate degree in social studies.
Contexts

As was the case for Joe and Elizabeth, a variety of contexts influence how Ann thinks about and enacts her CPI discussion teaching practice. In particular, the community in which she teaches, her school, its students, and the curriculum inform Ann’s approach to CPI discussions.

Community. Ann’s school is located in a large school district outside a major city in the Pacific Northwest. The community served by the school district is established, growing rapidly, and economically diverse. Students at Ann’s school are bused in from all over the 36-mile school district. Ann defines the community in very broad terms: “It’s a nice community, supportive community of the schools. Levies pass” (A.T. IV#1, p. 11). The parents who send their children to Ann’s school tend to be more affluent than is typical in the community, and less likely to move, causing Ann to describe the sub-community made up of the school’s parents and their children as “established and pretty stable.... Kids tend to be with us from the first grade through eighth grade” (A.T. IV#1, p. 10).

Douglas Middle School. Ann’s K-8 school is eleven years old and has 700 students, 150 of whom are in the middle school. There is a waiting list to get into the school. The two terms most frequently used to describe the school are “multi-age” and “non-graded,” which reflect how the school is an alternative to more traditional schools in the district. “Multi-age” means that students are in classes with children either younger or older than they are. For example, all of Ann’s classes combine students who in a traditional school would be in the seventh- and eighth-grades. “Non-graded” means that traditional letter grades are not used to report students’ progress to their parents. Instead, each teacher develops an elaborate report card based on the
student outcomes for her/his subject. Ann’s report card, for example, includes 33 specific outcomes organized into six categories. For each outcome, students are rated on a scale ranging from “unacceptable progress toward standard” to “exceeds standard” (A.T., ART).

Parents tend to be quite involved in school activities. Ann explains this generalization with a specific example: “I am able to have a service learning program that requires kids to be driven to service learning sites off campus twice a month. And I have close to 25 parents that come and take these kids. So I think that’s pretty phenomenal” (A.T. IV#1, p. 5). Parents also participate in the curriculum within the school walls, by serving as jurors in mock trials or facilitators for a simulation of the United Nations, for example.

**Students.** Ann teaches 150 students each day. She categorizes her students as those who are “really self-directed and capable” and those who are “very needy and their parents thought that a program that is not textbook-based, that is more individualized, would be better for them” (A.T. IV#1, p. 5).

The school district’s hearing impaired program is housed at Douglas Middle School, which contributes to diversity among the students, but in other categories, especially race and ethnicity, the student body is extremely homogenous. Ann estimates that 85% of the middle school students are white. Moreover, the students come from families with economic resources. Ann estimates that fewer than 10% qualify for the free or reduced-price lunch program. Ann is concerned about the homogeneity of her school’s student population: “What we’re now looking at is that our population is not diverse enough . . . so we really need to up the diversity” (A.T. IV#1, pp. 6-7).

**Curriculum.** Each day, Ann teaches four sections of multi-age social studies and one section of drama. Because Ann has both seventh- and eighth-
graders in the social studies classes, and they stay with her for two years, her
curriculum must change each year to avoid repetition. To accommodate this
unusual circumstance, Ann has designed a curriculum that spans a two-year
period. The fact that Ann teaches both seventh- and eighth-graders in the
same class creates two additional challenges: (1) it heightens the academic
differences within each class ("there is a big difference between a low seventh
and a high eighth") (A.T. IV#1, p.9), and (2) it broadens the curricular focus.

During any one school year, Ann's course focuses on an amalgam of
United States history, government, world geography, and current events. The
specific eras in history change from year to year, but overall Ann's students
study more 20th-century history than would typically be the case in other
middle school social studies courses in the district. The government
component of the course centers on the Constitution and other founding
documents, the form and structure of the federal government, with an
emphasis on the Supreme Court, and controversial public issues, especially
those that are scheduled to be on the state ballot as initiatives. The world
geography units in her course emphasize themes of geography (such as the
interaction between humans and their environment) and the role played by
world organizations, such as the United Nations. The study of current events
is carefully woven into the curriculum, both to enhance its relevancy to
students and to help them better understand the world in which they live.

Teachers in the school rarely use textbooks, and Ann is no exception to
this rule. Instead, she creates "original curriculum" that involves "hands on"
activities, such as historical re-creations, other simulations, and inquiry
learning (A.T. IV#1, p. 7). Because her students have been steeped in this type
of learning for many years, they have developed the skills necessary to
participate effectively in them. Ann says her students are “very used to getting up and presenting orally. That’s not something that I had to introduce at this level. It’s something that they feel pretty comfortable with” (A.T. IV#1, p. 8).

**Summary of Contexts.** Ann has been teaching middle school social studies for five years in a suburb of a major city in the Pacific Northwest. She is actively involved in a number of professional development programs, both as a student and a leader. The community in which Ann’s alternative school is located supports education, and the parents who elect to enroll their children in Ann’s school are extraordinarily involved in school activities. The K-8 school is “multi-age” and “non-graded,” reflecting its non-traditional approach. The social studies classes Ann has created focus on United States history, government, world geography, and current events. Because she teaches seventh- and eighth-graders in each class, the curriculum must change each year. All of these contexts (community, school, students) influence how Ann thinks about and practices her teaching. Additionally, her conceptions of a number of important factors also inform one aspect of her teaching, which is her quest to teach students to participate effectively in CPI discussions.

**Conceptions**

Recall, in this study I am defining conceptions to include a teacher’s knowledge, beliefs, thoughts, and images that may inform classroom practice (Shavelson & Stern, 1981). In this section, I focus on Ann’s conceptions of four factors: the purposes of social studies; the characteristics of effective democratic citizenship; the rationales for CPI discussions; and the characteristics of effective CPI discussions.
Conceptions of the Purposes of Social Studies. I learned about Ann's conceptions of the purposes of social studies by looking at a concept map she drew on the topic (see Chapter Three) and by listening to her describe and explain the map. In the center of the map was a stick figure representing a student. Lines connected the student’s head to four large circles labeled to represent what Ann wants her students to know, be able to do, and be disposed to do as a result of her social studies curriculum. The labels said: issues I’ll face in my future; citizenship; geography; and United States history (A.T., ART).

Unlike Joe and Elizabeth, Ann began her description of the concept map by describing the origins of her curriculum:

When I started my first year of teaching and I knew I would be making up my own curriculum, I didn’t start with these other categories of United States history, geography, and citizenship . . . I started by sitting down and saying to myself, “Okay. I have these kids and what are the most important things that I can equip them to deal with?” (A.T. IV#2, p. 1)

Ann answered this question by identifying four overarching issues she felt would be present throughout her students' lives: immigration, population pressures, environment, and technology. These issues became conceptual drivers as she created a curricular path for her students to follow in achieving the outcomes she valued. The issues were also curricular topics in and of themselves. For each one, she had three specific outcomes. First, she wanted her students to understand that these were issues they should care about. Second, she wanted them to be informed about multiple perspectives on the issues. Third, she wanted to influence their behavior: for example, “to
be able to think about and make smart choices in terms of resource use” (A.T. IV#2, p. 2). Ann was careful to point out that she recognizes the controversial nature of these issues and does not want her students to form a particular point of view on specific policy proposals. Instead, she wants them to recognize and care about each issue writ large. The following excerpt from the concept map interview explains this distinction:

Hess: I want to come back to these issues. Are these issues that you want kids to have a particular point of view on?

Twain: No, Um, hmmm. Now that I’ve said that, let me think about that. In a couple of cases, I’ll have to say “yes.” Environment. I want them to feel they have a role as a custodian of the environment. I don’t necessarily want them to feel that, you know, the Endangered Species Act is the way to go. . . . My hope is that in each of these cases [issues], with more knowledge, they will take an enlightened viewpoint, which is a responsible one.

Hess: What do you mean by “enlightened?”

Twain: In looking at the environment, [etc.], that they not be self-centered in their views and in their approaches. That they’re always looking at the whole. The collective whole, and not what’s good for them, but what’s good for people. What’s good for the world. What’s good for the community they live in. (A.T. IV#2, p. 6).

Although Ann’s identification of four big issues that drive her curriculum certainly has strong connections to citizenship education, on her concept map she created a special category labeled “citizenship” that had several components, including multiple perspectives, giving to the community, appreciating diversity, activism, and ethics. The citizenship
outcome is motivated by her opinion that “we want people who can be trusted with a really important role in society, which is to be a good citizen” (A.T. IV#2, p. 3).

Within the citizenship category, Ann distinguishes between giving to the community and activism, although both are purposes of her social studies teaching. Her definition of giving to the community is represented by a service learning program she has created for her students. Each year, her students are released from eight half-days of school to volunteer at various sites where their service is needed, such as nursing homes and elementary schools. By contrast, activism is taught to her students as a form of proactive problem solving. Ann wants her students to learn how to look at a “problem from the faucet where it starts and not the puddle on the floor that you have to mop up after it’s already flowed over” (A.T. IV#2, p. 3). An example of activism that Ann cited was, “my kids going to the city council and proposing a day of concern for youth violence, and proposing a recreation center for kids during the hours when there’s the most juvenile crime in our community” (A.T. IV#2, p. 3).

In addition to conceptualizing the purposes of social studies as knowing and caring about the four overarching issues and other citizenship outcomes, Ann also wants her students to learn content drawn from the disciplines of history, geography, and political science “so we’re all on the same field, and know we have the same equipment” (A.T. IV#2, p. 6). “Knowing” takes many forms in Ann’s conception of social studies: “I think that knowing can also have a human side . . . knowing what certain experiences are like. It may be knowing there is human knowing and not just fact-based knowing” (A.T. IV#2, p. 6).
Finally, Ann is concerned that social studies prepare her students to understand how to interpret and think about persisting issues of history that are present in today's world. Toward this end, she focuses on current events and selecting issues drawn from history and geography that require students to discuss tensions between such core values as individual rights and the common good.

**Conception of Effective Democratic Citizenship.** When describing and explaining how she thinks about citizenship, Ann distinguished between knowledge that effective democratic citizens should possess, skills they should be able to demonstrate, and actions in which they should be engaged. The knowledge category includes four components: understanding founding principles of the government in the United States (such as rule of law); being aware of the history related to national identity; understanding significant historical milestones, especially those related to past inequalities; and understanding historical examples of good and bad political leadership.

Ann identifies founding documents (such as the Constitution) as the primary source for the founding principles of government in the United States and believes that an understanding of those principles correlates positively with effective democratic citizenship. She links knowledge of founding principles to behavior that effective citizens would exhibit. For example, if people understand that "No one is above the law" (AT. IV#1, p. 20), they might be more likely to respect the law.

In addition to knowledge of founding principles, Ann also hopes that citizens will have historical knowledge about who lives in the United States and how different groups have influenced national identity. This knowledge can be used by people to address a key question: "Do we still share a vision
[of/for the United States]?” (A.T. IV#1, p. 20). Historical background knowledge can also help prevent repeating the mistakes of the past. In particular, Ann believes that citizens should know how past inequalities have shaped the issues facing the United States today. To support this point, Ann asked a rhetorical question: “How can we understand something like affirmative action, or the Civil Rights Act, if we don’t understand what wasn’t right about what was? And what issues are we still dealing with today because of what wasn’t right?” (A.T. IV#1, p. 21).

A final knowledge component in Ann’s conception of effective democratic citizenship deals with political leadership. Ann hopes that citizens will know examples of good and bad leadership over time so “we know what we should be striving for” (A.T. UV #1, p. 21).

Ann also expects that effective democratic citizens will have developed a number of specific skills they can employ in their citizenship role. Recognizing that we live in the information age, Ann wants citizens to be able to acquire knowledge drawn from multiple perspectives that is learned from multiple stake holders and represents multiple value orientations. Believing that not all information is equally valid, Ann also sees the need for citizens to have a “BS detector so they can sniff out bias and falsehoods” (A.T. IV#1, p 21). She also places considerable stock in individual agency and wants citizens to make up their own minds about the issues that confront them. Once these ideas have been formed, Ann adds another skill citizens need: “the ability to articulate your ideas orally and in writing so that other people can understand them” (A.T. IV#1, p. 21).

Ann further believes that effective citizens in a democracy, armed with citizenship knowledge and skills, should want to engage in numerous citizen
actions, including voting, volunteering, attending public meetings, and standing up for the rights of others, especially people who are oppressed. She places special importance on the last activity: "The one that I think is really important is to stand up for others and stand up to others and not to be silent. I think there's a real problem with bystander behavior in our society" (A.T. IV#1, p. 22). Linking silence to compliance, Ann continues, "There are times when silence is a really negative thing and something to be avoided. That's part of being a good citizen" (A.T. IV#1, p. 22). I asked her, "So what do you mean by that? What does it look like to stand up for and to others?" (Hess, IV#1, p. 22). Ann responded:

If someone is having their rights violated or someone who lacks a voice in our community, a person who is mentally ill, a senior citizen, a child who can't yet vote. Being able to help be a voice for people who maybe are oppressed or who don't have the skills to have a voice. And then standing up to others. . . . What are the ways that we, as citizens, can be heard? If we aren't the power holders, what can we do to have some power? (A.T. IV#1, p. 22).

Conception of the Rationales for CPI Discussions. Ann holds multiple rationales for teaching her students to participate more effectively in CPI discussions. Ann described and explained these rationales as she engaged in a card-sorting task that required her to rate and rank various rationales for CPI discussions that were drawn from the literature (see Chapter Three). Her conceptions of the rationales are summarized in Figure 10, but here I will explain in greater detail how Ann thinks CPI discussions improve democracy, activate youth's passion for issues, improve critical thinking skills, enhance
appreciation for diversity and perspective-taking, and inform personal decision-making.

<table>
<thead>
<tr>
<th>Rationales for CPI Discussions</th>
<th>Do CPI discussions accomplish this goal?</th>
<th>Reasons why CPI discussions do/do not accomplish this goal:</th>
<th>Caveats and/or further explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Improve interpersonal skills</td>
<td>yes</td>
<td>Four important skills: listening, understanding others, inviting participation, not monopolizing</td>
<td></td>
</tr>
<tr>
<td>Understand important democratic values</td>
<td>Maybe</td>
<td>Depends on the CPI, but some CPIs illuminate democratic values</td>
<td>Help students see democratic values as “living”</td>
</tr>
<tr>
<td>Participate in political life of democracy (e.g., voting, serve on jury)</td>
<td>yes</td>
<td>Discussions won’t cause people to vote or serve on a jury, but will make them better at doing those things</td>
<td>Discussions help people be informed voters and jurors</td>
</tr>
<tr>
<td>Learn important social studies content</td>
<td>yes</td>
<td>Brings content to life, helps students see content in 3-D</td>
<td>You “own” ideas when you articulate them</td>
</tr>
<tr>
<td>Learn how to think critically</td>
<td>yes</td>
<td>Helps students think critically, quickly, and thoughtfully</td>
<td>Causes discussants to learn how to use analogies, create relationships among ideas</td>
</tr>
<tr>
<td>Improve democracy</td>
<td>yes</td>
<td>Improves the marketplace of ideas that is fundamental to democracy</td>
<td>Truth and logic emerge from CPI discussions, divergent ideas are sometimes best</td>
</tr>
</tbody>
</table>

Figure 10. Rationales for CPI Discussions: Ann Twain (Note: Ann’s responses are from A.T. IV#3, pp. 1-15).
Identical to Joe and Elizabeth, the rationale that has the most power for Ann is the positive connection between CPI discussions and a healthy democracy. Linking the health of a democracy to the vibrancy of the “marketplace of ideas” within it, Ann said, “we progress as a society because new ideas are constantly getting aired and we are not stagnant” (A.T. IV#3, p. 4). Moreover, the marketplace of ideas has the potential to prevent or redress mistakes. Ann used a negative example to support this point:

We’re constantly re-evaluating what we’re all about. And as new issues come out, I mean it was Einstein, wasn’t it, with the atomic bomb technology, who said immediately, “This needs to be taken into a public forum and discussed by people.” (A.T. IV#3, p. 4)

CPI discussions help improve democracy because within the marketplace of ideas, the truth will emerge. Representing the value she places on multiple perspectives on CPIs, Ann says, “what is true, what is real, what is meaningful rises . . . and I think that’s crucial in a democracy. That’s what democracy is all about” (A.T. IV#3, p. 9). Ideas that are outside of the mainstream are necessary because progress depends on fresh ideas and “because occasionally it’s that divergent idea that is the right idea” (A.T. IV#3, p. 9).

Ann reaches into her childhood memories to explain another rationale that she holds for CPI discussions: activating youth’s passion for issues. “I felt clueless when I was an adolescent. Adults would be discussing things [issues] around me and it made me very passive about what was going on. I think that by inviting them into discussions of real issues, it’s very empowering to them” (A.T. IV#3, p. 7). Ann uses CPI discussions to make students passionate about issues in the world around them. Young people “can get so
darn apathetic about school, about the world. This [CPI discussions] lights a fire under them” (A.T. IV#3, p. 10).

Improving her students' critical thinking skills is another rationale Ann uses to support her emphasis on CPI discussions. Unlike Joe and Elizabeth, who often mentioned slowing down the pace of discussions to help students think better, Ann wants her students to think quickly and thoughtfully and uses CPI discussions to that end. Defining critical thinking skills to include questioning, applying information to new situations, creating analogies, and seeing relationships, Ann uses CPI discussions because she thinks they help students develop these skills and give them a forum for demonstrating their critical thinking prowess. She also recognizes that talking in a CPI discussion does not necessarily correlate positively to critical thinking. Explaining this point, she says, “I would argue that I have some kids who talk in town meetings [the model of CPI discussions she uses in class] and aren’t really thinking; they like to hear themselves talk” (A.T. IV#3, p. 12).

The CPI discussion model Ann uses requires the explanation of multiple perspectives on issues from the roles of multiple stake holders in the issue (see pages 144-145 for explanation of the model). This model reflects Ann's emphasis on developing appreciation for diversity. Ann defines what this means in three ways: students will (1) become more appreciative of multiple perspectives on issues, (2) appreciate the diverse groups in the United States, and (3) develop greater empathy for other people and their ideas. She thinks that CPI discussions can help students step outside of their own situations and backgrounds and appreciate that others have different experiences that inform their views on CPIs.
Finally, Ann sees a positive correlation between students' participation in CPI discussions and the personal decisions they make about how to lead their lives. For example, after participating in a CPI discussion on abortion policy, "undoubtedly kids went away and thought, 'How do I feel about this, personally?'" (A.T. IV#3, p. 6). The decisions students make as a result of CPI discussions could be either political or personal, both of which would be informed by a clarification of their values.

Conception of Effective Classroom Discussion. Ann's ideas about the characteristics of effective classroom discussion emerged as she explained her reactions to two videotaped excerpts of classroom discussions (see Chapter Three). Like Joe and Elizabeth, Ann has strong opinions about the elements of a good discussion. In an effective discussion, discussants are seated in a circle; if there is a facilitator, that person should say little. Participants should use evidence and logic and frame the issue at hand as an example of larger, more transcendent value conflicts. During the discussion, students should exhibit interpersonal skills, such as listening attentively to one another and not monopolizing.

1. Setting. Both Joe and Elizabeth recommended seating discussants in a circle. Ann shares that view, linking where discussants sit in relation to one another to the kind of interaction that occurs among them. When students are seated in rows facing the teacher, too much of the interaction is like a tennis match, with the teacher volleying back to the students. Conversely, if the students are seated in a circle so they can all see one another, interaction is more likely to occur among the students—which is what Ann desires.

2. Facilitator's Role. Ann reacts negatively to talkative facilitators. Because she places such a premium on interaction between students, a
facilitator who takes up too much airtime, by definition, diminishes the quality of the discussion. The facilitator still has an important role, however. Because Ann uses a formal discussion assessment process, she sees one role of the facilitator as clearly stating standards of what constitutes effective discussion. Explaining why she thinks it is important to formally assess discussion, Ann says, "Share with the kids what you’re assessing, so they know what that looks like. And then they can be successful if the criteria are unmasked and they’ll know that participation is really valued" (A.T. IV#2, p. 15).

3. Interaction Among Students. In an effective discussion, students should be interacting with one another, working to build and challenge previous statements. By doing this, the discussion is "something they are all creating" (A.T. IV#2, p. 20). Ann attends to how many students are participating and favors discussions that have a high rate of oral participation. Moreover, she wants the responsibility for the content, pace, and flow of the discussion to rest with the students--not the facilitator. Explaining why the Electoral College discussion was so impressive, Ann said, "It’s moving somewhere. It’s moved in a lot of directions. And, they’re moving it, there’s no teacher moving it, they’re moving it" (A.T. IV#2, p. 18).

4. Interpersonal Skills. Just like Elizabeth, Ann uses the word "civility" to summarize the goal of various interpersonal skills. Attentive to whether students are listening to one another, inviting others to participate, and taking care not to monopolize, Ann connects the exhibition of these skills to the overall quality of the discussion.

5. Ideas in Discussion. Ann wants to hear ideas talked about in several ways in discussion. She favors the use of historical examples and connections
to modern life, along with a liberal dose of evidence and logic. Moreover, the range of ideas that students discuss should be broad, illustrating the value she places on multiple perspectives. Reacting negatively to a forced-choice exercise the teacher in the Richard Wright discussion used, Ann said, "They [students] didn’t have a choice other than ‘yes’ or ‘no,’ I was wondering if she was going to say, ‘Or are you torn? ’ ‘Or are you somewhere in between?’" (A.T. IV#2, p. 4). Finally, Ann responds positively to discussions in which the talk focuses on the larger value conflicts represented by specific issues. For example, while viewing the section of the videotape of the Richard Wright discussion in which a student talks about the conflicts between individual morals and the moral code implicit in democratic governance, Ann says: "Excellent comment. Because he is getting at the whole notion of individual rights vs. the common good" (A.T. IV#2, p. 6).

Summary of Conceptions. Ann has multiple and complex conceptions of the purposes of social studies, the characteristics of effective democratic citizens, the rationales for using CPI discussions, and the characteristics of effective discussion. Like those of Joe and Elizabeth, Ann’s conceptualization of the many purposes of social studies is congruent with what she believes citizens should know, be able to do, and want to do. Coming from an activist bent, she does not see citizenship preparation only in the future tense. Instead, she envisions “kids as activists” who learn how to be good citizens in social studies classes through participation in citizenship activities. Consequently, her primary rationale for using CPI discussions is that they mirror the marketplace of ideas that exists in the world outside of school and are fundamental to the health of a democracy. Classroom discussions that are
especially effective involve interaction among the students across a broad range of ideas.

**Selecting Content for Discussion: Issues and Materials**

Thus far, the portrait of Ann has centered on her background, the contexts in which her teaching is situated, and her conceptions of factors that may influence how she teaches students to participate in CPI discussions. Now I begin describing and explaining Ann's CPI discussion teaching practice by focusing on how she selects the issues and materials for social studies curriculum. As in the other two portraits, this section addresses three questions: Who decides what content (i.e., issues) will be discussed? What criteria are used to select the content? Once an issue is selected, a third question is introduced: What materials will students interact with to prepare for CPIs discussions?

CPI discussions in Ann's social studies classes are structured using a model she calls Town Meetings. The topics of the Town Meetings are determined by Ann, using the criteria described in this section. While some of the Town Meeting topics are historical CPIs (such as, "Was it right to drop the bomb?") , most are contemporary CPIs.

**Criteria for Selecting Issues.** Ann and her students use five criteria when selecting the topics for the Town Meetings: the extent to which the CPI lends itself to multiple perspectives from multiple stakeholders; the availability of resources; relevance to students' lives and interests; connection to democratic values; and whether the CPI is currently a matter of public deliberation. These criteria emerged as Ann engaged in an issues-selection task. Recall, this task required Ann to sort ten CPI topics into three piles to reflect what she would include in her curriculum, what she might include,
and what she would not include (see Chapter Three). Figure 11 illustrates
Ann’s decision-making on issues selection.

<table>
<thead>
<tr>
<th>CPI Topics</th>
<th>Ann’s decision</th>
<th>Ann’s reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abortion</td>
<td>Yes</td>
<td>tough issue for students to depersonalize</td>
</tr>
<tr>
<td>Affirmative Action</td>
<td>Yes</td>
<td>historical connections, matter of current public discussion</td>
</tr>
<tr>
<td>Balanced Budget</td>
<td>No</td>
<td>issue is too flat, students would not be engaged</td>
</tr>
<tr>
<td>Gay Rights</td>
<td>Maybe</td>
<td>students too immature, concerned about gay students in class</td>
</tr>
<tr>
<td>Immigration</td>
<td>Yes</td>
<td>huge issue facing the United States</td>
</tr>
<tr>
<td>Legalizing Drugs</td>
<td>Yes, but focus on red flag with parents, Web site medical use question resources are questionable</td>
<td></td>
</tr>
<tr>
<td>Physician-assisted suicide</td>
<td>Yes</td>
<td>human element, matter of current public discussion</td>
</tr>
<tr>
<td>Trade policy</td>
<td>Yes, if focus is on human rights issues</td>
<td>human rights</td>
</tr>
<tr>
<td>Vouchers for private schools</td>
<td>Probably not</td>
<td>hot potato for parents</td>
</tr>
<tr>
<td>Welfare Reform</td>
<td>Yes</td>
<td>human dimension, personally affects some of their lives</td>
</tr>
</tbody>
</table>

Figure 11. Ann’s Decisions and Reasons About Which CPIs To Have Students Discuss
Multiple Perspectives from Multiple Stakeholders. CPIs Ann selects for her curriculum must correspond to the unique demands of the Town Meeting model; that is, the issues must be ones for which there are multiple perspectives held by different stakeholders. By multiple perspectives, Ann means a variety of different ways that people can analyze an issue and a variety of different views that people might hold about an issue. Although, in theory, multiple perspectives can be identified for all CPIs (because there is something "at issue"), Ann believes that some CPIs are more perspective-laden than others. For example, CPIs about immigration topics meet this criterion for Ann because such issues are loaded with economic, political, and social perspectives.

Ann defines stakeholders as "people who have an interest in the issue because it affects them" (A.T. IV#4, p. 7). As a general rule, the more people who are directly affected by an issue, the more likely Ann is to select it for class discussion. She mentioned physician-assisted suicide as an issue that potentially affects the lives of many people; she therefore included it as the topic of a Town Meeting in the school year before this study took place.

Availability of Resources. Because Ann's students read and view many different resources to prepare for Town Meetings, she selects CPIs for which she knows many high quality resources representing a range of perspectives are available. Ann explained this criterion by linking it to the previous one: "I'm looking for topics that lend themselves to multiple perspectives and perspectives that I can find resources on, such as guest speakers, Web sites, printed material, and video clips. I want topics that are in the news enough for kids to find information" (A.T. IV#4, p. 6).
Relevance to Her Students' Lives. A third criterion Ann uses to select CPIs for Town Meetings is whether they have a "kid hook" (A.T. IV#4, p. 7). Issues that "hook kids" do so because of their relevancy. Summarizing this criterion, Ann said, "I select issues that involve something that a middle school kid would care about. That has relevance in their lives or will have relevance in their lives in the future" (A.T. IV#4, p. 7). She believes her students are particularly attracted to issues that have a "human element" and cites the balanced budget issue as an issue that lacks in that element and would therefore not be selected. Conversely, Ann identifies immigration, abortion, and physician-assisted suicide as issues that possess the "human element" because they are fundamentally about people.

The "human element," however, sometimes makes an issue too difficult for her students to discuss. Explaining her reluctance to include gay rights as a Town Meeting topic, Ann said, "I am worried about opening the flood gates and having really intolerant comments come out. I don't want anybody to be exposed to that kind of ugliness from other kids. Because I've got some kids who, you know, might be gay. And I just don't want to go there" (A.T. IV#4, p. 12).

Connection to Democratic Values. Because Ann is interested in shaping the democratic values held by her students, she selects issues that are especially value-laden. Through CPI discussions, she hopes that students' "values, their democratic values . . . are being strengthened" (A.T. IV#4, p. 8). Distinguishing between causing her students to hold a particular view on a CPI and enhancing their overall democratic values, Ann said, "I think a person could still value equality in this country, but be opposed to current legislation on affirmative action" (A.T. IV#4, p. 9). She does, however, want
students to develop an understanding and appreciation for democratic values, such as equality, the absence of tyranny, and the freedom of thought and belief.

**Currently a Matter of Public Deliberation.** The final criterion Ann uses to select issues is whether they are currently matters of public debate in the world outside of school. In the state where Ann lives, CPIs are often on the ballot as initiatives. Each year, Ann selects at least one initiative issue for students to discuss because it provides them an opportunity to practice preparing to be informed voters. Additionally, these issues provide students a good opportunity to educate their parents. Ann explained how this worked when she said, "I think that [discussing ballot initiatives] also gets kids talking to their parents about the importance of being informed on an issue that you’re going to have to vote on" (A.T. IV#4, p. 10). Recalling how this has worked in the past, Ann described how her students carried their new understanding formed by a Town Meeting on property rights to their homes: "I truly believe that kids in my room were more informed than their parents and were going home telling their parents things. That models discussing and really getting into the meat of these issues and I don't think we can do that enough" (A.T. IV#4, p. 11).

**Materials for CPI Discussions.** Of the three teachers in the study, Ann uses the widest range of materials to prepare students for CPI discussions. In preparation for each Town Meeting, students read a packet of articles that provide common background, and specialized articles selected because they illustrate the particular perspective each student will represent in the discussion. Additionally, the students search the World Wide Web for additional information, view videotaped programs, listen to guest speakers
who have specialized knowledge about the issue, and sometimes have a conference call with experts on the issue. When the Town Meeting is on a ballot initiative, Ann’s students also read the exact wording of the initiative and state-prepared materials summarizing various arguments.

Similar to the demands placed on Elizabeth, the varying reading levels of students in Ann’s classes cause her to seek out materials that are written in simple language so students reading dramatically below grade level can access enough information to participate in the Town Meeting. One of the reasons Ann goes to the trouble of finding guest speakers and videotaped programs about an issue is to help students who read below grade level get a better grasp of the background and various perspectives on the issue.

**Summary of Issues Selection and Materials.** Ann and her students select the CPIs for Town Meeting discussions. The criteria they use when selecting issues are the extent to which the CPI lends itself to multiple perspectives from multiple stakeholders; the availability of resources; relevance to her students’ lives and interests; connection to democratic values; and whether the CPI is currently a matter of public deliberation. To prepare her students for the Town Meeting discussions, Ann recruits guest speakers, shows videotapes about the issue, and prepares reading packets. Her students also search the World Wide Web for additional information. All students read some core articles explaining the background of the issue and then read specialized packets to prepare for a specific role in the Town Meeting.

**Snapshot of a Town Meeting**

During the 1997-98 school year students participated in eight Town Meetings on issues ranging from whether it was right to drop the atomic
bombs to gun control. As previously mentioned, one criterion Ann uses to select issues is whether an issue is currently a matter of public deliberation. In the spring of 1998, it became apparent that an initiative banning local and state government affirmative action programs based on race and gender would be on the ballot in the fall. Ann decided it would be an especially good topic for a Town Meeting.

The affirmative action Town Meeting is showcased in this section of Ann’s portrait. First, I explain the unique Town Meeting discussion model that Ann has created and how she teaches the model to her students. Then I describe how her students prepare to participate in Town Meetings, and how Ann uses traits on an assessment rubric to remind them of the types of oral participation that contribute to successful Town Meetings. Next, I turn to what actually occurred during the Town Meeting, focusing on what the students said, Ann’s role as a facilitator, and how Ann assessed her students’ participation.

The Town Meeting Model. While a first-year teacher Ann developed the Town Meeting discussion model, which she defines as “a public forum where participants air their views on an important controversial issue as a way to either affect public policy, or educate others, or persuade others to come around to their point of view” (A.T. IV#5, p. 6). The primary reason she uses Town Meetings is her belief that it helps her students better understand multiple perspectives about the issue being discussed.

The Town Meeting model is a large group discussion in which each participant assumes the role of a person who has a particular perspective on the issue. The roles are crafted by Ann and her students to cover a broad spectrum of views on the issue. Students select their roles; Ann encourages
them to pick a role that represents a position other than the one they currently have on the issue. Additionally, Ann makes sure there is a relatively equal distribution of roles between the various points of view on the issue. That is, the students are not allowed to all select roles in favor of one position.

Before the first Town Meeting in the fall, Ann taught the model to her students by explaining the assessment rubric for the Town Meeting and showing them a videotape of an especially good Town Meeting from the previous year. Ann occasionally stopped the videotape and pointed out students' contributions that met the exemplary standards of the rubric. Thus, students first learned the model by viewing a positive example. Ann followed up with a negative example, showing a videotape of adults participating in a Town Meeting that was not effective. She had her students identify what the adults were doing wrong, such as monopolizing, not using evidence to support their opinions, and talking over one another.

Preparing for the Town Meeting. One week before the Town Meeting, Ann's students received a packet of background material on the affirmative action initiative to read. After one class period of didactic instruction on the issue, Ann and her students created the roles for the Town Meeting. Some of the roles were the Governor of the state, a university admissions officer, a newspaper reporter, a white business owner, a minority student, and representatives of education and advocacy organizations that had taken positions on the initiative. After selecting which role they wanted to adopt, each student was given a specialized reading packet that focused on the particular position of her/his role and a role sheet that required him/her to state the position and identify pro and con arguments. For the next three class
days, the students worked individually and in pairs reading the articles and preparing their role sheet. During this time, Ann also showed a video about affirmative action, opened her classroom to a guest speaker from an educational organization that had studied and taken a position on affirmative action, arranged for several students to call an advocacy organization that had funded the initiative drive, and gave students time to access the World Wide Web to search for additional information.

Classroom Arrangement. As Ann’s 29 seventh- and eighth-grade students enter the classroom, they immediately notice that the furniture has been reconfigured for the Town Meeting. The tables are arranged in a large circle and Ann has placed name tents listing the various roles on the tables. The students take their places. Ann intersperses the roles to ensure that a debate-like atmosphere is prevented. For example, at one table is the Governor (who opposed the initiative), while sitting at the next table are representatives of an advocacy organization that is supporting the initiative.

Explaining the Assessment Rubric. Because Ann assesses each student’s participation in the Town Meeting, she begins the class period by reminding students of what is on the assessment rubric. The three categories of traits on the rubric are: knowledge of subject matter, portrayal of role, and effectiveness as a participant. Ann briefly explains each trait while holding up the tally sheet she will use to mark throughout the Town Meeting.

Identifying Their Characters. The Town Meeting officially begins with each student stating her/his role and the position with which it corresponds. Ann does this as a verbal warm-up, as well as so that students will be reminded of the many roles. One student does not have a role because he has been absent.
Stating the Purpose of the Town Meeting. Next, Ann tells the students to "stand behind that character; give him the benefit of your voice" and states the purpose of the Town Meeting: "We're here to get the facts on how you feel about the initiative (A.T., FN). She then asks, "What does the initiative say? (A.T., FN). Several students quickly respond, explaining the major points of the initiative.

The Town Meeting. For the next ninety minutes all but two of the 29 students participate orally in the discussion about the affirmative action initiative. This is an unusually long Town Meeting; more typically they last fifty minutes. A class will occasionally have a discussion this lengthy due to a changing block period schedule.

Throughout the Town Meeting students raise their hands when they want to speak, and Ann calls on them, going back and forth between students who support and those who oppose the initiative. Because the initiative addresses affirmative action based on race and gender, the discussion alternates between statements specifically focused on each. For example, in the beginning of the discussion a student says, "The initiative says you can't discriminate [by preferring racial minorities and women]," and another student responds, "But sometimes people are naturally racist, really just trying to even out [the playing field] (FN)." A student immediately shifts the focus to gender when she says, "To add to that, 95% of management jobs in this state and in the nation are taken by men" (FN).

Bringing Value Tensions to the Fore. This shifting between race and gender continues throughout the Town Meeting, although several students contribute statements that contextualize the initiative in broader tensions between competing goods and values, such as equality vs. merit, and equality
vs. safety. One lengthy interchange about firefighters represents the latter tension. A student focuses attention on this tension by saying, “I don’t think we should risk people’s lives. Fire departments are forced to hire women because of affirmative action and they can’t do the job” (FN). Another student agrees and adds, “Many women couldn’t pass the physical tests that men had to pass to become firefighters so they changed the tests. Again, that is risking people’s lives” (FN). A few statements later, another student challenges this view by telling a long story about how women are discriminated against in the local fire department even when they score the highest on the tests. With this story she is shifting the discussion to the issue of merit, which is immediately reinforced by another students who says, “They [the fire department] should hire the person who is the most qualified” (FN).

Citing Factual Evidence. Ann’s students’ conflicting opinions on the affirmative action initiative parallels the public debate occurring outside of school. Some students think there is still a lot of discrimination against racial minorities and women, while others disagree. A student raises this issue by quoting from a study he has read: “A women or minority has only a 2% chance of being hired by a company that is run by white men” (FN). Another student immediately challenges that statistic and asks for where he read it. The first student points to an article he has in front of him. Another student uses stipulating language when he says, “Well, if what he said is true [the 2% statistic], that’s why we need to keep affirmative action” (FN). Several students stick to the topic, which finally causes one boy to ask, “Why do people discriminate when we’re all one race—the human race?” (FN) The simple wisdom implied in this rhetorical question momentarily silences the
entire class. A student then answers in a quiet voice, "I think people are afraid of what they don’t know" (FN).

The Influence of Their Roles. Throughout the Town Meeting, it is clear that the assumed roles are influencing the content of students’ comments in different ways. The student playing the Governor accurately and consistently represents what he has read about why the Governor of the state opposes the initiative. Conversely, some students quickly stray from their roles. For example, a student playing another Governor seems to change his mind midway through the Town Meeting and begins to advocate continuing affirmative action, which is not the position the real Governor supports. It appears that the roles played are not viewed as rigidly binding ideological directives by the students.

Ann’s Stance as a Facilitator. Ann is incredibly busy throughout the discussion. She directs discussion by calling on students who have their hands raised, assesses by marking on the tally sheet, and, on occasion, redirects the content of the discussion by asking clarifying questions and raising issues the students haven’t yet considered. When there are factual disputes that need to be clarified, Ann often inserts very short questions, such as "Are quotas legal?" and "Is the playing field level?" (FN). At other times, she helps students who are having difficulty clearly stating their comment or question by rephrasing it for the entire class. For example, after a student’s very murky explanation of the predictive value of test scores, Ann says, "So you are saying that test scores don’t predict job success" (FN). Midway through the Town Meeting, she stops the discussion and directs students to look at their role sheets. She then says, "Look for points that have not been made yet or that could be strengthened" (FN).
Only once during the ninety minute discussion does Ann need to intervene on a behavioral issue, and she does so without saying a word. About thirty minutes into the discussion, it becomes clear that three boys are not paying attention and are distracting others with whispered side conversations. She makes eye contact with them, gives them a classic “teacher look,” and points to the couch that is outside the circle, indicating that if they don’t start paying attention she will remove them from the Town Meeting. It works. They stop the whispering.

With a few minutes left in the period, Ann ends the Town Meeting by directing the students to turn to the person next to her/him and say anything about the initiative that he/she has not had time to contribute during the discussion. They do this, then hand in their role sheets to Ann and exit the room.

**Debriefing and Assessing the Discussion.** Similar to Elizabeth, Ann’s discussions often occur on Fridays, so the debriefing of the discussion does not occur until the following Monday. I did not attend the debriefing but Ann reported to me that two things typically occur after the Town Meetings. First, the students talk about what went well and what didn’t go so well. Second, Ann gives them her assessment of the Town Meeting, focusing on the traits listed on the rubric. Students also have their role sheets returned, with comments from her, find out how she assessed their participation in the Town Meeting, and sometimes get individual feedback on their participation.

**Summary of the Town Meeting.** Although the Town Meeting described in this snapshot was lengthier than is usual, in other ways it typified how Town Meeting discussions work in Ann’s classes. Ann usually solicits her students’ input when selecting the issue that will be the focus of
the Town Meeting and always co-creates the various roles with her students. Students select which role they want to adopt for the Town Meeting and spend several class periods learning about the issue, and the perspective represented by their role, by reading articles, searching Web sites, listening to guest speakers, viewing videotaped programs, and talking on the phone with experts on the issue. On Town Meeting day the students sit in a circle behind a name tag that states their role. Ann begins the discussion by reminding students of the traits on an assessment rubric and asking them to go around the circle stating their role and position. Students control the content of the discussion, although Ann does not let students speak unless until she calls on them. Throughout the discussion, most students participate orally, in part because Ann has signaled the value of talking in discussion through her assessment rubric. The next class session includes a short debrief of the discussion. Students also receive their role sheets back with feedback from Ann and find out how she assessed their participation in the Town Meeting.
Chapter Five

Propositions and Reflections on the Study

In this chapter, I state and reflect on propositions that capture theoretically what skillful teaching of CPI discussion entails. These propositions, and relationships among them, were induced from the CPI discussion conceptions and practice of Joe, Elizabeth, and Ann. Thus, understanding this chapter depends on reading the portraits of the three teachers in Chapter Four and the data on which the propositions and their relationships are based. As a reminder, however, I begin with a short summary of each teacher’s conceptions and practice. These are followed by four sections: (1) propositions that emerged from the study that, when related to one another, constitute an initial theory, (2) what this initial theory contributes to the literature on classroom discussion in the social studies and suggests for future research, (3) the implications of the initial theory for teacher educators, and teachers, and (4) limitations of the study. The chapter ends with a summary and some concluding thoughts about what I learned from doing the study that will inform my teaching and research about classroom discussion in the social studies.

The propositions cross the three research questions. As a reminder, however, the research questions undergirding the study were:

1. How do secondary social studies teachers who are skilled in the use of CPI discussions teach their students to participate effectively in such discussions?

2. What role do instructional strategies, issues, materials, and assessments play in this teaching process?
3. What conceptions account for these teachers’ approaches to CPI discussions? I investigated the teachers’ conceptions of four things: (a) democratic citizenship, (b) the purposes of social studies education, (c) what constitutes good discussion, and (d) their rationales for CPI discussions.

The Teachers: A Summary

Joe Park has taught middle and high school social studies for 22 years, the last five at a "break the mold" high school in a university community close to a major city in the western United States. Because his school has no set curriculum, Joe is able to create courses that closely mirror his conceptions of what knowledge and skills in social studies are most important for students to develop. One such course, Important Supreme Court Decisions, was taught for nine weeks in the fall and winter of 1997 and was the basis of my learning about how Joe uses CPI discussions. This course focused on landmark free speech decisions of the United States Supreme Court. Throughout the course, Joe’s 9th- through 12th-grade students each week read the decisions in one case, completed pre-discussion preparation activities, participated in a lengthy seminar discussion, debriefed the discussion, and wrote papers about their opinions on the Court’s decision. During the seminars, Joe is an active facilitator, asking many questions and working to keep his students focused on major ideas communicated in the Supreme Court decisions. Students are not formally graded for their oral participation in seminars, although they assess the seminar as a whole, and their individual participation, after each seminar has been completed. Joe also comments on their oral participation in the seminar on the narrative assessment he writes for each student at the end of the course.
Elizabeth Hunt has been teaching middle school language arts and social studies for 17 years. Her current middle school is located in a suburb outside of a large city in the West. Because her school is so large, students are divided into communities, each taught by a team of teachers. As the eighth-grade social studies teacher in one of the school's "communities," Elizabeth teaches a year-long course in American Studies that focuses primarily on civics and United States history. Throughout the course, Elizabeth selects CPIs for her students to discuss using a model, Public Issues Discussions, developed in the 1960s. Reflecting her interest in teaching her students the interpersonal skills necessary to participate effectively in such discussions, Elizabeth stresses equal participation and facilitates primarily to ensure that all students have the opportunity to participate orally. Using a formal assessment instrument, Elizabeth provides her students with feedback on their discussion skills and factors their preparation for and participation in discussion into their grades.

Ann Twain has been teaching social studies for five years at a magnet middle school in a suburb outside a large city in the Pacific Northwest. Because Ann's school has multi-age classes, she teaches social studies classes comprised of both seventh- and eighth-graders. The social studies class Ann has created combines United States history, civics, and world geography, laced with an extensive service learning program and whole-class discussions of CPIs in a format she has labeled Town Meetings. Ann and her students select the CPIs that will be the basis for the Town Meetings and then create a variety of roles representing people who would be interested in or affected by the CPI. After the students select roles to represent in the Town Meeting, they spend several days preparing by reading background articles, calling experts, and
viewing videotapes. During the Town Meetings, Ann facilitates with an eye toward the representation of multiple perspectives on the issue under discussion. Like Elizabeth, she uses a rubric to assess each student's oral participation in the discussion and counts both their preparation and participation as part of their formal grade.

Skillful Teaching of CPI Discussion: Beginnings of a Theory

From the similarities and differences among these teachers' CPI discussion conceptions and practice, one can induce statements that capture theoretically what skillful teaching of CPI discussion entails. The following propositions synthesize what can be learned from these cases. A proposition is something put forward for discussion; given the initial nature of this theory, this definition accurately represents how I want the propositions that make up this theory to be viewed. The use of the word "proposition" (instead of, for example, "hypothesis") is not meant to connote a lack of grounding for the ideas represented in each proposition. As discussed in Chapter Three, each proposition was induced from the data by following the tenets of grounded theory methodology.

Each proposition is described and explained in detail. I begin with an outline of the seven propositions. Please note that the word "teachers" in the wording of the propositions refers to skilled CPI discussion teachers. Following the outline is a description and explanation of each proposition.

Propositions

I. Teachers teach for, not just with, discussion. Discussion is both a method (of teaching students to create new knowledge, critical thinking skills, social studies content, interpersonal skills, and the like) and a desired outcome.
A. Teachers believe that discussion is a difficult outcome, so instructional time is devoted to preparing for, enacting, and debriefing discussions. Discussion is a priority in the teachers' curriculum.

B. Teachers teach their students discussion skills instead of presuming they already possess them.

II. Teachers work to make the discussions the students' forum.

III. Teachers select a discussion model and a facilitator's style that is congruent with their reasons for using discussion and their definition of what constitutes effective discussion. Thus, the selection and use of a discussion model is conception and rationale-driven. The selection and use of a particular discussion model creates tensions and tradeoffs that influence the type and quality of discussion in teachers' classes.

IV. Decisions about whether and how to assess students' participation in CPI discussions are influenced by an enduring tension between authenticity and accountability.

V. Teachers' personal views on CPI topics do not play a substantial, visible role in classroom discussion itself. However, teachers' views strongly influence the definition and choice of CPIs for discussion.

VI. Teachers engage in CPI discussion teaching practices that are informed by their conceptions of democracy.

VII. Teachers are receiving support for their CPI discussion teaching from school administrators, the overall culture of the school, and the school's mission. Thus, their CPI discussion teaching is aligned with, not in opposition to, what is expected in the school.
Description and Explanation of the Propositions

Teachers teach for, not just with, discussion. Discussion is both a method (of teaching students to create new knowledge, critical thinking skills, social studies content, interpersonal skills, and the like) and a desired outcome.

Across the conceptions and practices of the three teachers, there is evidence that an important reason they include discussion in their curriculum is to teach students how to participate effectively in discussions in other situations. Joe, for example, talks about the importance of scaffolding discussion instruction so students can participate in the "great conversations" of democratic society. Elizabeth’s emphasis on interpersonal skills, such as listening and yielding, is aimed at an outcome larger than the development of such skills. Recognizing that speaking and listening to others is critical to the success of a society in which CPIs are the norm, not the exception, she aims to inculcate in her students habits of communication that will allow them to participate effectively as citizens in a democratic society. Ann’s emphasis on preparing her students to participate actively in solving problems in their community drives her use of Town Meetings. Because she remembers what it felt like to be excluded from adult conversation about CPIs, she wants her students to feel they are citizens now—not just future citizens in training. Thus, an outcome of her use of CPI discussions is that students are familiar with and skilled at engaging in public discourse.

The primacy that these three teachers place on discussion as an outcome in its own right does not mean they are unconcerned about other outcomes that might result from students' participation in CPI discussions. Instead, it adds to the complexity of the rationales they hold for using discussion in the courses they teach. The breadth of their rationales for using
discussion and of the outcomes they hope their students will achieve as a result of participating in discussion also accounts for another phenomenon that extends across the conceptions and practices of the three teachers: their devotion of a large amount of class time to preparing for, participating in, and debriefing discussions.

If skilled participation in discussion is an outcome, it cannot be achieved through the use of other instructional strategies. For example, students cannot learn to participate more effectively in discussions by writing papers. Recall Joe’s statement about teaching students to talk and listen in discussion: “There has to be some sort of environment in society where kids practice doing that. They practice baseball batting, for God’s sakes, why can’t they practice talking?” (J.P. IV#2, p. 7). A result of viewing discussion as an outcome is that these teachers devote a generous amount of classroom time to teaching students how to prepare for discussions, how to participate in them, and how to debrief them so they will be improved. These teachers have elaborate lesson plans and direct the full battalion of their pedagogical content knowledge to the lesson planning process. Everyone (teachers and their students) does a lot of work to prepare for discussion. Clearly, there is an opportunity cost associated with devoting so much classroom time to discussion. Simply stated, time spent on this instructional goal is time lost from other educational activities that have value for students. For these three teachers, however, the value they place on discussion as both a method and an outcome outweighs the opportunity costs involved.

A further elaboration of the proposition that teachers teach for, not just with, discussion is seen in the ways that instruction is provided to students on the skills needed to participate effectively in discussion. All of the teachers
involve students in setting guidelines for what constitutes effective CPI discussion. Their practice is also similar in the way they use the debriefing that follows each discussion to focus students on what could be done as a group and as individuals to improve the quality of the next discussion.

There are also differences in how they approach the teaching of discussion skills. Not surprisingly, the teaching is more direct with the younger students. Elizabeth, for example, provides direct didactic instruction on various discussion skills, such as recognizing that other participants want to join the discussion. Ann has developed the most elaborate process for helping her students form an understanding of the differences between high and low quality discussions. By showing her students videotapes of effective and less effective discussions at the beginning of the school year, she aims to help her students create a common vision of what good discussion looks and sounds like. Joe provides more implicit instruction through including an "expert discussant" in most of the seminars and modeling the use of questions as the facilitator. Notwithstanding the different approaches the three teachers take to providing instruction on discussion skills, the three teachers do not assume that students already possess these skills. Quite simply, they think that discussion skills must be taught continually--it's not a one-time lesson.

In the process of teaching for and with discussion, the teachers are making many instructional decisions that bring to the forefront a key issue about the use of classroom discussion. The issue is one of ownership and control, and can be phrased as: Whose discussion is this? Addressed in the second proposition, evidence from this study suggests that the teachers are
concerned about structuring the use of discussion to create a forum for their students.

*Teachers work to make the discussions the students' forum.*

This proposition means that key decisions about CPI discussions are jointly made by the teachers and the students. For example, as previously mentioned, all three teachers involve students in creating guidelines that will be followed throughout the discussions. These guidelines are made public and referenced periodically as a way to both hold students accountable for following them and to remind students that the guidelines represent the group's will.

A second way that the three teachers work to make the discussions the students' forum is through involving them in selecting some, but not all, the issues to be discussed. The two middle school teachers were particularly responsive to issues that their students identified as interesting. For example, at the beginning of the school year Ann asked all her classes to brainstorm issues that would be the focus of future Town Meetings. While she did not select all of the issues for inclusion in the curriculum, some were selected, signifying to students that their concerns mattered to her. Similarly, Elizabeth included issues that were sparked by her students' knowledge of current events, such as a recent school shooting in Arkansas, and tensions between the United States and Iraq. While Joe's curriculum was more set prior to the beginning of the course, his students did have a choice between two Supreme Court cases as the topic for their final seminar.

The teachers' roles as facilitators of the discussions most clearly demonstrate how the three teachers work to enhance the likelihood that students will view the discussions as their own forum. Even though the
teachers' facilitation styles differ, the unifying emphasis is on encouraging the students to speak to one another, and not go through the teacher. Furthermore, while students are encouraged to hold and state opinions on the issues, the teachers' opinions are not explicitly stated. This finding will be elaborated in greater detail under the fifth proposition, but it is important evidence for this proposition as well. In all of the discussions I observed, viewed, or listened to, never did I hear a student ask the teacher his or her opinion on the issue, nor did I hear the teacher volunteer a position. More than any other single piece of evidence, this suggests to me that the teachers viewed CPI discussions not as a classroom soapbox on which they stood, but as a forum for their students.

The power of the students is limited in all three teachers' classrooms. The discussion model that the students learn to use (and use to learn) is selected by the teacher, without input from the students. Although all three teachers asked for students' feedback on how the model was working for them, they did not suggest that a different model for CPI discussions could be selected. One can imagine a curriculum in which students are introduced to various models for discussing CPIs and then given some choice about which model would be used for various issues. To take that route, however, would negate the strong linkage between the models selected by the teachers and what they are trying to teach. As explained in the first proposition, because all three teachers are teaching discussion as both a method and an outcome, it appears they are retaining the power to select a discussion model as a way to ensure their multiple outcomes are being met. Thus, there is a tension between the first two propositions. If one has multiple outcomes for using
discussion in the classroom, then there are limits to how much power students are given to influence the discussion model that will be used.

Given that the teachers did retain control to select the discussion model taught to their students, it is important to explain on what basis discussion models were selected. *Teachers select a discussion model and a facilitator's style that is congruent with their reasons for using discussion and their definition of what constitutes effective discussion. Thus, the selection and use of a discussion model is conception and rationale-driven. The selection and use of a particular discussion model creates tensions and tradeoffs that influence the type and quality of discussion in teachers' classes.*

This proposition suggests that a particular discussion model is selected because it concretizes and exemplifies how the teacher defines effective discussion and the rationales she/he holds for teaching for and with discussion. Moreover, once a model is selected, the style of facilitation that each teacher has created is also closely aligned with her/his definitions of and goals for discussion.

To illustrate this proposition, I will briefly restate each teacher's thinking about discussion and then link it to the model selected. First, recall that Joe believes that effective discussions involve a setting that promotes equality among participants, a sense that the participants "own" the content of the discussion, intense preparation on common content, active facilitation, and the creation of new ideas. Although he has many reasons for including discussion in his courses, he is most interested in promoting critical thinking and teaching students to participate in democratic discourse. His selection of the seminar model is so closely aligned to these conceptions of discussion to
be almost tautological. In seminar discussions, the creation of new ideas and understandings is paramount. But, unlike the models used by the other two teachers, the precise content that is being examined, challenged, analyzed, and extended must be shared in common. Instead of providing students with many different articles on the various free speech CPIs that formed the basis for the course we learned about in Chapter Four, Joe selected Supreme Court decisions that all students read. This common text is one of the primary attributes of the seminar model and is intrinsic to how Joe thinks about the primary goal of discussions, which is to teach students how to think. Without a common text, Joe fears that the students would not have anything in common about which to think.

Joe’s emphasis on improving students’ critical thinking skills is also linked to the active facilitation role he takes during seminars. Recall that Joe’s facilitation, primarily through questions, took up approximately 30% of the seminar air time. An apparent contradiction is indicated between Joe’s desire to have students view the discussion as their own and the fact that such a high percentage of the available time is taken by his questions. Joe, however, does not recognize any such contradiction because his active facilitation is designed to scaffold instruction on critical thinking. Joe sees his responsibility as the facilitator as both acting as a traffic cop and leading students in critically examining the ideas in the text.

Elizabeth and Ann also have selected discussion models and created facilitation styles that are congruent with how they define effective discussion and conceptualize the purposes for teaching for and with discussion. Elizabeth wants her students to develop interpersonal skills, an understanding of social studies content, critical thinking skills, and the ability
to participate in democratic discussion of difficult issues. The Public Issues Model she has selected is aligned to those goals. For example, as we learned in Chapter Four, this model separates issues into three types: factual, definitional, and ethical. Understanding the three types of issues is a significant part of the preparation that students engage in prior to the actual discussions. Her attraction to this model illustrates her interest in teaching the content of social studies, for the separation of issues into these sub-types reflects core social science content.

Through analyzing the choices Elizabeth makes about the role she will take as the facilitator, we see a clear intersection between what she values and what she does. Of the attributes that Elizabeth thinks effective discussion should have, two stand out as particularly important: the demonstration of interpersonal skills and equal participation among the discussion participants. Highly concerned with teaching her students to be civil discussants, virtually all of Elizabeth’s comments as a facilitator are directed toward that goal. When she does speak as a facilitator (which is rare), her comments typically focus on the need for students to listen, not monopolize, and to draw other participants into the conversation. This facilitation style creates a tension between deep analysis of particular social studies content versus widespread and relatively even participation in the discussion. Elizabeth is not aware of this tension because the twin goals of interpersonal skills and equal participation loom so large in her conceptions of discussion.

Finally, Ann’s conceptions and practice also support the proposition that the selection of a discussion model and a facilitation style is rationale-driven. Given that Ann specifically created a discussion model to implement her ideals for discussion, it is not surprising that such a tight link exists
between what she values and what she does. Particularly concerned about how multiple perspectives on issues are raised in discussions of CPIs, Ann has created a model that forces different perspectives to come into play in the different roles her students assume. While the students do not always stay "in role," they at least start that way and, as a result, the various ways that different people and groups would be influenced and affected by CPIs form the basis of CPI discussions in her classroom. When facilitating CPI discussions, Ann makes an effort to call on roles that have not yet been represented, again indicating a link between how she behaves and what she values. Ann's emphasis on multiple perspectives creates a tension not dissimilar to what is often seen in democratic discourse outside of the school. When multiple perspectives on an issue is the primary goal, the depth of exploration on particular parts of the issue may be sacrificed. Ann appears not to be aware of this tension.

In addition to those created by the discussion model and facilitator's role selected by each teacher, there are ways in which tensions between competing goods influence their use of CPI discussions. This was most evident in the decisions made by the teachers about the assessment of their students' participation in CPI discussions.

*Decisions about whether and how to assess students' participation in CPI discussions are influenced by an enduring tension between authenticity and accountability.*

One of the clearest differences in how the teachers approach CPI discussion teaching is seen in the decisions they make about assessment. I am defining assessment broadly to include how teachers and students find out and make judgments about students' progress toward desired educational
outcomes. This definition collapses the distinction that is often made between assessment as finding out and evaluation as judging (Parker & Jarolimek, 1997). All three of the teachers assess their students’ participation in discussion. Both of the middle school teachers assess their students’ preparation for and participation in CPI discussions through the use of codified rubrics. Both of them also count how their students prepare for and perform in discussion as a formal part of their grades. Holding students accountable for their preparation and performance in CPI discussions and rewarding oral participation in CPI discussions reflect the middle school teachers’ concern about aligning their assessment procedures to what is valued in their classrooms. Conversely, while Joe does assess his students’ seminar participation, he is adamantly opposed to the grading of seminar participation, because he believes that “paying kids to talk” is inauthentic. That is, it does not represent the way public discourse operates in the world outside of school. Moreover, Joe believes that grading oral participation would be at odds with the creation of effective seminars, in which participants should talk because they have something to say, not because they are being rewarded by an authority figure.

Thus, comparing and contrasting the assessment practices of the three teachers makes apparent a tension between accountability and authenticity. Ann and Elizabeth have chosen to privilege accountability because they believe that if they value discussion, assessing it in a fairly formal way with a rubric delivers the message of its importance. Additionally, formal assessment gives them the opportunity to provide specific feedback to students about what they do well and what they still need to improve. Explicitly then, both Ann and Elizabeth have made a choice about the issue of
whether students should be required to participate orally in CPI discussions. They have decided that requiring such participation (through formal assessment) is important because it communicates a message that democratic discourse is a critical outcome of their curriculum.

Joe, on the other hand, has chosen to privilege authenticity. Because he believes that "paying kids to talk" will destroy the genuineness of a seminar, he is willing to allow some students to remain silent. While students in Ann's and Elizabeth's classes can also make that choice, there is a cost involved. No such cost exists for students in Joe's class.

Just as the teachers' views on whether authenticity or accountability should be given more weight when making decisions about assessing CPI discussions, there are other ways in which teachers' views influence their CPI discussion teaching practices. In particular, the teachers' personal views on CPI issues mattered—but not in the way I expected they would.

*Teachers' personal views on CPI topics do not play a substantial, visible role in classroom discussion itself. However, teachers' views strongly influence the definition and choice of CPIs for discussion.*

The literature on CPI discussions suggests that what is important about teachers' personal views on issues is the way and extent to which they are directly communicated to their students in the classroom (Lockwood, 1995; Kelly, 1986). This proposition suggests that the disclosure of teachers' personal views on issues is not what is significant to the practice of these skilled discussion teachers. As mentioned in the explanation of proposition two, there was *not a single example* in the many discussions I observed, listened to, or viewed of a teacher being asked his or her personal opinion on an issue by a student. Moreover, none of the teachers *ever* volunteered their personal
opinion on an issue. While this lack of disclosure is evidence that the discussions are the students' forums, it does not mean that teachers' personal views on issues do not matter in their CPI discussion thinking and teaching. The teachers' personal views on what is a CPI in the first instance, and on specific CPIs, clearly influences what issues students are allowed to discuss and the materials they are exposed to when preparing for discussion.

As an example of how teachers' personal views about CPIs influence their discussion teaching, Figures 7, 9, and 11 illustrate the different reasons the three teachers gave for not including gay rights as a CPI in their curriculum. Note that Joe does not believe that gay rights issues are CPIs. Instead, he likens such issues to human rights issues on which there are no legitimate differing views. Thus, he does not select gay rights as CPIs because his personal value system directs him to treat such issues as moral issues with one clearly right position. About gay rights, Joe stated, "The correct answer is that people should not be discriminated against on the basis of race, gender, ethnicity, sexual preference, physical disability" (J.P. IV #6, p. 10). Likening the denial of equal rights for gays to historical abuses of human rights, such as slavery and Nazism, Joe advocates including gay rights in the curriculum, but as an example of the denial of civil rights, not as a CPI.

Elizabeth and Ann's decision not to include gay rights as a CPI discussion topic also reflects personal views, but their views are different from Joe's. Elizabeth's personal discomfort with gay rights issues keeps her from including them as CPIs open for discussion in her classroom. Additionally, teaching in a conservative community, she worries that including gay rights issues would spark too much controversy and community disapproval. Ann does not include gay rights issues because she
worries her students would not discuss them with sensitivity and that gay
students in her classes would feel uncomfortable.

Another way in which the teachers' views influence their discussion
teaching practice is found when exploring their conceptions of democracy and
social studies.

*Teachers engage in CPI discussion teaching practices that are informed by
their conceptions of democracy.*

All of the teachers in this study are teaching for and with discussion, at
least in part, because they see a connection between a healthy democracy and a
citizenry that can participate skillfully in discussions of CPIs. While there are
many similarities in the teachers' discussion conceptions and practice, there
are key differences as well, especially evident in the model of discussion used
in the classroom and the role the teachers take when facilitating and leading
discussions. One reason for these differences is that they represent differences
in how the teachers view what should happen to ideas in a democracy.

Joe believes that ideas should be collaboratively created by discussants.

Recall the bread-baking metaphor he used to explain what constitutes
effective discussion. Various ideas put on the discussion table were like
ingredients in the making of bread, and only through combining them would
true democratic discussion occur. In the process of this combination, ideas
would be challenged and new understandings would emerge. This
conception places a premium on the ability of discussants to create, which
represents his view that personal agency is what matters in a democracy.
People must feel they have the ability to create new understandings and new
solutions to problems. His selection of the seminar discussion model is
closely aligned with this understanding of what happens to ideas in a
discussion. That model demands the exploration and creation of ideas through a tightly-focused analysis of specific text. Moreover, his active role as a discussion leader is designed to facilitate his students' exploration of and creation of ideas. Recall that virtually all of his oral participation as a discussion facilitator involves asking questions and encouraging his students to challenge the ideas in the text and those created by their classmates. By taking this role, Joe believes he is scaffolding and modeling what citizens should do in a democracy--create new ideas and understandings through the analysis and challenging of ideas.

Elizabeth's discussion teaching is also informed by her conceptions of what should happen to ideas in a democracy. Foregrounding equality as a democratic principle, she is most interested in teaching her students that all discussants have both a right and a responsibility to share ideas about the CPI that is being discussed. Unlike Joe, who insists that students read and discuss a common text, Elizabeth carefully selects different materials geared to the reading ability of various students. In this way, she is hoping that all students will be able to contribute to the discussion. Thus, she is differentiating background information in order to achieve equal participation in the discussion. The discussion model she has selected represents her belief about the importance of sharing ideas in discussion. Because the model focuses on an issue instead of a specific text, all of her students are able to bring something to the discussion. Her primary concerns are that all students participate orally in the discussion and that they learn to do so in a civil manner. Her facilitation style further exemplifies the importance she places on the equal sharing of ideas in discussion. Recall that she intervenes rarely,
but when she does say something the goal is to encourage civil behavior and equal participation.

Like Elizabeth, Ann also believes that ideas should be shared in a democracy. Her concern with the representation of multiple perspectives, however, illustrates her interest in ensuring that the ideas that are shared reflect the interests and concerns of a broad range of people. Although Ann differentiates the materials students use to prepare for discussion, the materials are selected for both accessibility in terms of reading level and the extent to which they represent the views of various groups in society. In her creation of a discussion model, Ann’s interest in sharing ideas that represent multiple perspectives is also evident. The Town Meeting model that Ann has created mandates the representation of different views because students select roles designed to represent a wide range of opinions about a particular CPI. In her facilitation of the Town Meetings, Ann works to get a variety of perspectives represented, which results in fairly equal participation because the students are in different roles.

The final proposition shifts the focus away from how the teachers’ conceptions inform their practice to how the school context in which the teachers are working supports and influences their ability to teach CPI discussions.

*Teachers are receiving support for their CPI discussion teaching from school administrators, the overall culture of the school, and the school’s mission. Thus, their CPI discussion teaching is aligned with, not in opposition to, what is expected in the school.*

Representations of skilled teachers in popular culture, especially in film, often portray them as remarkable because they differ from what is
valued in their schools. This proposition suggests the opposite—these skilled discussion teachers are supported by the larger school environment and are teaching in alignment with what is valued by others in the school community.

Joe's school was created to be an alternative to the traditional high school. Formed on a foundation of constructivist learning theory, the school day and school week are purposely structured to encourage seminar discussions. While not all teachers use seminar discussions, they are not unusual. Consequently, some of Joe's students come into his classes with experience in participating in discussions and can act as role models for others without such experience. Moreover, the principal of Joe's school participates frequently and actively in seminar discussions in Joe's classes. This level of support from a principal is unusual and undoubtedly contributes to Joe's sense that what he is trying to teach his students is considered important by others in the school. Finally, Joe is able to create a unique curriculum that is particularly well-suited to the discussion of CPIs. The Important Supreme Court Cases class focuses just on First Amendment cases, allowing his students to analyze and deliberate about CPIs embedded in the First Amendment with more depth than would be typical in a high school curriculum. Joe has found a school that values what he thinks is important and, as a consequence, does not need to spend his time convincing others to let him teach in a particular manner. His time can be devoted to the difficult task of planning instruction on the discussion of CPIs.

Like Joe, Elizabeth has also found a school environment that matches what she thinks is important. Although her school is much more traditional than the one in which Joe teaches, it matches her ideas about the kind of
school experiences that are appropriate for middle school-age students. The curriculum that Elizabeth teaches is one that she helped to create and it allows her enough flexibility to select issues for discussion that are relevant to her students because they are current matters of public deliberation. Recognizing that the school and community are fairly conservative, Elizabeth is careful when selecting issues and shies away from those that would directly challenge community norms. Like Joe, Elizabeth is comfortable in the school and feels like she is a valued team member.

Ann has also found a school that is particularly well-suited to her conceptions of the type of education that young people should experience. Like Joe's school, hers was formed as an alternative to the traditional schools that are more common in her school district. Her school values innovation and provides support for the curriculum she has created. For example, the school's daily schedule can be rearranged to allow a longer period for Town Meetings. While her principal does not participate in Town Meetings as frequently as Joe's principal participates in seminars, she does participate in at least one per year. In their other classes, Ann's students are frequently asked to participate orally in various kinds of discussions, which results in a school-wide norm that supports the kinds of behaviors that Ann is teaching her students in the Town Meetings.

All of the teachers in the study are in the fortunate position of working in a setting that reinforces and supports their teaching practice. As is the case when buying real estate, the lesson here may be that one of the factors that is important to the skilled teaching of CPI teaching is location, location, location.
From Propositions to an Initial Theory

The seven propositions are the building blocks of an initial theory about skilled CPI discussion teaching. Here I will identify and describe ways in which the building blocks connect to one another toward the goal of building an initial theory. As recommended by Glaser and Strauss, this initial theory is based on the idea of "theory as process; that is, theory as an ever-developing entity, not a perfected product" (1967, p. 32).

Recall, in Chapter Two I presented the conceptual framework on which this study was based. Figure 12 is a revised conceptual framework that illustrates ways in which the propositions are linked to one another. These linkages form an initial theory because they suggest relationships among the propositions. The relationships are broader and more abstract than the propositions, giving them more explanatory power.

Figure 12. Revised Conceptual Framework
While many of the relationships are suggested in the explanations of the propositions, here I briefly describe each of the six major ways in which the propositions influence one another. The first relationship is marked "A" in the figure. It shows that propositions one and seven are linked. Recall, that the first proposition explains the multiple rationales that teachers have for teaching CPI discussions and further states that discussion is a priority for the teachers because there are so many rationales they have for teaching it to their students. Linkage "A" illustrates that the teachers have support from the school community and school mission for making discussion a priority in their curriculum.

The teachers' multiple rationales for CPI discussion are also linked to their conceptions of democracy, a relationship that is shown in "B" on the figure. Teachers believe that a healthy democracy depends on a citizenry that can effectively participate in discussions of CPIs. That is exactly what "teaching for discussion" is designed to create. Skillful discussion participation is one of the desired outcomes because the teachers believe it will lead to a healthier democracy.

A third way in which the teachers' rationales for CPI discussion are important is seen in the relationship marked "C" which illustrates how the teachers' instructional plans and practices are informed by their rationales for teaching discussion. The most specific relationship in this category was previously explained in proposition three. Recall, that proposition states that teachers select a discussion and a style of facilitation based on their rationales for using discussion. There are, however, other ways in which rationales influence plans and practice. For example, if a teacher is especially concerned
with using CPI discussions to teach interpersonal skills, the assessment of a students' discussion ability would focus on those skills.

Teachers' conceptions of democracy influence their CPI discussion teaching plans and practices, labeled relationship "D" in the figure. As previously explained in proposition six, what the teachers think should happen to ideas in a democracy affects the selection of a discussion model and the style of facilitation used during discussions. What the teachers think about democracy also influences their interest in working to make the discussion the students' forum--seen at its clearest in the unexpected finding that the teachers never disclosed their personal views on a CPI to their students.

Within the category of instructional plans and practices are relationships between students' forum and facilitation style, students' forum and issue selection, and students' forum and model selection. Marked as "E" in the figure, these relationships have already been explained within the propositions. It is important to reiterate, however, that the tension between creating a students' forum and selecting a discussion model represents yet another way in which the teachers' multiple rationales for discussion are important. That is, teachers select a discussion model that is best suited for the many rationales they hold for CPI discussions. The teachers' "voice" is strongest with this decision. Perhaps to balance the power somewhat, students are given more of an influence when selecting issues.

The final relationship illustrated in the revised conceptual framework figure is marked "F" and connects the larger school environment to the teachers' CPI instructional plans and practices. Beginning with the big picture, the very fact that the teachers can devote so much time to teaching CPI
discussions without getting any administrative or peer disapproval is important, for these are schools in which the teachers and administrators talk about the curriculum. That is, these schools are not like some where the twin norms of privacy and autonomy result in teachers and administrators not knowing what others are doing. In more specific ways, the teachers are also receiving support from others in the school community for their CPI discussion teaching. For example, in two of the schools, participation in CPI discussions is represented on the social studies portion of the report card sent home to parents. Another example is the involvement of two of the schools' principals as participants in CPI discussions.

Taken together, the relationships between the propositions form an initial theory that explains how skilled CPI discussion teachers conceptualize and practice with respect to CPI discussions. What this theory contributes to the relevant literature, and how it could inform the work of teachers, teacher educators, and researchers is discussed in the next section.

Contributions to the Literature and Implications for Research

This study elaborates, specifies, and challenges existing literature related to classroom discussion in social studies in three areas: teachers' conceptions of discussion, assessment of discussion, and the influence of teachers' personal views on CPIs. First, I analyze the relationship between the initial theory presented in this study and what previous research has shown about teachers' conceptions of discussion and what influences those conceptions. Second, I discuss the tension between authenticity and accountability in light of recent literature on authentic assessment. Third, in the area of curriculum content, I explain how this study challenges what previous literature states is important about teachers' personal views on CPIs.
Following each explanation of the contribution this study makes to the literature, I suggest further research related to each of the three areas.

**Teachers' Conceptions of Discussion**

Recent research on teachers' conceptions of classroom discussion has shown that high school social studies teachers have multiple conceptions of discussion and that these conceptions are variously implemented based on the objectives of a lesson (Larson, 1995). Moreover, teachers use discussion to accomplish varied goals, which can be separated broadly into process and product (or outcome) categories (Larson, 1997). This study confirms, challenges, and adds to these important empirical findings on classroom discussion. As explained in the first proposition of the theory, teachers use CPI discussions as both an avenue to accomplish multiple objectives (such as critical thinking and interpersonal skills) and an outcome. This is exactly what Larson found in his study.

Larson's (1995) finding suggests that teachers use a variety of types of discussion, selected to achieve different goals. This study found the opposite—that each teacher uses just one model of CPI discussion, selected based on the teacher's rationales for discussion and definitions of what constitutes effective discussion. There are at least two possible explanations for the differences in Larson's findings and my own. One is that this study concentrated on discussions of CPIs, whereas when Larson selected his sample of six high school social studies teachers, he sought those who used classroom discussion of various topics, not just CPIs. This narrowing of discussion topics to just CPIs, by definition, may have resulted in the selection of teachers who were more prone to concentrate on just one type of discussion. Another explanation is that teachers who are selected because they are considered
skilled discussion teachers may be more likely to use just one model because it allows multiple practice opportunities for their students. If students use the same model time and time again, it is more likely they will become fluent in participation in that type of discussion. Larson’s sample was selected based on their principal’s nomination of them as thoughtful and effective teachers, which does not necessarily mean they were especially skilled at teaching discussion.

What this study adds to Larson’s (1995) findings is another explanation of what accounts for teachers’ various conceptions of discussion. As explained in proposition five, teachers’ conceptions of discussion (and, by extension, what they consider effective discussion) are informed by their conceptions of democracy. Given that the teachers in this study are using CPI discussions, at least in part, because they see a connection between participation in such discussions and a healthy democracy, it logically follows that their ideas about democracy will inform how they conceptualize discussion. The teachers’ differing notions about democracy, then, help to explain why each selected a particular discussion model, and what kind of communication is valued in discussion. This finding adds to Larson’s (1995) explanation of how lesson objectives influence teachers’ conceptions of discussion. In his study, teachers used various types of discussion based on their particular objectives for a lesson. In this study, teachers’ lesson objectives for discussion were informed by broader, overall objectives based on their conception of democracy.

Further research could broaden and deepen the emerging theoretical framework on teachers’ conceptions of discussion. Working with the distinction between teaching for and with discussion, two questions are important for further research. First, how does conceptualizing discussion as
both a democratic outcome and a method relate to skilled discussion teaching? Given that the teachers in Larson's study were not selected because of their discussion teaching prowess, and that only three teachers participated in this study, it is necessary to explore further whether the connection that I am suggesting between teachers' conceptions and skilled discussion teaching holds up or is repudiated or modified when additional skilled teachers are studied.

In addition to increasing the number of teachers who are studied, it is also important to investigate whether the hypothesized connection between skilled discussion teaching and for/with conceptions applies to teachers working with a more racially diverse student population. This initial theory was formed based on the thinking and teaching of three white middle-class teachers who teach, for the most part, white middle-class students. Thus, the initial theory begs to be "tested" in other contexts that are more representative of the socioeconomic, racial, and ethnic diversity that exists in the United States, and in which CPI discussion teaching may take on new dimensions.

Assessment of Discussion

Literature on classroom assessment suggests that the most powerful assessments of students' learning are classroom-based (as opposed to district or state level) and tightly aligned to curriculum and instruction (Miller & Singleton, 1997; Stiggins, 1997). Additionally, assessment experts (Martin-Kniep, 1998; Newmann & Wehlage, 1995) recommend that teachers assess students' progress toward goals that are valued in the world beyond school, which is often called authentic assessment. Educators who specialize in the assessment of CPI discussions (Harris, 1996) recommend formal assessment of students' participation in discussion as a way to communicate to students that
discussion is valued and to provide students with the specific feedback they need to improve their discussion skills. As explained in proposition four, skilled discussion teachers vary in how they approach the assessment of students’ participation in CPI discussions. Framed as a tension between accountability and authenticity, this proposition both reinforces and challenges the literature on classroom assessment.

The teachers in this study who gave precedence to accountability over authenticity did so because they felt formal assessment (i.e., using rubrics and grading) of discussion participation communicates the importance they place on discussion to their students. It also provides their students with a sense of how they are progressing toward the discussion goals the teachers had identified and codified in the discussion rubric. Equally important, however, is the explicit decision the teachers who are formally assessing have made about whether oral participation is required of all students. In short, all students must talk or pay a price for their silence. The advantage of this stance toward the “choice” issue (i.e., whether students may choose to be silent without penalty) is that it reinforces high and common standards. Unlike some classrooms where only the already-verbally proficient students participate orally in classroom discussion, these teachers recognize the connection between practice and progress. Grading students’ participation orally in discussions is, therefore, an example of the connection assessment experts see between what is assessed and what is communicated about the importance of all students’ learning.

But, there is a downside to formal assessment of discussion participation, which is captured in the phrasing of proposition four as a tension between authenticity and accountability. Recall that Joe refuses to
formally assess students’ oral participation in discussion because to do so could jeopardize the authenticity of the discussion. This reasoning directly challenges the literature on authentic assessment because it underscores the problems associated with common standards for students. Common standards only work if there is agreement about what good performance looks and sounds like, and it may be that discussions, especially those that occur in a large group, work best if participants are behaving in different ways. Authentic examples of CPI discussions that occur in the world beyond school do not demand that all participate in the same manner. Think of a particularly good discussion among community members about how to solve a public problem. We would expect that some people would talk more, and some less. We would expect that some people would use analogies to explore the problem, while others would use statistical evidence. We would expect that some people would ask many questions, while others would use examples from their personal history to explore the problem or suggest solutions. In short, we would expect difference. Yet, discussion rubrics that are specific enough to be helpful to students do not allow for difference. They explicitly identify common ways that people should behave in a discussion. Thus, proposition four, which identifies a tension between accountability and authenticity, can be viewed as a direct challenge to the advantage of high and common standards that are assessed in a meaningful manner.

Assessment experts (Martin-Kniep, 1998; Newmann & Wehlage, 1995) are clear that two of the purposes of classroom assessment are to improve teaching and learning. With respect to the assessment of students’ participation in CPI discussions, further research is needed to examine whether the tension between accountability and authenticity is serious
enough to inform decision-making about whether and how to assess classroom discussion. A study of what influence various kinds of assessment practices have on students’ abilities to participate effectively in classroom discussion of CPIs is needed. It may be that a student study could shed light on whether losing some authenticity is an acceptable trade-off if the result is enhanced student learning.

**Teachers’ Views of CPI Issues**

Little is known about how teachers select CPIs for classroom discussion: accordingly another contribution this study makes to the literature is to demonstrate/identify that, at least for some teachers, their personal views on CPIs inform their selection of issues. As was explained in proposition five, these skilled discussion teachers do not believe that all CPIs have equal curricular value. In fact, just the opposite was found. These teachers select CPIs based on a variety of factors—one of which is their own view about whether the issue is really a CPI, and, if so, whether it meets enough of the criteria they have created for content selection to warrant inclusion in the curriculum. Recall, that some issues were also not selected because of the teachers’ personal discomfort with the issue, concerns about whether students could discuss particular issues in a sensitive manner, or worries of community disapproval.

This is a particularly important finding because of its potential to influence the focus of scholarly discussion on how teachers’ personal views on CPIs influence their discussion practice. Previously, most of the literature on teachers’ personal views on CPIs focused on whether they should be shared with their students (Lockwood, 1995; Kelly, 1986). But, that question presumes that a CPI has already been selected for discussion. This study
suggests that an equally important question for teachers and researchers is how teachers' personal views inform the decisions they make about what is discussed in the first instance.

Further research is needed on how teachers select CPIs for classroom discussion. In particular, a more in-depth exploration of how teachers define a CPI is necessary in order to deepen understanding or what teachers consider legitimate controversial content and how that content should be learned by students. Recall, that Joe conceptualized gay rights not as a CPI, but as a human rights issue. His personal views clearly influenced his definition of what constitutes a legitimate CPI. A larger and more diverse sample of teachers is needed to further investigate whether and how the distinction between human rights issues and CPIs is informing the content selection and pedagogical decisions made by teachers.

**Additional Research**

In addition to the research that stems directly from this study's findings with respect to teachers' conceptions, assessment, and issue definition and selection, there are other studies for which this dissertation has established a need. Here I briefly describe three additional studies.

This study sheds no light on the question of what, in fact, the students of these teachers learn from participating in CPI discussions, or whether the discussions do, in fact, improve their ability to participate, relative to their skills before discussion instruction. A study of what the students of skilled teachers learn from discussion, how they learn to participate more effectively in discussions, and how they experience their learning would contribute to the literature on classroom discussion.
If other researchers think, as I do, that these teachers are skillful, then the question is raised: How did they learn to teach in this manner? There has been virtually no research on the effect of teacher education or professional development programs on teachers’ ability to teach for and with discussion. A study that investigated the effect of such programs on teachers’ ability to teach discussion would be a particularly helpful contribution to the literature. A research question for this study is: In what ways do various professional development programs on classroom discussion influence teachers’ discussion teaching practice?

Finally, another way to develop the initial theory presented in this study would be to have the three teachers whose conceptions and practices the theory was based on work together to examine, challenge, and extend the discourse among them about their conceptions and practices. The initial theory might thereby result in a revised theory that is richer and more satisfying than the one that I, as a single researcher, have constructed. I envision a research process that begins with sharing the entire dissertation with each of the three teachers. Then, each teacher would be asked to videotape a classroom discussion that is fairly representative of how CPI discussions work in his or her classroom. The videotapes would be viewed by all three teachers. Through the use of focus group methodology, the three teachers and I would work together to examine discussion practice and develop a revised theory.

Implications of the Theory for Practice

This study can contribute insights to the practice of teacher educators and secondary social studies teachers.
Implications for Teacher Educators

The problem this study was designed to address is that few secondary social studies students are given opportunities to learn to participate effectively in classroom discussions of CPIs. Teaching tomorrow’s secondary social studies teachers how to teach students to discuss such issues is one way of addressing the problem. Teacher educators, then, have a responsibility to teach preservice students about classroom discussion. In what ways might this study help them fulfill that responsibility? Here I suggest three specific ways that teacher educators may want to use this study to inform their teaching of preservice students.

The portraits could be used to counteract the “apprenticeship of observation” (Lortie, 1975) that causes many teacher education students to lack a conception of what constitutes effective classroom discussion and/or to understand that discussion must be taught (Parker & Hess, forthcoming). One of the difficulties teacher educators face is that there are few models of skillful classroom discussion available to use when teaching preservice students. Although there are some videotapes showcasing the work of skilled discussion teachers, a finding from this study is that much of the work of discussion teaching occurs in the planning and preparation for discussion. Thus, viewing a videotape of a classroom discussion does not adequately explain the many steps that the teacher and students go through to prepare for discussion. Teacher educators may want to have their preservice students read and discuss the portraits as a way to bring to the fore the elaborate lesson plans that each teacher has developed.

The lesson plans embedded in the portraits could also be used by teacher educators who, themselves, lack confidence in how to teach their
teacher education students to participate effectively in classroom discussions. For example, if a teacher educator was interested in teaching one of the discussion models showcased in the portraits, the teachers' lesson plans could provide useful guidance in how to prepare for and structure the discussion. It may be that combining demonstration lessons using a particular model with reading the accompanying portrait could be a helpful way of engaging preservice teachers in an analysis of both how they experienced the model and how it was used with secondary social studies students.

In addition to using the portraits, teacher educators may want to share the original and revised conceptual frameworks in this study with preservice students to help them think about possible interactions between teachers' conceptions, instructional plans and strategies, instructional practice, and contexts. By doing so, teacher educators would be working to complexify preservice students' ideas about the purposes and teaching of classroom discussion. My experience suggests this would be a helpful because many preservice teachers fail to grasp the reality that teaching students to participate effectively in classroom discussion is a challenging enterprise that requires intense preparation and focus.

Implications for Secondary Social Studies Teachers

The three portraits, and the propositions and initial theory that emerged from them, can help teachers reflect on, challenge, and strengthen their practices. By reflect on, I mean they can be used as "teaching cases" to bring to the fore issues related to teaching for and with discussion. For example, teachers may want to compare and contrast the cases as an avenue toward deeper understanding of important instructional decisions.

Considering, for example, the difference between how Joe, Elizabeth, and Ann
approach the assessment of discussion may help teachers become aware of the tensions and tradeoffs embedded in this process.

By challenge, I mean the portraits and initial theory can be used to help teachers examine their own practices in relation to what these three teachers are doing. For example, one of the common problems in classroom discussion is that many students choose not to participate orally (Bickmore, 1991). Yet, in all three of these teachers' classes, few students are making that choice—in fact, quite the opposite. Teachers may want to analyze the three portraits as a way to challenge the oft-stated view that some students will just never participate orally in classroom discussion.

By strengthen, I mean that the portraits and the initial theory can be used to improve teachers' thinking about, and practice of, classroom discussion. For example, if a teacher is intrigued by seminar discussions, she may want to experiment with the detailed lesson plan that Joe has developed to teach his students to "do the work of" seminars. A teacher who is interested in foregrounding multiple perspectives in the social studies curriculum may want to adopt and/or adapt Ann's Town Meeting model.

**Methodological Strengths, Limitations, and Issues**

Focusing on and evaluating the credibility of this study illuminates its strengths and limitations. In making judgments about this study, I rely on how experts in qualitative research discuss the term "credible." Credibility is broadly synonymous with trustworthiness (Lincoln & Guba, 1985, p. 301) and is to the qualitative researcher what internal validity is to the quantitative researcher. As Miles & Huberman stated, credibility is concerned with truth value and can be assessed by asking such questions as: "Do the findings of the study make sense? Are they credible to the people we study and to our
readers? Do we have an authentic portrait of what we were examining?" (1994, p. 278)

Assessing the Credibility of This Study

In assessing the credibility of this study, I focus on two large categories—the quality of the data and how it was interpreted.

Data Quality. Beginning with the data on which the entire study is based, it is important to analyze whether the data was accurate and complete. As Maxwell states, "The main threat to valid description, in the sense of describing what you saw and heard, is the inaccuracy or incompleteness of the data" (Maxwell, 1996, p. 89). I assess the accuracy of the data in this study as high and the completeness of the data as relatively low.

I rate the accuracy of the data as a strength of this study because of the techniques I used to capture the data, the ways in which I triangulated the data, and the use of member checks to correct for mistakes in the data. I discuss each in turn. As explained in Chapter Three, various types of data were collected. For example, I interviewed the teachers, collected artifacts of their CPI discussion conceptions and practices, and observed many CPI discussions. All of the interviews were audiotaped and transcribed, which obviously allowed for a precise record of what was said during the interviews. All of the classroom discussions I observed were also either audiotaped or videotaped, and I took field notes as well. Thus, I had an accurate record of exactly what transpired during the discussions. Taping the discussions was an important check on the accuracy of my field notes. As a novice researcher, I recognize that taking good field notes is a craft that I must continue working to develop. By relying on the tapes along with the field notes, I could recreate
a more accurate and complete picture of what happened during the discussions.

Furthermore, the multiple types of data collected allowed for triangulation of the data. Triangulation, in a qualitative study of this type, means that different methods are used to collect data from the same source (Lincoln & Guba, 1985, p. 306). As a measure of credibility, triangulation is important because it decreases the possibility that the data collected is not representative of what one is trying to understand. For example, in this study understanding how the teachers defined effective classroom discussion was important. I first had the teachers define effective discussion through watching the videotapes of discussions in other teachers' classes. This strategy was designed to reduce the tendency for teachers to create a definition that fits their own practice. Additionally, in later interviews, I asked the teachers to assess the discussions I had observed, viewed, or listened to. Thus, I used different methods to solicit data that could help me understand what these teachers thought constituted effective discussion.

A third reason I rate the accuracy of the data in this study positively is because of the use of member checks. Member checks, "whereby data, analytic categories, interpretations, and conclusions are tested with members of those stakeholding groups from whom the data were originally collected, is the most crucial technique for establishing credibility" (Lincoln & Guba, 1985, p. 314). In this study, I used member checks by sharing with each teacher a draft of his/her portrait and having a conversation with each about the extent to which the portrait was accurate, or "rang true." As described in Chapter Three (p. x), the teachers' feedback was critical to the correction of some factual
errors which, if allowed to remain unchanged, would have decreased the credibility of the study.

One challenge to the accuracy of the data in this study was reactivity, defined by Maxwell as, “the influence of the research on the setting or the individual studied” (1996, p. 91). I felt this was particularly a problem for Elizabeth because I had been her teacher. Thus, I was concerned that she might parrot back what I had said while teaching the discussion class. If, in fact, she was simply describing her practice, that would not be a problem, but if she was changing the description of her practice to please me by incorporating what she had been taught in the discussion class, then the credibility of the study would be seriously impaired. I dealt with this in two ways. First, I told her I thought this might be a concern. We discussed why it was important for her to describe her conceptions and practice, not what she had heard from me when I was her teacher. Second, I carefully monitored my tone of voice and the phrasing of questions to guard against being perceived as judgmental.

Notwithstanding the overall accuracy of the data, its completeness is less impressive, thus raising what I consider to be a major limitation of the study. As discussed in Chapter Three, I learned about how the teachers enact CPI discussions through interviews and observations. The observations were critical to understanding what really happens in the discussions, as opposed to just relying on teachers’ accounts. In a best-case scenario, I would have observed discussions in each of the teachers’ classes. This was not possible, however, because Joe’s class ended before the study began. As an alternative, I listened to audiotapes of all of the CPI discussions that occurred during Joe’s nine-week class. While the audiotapes allowed me to learn what Joe and his
students said during discussions, I clearly missed other discussion-related behaviors that must be seen to be understood. For example, I could not make inferences about the students' body language or tell whether silent students appeared to be engaged or bored. Due to this lack of observational data for Joe's classes, I was unable to generate any propositions that dealt broadly with the types of student behaviors that must be seen to be understood.

There was a second way in which the data was not as complete as I would have liked. Interviews with the teachers suggested that much of the discussion-related teaching in the three teachers' classes occurs at the beginning of the courses. Given that I wanted to understand how the teachers were teaching their students to participate more effectively in CPI discussions, it was important to observe this instruction. This was not possible for all three teachers, however, because two of the classes had been in session for more than half of the school year when the study began. Therefore, it is logical to presume that some of what was important about the discussion teaching practice of two of the teachers was not observed—obviously a major limitation of the study. I addressed this limitation by asking the teachers to describe how they introduced CPI discussions to their students and by analyzing artifacts from the beginning of the school year.

Data Reduction and Interpretation. In addition to assessing the accuracy and completeness of the data, another way to judge the credibility of a study is to focus on how the data was reduced and interpreted. Recall that the data in this study was reduced and interpreted in two ways. First, I wrote portraits of each teacher's conceptions and practice and, second, I induced an initial theory consisting of seven propositions and relationships that exist among them. Here I will focus on the first level of interpretation, the writing of the
portraits. Miles and Huberman (1994, p. 279) suggest many questions that can be used to assess the credibility of how data is described. Two of them are: How context-rich and meaningful ("thick") are the descriptions? Does the account "ring true," make sense, seem convincing or plausible, enable a "vicarious presence" for the reader?

As descriptions of the teachers' CPI discussion thinking and teaching, the portraits are fairly thick, and offer context-rich first-level interpretations of the data. In using the term "thick," I rely on Maxwell's (1996) explication of the term made famous by Geertz (1973): "... I mean data that are detailed and complete enough that they provide a full and revealing picture of what is going on" (p. 95). To assess the thickness of the descriptions of the data in the portraits, I relied primarily on the reactions of people who read them. In addition to members of my dissertation committee, the portraits were read by a high school teacher interested in classroom discussion, a social studies expert, two adults who are non-educators, and the teachers themselves. All responded positively to my question about whether the portraits were detailed enough to provide the reader with a good sense of the teachers' conceptions and practices. Criticism of the portraits, in fact, suggested that they were too thick. One reader, for example, said a portrait was "cruelly long" and therefore less interesting than would be the case with a more succinct description. The teachers, of course, were most qualified to assess whether the portraits "rang true." As previously mentioned, through the use of member checks, I purposely solicited their feedback on whether they thought the portraits accurately captured and described their thinking and teaching relative to CPI discussions. Without exception, the teachers stated that the
first-level interpretations in the portraits were accurate portrayals of their conceptions and practices.

In addition to the first-level interpretation that occurred throughout the portrait writing stage of this study, the initial theory consisting of seven propositions also involved interpretation or meaning-making. Recall that Chapter Three explained the tenets of the grounded theory methodology used in this study to generate an initial theory about the CPI discussion conceptions and practices of skilled discussion teachers.

By describing the steps I undertook to create the initial theory (see Chapter Three), readers can evaluate how I moved from the raw data to the seven propositions. One of the limitations of this study is that it is difficult to describe a non-linear process (the constant comparative technique) in a linear way. By that, I mean that the recursive nature of data collection and data interpretation that is the hallmark of grounded theory does not lend itself to clear description. While I created a flowchart (Figure 5) to illustrate the steps I used in the study, I found it difficult to capture accurately the inherently complicated nature of generating grounded theory.

Ethical Issue

Disciplined inquiry, by definition, involves ethical issues that should be paramount to the researcher. Throughout the study, I was conscious of and concerned about creating a reciprocal relationship with the teachers. Although all three teachers indicated interest in participating in the study because they felt it would help them better understand their teaching, they all voiced frustration that the interviews were not conversations. On a few occasions, a teacher asked, "What do you think?" and then immediately said, "Oh, I forgot, I know you are not supposed to say." By not participating in a
genuine two-way conversation, I was trying to avoid influencing the teachers’ views. After all, I already knew what I thought about CPI discussions—I was interested in learning what they thought. Toward the end of the interview schedule, two of the three teachers directly asked my advice about a part of their CPI discussion teaching practice that was troubling to them. Although some qualitative researchers, such as Grossman (1990), recommend not giving any "advice" to informants until after the study is completed, refusing to respond felt parasitic to me. When the two teachers asked my opinion about certain elements of their practice, I responded. However, this did not occur until the final interview, and I was very careful about what I said. Still, it raises an issue about the tension between reactivity and responding to the needs of the teachers. To avoid exploiting the teachers as informants (Glesne & Peshkin, 1992), I might have enhanced reactivity.

**Conclusion**

To close this dissertation, I take the opportunity to reflect on what I have learned about how secondary social studies teachers conceptualize and practice CPI discussion teaching and how I plan to use what I have learned in the future. At the beginning of this study, I expected to find that teaching students to participate effectively in CPI discussions is an extremely difficult and time-consuming enterprise. I expected this finding, based on my own experiences teaching students and teachers to engage in CPI discussions and based on a review of the literature showing that few teachers teach with, let alone for, discussion. Perhaps so few teachers use CPI discussions, I thought, because they are just so difficult.

While I still believe that it is not easy to teach with and for discussion, I have moderated my assessment of the difficulty of CPI discussions as a result
of what I have learned from the three teachers in this study. I do believe these are teachers who bring to the classroom a host of skills that some teachers may not possess. Notwithstanding this assessment of their abilities, it is apparent to me that one of the factors that matters most in their teaching of CPI discussions is the elaborate and sophisticated lesson plans they use when teaching discussion to their students. One of the factors that makes these teachers skillful is not any kind of classroom wizardry, but well thought-out and thorough lesson plans. While teaching people to be classroom wizards may be extremely difficult, learning how to develop sound lesson plans for CPI discussions is not. For those educators who believe that participating in CPI discussions is an important goal for their students, this is clearly good news. Thus, in my own teaching of preservice and inservice teachers, I plan to focus more specifically on lesson planning as it relates to CPI discussions.

I have also come to believe that it is important for educators to constantly reflect on what they value, for their values influence what they are teaching and what students are learning. The practice of the teachers in this study are clearly in line with what is important to them. I did not find this surprising. In fact, I expected it, given my personal experiences with other skilled teachers. What became more apparent to me, however, is that many tensions and tradeoffs are embedded in the practice of CPI discussion teaching. For example, I observed tensions between authenticity and accountability, and between equal participation and the focus on important ideas. These tensions are important reminders that teaching involves complicated choices between competing goods. Thus, in my own teaching and research about CPI discussions, I will be careful to bring to the fore these tensions so teachers are aware of the choices they are making.
Finally, I was heartened by how much the students in the classes of these three teachers seemed to enjoy the discussions. Often I heard students complain at the end of a discussion that they wished it did not have to end. They were having fun because they were engaged in talking about topics that are difficult and important. As a citizen in a democracy that relies on citizens’ discussions of public problems, I found in the students’ enjoyment of CPI discussions a bright ray of personal hope.
References


Engel, S. L. (1993). Attitudes of secondary social studies and English teachers toward the classroom examination and treatment of controversial


Issues-Centered Education. (1989). The Social Studies, 80, (5).


Appendix A
Consent Form

Teaching Controversial Public Issues Discussions:
A Grounded Theory Study

Diana Hess, Ph.D. Candidate, Principal Investigator (206-523-6563)
College of Education, University of Washington

Purpose and Benefits
I seek your participation as well as your consent for participation in a research
project aimed at understanding the instruction, and the thinking
undergirding the instruction, of secondary social studies teachers who are
skillfully teaching their students to participate more effectively in discussions
of controversial public issues. I believe your participation will help you and
other teachers and teacher educators better understand the characteristics of
effective classroom discussion of controversial public issues. I have chosen to
conduct this study in partial fulfillment of the requirements for a graduate
degree in Education.

Procedures
I will interview you several times between in the winter and spring of 1998.
The interviews will take place at your school, or another site you select. Each
interview will last about an hour and a half, and will be tape recorded. In the
interviews I will ask you questions about how your conceptions of social
studies, democracy, and effective classroom discussions of controversial
public issues; what issues you would and would not include in your
curriculum and why; and for your reactions to videotaped excerpts of
classroom discussions. I will observe (or listen to audio tapes of) three
discussions your classes. After each observation, I will ask you questions
about how you planned for the discussion, the instruction, and your
assessment of the quality of the discussion. For these interviews, I will ask
you to bring copies of any materials you assigned to your students to prepare
for the discussion and any assessment instruments you used to evaluate the
discussions.

Risks, Stress, and Discomfort
In any interview situation there is the possibility that some questions may
cause confusion or stress. If you feel that any question is inappropriate or you
do not wish to respond, you are free to refuse to answer.
Other Information
You will be assigned a pseudonym to protect your privacy. You have the right to review and delete any portion of the audio tapes made of the interviews. I will destroy the audio tapes on July 1, 1999. No one other than the investigator will have access to the audio tapes or to any data from the study that identifies you by name.

Signature of Investigator________________________ Date_______

The study described above has been explained to me. I voluntarily consent to participate in this activity. I have had an opportunity to ask questions. I understand that future questions I may have about the research or about my rights as a subject will be answered by the investigator listed above.

Signature of Subject_________________________ Date_______

cc:
Participant
Diana Hess
Appendix B

Interview Protocols for Semi-Structured Interviews

Interview One: Context, Purpose of Social Studies, Democratic Citizenship. I will explain the purpose of the study, how confidentiality will be maintained, and the consent form. I will ask the participant if she/he has any questions about the study or the form. I will ask the participant to sign the form. Then, I will set up the tape recorders, explain what this interview will focus on, and that she/he should feel free to not answer any question that makes her/him uncomfortable, that she/he has the right to erase any part of the tape, and then begin the interview.
(Note to readers: The majority of these questions were followed up by others designed to elicit an understanding of the initial response. As such, the content of the follow-up questions varied).

1. Please describe what caused you to become a social studies teacher.
2. What kind of teacher education program did you go through? How would you characterize your teacher education experience?
3. How long have you been teaching? In what schools have you taught? If you changed schools, why did you do so?
4. Please describe and explain the school in which you currently teach.
5. Tell me about the students in your school.
6. Now, let's shift the focus from your school to the community in which it is located. Please describe the community.
7. Moving to the social studies curriculum in your school, please describe its design with a specific focus on the course(s) you teach.
8. Please draw a concept map picture (or some kind of visual representation) that illustrates what you want your students to know, be able to do, and want to do as a result of the social studies class you teach. Probe for explanations of what is on the concept map.

9. Now, draw a concept map (or some kind of visual representation) that shows what you think effective citizens in the United States should know, be able to do, and want to do. Probe for explanations of what is on the concept map.

Interview Two: Characteristics of Good Discussion. In an adaptation of the think-aloud method, the teachers view two short teaching tapes that showcase class discussion and explain their reactions to the discussions. The first videotape is of a high school United States History class discussing a moral dilemma drawn from an excerpt of Richard Wright's autobiography in Reasoning with Democratic Values (Lockwood & Harris, 1985). This class discussion is on the film Minds on Social Studies (National Council for the Social Studies, 1990). The second discussion is of six students in a high school American Government class discussing whether the electoral college should be abolished or reformed. This small group discussion is included in the film series Preparing Citizens (Social Science Education Consortium, 1997).

1. This interview will focus on how you conceptualize good discussion and the purposes of discussion in social studies. Together we will watch two short excerpts of video-taped discussions in middle and high school classes. During the viewing, I want you to think aloud about your reactions to what you are viewing. This is a kind of interactive viewing, where you can talk back about or comment on what you are viewing. At this point, I provide the teacher
with an opportunity to practice the think-aloud technique by showing a short excerpt of a kindergarten classroom. Practice continues until the teacher feels comfortable with the think-aloud process.

2. Introduce the first discussion the teacher will view by explaining what the discussion is about, where it takes place, the age of the students, the type of class, and the length of the discussion. As the teacher views the discussion, he/she will hold a remote control devise so the tape can be stopped to make comments. I will probe with follow-up questions designed to elicit a better understanding of what the teacher is saying throughout the think-aloud process. Repeat the entire process with the second discussion.

**Interview Three: Rationales for CPI Discussions.** Using labels drawn from the rationales for CPI discussions in the literature, the teachers rank those rationales to reflect the reasons they use CPI discussions.

1. We're going to focus on the rationales for including classroom discussions in social studies. Specifically, we're going to look at discussions of controversial public issues. I'm going to show you a number of index cards. On each card there is a different rationale for classroom discussion of controversial public issues. As you look at each one, I want you to first read what's on the card so we get a good transcript. Then, explain what this rationale means, in your own words, regardless of whether you agree with it or not. Probe for more fully elaborated explanations of each rationale.

2. Now I would like you to rank order the rationales from the one that most reflects why you use CPI discussions to the one that least reflects why you use CPI discussions. Probe for more elaborated explanations of the rankings.

3. On each index card, is one of the following rationales:
Improve interpersonal skills
Understand important democratic values
Participate in political life of democracy (e.g., voting, serve on jury)
Learn important social studies content
Learn how to think critically
Improve democracy

Interview Four: Issues Selection. This interview involves another selection and ranking exercise. The teacher organizes various CPI topics in terms of the likelihood they would be included in his/her curriculum.

1. I am going to give you another set of index cards to work with -- written on each one is a CPI topic. First, I would like you to shuffle through these and place them in two piles: those that you would include in your curriculum and those you would not. As you are doing so, please explain your reasons to me.

2. Now, I would like you to rank order the cards to reflect the issues that you are most and least likely to include in your curriculum. Again, as you are sorting them, please explain your reasons to me.

3. The CPI topics written on the index cards are:
   Abortion
   Affirmative Action
   Balanced Budget
   Gay Rights
   Immigration
   Legalizing Drugs
   Physician-Assisted Suicide
   Trade Policy
   Vouchers for Private Schools
   Welfare Reform
CURRICULUM VITAE

Diana Hess

PRESENT POSITION

Assistant Professor, Social Studies Education (appointment begins January, 1999)
University of Wisconsin-Madison
Department of Curriculum and Instruction
Teacher Education Building
225 North Mills Street
Madison, Wisconsin, 53706-1795

ACADEMIC PREPARATION

1998 Ph.D. Education, University of Washington, Seattle, WA.

Fields of concentration:
Curriculum & Instruction in Social Studies, Educational Policy, Culture and Citizenship, and Law.

Dissertation:
Teaching Controversial Public Issues Discussions: Learning from Skilled Teachers investigates how three secondary school social studies teachers teach their students to participate effectively in classroom discussions.

1995 M.A., Educational Policy, University of Illinois.

1983 N.E.H. program on political philosophy, Harvard University.

1979 B.A., Political Science, Western Illinois University.

PROFESSIONAL EXPERIENCE

1998 Lecturer during the fall quarter, Social Studies in the Secondary School, University of Washington.
Graduate student, teaching assistant for Social Studies in the Secondary School, developer and instructor of technology laboratory component of Social Studies in the Secondary School, lead teaching assistant (1997), research assistant on Historical Sense-Making Project at the University of Washington.

Instructor at Supreme Court Institute, a professional development program for high school teachers, co-sponsored by Street Law, Inc. and the Supreme Court Historical Society at Georgetown Law Center and the United States Supreme Court.

Developed and taught graduate course: Teaching with Discussion, University of Washington.

Developed and taught graduate course: Discussion for Democracy, University of Colorado at Denver.

Associate Director of the Constitutional Rights Foundation Chicago. Responsibilities included developing political science, law-related and service learning curriculum and program materials, writing over four million dollars of funded grants, developing and implementing a national training of trainers program, designing and leading courses, institutes, and workshops for teachers, administrators, lawyers, judges, and police officers, providing technical assistance to national, state, and local civic education organizations, and hiring/supervising staff.

Adjunct Instructor: Developed and taught graduate level social studies methods courses for DePaul University and Roosevelt University, Chicago, IL

Social Studies teacher: Taught courses in political science, law, US. History, global education, coordinator of the Student Mock Trial team and developer/coordinator of the Government Internship Program. Community High School District #99 South, Downers Grove, IL.

President of Downers Grove Education Association, IEA/NEA. Responsibilities included organizing, bargaining, lobbying, enforcement of contract, and developing/supervising various association programs.
1976 Congressional intern for US Representative T. Railsback, Washington, D.C. Researched legislative issues, and responded to constituents' requests for information and service.

PUBLICATIONS


Hess, D., and students at East High School, Denver, CO. (1997) This video publication, Preparing Citizens, features students in a high school law class participating in a discussion using the public issues approach on the question, "Should physician-assisted suicide be legal in Colorado?" Hess facilitates the discussion. Available from the Social Science Education Consortium, Boulder, CO.


**HONORS**


Liberty Bell Award (1995), awarded annually by the Chicago Bar Association to a non-lawyer who has rendered service which strengthens the effectiveness of the American system of freedom under the law.

Nominated for the Isidore Starr Award (1995), awarded annually by the American Bar Association for national leadership in the field of law-related education.

Arkansas Traveler Award from Bill Clinton (1989), Governor of Arkansas, for working to create a law-related education center in Arkansas.

**NATIONAL AND STATE CONFERENCE PRESENTATIONS** (selected)


Hess, D. (1994) "Developing Assessments for Moot Court Simulations: The Experiences of Three High School Teachers." Paper presented at the meeting of the College and University Faculty Assembly of the National Council for the Social Studies, Phoenix, AZ.


CONSULTATIONS (selected)

1997 Developed and facilitated professional development program on authentic assessment in K-12 social studies for the Edmonds School District, Lynnwood, WA. The program helps teachers form collaborative partnerships to improve curriculum, instruction, and assessment in social studies. A videotape showcasing the project and teacher-produced model assessments developed as a result of the program will be disseminated by the Edmonds School District in the summer of 1998.

1997 Developed and facilitated a professional development program for the education faculty at the University of Puget Sound, Tacoma, WA. The program focuses on how to improve class discussions in teacher education courses.

1992-1997 Provided input on program planning, presented at workshops, and provided feedback to teachers through the Authentic Assessment in Civic/Law-Related Education Project, Social Science Education Consortium, Boulder, CO.

1997 Worked with a team of educators to develop and implement Project REAL, an assessment program for social studies and science teachers, University of Washington, Seattle, WA.
1994-1995  Designed and implemented an evaluation of the Iowa Courts Curriculum Project for the Iowa Center for Law and Civic Education, Drake University, Des Moines, IA.

1994-1995  Worked with teachers and administrators to re-design the required US History course and create a Service Learning course for Mundelein High School, Mundelein, IL.


1991-1992  Curriculum design for an after-school enrichment program on law-related education developed by the National Council of La Raza, Los Angeles, CA.


PROFESSIONAL MEMBERSHIPS

National Council for the Social Studies
Washington Council for the Social Studies
Association for Supervision and Curriculum Development
American Educational Research Association

COMMITTEE SERVICE

Representative Fawell’s (IL) Education Advisory Committee (1984-1995)
National Education Association Congressional Contact Team (1985-1987)
Illinois Education Association Women’s Leadership Cadre (1985-1987)

References Available Upon Request
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Dynamical Systems

by

Chris Hillman

A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

University of Washington

1998

Approved by

Boris Solomyak

(Chairperson of Supervisory Committee)

Program Authorized
to Offer Degree Mathematics

Date August 21, 1998
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Signature

Date 11/27/1998
Let $W$ be a $p$-dimensional subspace of $\mathbb{R}^d$, where $d = p + q$ and $p, q \in \mathbb{N}^+$. Sturmian tilings (also known as generalized Penrose tilings) $T(x + W)$ arise by passing $p$-dimensional cutplanes $x + W$ through a periodic tiling of $\mathbb{R}^d$ called the oblique tiling $O(W)$. The set of such tilings is the Sturmian system $S(W)$. Each tiling $T(x + W)$ is associated with a number of $\mathbb{Z}^p$ shift spaces called Robinson shifts $X_T(X)$, which generalize the classical Sturmian shifts (which are $\mathbb{Z}$ shift spaces); each point in a Robinson shift is an infinite $p$ dimensional array of symbols drawn from a finite alphabet. The Sturmian system $S(W)$ also defines a Sturmian tiling dynamical system $T(W)$ under the obvious translation action by $\mathbb{R}^p$; in addition some oblique tilings $O(W)$ permit compositions which induce compositions (also called inflation rules or substitution rules) on $T(W)$ and a (very different!) action by a composition group. Thus, Sturmian systems possess a dynamical character as well as a geometric character and a combinatorial or symbolic character. In this thesis, we study how a variety of probabilistic, combinatorial, and dynamical properties of the
tilings, the shift spaces, and the Sturmian system $S(W)$ as a whole change as we vary the position of the subspace $W$. It turns out that, roughly speaking, these properties depend upon how "rationally" $W$ is embedded in $\mathbb{R}^d$; in fact, the bifurcations in these properties are organized by a nested partitioning of the Grassmannian $G(p, q)$, which generalizes the Stern-Brocot-Farey tree which organizes simple continued fraction expansions. A more detailed description of the contents of this thesis will be found in the first chapter.
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i, j, k \in \mathbb{N}$</td>
<td>non-negative integers</td>
</tr>
<tr>
<td>$m, n, \in \mathbb{Z}$</td>
<td>integers</td>
</tr>
<tr>
<td>$s, t \in \mathbb{R}$</td>
<td>real numbers</td>
</tr>
<tr>
<td>$z, w \in \mathbb{K}$</td>
<td>complex numbers of modulus one</td>
</tr>
<tr>
<td>$m, n \in \mathbb{Z}^d$</td>
<td>integer vectors</td>
</tr>
<tr>
<td>$x, y, z \in \mathbb{R}^d$</td>
<td>real vectors</td>
</tr>
<tr>
<td>$U, V, W$</td>
<td>linear subspaces of $\mathbb{R}^d$</td>
</tr>
<tr>
<td>$e_j, 1 \leq j \leq d$</td>
<td>the standard basis vectors for $\mathbb{R}^d$ (and $\mathbb{Z}^d$)</td>
</tr>
<tr>
<td>$x_j$</td>
<td>the corresponding Euclidean coordinates</td>
</tr>
<tr>
<td>$J, K$</td>
<td>subsets of coordinates</td>
</tr>
<tr>
<td>$A, B, C$</td>
<td>matrices</td>
</tr>
<tr>
<td>$[v_1 v_2 \ldots v_p]$</td>
<td>matrix whose columns are $v_1, v_2, \ldots v_n$</td>
</tr>
<tr>
<td>$A'$</td>
<td>transpose of $A$</td>
</tr>
<tr>
<td>$A(J, \cdot)$</td>
<td>submatrix obtained from $A$ by selecting rows from $J$</td>
</tr>
<tr>
<td>$A(\cdot, K)$</td>
<td>submatrix obtained from $A$ by selecting columns from $K$</td>
</tr>
<tr>
<td>$A(J, K)$</td>
<td>submatrix obtained from $A$ by selecting rows from $J$ and columns from $K$</td>
</tr>
<tr>
<td>$I_d$</td>
<td>$d$ by $d$ identity matrix</td>
</tr>
<tr>
<td>$P_W$</td>
<td>orthogonal projection onto $W$ (often abbreviated $P$)</td>
</tr>
<tr>
<td>$Q_W = I_d - P_W$</td>
<td>orthogonal projection onto $W^\perp$ (abbreviated $Q$)</td>
</tr>
<tr>
<td>$p_j$</td>
<td>the vectors $P_W(e_j)$, where $W$ is some fixed subspace</td>
</tr>
<tr>
<td>$q_j$</td>
<td>the vectors $Q_W(e_j)$</td>
</tr>
<tr>
<td>$p, q \in \mathbb{N}$</td>
<td>$\dim W$, $\dim W^\perp$, where $W$ is some fixed subspace and where $p + q = d$</td>
</tr>
</tbody>
</table>

- **col**: column space (of a matrix)
- **nul**: null space (of a matrix)
- **span**: linear span
- **con**: convex hull
- **rat**: rational closure, $\text{rat } W = (W^\perp \cap \mathbb{Z}^d)^\perp$

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(p, q)$</td>
<td>Grassmann manifold of $p$-dimensional subspaces of $\mathbb{R}^d$</td>
</tr>
<tr>
<td>$\Gamma(\mathbb{R}^d)$</td>
<td>the Grassmann algebra for $\mathbb{R}^d$</td>
</tr>
<tr>
<td>$\Lambda^p(\mathbb{R}^d)$</td>
<td>the space of $p$-multivectors for $\mathbb{R}^d$, where $0 \leq p \leq d$</td>
</tr>
<tr>
<td>$\Lambda^pT$</td>
<td>operator on $\Lambda^p(\mathbb{R}^d)$ induced by operator $T$ on $\mathbb{R}^d$</td>
</tr>
<tr>
<td>$A, B, C$</td>
<td>multivectors</td>
</tr>
<tr>
<td>$A \wedge B$</td>
<td>exterior product (Chapter 3)</td>
</tr>
<tr>
<td>$E_J$</td>
<td>the standard basis multivector $\wedge_{j \in J} e_j$</td>
</tr>
<tr>
<td>$P_J$</td>
<td>the multivector $\wedge_{j \in J} p_j$, where $p_j = P_W(e_j)$ as above</td>
</tr>
<tr>
<td>$\Psi(W)$</td>
<td>the Plücker line for $W$, $\Psi(W) = \text{span}{p_J}$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>sublattice of $\mathbb{Z}^d$ (perhaps contained in some subspace of $\mathbb{R}^d$)</td>
</tr>
<tr>
<td>Symbol</td>
<td>Meaning</td>
</tr>
<tr>
<td>--------</td>
<td>---------</td>
</tr>
<tr>
<td>$\Lambda^*$</td>
<td>dual lattice</td>
</tr>
<tr>
<td>$M(\Lambda)$</td>
<td>integer matrix giving generators for $\Lambda$</td>
</tr>
<tr>
<td>$G(\Lambda)$</td>
<td>Gramm matrix of $\Lambda$</td>
</tr>
<tr>
<td>ann $H$</td>
<td>Annihilator of $H$ (where $H$ is a subgroup of some locally compact abelian group)</td>
</tr>
<tr>
<td>$X, Y, Z$</td>
<td>tiling spaces (shift spaces)</td>
</tr>
<tr>
<td>$S : X \to X$</td>
<td>shift map $Sx(n) = x(n + 1)$</td>
</tr>
<tr>
<td>$T^x$</td>
<td>translation by $x$</td>
</tr>
<tr>
<td>$\alpha, \beta, \gamma$</td>
<td>patches (blocks)</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>tilings (sequences)</td>
</tr>
<tr>
<td>$P(x)$</td>
<td>patches (words) appearing in tiling (sequence) $x$</td>
</tr>
<tr>
<td>$Z(\alpha)$</td>
<td>cylinder over $\alpha$</td>
</tr>
<tr>
<td>$E(\alpha)$</td>
<td>empire of $\alpha$</td>
</tr>
<tr>
<td>$K(\alpha)$</td>
<td>kingdom of the word $\alpha$</td>
</tr>
<tr>
<td>frq</td>
<td>frequency (of a protopatch or prototerm)</td>
</tr>
<tr>
<td>cov</td>
<td>coverage (by a protopatch or prototerm)</td>
</tr>
<tr>
<td>vol$_p$</td>
<td>$p$-dimensional volume (of a patch)</td>
</tr>
<tr>
<td>skel$_k$</td>
<td>$k$-skeleton (of a cell)</td>
</tr>
<tr>
<td>$I^d = [0, 1]^d$</td>
<td>standard $d$-dimensional unit cube</td>
</tr>
<tr>
<td>$T^d = (S^1)^d$</td>
<td>standard (geometrically flat) $d$-dimensional torus</td>
</tr>
<tr>
<td>$O(W)$</td>
<td>Oblique tiling defined by the $p$-dimensional subspace $W$ of $\mathbb{R}^d$</td>
</tr>
<tr>
<td>$O^{(k)}(W)$</td>
<td>$k$-th higher block oblique tiling</td>
</tr>
<tr>
<td>$S(W)$</td>
<td>Sturmian system defined by $W$</td>
</tr>
<tr>
<td>$X_j(W)$</td>
<td>$J$-th Robinson shift space, with action by $\mathbb{Z}^p$</td>
</tr>
<tr>
<td>$T(W)$</td>
<td>Sturmian tiling space defined by $W$, with action by $\mathbb{R}^p$</td>
</tr>
<tr>
<td>$C_J$</td>
<td>$J$-th protocell of $O(W)$</td>
</tr>
<tr>
<td>$F_J$</td>
<td>$J$-th protofacet for digital approximations to $W$</td>
</tr>
<tr>
<td>$T_J$</td>
<td>$J$-th prototile of $S(W)$ or $T(W)$</td>
</tr>
<tr>
<td>$D(x + W)$</td>
<td>Digital approximation to the cutplane $x + W$</td>
</tr>
<tr>
<td>$T(x + W)$</td>
<td>Sturmian tiling obtained by projecting $D(x + W)$ into $x + W$</td>
</tr>
<tr>
<td>$G(x + W)$</td>
<td>multigrid corresponding to $T(x + W)$</td>
</tr>
<tr>
<td>$S_J(x + W)$</td>
<td>$J$-th Robinson supertiling of $T(x + W)$</td>
</tr>
<tr>
<td>$H_J(n)$</td>
<td>hyperplane with equation $x_j = n$</td>
</tr>
<tr>
<td>$E_J(n, x + W)$</td>
<td>terrace corresponding to the plane $H_J(n)$</td>
</tr>
<tr>
<td>$W_J(n, x + W)$</td>
<td>wall for $E_J(n, x + W)$</td>
</tr>
<tr>
<td>$L_J(n, x + W)$</td>
<td>lane corresponding to $W_J(n, x + W)$</td>
</tr>
<tr>
<td>$G_J(n, x + W)$</td>
<td>gridplane</td>
</tr>
<tr>
<td>$T_W(x)$</td>
<td>Sturmian tiling in $T(W)$ (a translate in $W$ of $T(x + W)$)</td>
</tr>
<tr>
<td>$L$</td>
<td>composition operator on $O(W)$ or $S(W)$</td>
</tr>
</tbody>
</table>
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This work could never have been completed, or the work even begun, without the assistance of a number of people.

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Chapter 1

Overview

1.1 The Objects of Study

In this thesis we study a class of dynamical systems, the Sturmian systems, which can be characterized (roughly speaking) as the most "rigid" and "predictable" systems other than periodic orbits.

Each "point" in a Sturmian system is a very special type of tiling of some Euclidean space $\mathbb{R}^p$. These Sturmian tilings arise, intuitively speaking, by taking digital approximations to $p$-dimensional affine planes, or $p$-flats, in some higher dimensional Euclidean space, $\mathbb{R}^{p+q}$. Fig. 1.1 illustrates the idea of such an approximation. Every tile in a Sturmian tiling is a translate of one of a finite set of prototiles$^1$, and every prototile is, geometrically speaking, a zonotope$^2$. See Fig. 1.2 for some more examples of Sturmian tilings.

Suppose $W$ is a $p$-dimensional linear subspace of $\mathbb{R}^{p+q}$. Then each flat parallel to $W$ has the form $x + W$, where $x \in \mathbb{R}^{p+q}$ may be chosen to lie in $W^\perp$. Thus, the set of those $p$-flats which are translates of $W$ is in bijection with $W^\perp$. We will denote the tiling of $\mathbb{R}^p$ obtained as the digital approximation to $x + W$ by $T(x + W)$, where it is understood that $x + W$ is identified with $\mathbb{R}^p$ by translating it back along $W^\perp$ to $W$, and where $W$ itself is identified with $\mathbb{R}^p$ in any way which puts the origin at zero. We write the set of all such tilings as $S(W)$, and we say the tiling in $S(W)$ have genus $(p, q)$ and species $W$.

Sturmian tilings have a fundamentally hierarchical or "recursive" aspect, in the

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$^1$The fundamental terminology of tiling theory (e.g. prototile, protopatch, prototiling versus tile, patch, tiling) will be introduced in Chapter 2.

$^2$Zonotopes are the multidimensional analogs of rhombi; they are defined in Section 3.3.
Figure 1.1: A Sturmian tiling of genus (2, 1). Top: the digital approximation to the 2-flat $0.1q_1 + 0.2q_2 + W$, where $W$ is the indicated two dimensional subspace of $\mathbb{R}^3$. Three families of terraces, corresponding to planes of form $x_1 = n_1$, $x_2 = n_2$, and $x_3 = n_3$, respectively, are evident (or if not, see the figures in Section 4.3). Bottom: the corresponding Sturmian tiling of $\mathbb{R}^2$ is obtained by projecting this digital approximation into $W$; notice that the lower edge in this picture corresponds to the upper edge in the top picture.
1.1. THE OBJECTS OF STUDY

Figure 1.2: Some two dimensional Sturmian tilings. Each tiling is drawn from the space $S(W)$ defined by taking $W$ to be the column space of the indicated matrix. Top: two genus $(2, 2)$ tilings, each having 6 prototiles and 4 families of "parallel" walls. Bottom: two genus $(2, 3)$ tilings, each having 10 prototiles and 5 families of "parallel" walls. Notice how the multidimensional structure fairly "leaps out at" the viewer!
sense that each tiling $T(x+W)$ in effect decomposes into families of simpler Sturmian tilings. For example, the digital approximation shown in Fig. 1.1 naturally decomposes (in three ways) into "layers" called terraces which are separated by short "retaining walls", or walls for short. Indeed, every $(p,q)$ Sturmian tiling decomposes into $p+q$ distinct families of "parallel" walls. The remarkable fact, which may already be apparent to the reader from examination of Fig. 1.1, is that each wall is combinatorially isomorphic to a Sturmian tiling of genus $(p-1,q)$. Thus, a simple induction shows that we can consider any $(p,q)$ Sturmian tiling to be "woven" from interlaced $(1,q)$ Sturmian tilings, called ribbons; see Fig. 1.3. In general, there is a whole hierarchy of subwalls from ribbons, which are $(1,q)$ tilings, up to walls, which are $(p-1,q)$ tilings. (We will study this hierarchy in Section 4.3.)

Sturmian tilings have a dual nature. On the one hand, they are geometric tilings. On the other, each tile can fit together with its neighbors in only a finite number of ways, and there are only finitely many prototiles, so we can regard prototiles as being analogous to letters, protopatches to words, and the combinatorics of patches as Sturmian grammar. Indeed, each $(p,q)$ Sturmian tiling defines a number of $p$-dimensional tabular arrays of symbols, as was first noticed by Robinson [114]. This is easiest to see in the case of $(1,q)$ tilings; see for instance Fig. 1.5, where the projections of the three types of edges $(x_1, x_2, x_3)$ in the $\mathbb{Z}^3$ lattice of $\mathbb{R}^3$ gives the three prototiles in the resulting Sturmian tilings of $\mathbb{R}$. In higher dimensions, it is only necessary to notice that any choice of $p$ of the $p+q$ families of walls defines an $p$-dimensional "table" of symbols. In this way, from each system of Sturmian tilings we obtain symbolic dynamical systems called Robinson shifts. (We will discuss Robinson shifts again in Section 3.7.)

There are three (completely equivalent) methods of constructing Sturmian tilings: the projection method, the multigrid method, and the oblique tiling method. In this thesis, we will for the most part employ the third method, which turns out to extremely convenient for our purposes. Given a $p$-dimensional subspace $W$ of $\mathbb{R}^{p+q}$,
Figure 1.3: Part of a Sturmian tiling from the (3, 2) system $S(W)$. There are altogether 10 prototiles, 5 families of "parallel" walls and 10 families of "parallel" ribbons. The picture shows one example of each of the ten families of ribbons. Each ribbon is essentially a Sturmian tiling of genus (1, 2), and each family of ribbons shares 3 prototiles, whereas each wall (see Fig. 1.4) is essentially a Sturmian tiling of genus (2, 2), and each family of walls shares 6 prototiles.

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{11} \\ \sqrt{3} & \sqrt{2} & \sqrt{7} \end{bmatrix}$$
Figure 1.4: These pictures depict the same Sturmian tiling as in Fig. 1.3. Top: the wall defined by the coordinate 4-flat in $\mathbb{R}^5$ with Cartesian equation $z_2 = 0$. Middle: the wall defined by the 4-flat $z_5 = 0$. Bottom: the ribbon defined by the intersection of these two walls, i.e. by the coordinate 3-flat $z_2 = 0, z_5 = 0$. The upper surface of the ribbon is visible in the middle picture and its side is visible in the upper picture.
Figure 1.5: A Sturmian tiling of genus $(1, 2)$. Top: the digital approximation (bold broken line) to the flat $x + W$ (diagonal line), (where $x = 0.1q_{11} + 0.2q_{22}$, and where $W$ is the column space of the indicated matrix; $\alpha = (3 + \sqrt{17})/4 \approx 1.78078$ is the largest root of $2\alpha^2 - 3\alpha - 1 = 0$). Middle: the symbolic sequence for the corresponding tiling $T(x + W)$ (read bottom to top, where $A$ denotes the tile obtained by projecting $e_1$, $B$ that obtained by projecting $e_2$, and $C$ that obtained by projecting $e_3$). Bottom: the corresponding vertices of $T(x + W)$; note that the prototiles in increasing order of length are $A, B, C$. 

... CBACBCCBACBCACBCBC ...
the basic idea of this method is to first construct a periodic tiling in $\mathbb{R}^{p+q}$, the oblique tiling $O(W)$, and to obtain a system $S(W)$ of $(p, q)$ Sturmian tilings by taking slices parallel to $W$ through this higher dimensional tiling; see Fig 1.6. In this context, the flats $x + W$ are often called cutplanes. (The construction of Sturmian tilings is studied in detail in Chapter 3.)

The system $S(W)$ of $(p, q)$ Sturmian tilings constructed in this way is not quite what we mean by a Sturmian tiling dynamical system, however, because the vertices of the tilings in $S(W)$ are restricted to lie in the projections to $W$ of vertices in $O(W)$, and these are in general only a dense subset of $\mathbb{R}^p$. However, a slight modification of our definitions produces a space $T(W)$ of tilings consisting of all translates by $\mathbb{R}^p$ of the tilings in $S(W)$. We make $T(W)$ into a topological space by imposing the standard tiling metric; roughly speaking, two tilings are close in this metric if after a small translation they agree perfectly on a large ball around the origin of $\mathbb{R}^p$. With this added structure, $T(W)$ is a topological dynamical system under the translation action of $\mathbb{R}^p$.

While such a continuous translation action provides the most obvious way of thinking about Sturmian tilings as dynamical systems\(^3\), it is not the only one. In particular, Robinson shifts form dynamical systems whose points are infinite $p$ dimensional symbolic "tables" with an action by $\mathbb{Z}^p$; that is, they are $\mathbb{Z}^p$-shifts. In addition, some oblique tilings $O(W)$ admit a composition operator $L$, which is simply an invertible integer matrix with eigenspaces respecting the decomposition $\mathbb{R}^{p+q} = W \oplus W^L$; if such a matrix exists, it makes $S(W)$ (and $T(W)$) into a dynamical system with an action by $\mathbb{Z}$. (We will study compositions in Chapter 5.)

Sturmian tilings of genus $(p, q)$ are usually almost periodic under translation by $\mathbb{R}^p$, in the sense that given any $\epsilon > 0$, we can find a displacement (a vector in $\mathbb{R}^p$) such that shifting the tiling against itself by this displacement gives perfect agreement

\(^3\)A concise introduction to dynamical systems is given in Chapter 2.
Figure 1.6: These pictures show the oblique tiling $O(W)$ and a typical cut line $x + W$ for two different one dimensional subspaces $W$ of $\mathbb{R}^2$. In each picture, the cells pierced by the cut line are shaded; the tiles of the corresponding Sturmian tiling $T(x + W)$ are simply the intersection with $x + W$ of these cells.
on patches covering all but $\epsilon$ of $\mathbb{R}^p$. Consequently, $\mathcal{T}(W)$ usually decomposes into a collection of minimal (in fact, strictly ergodic) subsystems; if $\mathcal{T}(W^\perp)$ has $k$ independent periods, this will be a $k$ parameter family of subsystems, and vice versa (see Section 4.7). Almost periodicity implies that the diffraction spectrum of the vertices of Sturmian tiling exhibit Bragg peaks, and thus resemble the diffraction spectra of periodic crystals; this has led to their being proposed as mathematical models of quasicrystals (see for instance [124]).

While almost periodicity (or minimality) is perhaps the best known feature of Sturmian systems, our main focus will be two other features.

First, an inherent "defect" or "unwanted complication" in the oblique tiling method of constructing Sturmian tilings is that certain cutplanes contain patches where several overlapping tiles "are trying to coexist", and thus do not form true tilings at all. For instance, in Fig. 1.6, consider what happens if the cut line $x + W$ happens to run right into one of the line segments in $\mathcal{O}(W)$ which is parallel to $W$. These "not quite tilings" are called singular tilings, and they turn out to be equally unavoidable in the projection and multigrid construction methods; they are an intrinsic feature of Sturmian systems. We shall discuss Sturmian singularities again in Section 4.2, and study them in detail in Chapter 6.

Second, Sturmian systems exhibit a very stringent "rigidity" which is brought into sharp focus by studying their empires. The notion of empire is quite general and is dual to the familiar (and extremely important) notion of a cylinder. The cylinder over a patch is the set of all tilings in which that patch appears. The empire of a patch is the set of all tiles forced by that patch; that is, whenever $P$ occurs in some tiling $T(x + W)$, so do certain other tiles at particular locations. Cylinders and empires form a pair of lattices in Galois duality. (This fundamental duality, which appears to be new, is established in Chapter 2.)

We shall see that the empire of patch in a $(p, q)$ Sturmian system usually is infinite; indeed, one can find relatively small patches whose empires cover almost all of $\mathbb{R}^p$. To
see how astonishing this property is, consider the empires of a tiling, or equivalently, a symbolic sequence of long and short tiles, in a typical \((1, 1)\) Sturmian system. For example, the empire of the word \(AAA\) in the system illustrated in Fig. 1.6 (we let \(A\) denote a long tile and \(B\) a short tile) looks like this:

\[
\ldots \,**AA**AABAA**AA**AABAAABA**A**AABAA**AA** \ldots 
\]

(where the word \(AAA\) has been underlined, and where ** means “either \(AB\) or \(BA\)”). Now, the word \(AAA\) occurs infinitely often in each tiling in this system, and each time we see that word, we must see the entire empire, as suggested by the following example:

\[
\ldots \,**AA**AABAA**AA**AABAAABA**AA**AABAA**AA** \ldots \\
\ldots \,AAABAAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAABAAB
In 1940, Marston Morse and Gustav Hedlund wrote a paper [98] in which they introduced the classical Sturmian shifts; these are simply the symbolic tile sequences associated with Sturmian systems of genus (1, 1). However, the connection with tilings was not noticed until much later. (The classical theory of these shifts is summarized in Chapter 2.)

Around 1970, Roger Penrose devised an aperiodic tiling of the plane by rhombi (obeying certain matching rules); however, he did not at first fully appreciate the value of his achievement. Credit for recognition of the importance of Penrose's construction must go to Martin Gardner (see [56]), who introduced these Penrose tilings to the world in his famous article [41]. The fat and thin rhombic prototiles for the most popular version of Penrose tilings (and the only version we shall consider; see [53] for some of the others) are often called Penrose rhombs.

Meanwhile, J.H. Conway (who happened to be staying at Martin Gardner's home during the preparation of [41]) had discovered (in the special case of Penrose tilings) two phenomena of fundamental importance in the study of Sturmian systems, namely empires and singularities. Unfortunately, he too failed to appreciate the importance of his work (possibly because he apparently overlooked the fundamental nature of the empire concept), and he never published it. However, Gardner described some of his results in [41] and Shephard later gave some more details in [53].

A few years later, Robert Ammann introduced another type of matching rule, now called Ammann bars, for Penrose tilings. Around the same time, Ammann devised aperiodic tilings, here called the Ammann octagonal tilings, which are in many ways analogous to Penrose tilings, but in some ways even simpler. He also attacked the problem of enumerating the empires of the vertex neighborhoods of Penrose tilings (compare [53] with Chapter 7). Unfortunately, Ammann, who was a remarkable amateur mathematician, never published his work and seems to have vanished into obscurity (some details later appeared in [53]).

Meanwhile, in 1973, Coven and Hedlund [20] proved an important characterization
of the classical Sturmian shifts: they are the unique one dimensional shift spaces having precisely \( n + 1 \) words of length \( n \), which makes them the aperiodic shifts with the “least complex language”.

The whole field of tilings was enriched and stimulated by the publication of the book by Grünbaum and Sheppard [53] (the book was finished in 1977 but only published after an unfortunate ten year delay due to the fact that original publishers were unable to reproduce the high quality figures provided by the authors).

Nonetheless, few mathematicians studied aperiodic tilings until the astonishing 1984 discovery by Schechtman et al. [122] of an alloy exhibiting a “forbidden” ten fold rotational symmetry. This is impossible according to the standard theory of crystallography (see for instance [124]); fortunately, owing to the foresight of Martin Gardner in popularizing the Penrose tiling, it was quickly noticed that the diffraction spectra of the set of vertices of a Penrose tiling exhibit precisely the same ten-fold rotational symmetry as the quasicrystalline alloy of Schechtman et al., which led to the suggestion that Sturmian tilings can serve as mathematical models of quasicrystals. This immediately led to an explosion of work on such models by an interdisciplinary group of mathematicians, crystallographers, and physicists.

The crucial step facilitating almost all subsequent work on Sturmian systems was taken in 1981 by N. G. de Bruijn [26], who introduced two new methods (the projection and multigrid methods) of defining Penrose tilings other than by matching rules. As de Bruijn and others quickly realized, the new methods applied to a much larger class of tiling spaces, called here the Sturmian systems, but more generally known by the cumbersome term “spaces of generalized Penrose tilings”. (The introduction of the oblique tiling construction is quite recent and is due to Oguey, Duneau, and Katz [103].) Not long after de Bruijn’s work, Daniel Rudolph [119] introduced the concept of a tiling dynamical system.

Yvves Meyer [91][92] had introduced as early as 1969 the notion of what are now called model sets in connection with ideas which would eventually grow into the
modern theory of wavelets, but their relevance to quasicrystals was not recognized for almost twenty five years; this recognition has probably represented the most significant advance in the subject since the work of de Bruijn. See [96] for a fine introduction to model sets and their applications to quasicrystals.

Early examples of physical quasicrystals were full of impurities and defects in the aperiodic array of atoms, but by 1989 examples were known of physical quasicrystals whose diffraction spectra indicated that they are as defect-free as the purest crystal. In that year, Penrose [106] pointed out a very serious problem in modeling quasicrystal growth using aperiodic tilings defined with matching rules: despite placing tiles strictly in accordance with local rules, one continually finds that one has made “global mistakes” which necessitate removing large numbers of tiles and starting all over again. This is in strong contrast to real quasicrystals, which grow to large sizes without hesitation. However, George Onoda [28][127] soon devised a very interesting partially deterministic and partially nondeterministic algorithm for growing Penrose tilings and quite recently, Steinhardt and Jeong [128] have conjectured that one can characterize Penrose tilings (among all tilings built using the Penrose rhombs) by saying that they possess the maximal density of occurrence of a certain patch of tiles. Indeed, such rules (which presumably model free energy minima) may serve to characterize both crystal and quasicrystal growth.

Meanwhile, Petra Gummelt [54] has shown how to construct Penrose tilings using a single marked decagon which is placed with overlaps, obeying matching rules. This is particularly important because it shows that, like crystals, at least some quasicrystals can be built from a single repeating unit (provided one permits overlaps, which is not physically unreasonable). Recently, a physical quasicrystal has been found whose structure appears to involve just such overlapping decagons [128].

It is still too early to assess the impact of the recent contributions by Steinhardt, Jeong, and Gummelt, which may result in an interesting “paradigm shift” from quasiperiodic tilings defined by matching rules or inflations (say) to quasiperi-
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Periodic coverings defined by more physically reasonable energetic criteria, but one can point out certain trends or emphases in most work to date. First, the vast majority of work by mathematicians, physicists, and crystallographers has focused on the spectral and symmetry properties of quasicrystals; the recent book by Senechal [124] is devoted almost entirely to such questions. Second, a large percentage of this work assumes the existence of some sort of composition or self-similarity. However, recent work by Brown [13] and others, which shows that among the classical Sturmian shifts at least, very few Sturmian systems possess such compositions, suggests that this emphasis may perhaps be misplaced. Third, a lesser but still considerable percentage assumes the existence of some sort of matching rules. Le Tu Quoc Thang and his colleagues [85] [124] have determined which Sturmian tiling spaces can be defined by means of matching rules (the generic Sturmian tiling space cannot be so defined). Fourth, a number of mathematicians and computer scientists have studied various aspects of the words which can appear in Sturmian shifts (and related sets of words) [3][8][9][93]; this work is summarized in the survey article by Berstel [7], and in a forthcoming book by Shallit and Allouche.

For the most part, in this work we shall study rather different questions, as summarized in the next section.

No history of Sturmian systems would be complete without mention of the bizarre case of Penrose versus Kimberly-Clarke. It seems that Penrose managed to copyright his tilings some twenty years ago. In ignorance of this fact (we presume), some engineer wrestling with the problem that sheets of quilted toilet paper with printed patterns tend to stick together wherever the patterns agree, thought of using Penrose tiling pattern instead of a periodic pattern, presumably with the idea that this would prevent an exact overlap. (Actually, the choice of Penrose tilings in particular, rather than just any old Sturmian tiling, is really rather clever for reasons we shall explain in Section 4.6).

In any event, Penrose noticed Kimberly-Clarke marketing “quilted” toilet paper
embossed with one of the copyrighted patterns and filed suit. As Penrose's lawyer explained to the Times of London:

So often we read of very large companies riding roughshod over small businesses or individuals, but when it comes to the population of Great Britain being invited by a multinational (company) to wipe their bottoms on what appears to be the work of a knight of the realm without his permission, then a last stand must be made.

1.3 The Problems

Sturmian systems \( S(W) \), and their close cousins the Sturmian tiling spaces \( T(W) \), are defined as we have seen by a choice of some \( p \)-dimensional subspace \( W \) in \( \mathbb{R}^{p+q} \). The set of all such subspaces (for fixed \( p \) and \( q \), i.e. fixed genus) forms a smooth manifold, in fact a homogeneous space, the Grassmannian\(^5\) \( G(p,q) \). The central question studied in this thesis is: how do various properties of \( S(W) \) change as we vary \( W \) over the Grassmannian? Some such properties are:

1. At the level of a single tiling:

   (a) the geometry of the oblique tiling (Section 3.3; see also Fig. 1.6 for some immediate intuition),

   (b) degeneracy (Section 3.5),

   (c) walls (Section 4.3),

   (d) periods (Section 4.4),

   (e) almost periods (Section 4.6),

   (f) natural parsing into a “supertiling”,

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\(^5\)The theory of Grassmannians is summarized in Section 3.2; note our notation is slightly non-standard.
1.3. **THE PROBLEMS**

(g) the combinatorics and statistics of the occurrence and recurrence of patches within a tiling (Sections 4.8, 4.10, 6.4),

(h) singular patches and faults (Section 4.2 and Chapter 6),

(i) diffraction patterns (see [71][124]).

2. At the level of a tiling space:

(a) the language and grammar of patches (Chapter 7),

(b) dynamical subsystems (Section 4.7),

(c) unfolding of singular tilings (Section 4.2),

(d) the combinatorics and statistics of the occurrence and recurrence of patches within a Sturmian system (Sections 4.8, 4.10, 6.4),

(e) the empire of a patch (Chapter 7),

(f) dynamical spectra (see for instance [111]),

(g) projective homotopy groups (Chapter 8),

(h) the micropolar operation (Chapter 9),

(i) composition of the oblique tiling, if any, and the induced composition of Sturmian tilings (Chapter 5).

3. At a higher level:

(a) approximation of aperiodic tiling spaces by periodic ones (Section 4.4),

(b) nonlinear perturbation of Sturmian tiling spaces (see [60]).

Needless to say, upon closer study the rough distinctions between levels of structure indicated above tend to become blurred. We hasten to assure the reader that the technical terms mentioned above will be defined in the appropriate sections.
1.4 The Mathematical Themes

There are a number of recognizable mathematical themes which arise naturally in the study of Sturmian systems, including:

1. multiple levels of structure (e.g. blocks, sequences, systems, and parameter spaces of systems),

2. the central role of number theory; indeed, Sturmian tilings may well be regarded as ingenious geometric visualizations of problems in Diophantine approximation,

3. the role of the duality between $S(W)$ and $S(W^\perp)$,

4. the role of symmetry, both algebraic and geometric,

5. the tight interconnection between the combinatorial, probabilistic, geometric. and even algorithmic structure of Sturmian systems,

6. the transition from local to global order (e.g., empires),

7. approximations (e.g., periodic Sturmian tilings approximate aperiodic ones. SFT’s and Markov shifts approximate a Sturmian system)

8. linearization (e.g., the Grassmannian is replaced by flat simplicial faces)

9. singularities are best viewed not as an unwanted complication but as a fundamental phenomenon to be exploited in understanding other phenomena.

Somewhat more informal themes of this thesis include:

1. the conceptual superiority of the oblique tiling method to its rivals,

2. the importance of working at the right level of structure and generality,
3. spectral questions have been overemphasized; the empire phenomenon is equally fundamental, and conceptually simpler.

1.5 The Audience

Sturmian systems are currently of intense interest in physics and crystallography because they provide a mathematical idealization of quasicrystals. They are of interest to mathematicians as important examples of "rigid" dynamical systems where what is happening locally can influence events arbitrarily far away; our study of the empires of Sturmian systems makes this surprising phenomenon quite explicit.

Accordingly, we have made every attempt to reach a wide audience including not only mathematicians but also physicists and crystallographers. In places we necessarily assume some mathematical maturity and in particular some knowledge of first year graduate analysis (including some measure theory) and algebra (including some multilinear algebra). However, many of our arguments are highly pictorial in nature, and we always attempt to describe our results in clear and simple terms, often with the aid of examples and carefully chosen illustrations. Thus, we hope that most of this work will be found quite readable even by persons whose background does not meet these fairly modest standards.

1.6 Some Highlights

For the convenience of the busy specialist in symbolic and tiling dynamical systems, we provide here a brief guide (with section references) to the results, insights, and methods most likely to interest them.

1. a connection between Plücker lines, generalized Euler angles (Sections 3.1 and 3.2), and the volumes of the prototiles of $S(W)$ (Section 3.4),

2. the slide rule construction ought to be folklore but might be new (Section 3.8),
3. each wall of a \((p, q)\) Sturmian system is "essentially" a specific tiling in a specific \((p - 1, q)\) Sturmian system, so that every Sturmian tiling is "woven" from one dimensional Sturmian tilings (Section 4.3),

4. \(\mathcal{T}(W)\) is uniquely ergodic iff \(\mathcal{T}(W^\perp)\) is aperiodic (Section 4.7),

5. the ergodic decomposition of \(\mathcal{T}(W)\) is determined by the periods of \(\mathcal{T}(W^\perp)\), and vice versa (Section 4.7),

6. the enumeration and determination of alternatives for tile recurrence (Section 4.10),

7. the determination of which \((1, 1)\) tilings have compositions coming from a toral automorphism, and connections with Pell's equation and continued fractions (Section 5.1),

8. the introduction of the composition group and an example with a composition group which is neither trivial nor isomorphic to \(\mathbb{Z}\) (Section 5.3),

9. the classification of elementary singularities (Section 6.1),

10. the concepts of recurrence vectors and fault lines (quasiperiodic linear arrays of recurring singularities), and the classification of fault lines using Young diagrams (Section 6.2),

11. the existence of a number of previously overlooked phenomena involving singularities:

   (a) the faults (if any) of a one dimensional Sturmian tiling space correspond to the periodic ribbons (if any) of its dual (Section 6.3),
1.7. *SOME UNANSWERED QUESTIONS*

(b) the Penrose worm is actually a superposition of two types of worm (Section 6.4),

(c) it is possible to have singularities scattered over the entire tiling (Section 6.5).

12. the determination of all Markov and SFT approximations to (1, 1) tilings, using the new concept of “higher block oblique tilings” (Chapter 7),

13. the determination of all natural parsings into supertilings for (1, 1) tilings (Chapter 7 and [66]),

14. the determination of the kingdoms and empires of (1, 1) tilings; there is a close connection with the Farey tree and continued fractions (Chapter 7 and [66]),

15. the correction of Ammann’s assertions about empires of Penrose tilings (Chapter 7),

16. the Ledrappier shift is projectively connected and apparently has an infinitely generated projective homotopy group (Section 8.3).

17. the introduction of the micropolar operation on rhombic tilings (Chapter 9),

In the category of forthcoming work by the author which we hope will also be of interest to tiling theorists, we might mention the introduction of Arnold tilings, which are to Sturmian tilings as Arnold’s sine-circle map is to circle rotations [60].

1.7 Some Unanswered Questions

The author is confident he has by no means exhausted the riches of the point of view here adopted, or the new ideas introduced. In particular, we hope that some graduate student will be inspired to work on various problems and conjectures scattered
throughout the text (particularly Chapters 5 to 9), which ought to yield interesting results whether or not one ever answers the original question!

This a good place to point out that there is a natural “plan of attack” for any problem involving Sturmian tilings:

1. study the classical Sturmian systems, the \((1, 1)\) tiling spaces.

2. generalize to \((1, q)\) tilings,

3. “dualize” to \((p, 1)\) tilings,

4. use the fact that the walls of \((p, q)\) tilings are essentially \((p - 1, q)\) tilings to “induct" toward the general case.

Unfortunately, many phenomena (such as periods or “transverse faults”; see 6.5) are not susceptible to this line of attack.

1.8 Organization

This thesis is written according to a “helical” plan; many ideas are first introduced somewhat informally in early chapters and studied in depth in a later section. The author has provided copious internal references to help the reader follow these various plot twists.
Chapter 2

Symbolic Dynamics

In this chapter, we offer a sketch of certain background material and provide some references where the reader may find a more complete introduction to the concepts we will need in the sequel. We begin in Section 2.1 with a very sketchy introduction to the general notion of a dynamical system. In Section 2.2, we introduce some background material on lattices, particularly sublattices of the form $V \cap \mathbb{Z}^d$, where $V$ is a linear subspace of $\mathbb{R}^d$. In Section 2.3 we discuss toral rotations and in Section 2.4 we provide some references for toral endomorphisms. In Section 2.5–2.8 we briefly describe the notion of a shift dynamical systems and also discuss some examples. In Section 2.9 we sketch the notion of a tiling dynamical system.

2.1 Dynamical Systems

A dynamical system is a space $X$ of "points" together with a way to move from one point to another, called the dynamic. In the simplest situations, the dynamic can be described by a single mapping $T : X \to X$; this gives a discrete dynamical system. More generally, if $G$ is a group which acts on $X$ then the action by $G$ defines a dynamic. More generally still, if $M$ is a monoid which acts on $X$, then this action gives a dynamic. (See [10] for the definition of a group and some elementary theory. A monoid is "a group in which inverses might not exist"; see [35]. See Appendix B and [99] for group actions.)

In particular, starting with a mapping $T : X \to X$, the monoid $\mathbb{N}$ acts on $X$ by $n \mapsto T^n$, where $T^2 = T \circ T$ is obtained by composition, and so forth. If $T$ happens to be invertible, it defines an action by the group $\mathbb{Z}$. If we have two mappings $S, T : X \to X$, we obtain an action by the free product $\mathbb{N} \ast \mathbb{N}$, or (if they are invertible) by $\mathbb{Z} \ast \mathbb{Z}$. If
$S \circ T = T \circ S$, we obtain actions by $\mathbb{N}^2$ and $\mathbb{Z}^2$ respectively.

An action by $\mathbb{R}$ is often called a flow; in this case we have an invertible mapping $T_t : X \to X$ for each $t \in \mathbb{R}$, such that $T_{s+t} = T_s \circ T_t$. Ordinary differential equations in $x, x', x'', \ldots$ define such flows in $\mathbb{R}^2$ (identified with the $(x, t)$ plane).

The monoid $M$ or group $G$ may have additional structure; for instance $G$ may be a topological group or a Lie group. In general, actions by discrete monoids or groups are called discrete dynamical systems or cascades, whereas actions by continuous monoids or groups are called continuous dynamical systems, and actions by Lie groups are called smooth dynamical systems or flows. (See for instance [107].)

Now suppose that $X$ is not merely a set but rather an object in some category, such as the category of topological spaces or the category of probability measure spaces. (See [61] for the elements of category theory.) Because, for each monoid $M$ and each group $G$, the categories of $M$-sets (that is, $M$-equivariant mappings between sets acted upon by $M$) and the category of $G$-sets (that is, $G$-equivariant mappings between sets acted upon by $G$) form topoi, we are guaranteed that by assuming that the mappings arising from the actions are morphisms (or arrows), we will obtain a new category of actions by morphisms. (See [61] for an introduction to the theory of topoi.)

Many dynamical systems have interesting subsystems; i.e., closed subsets $Y \subset X$ which are invariant under $T$ in the sense that $T(Y) \subset Y$. In other words, $Y$ is a subsystem if $Y$ is closed and if the dynamic moves points of $Y$ only to new points of $Y$. A simple but important example: let $x$ be any "point" and consider the set of all points which can be obtained from $x$ under the dynamic, e.g. $T^{11}(x), T^{-5}(x)$ (if $T$ is invertible), etc. The closure of this set, called the orbit of $x$, is always a subsystem. A dynamical system which has no subsystems is called minimal. (See for instance [137].)

A given space of points $X$ can often possess two or more "interesting" dynamics; some very interesting mathematics arises in such cases.

For more background on dynamical systems in general we particularly recom-
mend [79] and [137]. (A very readable and less advanced alternative is [58].)

2.2 Lattices

It is an awkward fact that there are two uses of the word lattice in mathematics and we will have some occasion in this thesis to refer to both of them.

For the definition of a (geometric) lattice $\Lambda$, the reader may search in vain through [19] (I did). It is often convenient to list the generators of $\Lambda$ (as an abelian group) as the column vectors of an integral matrix $A$ (not necessarily square). The Gramm matrix $G(\Lambda) = A'A$ records the pairwise inner products of the generators and defines a quadratic form associated with $\Lambda$; see [19][124]. From $G(\Lambda)$ we can obtain a triangular square matrix of generators $M(\Lambda)$ (wrt basis for $V$). We assume the reader is familiar with the Dirichlet cells or unit cells of a lattice, which are zonotopes; we will not need the concept of the Voronoi cell. (See, however [19][124].) We also assume the reader is familiar with the concept of the dual lattice $\Lambda^*$ (see [19][124]), and we note the following useful formulae:

$$G(\Lambda^*) = G(\Lambda)^{-1}$$

$$M(\Lambda^*) = (M(\Lambda)^{-1})'$$

where $'$ denotes transpose.

In Section 4.4 and elsewhere we will need the concept of the Smith form of an integer matrix, which we briefly review here.

Definition 1 An integer matrix $S$, having $d$ rows and $r \leq d$ columns, is in Smith form if $S = \begin{bmatrix} D \\ 0 \end{bmatrix}$, where $D$ is an $r$ by $r$ diagonal matrix with diagonal entries $d_1 | d_2 \cdots | d_r$, and where 0 is a $d - r$ by $r$ zero matrix.

Notice that this means that any ones among the diagonal entries $d_j$ occur at top left and any zeros among the $d_j$ occur at bottom right.
There is a standard algorithm for computing the Smith normal form of a given integer matrix $A$; see [39][78][100]. This procedure uses three types of operations:

1. **swap**: permute two rows or columns,

2. **invert**: multiply a row or column by $-1$,

3. **combine**: add an integer multiple of one row (column) to another row (column).

Notice each such operation can be effected (for the column operations) by multiplying $A$ on the left by an appropriate unimodular matrix and (for the row operations) by multiplying $A$ on the right by an appropriate unimodular matrix. The row operations correspond to changing the basis of $H$ while the column operations correspond to changing the basis of $G$. This algorithm (see [39][78][100] for a complete description) yields the following result:

**Proposition 2.1** Given any integer matrix $A$, there are unimodular matrices $U, V$ such that $S = U A V^{-1}$, where $S$ is in Smith form. Moreover, $S$ is unique.

The point is that the diagonal entries $d_1 | d_2 \cdots | d_r$ are precisely the invariant factors of $G/K$; that is, the torsion subgroup of $G/K$, $H/K$, is isomorphic to the additive abelian group $\mathbb{Z}^{d_1} \oplus \mathbb{Z}^{d_2} \cdots \oplus \mathbb{Z}^{d_r}$. But each element of the torsion group plainly corresponds to a point of $G$, lying in $W$, which is inside the Dirichlet cell of $H \subset W$ defined by $A$, and vice versa.

### 2.3 Toral Rotations and $W$-Flows

Before discussing tori, we discuss a simpler special case: the circle $S^1$. As was discovered by Euler, there are two fundamental ways of defining $S^1$, as the group of complex numbers of modulus one (where the group operation is multiplication of complex numbers):

$$S^1 = \{ z \in \mathbb{C} : |z| = 1 \} = \{ \exp 2\pi i x : x \in \mathbb{R} \}$$
or as the quotient group of the abelian group $\mathbb{R}$ (where the group operation is addition of real numbers) by the normal subgroup $\mathbb{Z}$,

$$K = \mathbb{R}/\mathbb{Z}$$

Here, given $x \in \mathbb{R}$, we pass to the quotient group $K$ by the quotient map

$$x \to x + \mathbb{Z}$$

It is customary to regard the coset $x + \mathbb{Z}$ as the fractional part of $x$, which is often written (by abuse of notation) as $x \mod 1$. From the standpoint of measure-theory (but not that of Lie groups or topological spaces!), $K$ is essentially just the half open interval $[0, 1)$.

$K$ and $S^1$ are both Lie groups (groups which are smooth manifolds; see [11][15] for an introduction to the theory of Lie groups), and the point is that they are isomorphic as Lie groups (see [61] for an introduction to the concepts of categories and morphisms). The Lie group isomorphism $K \to S^1$ is given by $x \to \exp 2\pi i x$ (note that this doesn’t depend on the choice of representative $x$, since $x + 1$ maps to the same complex number; note too that $x$ plays the role of an angle in the Argand representation of complex numbers). The definition in terms of complex numbers is often referred to as multiplicative notation for the circle, while the quotient group definition is referred to as additive notation for the circle.

Next, given a fixed angle $\alpha$, the rotation by $\alpha$ of $S^1$ is the smooth map $R_\alpha : S^1 \to S^1$ defined by

$$R_\alpha(z) = z \exp 2\pi i \alpha$$

(in multiplicative notation) or

$$x \to x + \alpha \mod 1$$

(in additive notation). The interesting thing about $R_\alpha$ is that its dynamical properties depend on a simple number-theoretic property of $\alpha$.

\footnote{More properly, this equation should read $x + \mathbb{Z} \to x + \alpha + \mathbb{Z}$.}
1. If $\alpha$ is rational, $R_\alpha$ is periodic, i.e. there is some $n$ such that iterating the map $n$ times gives the identity map on $S^1$. In this case, the orbit of any $z \in S^1$ is a finite closed set.

2. If $\alpha$ is irrational, $R_\alpha$ is almost periodic, i.e. given $\epsilon > 0$, there is some $n$ such that $R_\alpha$ iterated $n$ times moves no point further than $\epsilon$ from its original position. In this case, the orbit of each point is dense. As a dynamical system, $(S^1, R_\alpha)$ is in this case minimal (see [79]).

This is our first hint of a general phenomenon in the theory of dynamical systems in general and Sturmian systems in particular (see [82] for a fine exposition of the role played by number theory in dynamical systems).

**Definition 2** The $d$-torus $T^d$ is the Cartesian product $\Pi_{j=1}^d S^1$; that is

$$T^d = \{(z_1, z_2, \ldots, z_d) : z_j \in \mathbb{C}, |z_j| = 1\}$$

Equivalently, in additive notation we may define the $d$-torus as $K^d = \mathbb{R}^d/\mathbb{Z}^d$, as before $K^d$ and $T^d$ are isomorphic as Lie groups. $T^d$ (or $K^d$) is a compact abelian Lie group equipped with the canonical covering map

$$\pi(x_1, x_2, \ldots, x_d) = (z_1, z_2, \ldots, z_d)$$

where $z_j = \exp 2\pi i x_j$. Note that the fundamental domain of this covering map is the unit cube $K = [0, 1]^d$; in other words $[0, 1)^d$ plays the same role here as $[0, 1)$ plays for the circular theory. (See [11],[89] for covering space theory.)

**Definition 3** Given a p dimensional subspace $W$ of $\mathbb{R}^d$, define the $W$-flow on $T^d$ to be the $\mathbb{R}^p$ action defined by the rotations

$$T^w(z_1, z_2, \ldots, z_d) = (w_1 z_1, w_2 z_2, \ldots, w_d z_d)$$

where $\pi(w) = (w_1, w_2, \ldots, w_d)$ and where we have identified $\mathbb{R}^p$ with $W$ (considered as an abelian group).
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If we write \( \pi(x) = (z_1, z_2, \ldots, z_d) \) this becomes

\[
T^w(\pi(x)) = \pi(x + w)
\]

(2.1)

Several books (e.g. [79][137]) discuss the theory of \( W \)-flows in the special case of one-dimensional \( W \); however, we need the theory of the general case.

**Definition 4** A subtorus of \( T^d \) is a submanifold which is homeomorphic to a torus of smaller dimension.

**Definition 5** Let \( W \) be a subspace of \( \mathbb{R}^d \). If \( W \) is the column space of some integer matrix, it is said to be rationally situated (or \textit{"rational" for short}). At the opposite extreme, if \( W \) does not contain any integer vectors, it is called \textbf{irrationally situated} (or \textit{"irrational"}).

**Proposition 2.2** \( W \) is rational iff \( W^\perp \) is rational.

**Proof:** Suppose \( W \) is rational; then \( W \) is the column space of some integer matrix \( A \). The null space of the transpose \( A' \) is \( W^\perp \), so \( W^\perp \) is also the column space of an integer matrix. Since \((W^\perp)^\perp = W\), this argument also shows the converse implication.

It is now tempting to guess that \( W^\perp \) is irrational exactly in case \( W \) is, but this is not true. For example, if \( \tau \) is the Golden Ratio, then

\[
W = \text{col} \begin{bmatrix} 1 \\ \tau \\ \tau^2 \end{bmatrix}
\]

is irrational, but since \( \tau^2 = \tau + 1 \),

\[
W^\perp = \text{col} \begin{bmatrix} \tau & \tau^2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \text{col} \begin{bmatrix} \tau & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}
\]

is not.

**Lemma 2.3** Every \( p \)-dimensional subtorus of \( T^d \) is a coset of the closed connected subgroup \( \pi(V) \), where \( V \) is a rationally situated \( p \)-dimensional subspace of \( \mathbb{R}^d \).
Several representative subtori of $T^3$ are pictured in Fig. 2.1. Note that the integer matrices are in Smith Form.

Our major goal is to describe the orbit closure under the $W$-flow for an arbitrary subspace $W$. The following elementary operation which proves surprisingly useful in several places in our subsequent work; it was apparently first introduced by Thang Le [86].

**Definition 6** Let $W$ be a linear subspace of $\mathbb{R}^d$. Then the rational closure of $W$ is

$$\text{rat } W = \text{span}(W^\perp \cap \mathbb{Z}^d)^\perp.$$  \hspace{1cm} (2.2)

Note that $W^\perp \cap \mathbb{Z}^d$ is the largest set of integer vectors contained in $W^\perp$, and $\text{span}(W^\perp \cap \mathbb{Z}^d)$ is the largest rational subspace of $W^\perp$, so $\text{rat } W$ is the *smallest* rational subspace containing $W$. 
Lemma 2.4 Rational closure $\text{rat}(\cdot)$ defines an algebraic closure operator; that is,

1. If $W$ is any subspace of $\mathbb{R}^d$, then $W \subset \text{rat} W$,

2. If $W \subset V$ are subspaces of $\mathbb{R}^d$, then $\text{rat} W \subset \text{rat} V$,

3. If $W$ is any subspace of $\mathbb{R}^d$, then $\text{rat} \text{rat} W = \text{rat} W$.

Proof: The proof amounts to routine computation. To wit: if $w \in W$, then $w$ is orthogonal to $W^\perp$ and thus to $W^\perp \cap \mathbb{Z}^d$, so $W \subset \text{rat} W$, which proves the first claim. If $W \subset V$, then $W^\perp \supset V^\perp$, so $W^\perp \cap \mathbb{Z}^d \supset V^\perp \cap \mathbb{Z}^d$, so $\text{rat} W \subset \text{rat} V$, which proves the second claim. And

$$
\text{rat} \text{rat} W = ((\text{rat} W)^\perp \cap \mathbb{Z}^d)^\perp \\
= ((W^\perp \cap \mathbb{Z}^d)^\perp \cap \mathbb{Z}^d)^\perp \\
= (W^\perp \cap \mathbb{Z}^d)^\perp \\
= \text{rat} W
$$

which proves the third claim.

(See [25] for more about algebraic closure operators; other examples of an algebraic closure operator include topological closure, linear span, and convex hull.)

Note that we always have $W \subset \text{rat} W \subset \mathbb{R}^d$; the two extremes are achieved under the following circumstances:

1. $\text{rat} W = W$ iff $W$ is rational (equivalently, $W^\perp$ is rational),

2. $\text{rat} W = \mathbb{R}^d$ iff $W^\perp$ is irrational.

Next, we recall some notions of Pontryagin duality theory. Recall that if $G$ is a local compact abelian group, its character group is

$$
G^\dagger = \{ \text{continuous homomorphisms } \chi : G \to S^1 \} 
$$
In particular, every character of \( T^d \) has the form (see for instance Theorem 0.14 of [137])
\[
\chi_n(z_1, z_2, \ldots, z_d) = \prod_{j=1}^{d} z_1^{n_1}z_2^{n_2} \cdots z_d^{n_d}
\]
where \( n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d \), and the character group of \( \mathbb{R}^d \) is homeomorphically isomorphic to \( \mathbb{Z}^d \). If we write \( (z_1, z_2, \ldots, z_d) = \pi(x) \) where \( x \in \mathbb{R}^d \), we compute
\[
\chi_n(\pi(x)) = \prod_{j=1}^{d} (\exp 2\pi i x_j)^{n_j} = \prod_{j=1}^{d} \exp 2\pi i x_j n_j = \exp 2\pi i \sum_{j=1}^{d} x_j n_j
\]
whence we have the elegant formula
\[
\chi_n(\pi(x)) = \exp 2\pi i n \cdot x \quad (2.3)
\]

If \( H \) is a subgroup of \( G \), its annihilator subgroup is
\[
\text{ann } H = \{ \chi : \chi(h) = 1 \text{ for every } h \in H \}
\]
The annihilator of \( H \) agrees with the annihilator of its closure \( \overline{H} \). Moreover (see for instance Theorem 27 of [97])
\[
H^\dagger = G^\dagger / \text{ann } H
\]
Also, the famous Pontryagin duality theorem implies that if \( H \) is closed then
\[
(H^\dagger)^\dagger = H
\]

**Lemma 2.5** Let \( W \) be any subspace of \( \mathbb{R}^d \). Then
\[
\text{ann } \pi(W) = \{ \chi_n : n \in W^\perp \cap \mathbb{Z}^d \}
\]
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Proof: By definition

\[ \text{ann } \pi(W) = \{ \chi_n : \chi_n(\pi(w)) = 1 \text{ for all } \pi(w) \in \pi(W) \} \]
\[ = \{ \chi_n : \exp 2\pi i \, n \cdot w = 1 \text{ for all } w \in W \} \]
\[ = \{ \chi_n : n \cdot w \in \mathbb{Z} \text{ for all } w \in W \} \]
\[ = \{ \chi_n : n \cdot w = 0 \text{ for all } w \in W \} \]

where the last equation follows from taking \( w \) such that \( n \cdot w \) is some integer, and considering \( tw \) where \( t \) is a small positive number. Thus,

\[ \text{ann } \pi(W) = \{ \chi_n : n \in W^\perp \cap \mathbb{Z}^d \} \]

as claimed.

\[ \square \]

Corollary 2.6 Let \( W \) be any subspace of \( \mathbb{R}^d \) and let \( V = \text{rat } W \). Then \( \pi(V) \) is the smallest closed subgroup containing \( \pi(W) \).

Proof: We have

\[ \text{ann } \pi(W) = \text{ann } \pi(W) \]
\[ = \{ \chi_n : n \in W^\perp \cap \mathbb{Z}^d \} \]
\[ = \{ \chi_n : n \in V^\perp \cap \mathbb{Z}^d \} \]
\[ = \text{ann } \pi(V) \]

Therefore the character groups of \( \pi(W) \) and \( \pi(V) \) agree, but these are both closed subgroups, so taking the character group again we find \( \pi(W) = \pi(V) \) as claimed. \( \square \)

It follows at once that the closures of the cosets of \( \pi(W) \) are precisely the cosets of \( \pi(V) \). Since the orbits of \( T^d \) under the \( W \)-flow are the cosets of \( \pi(W) \) we have proven the following.
**Theorem 2.7** Let $W$ be a subspace of $\mathbb{R}^d$, and let $V = \text{rat } W$. Then the orbit closures under the $W$-flow are the cosets of the closed connected subgroup $\pi(V)$; that is, they are "parallel subtori".

It follows that

1. the orbits of the $W$-flow are already closed iff $W$ is rationally situated,

2. the $W$-flow is minimal iff $W^\perp$ is irrationally situated.

**Lemma 2.8** Let $W$ be a subspace of $\mathbb{R}^d$. Then the following are equivalent:

1. the $W$-flow on $T^d$ is ergodic.

2. the $W$-flow on $T^d$ is uniquely ergodic.

3. the $W$-flow on $T^d$ is minimal.

For a proof see Theorem 3.2 of [70].

It follows from this that the $W$-flow on $T^d$ is strictly ergodic (that is, both minimal and uniquely ergodic) if $W$ is irrationally situated; otherwise the ergodic decomposition with respect to Haar measure on $T^d$ decomposes the flow into parallel subtori, the cosets of the closed connected subgroup $V = \text{rat } W$, on each of which the flow is uniquely ergodic.

For example, if

$$W = \text{col } \begin{bmatrix} 1 \\ \tau \\ \tau^2 \end{bmatrix}$$

where $\tau$ is the Golden Ratio, then, as we have seen,

$$W^\perp = \text{col } \begin{bmatrix} \tau & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

so that if we write $V = \text{rat } W$, then

$$V^\perp = \text{col } \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
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$$V = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$

The orbits of the $W$-flow are subspaces (topological lines), the cosets of the nonclosed subgroup $\pi(W)$, which densely fill up a family of subtori, namely the cosets of the closed subgroup $\pi(V)$. The flow is strictly ergodic on each subtorus.

2.4 **Toral Endomorphisms**

We assume the reader is familiar with the notion of a toral endomorphism and their association with integer matrices (see for instance [78] and [79]), and also with the notion of a Markov partition (see [1][79][107]).

2.5 **Shift Systems**

An important example of a discrete dynamical system arises from placing a dynamic upon the space of two-sided binary sequences $\mathbb{B}^\mathbb{Z}$, where $\mathbb{B} = \{0, 1\}$. Note that a “point” in this space is a function $x : \mathbb{Z} \to \mathbb{B}$; of course, $x$ is nothing but a binary sequence indexed by the integers $\mathbb{Z}$, but it turns out to be useful and suggestive to think of it as a function. The simplest way of defining a dynamic on this space of sequences is to define the shift map which takes $x$ to $Tx$, where $Tx(n) = x(n + 1)$; that is, $T$ simply “shifts” each binary sequence one place to the right. Note that the shift map on two-sided sequences is invertible (the inverse of course shifts one place to the left), so it defines an action by $\mathbb{Z}$ on $\mathbb{B}^\mathbb{Z}$. This dynamical system is called the full two-sided shift system.

On the other hand, consider the space of one-sided binary sequences $\mathbb{B}^\mathbb{N}$; we can still define the shift map, but this will no longer be invertible. Thus, we obtain an action by the monoid $\mathbb{N}$ on $\mathbb{B}^\mathbb{N}$ which we call the full one-sided shift system.

We can define a distance between two-sided sequences $x, y$ by $d(x, y) = 2^{-n}$ where $|n|$ is the least integer such that $x(n) \neq y(n)$, and similarly for one-sided sequences.
These definitions make $\mathbb{B}^Z$ and $\mathbb{B}^N$ into metric spaces; in both cases the metric topology is equivalent to the product topology (where $\mathbb{B}, \mathbb{N}, \mathbb{Z}$ are all given their discrete topologies). Interestingly enough, $\mathbb{B}^Z$ and $\mathbb{B}^N$ are each homemorphic to the Cantor set! See for instance [137] for details.

However, as continuous dynamical systems, $\mathbb{B}^Z$ and $\mathbb{B}^N$ are in some respects very different. The subsystems of $\mathbb{B}^Z$ are called \textbf{two-sided shift systems}; similarly the subsystems of $\mathbb{B}^N$ are called \textbf{one-sided shift systems}. We assume the reader is familiar with the \textbf{balls} of one-sided and two-sided shifts (see for instance [27]).

A fundamental principle of language is that one must be able to recognize a word regardless of its position in a sentence, and also to realize that the varying the position of a word may well change the meaning of a sentence in which it appears. We refer to a block of consecutive symbols in a particular position as a \textit{block}; a \textit{protoblock} is the equivalence class (under translation) of a some block.

Every shift system is defined by some set (possibly) infinite of forbidden protowords. For a duality between the forbidden protoblocks and subshifts, see Appendix A.

\subsection{2.6 Shifts of Finite Type}

We will assume that the reader is familiar with the most important type of subshift, the finite type subshifts, or \texttt{sft} for short, as well as the role of matrices in defining \texttt{sft}'s and in counting the length of words, and the fundamental criteria for mixing, transitive, and minimal shifts. See for instance [27][81][87] for all of these notions. Note that \texttt{sft}'s are called topological \texttt{Markov} chains in [79]. For some comments on levels of structure and the entropy theory of shifts of finite type, see [59].
2.7 Markov Shifts

The notion of a Markov shift is very clearly described in [126]. We will assume the reader is familiar with their definition in terms of matrices of transition frequencies, and with the fundamental criterion for ergodicity in terms of such a matrix. See for instance [6][27].

2.8 Classical Sturmian Shifts

Sturmian shifts were one of the first classes of symbolic dynamical systems studied in mathematics, and they remain important because they provide some of the simplest examples of quasiperiodic dynamical systems. Their importance for our work is that they are in fact the simplest case of a Sturmian system; moreover, the relation between Sturmian shifts and circle rotations suggests the relation between more general Sturmian systems and toral flows which was discovered by Robinson [114] (see Section 3.7).

We will discuss three equivalent definitions of Sturmian shifts in this section. The first is closest to the underlying circle rotations. Let $R_\alpha : S^1 \to S^1$ be the rotation by an irrational angle $\alpha$ as in Section 2.3. The central idea of symbolic dynamics is this: given a map on a space and a finite partition of that space, associated to each point in the space its symbolic itinerary [87]. The whole art lies in choosing the right partition for a given map.

In this case, the appropriate partition turns out to be

$$I_A = \{ \exp 2\pi i \beta : \beta \in [0, \alpha) \}$$
$$I_B = \{ \exp 2\pi i \beta : \beta \in [\alpha, 1) \}$$
Given \( z \in S^1 \), define a corresponding symbolic sequence \( x : \mathbb{Z} \to \{A, B\} \) by

\[
x(n) = \begin{cases} 
A & R^{n}_\alpha(z) \in I_A \\
B & R^{n}_\alpha(z) \in I_B 
\end{cases}
\]  

(2.4)

where \( R^2_\alpha = R_\alpha \circ R_\alpha \), etc., and where we assume that \( z \in S^1 \) is not in the preimage of the identity element \( 1 \in S^1 \); that is, we assume

\[
z \notin \bigcup_{N=-\infty}^{\infty} R^{-N}_\alpha(1)
\]

Suppose on the other hand that \( R^N_\alpha(z) = 1 \); a moment’s thought show this is equivalent to saying that \( R^{N+1}_\alpha(z) = \exp 2\pi i \alpha \). In other words, if \( z \in \bigcup_{N=-\infty}^{\infty} R^{-N}_\alpha(1) \), then precisely two points in the orbit of \( z \) under \( R_\alpha \) fall on the two endpoints separating the intervals \( I_A, I_B \). In this case, we define two sequences \( x_+ \) and \( x_- \), each given by the above formula except for the two exceptional indices \( N, N+1 \), where we have

\[
x_+(N) = A, \quad x_+(N + 1) = B \\
x_-(N) = B, \quad x_-(N + 1) = A
\]

(2.5) \hspace{1cm} (2.6)

Thus, \( x_+, x_- \) agree except at the two exceptional indices. (In Section 4.2 we will encounter a more geometric way of understanding this.)

The closure of this set of sequences defines a **Sturmian shift** \( X(\alpha) \), and we have a factor map \( \pi : X(\alpha) \to S^1 \) which is one-one except on the \( z \) of form \( R^N_\alpha(z) = 1 \) (for some \( N \)), for which it is two to one; this situation can be briefly described by saying \( \pi \) is **almost one-one** and has thickness two (see [114]). Like the underlying circle rotation \( R_\alpha \), the Sturmian shift \( X(\alpha) \) is minimal [87][98], and in fact strictly ergodic. (We shall generalize this in Section 4.7.) Both \( S^1 \) and \( X(\alpha) \) can be considered \( \mathbb{Z} \)-sets in the sense of Appendix B, and the factor map \( \pi : X(\alpha) \to S^1 \) is a \( \mathbb{Z} \) hom, which gives the action by \( R_\alpha \) on \( S^1 \) as a factor of the action by shift map on \( X(\alpha) \).
Our second definition of Sturmian shifts arises by “lifting” \( R_\alpha : S^1 \to S^1 \) to the covering space \( \mathbb{R} \) to obtain a map \( \hat{R}_\alpha : \mathbb{R} \to \mathbb{R} \), given by \( x \to x + \alpha \). Notice this corresponds to additive notation without the “mod 1”, and that \( \varepsilon \circ \hat{R}_\alpha = R_\alpha \), where \( \varepsilon : \mathbb{R} \to S^1 \) is given by \( \varepsilon(x) = \exp 2\pi ix \). We wish to understand iterations of \( R_\alpha \), so let us make copies of \( \mathbb{R} \) (the covering space of \( S^1 \)) and represent them as the vertical lines \( x_1 = n \), where \( n \in \mathbb{Z} \). The effect of \( \hat{R}_\alpha \) is to shove each successive line up by an additional increment of \( \alpha \). Suppose we write down an \( A \) every time the line \( x_2 = \alpha x_1 + \beta \) crosses a vertical segment, and that we write down a \( B \) every time it crosses a horizontal segment; see Fig. 2.2. A little thought shows that we have the alternate definition

\[
y(n) = \begin{cases} 
AB & 0 < n\alpha + \beta < \alpha \\
B & \alpha < n\alpha + \beta < \alpha + 1 
\end{cases} \tag{2.7}
\]

where again a modified definition must be used in the exceptional case that \( x_2 = \alpha x_1 + \beta \) runs through an endpoint of the partition.

Our third definition uses a sliderule consisting of two pieces which can be slid parallel to one another; the \( B \) piece has ticks separated by the distance \( \sqrt{1 + \alpha^2} \)
and the $A$ piece has ticks separated by $\sqrt{1 + \alpha^{-2}} = \frac{\sqrt{1 + \alpha^{-2}}}{\alpha}$. To obtain a given Sturmian sequence in this particular shift, the slides are moved to the correct relative position, and the tick marks are read off from left to right; see Fig. 2.3. Note that the “wavelength ratio” of $B$ to $A$ is $1 : \frac{1}{\alpha}$, corresponding to the “frequency ratio” $1 : \alpha$.

(In Section 3.8 we generalize this construction to $(1,q)$ Sturmian systems, and Section 3.6 we shall discuss the analogous multigrid construction of $(p,q)$ Sturmian systems, where $p > 1$.)

In the special case of $\beta = 0$, i.e. the line $x_2 = \alpha x_1$, our second definition can be re-expressed in terms of chain codes (see [82][90][139]) like this:

$$x_+(n) = \begin{cases} 
A & \left\lfloor \frac{na}{1+\alpha} \right\rfloor = \left\lfloor \frac{(n+1)a}{1+\alpha} \right\rfloor \\
B & \text{else}
\end{cases}$$

$$x_-(n) = \begin{cases} 
A & \left\lfloor \frac{na}{1+\alpha} \right\rfloor = \left\lfloor \frac{(n+1)a}{1+\alpha} \right\rfloor \\
B & \text{else}
\end{cases}$$

In the case $\alpha = 1/\tau$, where $\tau$ is the golden ratio, this yields a sequence which can also be constructed by the substitution

$$A \rightarrow B$$

$$B \rightarrow AB$$

It turns out that factor map $\pi : X(1/\tau) \rightarrow S^1$ now maps the corresponding composition map $X(\alpha) \rightarrow X(\alpha)$ to an endomorphism $S^1 \rightarrow S^1$.

### 2.9 Tiling Spaces as Dynamical Systems

Think of a black and white “picture” in the plane as a function $x : \mathbb{R}^2 \rightarrow \mathbb{B}$ where of course the value $x(s,t) = 1$ indicates the presence of ink at the point $(s,t) \in \mathbb{R}^2$. 
Given \((u, v) \in \mathbb{R}^2\), we can take \(x\) to the new function

\[
(T_{(u,v)}x)(s,t) = x(s+u, t+v)
\]

This defines an action by \(\mathbb{R}^2\) on the set of "pictures". We can make this set into a metric space by imposing a metric analogous to the metric on binary sequences introduced in Section 2.5.

The point is that given some tiling of the plane, thought of simply as a picture with ink present exactly on the vertices and edges of the tiling, the translation orbit closure of this picture is a subsystem of the dynamical system of pictures. Any such closed invariant subspace is called a tiling dynamical system. See [111][115] for more precise definitions and for the basic theory of tiling dynamical systems.

We will assume that the reader is familiar with the concepts of tiles and prototiles as introduced in [53]. In analogy with this terminology we will also refer to patches and protopatches.

The "dual" concepts of empires and cylinder sets play a fundamental role in Chapter 7 of this thesis. As it happens, the author has described, in an expository paper, the general concept of empire as an example of the general theory of concepts, along with several other pairs of dual concepts which are fundamental to symbolic dynamics. Since this paper places these notions in a wider context and hopefully makes entertaining reading, rather than excerpting the relevant sections here we refer the reader to the complete paper, *What is a Concept?*, which is included here as Appendix A.
Chapter 3

The Construction of Sturmian Systems

In this chapter we begin our study of Sturmian systems. We begin with two background sections (Sections 3.1 and 3.2) establishing various algebraic and geometric facts we'll make use of later in the chapter. We then (Section 3.3) describe the construction of a periodic tiling of $\mathbb{R}^d$, the oblique tiling. In Section 3.4 we use this to construct systems of Sturmian tilings by the so-called “cut-plane method” (due to Oguey, Katz, and Duneau). In Section 3.5 we discuss situations where this construction behaves badly, and give a simple condition which guarantees good behavior. In Section 3.6 we sketch an alternative construction of Sturmian systems, the so-called “multigrid method” (due to de Bruijn). In Section 3.7 we show how to obtain $\mathbb{Z}^p$-shifts from Sturmian systems (this construction is due to E. A. Robinson). In Section 3.9 we construct tiling dynamical systems with a continuous translation action. Finally, in Section 3.10 we show how to construct Sturmian systems with geometrically symmetrical prototiles (this construction is essentially due to de Bruijn).

While most of what we say here will be well known to experts, the material on direction cosines for $p$-dimensional subspaces (Section 3.2) and the corresponding formulae for tile volumes (Section 3.4) may be new.

3.1 Zonotopes

The following simple operation [5] is surprisingly useful:

Definition 7 Let $A, B \subset \mathbb{R}^d$. Then the Minkowski sum of $A, B$ is

$$A + B = \{a + b : a \in A, b \in B\}$$

We recall the following definition:
3.1. ZONOTOPOES

Definition 8 Given two points $x, y \in \mathbb{R}^d$, define the segment between them to be

$$[x, y] = \{sx + ty : s, t \geq 0 \text{ and } s + t = 1\}$$

A subset $A \subset \mathbb{R}^d$ is convex if whenever $a_1, a_2 \in A$, we have $[a_1, a_2] \subset A$.

It is easily verified that the Minkowski sum of two convex sets is convex.

Definition 9 An $p$-zonotope is a subset of $\mathbb{R}^d$ which is the Minkowski sum of $p$ line segments, where $p \leq d$.

Cubes are zonotopes; indeed, zonotopes may be thought of as the natural higher dimensional generalization of rhombs. It is easy to see that zonotopes are always convex. Sums of zonotopes are a natural higher dimensional generalization of convex polygons.

Lemma 3.1 Suppose $Z$ is a $p$-zonotope which is a translate of the Minkowski sum $[0, v_1] + [0, v_2] + \ldots + [0, v_p]$. Let $A = [v_1 v_2 \ldots v_p]$ be the matrix whose columns are the given vectors. Then the volume of $Z$ is given by

$$\text{vol}_p Z = \det(A'A)$$

where $A'$ is the transpose of $A$.

For an elementary proof, see [10].

Suppose $A$ is an $m$ by $n$ matrix and suppose $J, K$ are sets of $p$ row or column indices. Given a matrix $A$, we write $A(J, K)$ for the submatrix obtained by selecting the rows in $J$ and the columns in $K$; similarly, we write $A(J, \cdot)$ for the submatrix obtained by selecting the rows from $J$, and $A(\cdot, K)$ for the submatrix obtained by selecting the columns from $K$. For example, if $J = \{1, 4\}$ and $K = \{2, 3\}$, and $A$ is a four by four matrix, then

$$A(J, \cdot) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$
and

\[ A(J, K) = \begin{bmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{bmatrix} \]

Lemma 3.2 (Cauchy-Binet formula) Suppose \( A \) is a \( p \) by \( d \) matrix and \( B \) is a \( d \) by \( p \) matrix, so that the product \( AB \), as well the matrices \( A(\cdot, J) \) and \( B(J, \cdot) \) are all \( p \) by \( p \) square matrices. Then

\[ \det(AB) = \sum_{|J|=p} \det A(\cdot, J) \det B(J, \cdot) \]

For an elementary proof, see [109].

Corollary 3.3 (Pythagorean theorem) Let \( Z \) be a \( p \)-zonotope in \( \mathbb{R}^d \). Then the squared volume of \( Z \) is the sum of the squared volumes of the orthoprojections of \( Z \) on the \( p \)-dimensional coordinate planes of \( \mathbb{R}^d \).

In the case \( p = 1 \), this is the usual Pythagorean theorem.

Proof: Without loss of generality we can assume that \( Z \) is the Minkowski sum \([0,v_1] + [0,v_2] + \ldots [0,v_p] \). Let \( A = [v_1v_2\ldots v_p] \). Then the \( p \) by \( p \) matrix \( A(\cdot, J) \) gives the coordinates of the orthoprojection on the \( J \)-th coordinate plane of dimension \( p \). By Lemma 3.1, the squared volume of this projection is \((\det A(\cdot, J))^2\), whereas the squared volume of \( Z \) is \( \det A' A \). But by Lemma 3.2,

\[
\begin{align*}
\det A' A &= \sum_{|J|=p} \det A'(\cdot, J) \det A(J, \cdot) \\
&= \sum_{|J|=p} (\det A(J, \cdot))^2
\end{align*}
\]

We are often interested in zonotopes having a given direction; that is, lying in some translate of a given subspace \( W \). Up to translation, we have seen that any \( p \)-zonotope can be written as the (Minkowski) sum

\[ Z = [0, v_1] + [0, v_2] + \ldots [0, v_p] \]
3.1. ZONOTOPEs

For volumetric purposes, it is convenient to represent the set of all $p$-zonotopes with the same direction and $p$-volume by the exterior product $v_1 \wedge v_2 \wedge \ldots v_p$.

We refer the reader to a book such as [141] for the basic definitions of exterior algebra. We write the space of $p$-multivectors in $\mathbb{R}^d$ as $\Lambda^p(\mathbb{R}^d)$.

Let $\{e_j : 0 \leq j \leq d\}$ be the standard basis of $\mathbb{R}^d$; then the corresponding standard basis of $\Lambda^p(\mathbb{R}^d)$ is

$$\{ \mathcal{E}_J = \wedge_{j \in J} e_j : |J| = p \}$$

Thus, $\dim \Lambda^p(\mathbb{R}^d)$ is the binomial coefficient $d!/(p! q!)$, where $p + q = d$.

A $p$-multivector is simple if it is the exterior product of $p$ vectors. A fundamental fact about simple multivectors is the following:

**Lemma 3.4** Suppose $A = [v_1 v_2 \ldots v_p]$. Then

$$v_1 \wedge v_2 \wedge \ldots v_p = \sum_{|J|=p} \det A(J, \cdot) \mathcal{E}_J$$

See for instance [109] or [141].

We will find it very useful to introduce the euclidean inner product on $\Lambda^p(\mathbb{R}^d)$ given in terms of the $\mathcal{E}_J$. That is, if $A = \sum_{|J|=p} A_J \mathcal{E}_J$ and $B = \sum_{|J|=p} B_J \mathcal{E}_J$, then the inner product of $A$ and $B$ is

$$\langle A, B \rangle = \sum_{|J|=p} A_J B_J$$

We also define the norm or length of $A$ to be

$$\|A\| = \sqrt{\langle A, A \rangle}$$

**Lemma 3.5** Let $Z$ be the zonotope $[0, v_1] + [0, v_2] + \ldots [0, v_p]$. Then the volume of $Z$ is just the length of the multivector $\wedge_{j=1}^p v_j$; that is

$$\text{vol}_p Z = \| \wedge_{j=1}^p v_j \|$$

**Proof:** We have $\text{vol}^2 Z = \det(A'A) = \sum_{|J|=p} (\det A(J, \cdot))^2 = \| \wedge_{j=1}^p v_j \|^2$, where the last equality follows Lemma 3.4 and the definition of the inner product. \[\Box\]
3.2 Subspaces and Plücker Lines

As was noted in Chapter 1, in this thesis we define a system of Sturmian tilings, $S(W)$, for every $p$-dimensional subspace $W$, and study how various properties of these systems change as we vary $W$. It follows that since we are parameterizing our systems by the subspaces $W$, we will need some way to parameterize the subspaces themselves. The appropriate notion is the **Grassmann manifold** $G(p, q)$, which is a smooth manifold whose points are precisely the $p$-dimensional subspaces of $\mathbb{R}^d$, where $d = p + q$. (See [11] for the elementary theory of smooth manifolds.) We will regard $G(p, q)$ as the parameter space for our Sturmian systems; in this section we establish some basic facts about this manifold which we shall need later on.

The easiest way to begin to understand $G(p, q)$ is by means of its **affine coordinate patches** [42]. We can regard the generic $p$-dimensional subspace $W$ of $\mathbb{R}^d$ (with codimension $q = d - p$) as the graph of a linear transformation $T : \mathbb{R}^p \rightarrow \mathbb{R}^q$. If $B$ is matrix of $T$ with respect to the standard bases on $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, then the $pq$ components of $B$ give affine coordinates for the subspace $W$. Of course, we don't quite get **every** subspace in this way; the subspace corresponding to $\mathbb{R}^q$ itself is obviously omitted. We do, however, get a well defined coordinate system on an open subset of $G(p, q)$. Indeed, in this way we can easily find affine coordinate patches covering the entire manifold. This shows, incidently, that $\dim G(p, q) = pq$.

For example, let $d = 4$. If $p = 1$, we have a patch corresponding to the parametric equations

$$y_1 = b_1 x, \quad y_2 = b_2 x, \quad y_3 = b_3 x$$

and $\dim G(1, 3) = 3$. If $p = 2$, we have a patch corresponding to the parametric equations

$$y_1 = b_{11} x_1 + b_{12} x_2, \quad y_2 = b_{21} x_1 + b_{22} x_2$$

and $\dim G(2, 2) = 4$. 
3.2. SUBSPACES AND PLÜCKER LINES

In general \( G(1, q) \) is the space of one dimensional subspaces in \( \mathbb{R}^{1+q} \), or real projective space \( \mathbb{R}P^q \), which can be obtained from \( \mathbb{R}^{1+q} \setminus 0 \) by identifying nonzero scalar multiples [5]. As is well known, for many purposes it is convenient to use homoge-
neous coordinates for \( \mathbb{R}P^q \). For example, in the case of \( G(1, 3) \), we obtain \( \mathbb{R}P^3 \) by taking nonzero vectors in \( \mathbb{R}^4 \), with the identifications

\[
(x_1, x_2, x_3, x_4) \sim (tx_1, tx_2, tx_3, tx_4), \quad t \neq 0
\]

The homogeneous coordinates of a point in \( \mathbb{R}P^3 \) can be taken to be \((x_1, x_2, x_3, x_4)\) or any scalar multiple thereof; thus, in general the homogeneous coordinates for \( \mathbb{R}P^q \) contain one parameter's worth of "redundancy". In our example, assuming that the first component is nonzero, we can pick out a unique representative \((1, b_1, b_2, b_3)\), where the \( b_j \) correspond to the affine patch above.

One generalization of homogeneous coordinates to the Grassmannian \( G(p, q) \), \( q > 1 \), is obtained by using the Stiefel coordinates [42] defined in terms of the (real) Stiefel manifold \( S(p, q) \), whose points are the \( d \) by \( p \) real matrices \( A \) of full rank. We can identify the column spaces of the matrices \( A \) (the points of the Stiefel manifold) with the points of the Grassmann manifold. Many of these matrices have the same column space; to remove the redundancy, note that given any \( W \) in the affine coordinate patch defined above, we can find a unique representative of form

\[
A = \begin{bmatrix} I \\ B \end{bmatrix}
\]

where \( I \) is the \( p \) by \( p \) identity matrix and \( B \) is a \( q \) by \( p \) matrix; the components of \( B \) are precisely the affine coordinates already defined. This shows that in general we have \( p^2 \) real parameter's worth of redundancy in the \( dp \) Stiefel coordinates.

There is another way to generalize homogeneous coordinates to the general \( G(p, q) \), due to Plücker. We recall the following fact (see for instance [109]):

**Lemma 3.6** Let \( B \) be a \( p \) by \( p \) matrix and suppose \( v_j = Bu_j, \ 1 \leq j \leq p \). Then

\[
\wedge_{j=1}^p v_j = \det B \wedge_{j=1}^p u_j
\]
It follows that if $\land_{j=1}^p u_j$ is a scalar multiple of $\land_{j=1}^p v_j$, then

$$\text{span}\{u_1, u_2, \ldots u_p\} = \text{span}\{v_1, v_2, \ldots v_p\}$$

This suggests the following definition:

Definition 10 Given a $p$-dimensional subspace $W$ of $\mathbb{R}^d$, the Plücker line of $W$ is

$$\mathfrak{P}(W) = \{w_1 \land w_2 \land \ldots w_p : w_j \in W, 1 \leq j \leq p\}$$

Note that $\mathfrak{P}(W)$ is a one dimensional subspace of $\Lambda^p(\mathbb{R}^d)$. The Plücker coordinates for $G(p, q)$ are the components (with respect to the $e_j$) of the Plücker lines; of course, scalar multiples of simple multivectors are to be identified. Note too that our discussion shows that simple $p$-multivectors can be interpreted as representing the "direction" and volume of a class of $p$-zonotopes; however, we stress that most $p$-multivectors are not simple and do not have such an interpretation. (One interesting application of Plücker coordinates is to give an explicit embedding of $G(p, q)$ as an algebraic submanifold of $\mathbb{R} P^{n-1}$, where $n = d!/(p! q!)$; see [109][141] for details.)

We summarize the most important facts about Plücker lines in the following theorem.

Theorem 3.7 Every simple multivector is an element of some Plücker line. Thus, there is a bijection between the spans of simple $p$-multivectors in $\Lambda^p(\mathbb{R}^d)$ and $p$-dimensional subspaces $W$ of $\mathbb{R}^d$, taking each subspace to its Plücker line. Moreover, if $U, V$ are subspaces of $\mathbb{R}^d$ (not necessarily of the same dimension),

1. $U \subset V$ iff there is a multivector $A$ such that $\mathfrak{P}(U) \land A = \mathfrak{P}(V)$.
2. $U \cap V = \emptyset$ iff $\mathfrak{P}(U) \land \mathfrak{P}(V) \neq 0$, in which case $\mathfrak{P}(U) \land \mathfrak{P}(V) = \mathfrak{P}(U \oplus V)$.
3. Suppose $U, V$ are two distinct $p$-dimensional subspaces such that there is a linear isometry taking $U, V$ to distinct $p$-dimensional coordinate subspaces. Then $\mathfrak{P}(U), \mathfrak{P}(V)$ are orthogonal lines.
For proofs of all but the last assertion, see [141]. In order to prove the last assertion, we first need some additional definitions.

**Definition 11** Given the linear operator $T$ on $\mathbb{R}^d$, the induced operator of $\Lambda^p(\mathbb{R}^d)$ is defined by

$$\Lambda^p(\mathcal{E}_J) = \wedge_{j \in J} T e_j, \ |J| = p$$

The fundamental fact concerning the induced operator is the following:

**Lemma 3.8** Suppose that, with respect to the standard basis vectors $e_j$ of $\mathbb{R}^d$, the operator $T$ has the matrix $A$. Then, with respect to the standard basis vectors $\mathcal{E}_J$ on $\Lambda^p(\mathbb{R}^d)$, the matrix of $\Lambda^p T$ is the $p$-th compound of $A$; that is, the matrix $A^{(p)}$ whose $J, K$ entry is the $p$ by $p$ minor $\det A(J, K)$. In other words

$$(\Lambda^p T)(\mathcal{E}_K) = \sum_{|J| = p} \det A(J, K) \mathcal{E}_J$$  \hspace{1cm} (3.1)$$

Notice that Eq. 3.1 is the exact analog of the familiar expression $T e_k = \sum a_{jk} e_j$ giving the $k$-th column of $A$. For a proof see [109] or [141].

**Proposition 3.9** If $T$ is a linear isometry, so is $\Lambda^p T$.

**Proof:** Suppose that the matrix of $T$ with respect to the standard basis is $A$. Write $M = A^{(p)}$ for ease of notation. Then it suffices to show that $M'M = I$, the $m$ by $m$ identity matrix, where $m = d!/(p!q!)$. Notice that $A'A = \sum_{\ell} a_{\ell j} a_{\ell k}$ and likewise $M'M = \sum_{L} \det A(L, J) \det(L, K)$. Now fix $J, K$ and let $B = A'(J, \cdot) = A(\cdot, J)$ and let $C = A(\cdot, K)$ On the one hand, by Lemma 3.2,

$$\det BC = \sum_{L} \det B(\cdot, L) C(L, \cdot) = \sum_{L} \det A(L, J) \det B(L, K)$$

On the other hand, since $T$ is an isometry, we have $A'A = I$. Therefore, $BC = I$ if $J = K$ and otherwise $BC$ is singular. (The easiest way to see this is to explicitly write out the products $BC$ for a small example.) Thus, the $J, K$ entry of $M'M$ is one
if \( J = K \) and zero otherwise.

(I am grateful to Boris Solomyak for this argument.)

The last assertion in Theorem 3.7 now follows since if \( U, V \) are orthogonal subspaces of dimension \( p \), let \( T \) take \( U, V \) to the \( J, K \) coordinate \( p \)-planes, respectively, of \( \mathbb{R}^d \). Then \( (\Lambda^p T)(\mathcal{P}(U)) = \mathcal{P}(TU) = \text{span}\{\mathcal{E}_J\} \) and \( (\Lambda^p T)(\mathcal{P}(V)) = \mathcal{P}(TV) = \text{span}\{\mathcal{E}_K\} \) are orthogonal lines. But \( \Lambda^p T \) is itself an isometry, so \( \mathcal{P}(U) \) and \( \mathcal{P}(V) \) must be orthogonal lines.

We note here for later reference several useful facts.

**Lemma 3.10** Let \( T \) be the orthoprojection on \( W = \text{col} A \). Then the matrix of \( T \), with respect to the standard basis vectors \( e_j \), is given by

\[
P = A(A'A)^{-1}A' \tag{3.2}
\]

where \( A' \) is the transpose of \( A \).

For an elementary proof, see [130].

The matrix of the orthoprojection onto \( W^\perp \) is

\[
Q = I_d - P \tag{3.3}
\]

where \( I_d \) is the \( d \) by \( d \) identity matrix.

In this situation, we let \( p_j = Pe_j \) and \( q_j = Qe_j \). Notice that \( p_j \) and \( q_j \) are orthogonal, for each \( 1 \leq j \leq d \).

It is often convenient to observe (as suggested our discussion in Section 3.2 of Stiefel coordinates) that by elementary column operations (plus possibly permuting the rows, i.e. renumbering the coordinates), \( A \) may\(^1\) be brought into the form

\[
\tilde{A} = \begin{bmatrix} I_p \\ E \end{bmatrix} \tag{3.4}
\]

\(^1\)Subject to a mild nondegeneracy condition discussed in Section 3.5.
3.2. SUBSPACES AND PLÜCKER LINES

where \( I_p \) is the \( p \) by \( p \) identity matrix and \( E \) is some \( q \) by \( p \) matrix. where \( q = d - p \) is the codimension of \( W \). Then \( W^\perp \) is the column space of the matrix

\[
\tilde{B} = \begin{bmatrix} -E' \\ I_q \end{bmatrix}
\]  

(3.5)

where \( I_q \) is the \( q \) by \( q \) identity matrix. This claim is easily verified by observing that

\[
\tilde{B}'\tilde{A} = [-E \ I_q] \begin{bmatrix} I_p \\ E' \end{bmatrix} = [-E + E]
\]

is the \( q \) by \( p \) zero matrix while

\[
\tilde{A}'\tilde{B} = [I_p \ E'] \begin{bmatrix} -E' \\ I_q \end{bmatrix} = [E' - E']
\]

is the \( p \) by \( q \) zero matrix.

The following proposition, together with Prop. 3.9, tells us that the geometry of operators on \( \mathbb{R}^d \) carries over nicely to the Grassmann algebra \( \Gamma \mathbb{R}^d \): not only do isometries induce isometries, but the orthoprojection on any subspace induces the orthoprojection on the corresponding Plücker line.

**Proposition 3.11** If \( T \) is the orthoprojection on \( W \), \( \Lambda^p T \) is the orthoprojection on \( \mathcal{P}(W) \).

**Proof:** For each \( J \) with \(|J| = p\), let \( \mathcal{P}_J = \bigwedge_{j \in J} p_j = (\Lambda^p T)(\mathcal{E}_J) \). We must show that \( \mathcal{P}_J \) is the orthoprojection of \( \mathcal{E}_J \) on \( \mathcal{P}(W) \). For ease of notation, we perform the necessary computation in a special case, for (say) a two dimensional subspace of \( \mathbb{R}^5 \).

Fix \( J = \{1, 2\} \), Then

\[
e_1 \wedge e_2 - p_1 \wedge p_2 = (p_1 + q_1) \wedge (p_2 + q_2) - p_1 \wedge p_2
\]

\[
= p_1 \wedge q_2 - p_2 \wedge q_1 + q_1 \wedge q_2
\]

Now, \( p_1 \wedge p_2 \) and \( p_1 \wedge q_2 \) represent the Plücker lines of subspaces which can be moved by an isometry onto distinct coordinate planes of \( \mathbb{R}^d \), because \( p_2 \) and \( q_2 \) are orthogonal. Thus the Plücker lines must be orthogonal. Similarly for \( p_1 \wedge p_2 \) and \( p_2 \wedge q_1 \)
and for \( p_1 \wedge p_2 \) and \( q_1 \wedge q_2 \). Thus, \( e_1 \wedge e_2 - p_1 \wedge p_2 \) is orthogonal to \( p_1 \wedge p_2 \); that is, \( p_1 \wedge p_2 \) is the orthoprojection on \( \mathfrak{P}(W) \) of \( e_1 \wedge e_2 \). Similarly for other \( J \).

We immediately obtain the following consequence:

**Corollary 3.12** The length of \( \mathcal{P}_J \) is \( \cos \theta_J \), where \( \theta_J \) is the angle between \( \mathcal{E}_J \) and \( \mathfrak{P}(W) \).

**Proof:** Since \( \mathcal{P}_J \) is the orthoprojection of \( \mathcal{E}_J \) on \( \mathfrak{P}(W) \), i.e. \( \langle \mathcal{E}_J - \mathcal{P}_J, \mathcal{P}_J \rangle = 0 \), we have \( \langle \mathcal{E}_J, \mathcal{P}_J \rangle = \langle \mathcal{P}_J, \mathcal{P}_J \rangle \) so

\[
\cos \theta_J = \frac{\langle \mathcal{P}_J, \mathcal{E}_J \rangle}{\| \mathcal{P}_J \|} = \frac{\langle \mathcal{P}_J, \mathcal{P}_J \rangle}{\| \mathcal{P}_J \|} = \| \mathcal{P}_J \|
\]

Note that Corollary 3.12 generalizes the usual notion of direction cosines for lines to subspaces of any dimension. This suggests defining the **dihedral angle** between any two \( p \)-dimensional subspaces \( U, V \) to be the angle between their Plücker lines.

### 3.3 The Oblique Tiling

**Definition 12** The standard unit cube in \( \mathbb{R}^d \) is

\[
I^d = \left\{ \sum_{j=1}^{d} t_j e_j : t_j \in [0, 1], \ 1 \leq j \leq d \right\}
\]

For each selection \( J \) of \( p < d \) coordinates in \( \mathbb{R}^d \), the set

\[
F_J = \left\{ \sum_{j \in J} t_j e_j : t_j \in [0, 1], \ j \in J \right\}
\]

is called an \( p \)-facet of \( I^d \). The collection of all the \( p \)-facets of the cube \( I^d \),

\[
\text{skele}^p \ I^d = \bigcup_{|J|=p} \bigcup_{K \subseteq J^c} \left( \sum_{k \in K} e_k \right) + F_J
\]

is called the \( p \)-skeleton of \( I^d \).
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In this definition, note that for each \( F_J \), given any \( K \subset \mathcal{J}^c \) we translate \( F_J \) by the vector \( \sum_{k \in K} e_k \), obtaining in this way the set of all the facets in the skeleton which are parallel to the "canonical" facet \( F_J \), namely

\[
\bigcup_{K \subset \mathcal{J}^c} \left[ \left( \sum_{k \in K} e_k \right) + F_J \right]
\]

Indeed, if \( q = d - p \), then for each \( J \) with \( |J| = p \), there are \( 2^q \) translates of \( F_J \) in \( \text{skel}^p \mathcal{I}^d \), corresponding to the \( 2^q \) subsets of \( \mathcal{J}^c \).

Two special cases are worthy of attention: the \( (d - 1) \)-skeleton \( \text{skel}^{d-1} \mathcal{I}^d \) is just the usual notion of the "skin" or surface of the cube, and the 1-skeleton \( \text{skel}^1 \mathcal{I}^d \) is the set of edges.

Fix a \( p \)-plane \( W \) and let \( P, Q \) be the orthoprojections on \( W, W^\perp \) respectively. Write

\[
p_j = P(e_j), \quad q_j = Q(e_j), \quad 1 \leq j \leq d
\]

For each selection \( J \) of \( p \) coordinates in \( \mathbb{R}^d \), where \( d = p + q \), let \( J^c \) be the complementary subset of \( q \) coordinates. Write

\[
T_J = P(-F_J) = \left\{ - \sum_{j \in J} t_j p_j : t_j \in [0,1], \ 1 \leq j \leq d \right\}
\]

\[
T_{J^c} = Q(F_{J^c}) = \left\{ \sum_{j \notin J} t_j q_j : t_j \in [0,1], \ 1 \leq j \leq d \right\}
\]

Note that \( T_J \) and \( T_{J^c} \) are zonotopes.

**Definition 13** Define\(^2\)

\[
C_J = -T_J + T_{J^c} = \left\{ - \sum_{j \in J} t_j p_j + \sum_{j \notin J} t_j q_j : t_j \in [0,1], \ 1 \leq j \leq d \right\}
\]

\(^2\)In the original paper [103], \( C_J \) is defined by \( C_J = T_J - T_{J^c} \); the convention adopted here (for a reason which seemed good at the time) changes nothing essential.
The oblique tiling $\mathcal{O}(W)$ is the collection

$$\mathcal{O}(W) = \{ n + C_J : n \in \mathbb{Z}^d, |J| = p \}$$  \hspace{1cm} (3.12)

We say that $\mathcal{O}(W)$ is said to have genus $(p, q)$, meaning that $W$ is has dimension $p$ and codimension $q$ (that is, $W^\perp$ has dimension $q$.) The $C_J$ are called the protocells of $\mathcal{O}(W)$, and integer vector translates of these, $n + C_J$, are called the cells of $\mathcal{O}(W)$. We say that $n + C_J$ is based at the lattice point $n$.

Note that since $T_J \subset W$ and $T_J^\perp \subset W^\perp$, each cell $C_J$ is a Cartesian product $T_J \times T_J^\perp$. The cells are zonotopes of full dimension\(^3\). In general, an oblique tiling of genus $(p, q)$ has $C(d; p)$ protocells, where $d = p + q$ and where $C(d; p)$ is the binomial coefficient

$$C(d; p) = C(d; q) = \frac{d!}{p!q!}$$

We postpone the proof that $\mathcal{O}(W)$ is indeed a tiling of $\mathbb{R}^d$ to the end of this section, on the grounds that since the oblique tiling is of paramount important in this thesis, it is worth discussing in detail several special cases in order to obtain the maximal amount of geometric intuition for how the cells of $\mathcal{O}(W)$ fit together, especially in the higher dimensional cases.

Consider first the simplest case, the oblique tilings of genus $(1, 1)$. (See Figure 3.1 for an example.) We have

$$A = \begin{bmatrix} 1 & \end{bmatrix}$$  \hspace{1cm} (3.13)

$$P = \frac{1}{\Delta_{11}^2} \begin{bmatrix} 1 & v \\ v & v^2 \end{bmatrix}$$  \hspace{1cm} (3.14)

$$Q = \frac{1}{\Delta_{11}^2} \begin{bmatrix} v^2 & -v \\ -v & 1 \end{bmatrix}$$  \hspace{1cm} (3.15)

where $\Delta_{11} = \sqrt{1 + v^2}$ (we assume that $v \neq 0$). The columns of $P$ are $p_1, p_2$ and those

---

\(^3\)We shall consistently use the word "cells" when referring to zonotopes of full dimension and the word "tiles" when referring to the zonotopes of smaller dimension; see Section 3.4.
3.3. THE OBLIQUE TILING

\[ W = \text{col} \left[ \frac{1 + \sqrt{5}}{2} \right] \]

Figure 3.1: The $(1, 1)$ oblique tiling (solid lines) $O(W)$, where $W$ is the indicated subspace. The oblique tiling $O(W)$ has the same periodicity as the lattice $\mathbb{Z}^2$ (dotted lines). The protomulticell is shaded.

of $Q$ are $q_1, q_2$. There are only two protocells:

\[
C_1 = -P(F_{(1)}) + Q(F_{(2)}) \\
C_2 = -P(F_{(2)}) + Q(F_{(2)})
\]

Since $\|p_1\| = \|q_2\| = 1/\Delta_{11}$ and $\|p_2\| = \|q_1\| = |v|/\Delta_{11}$, both protocells are squares ($C_1$ is larger than $C_2$). The fact that $O(W)$ is indeed a tiling in the case of genus $(1, 1)$ is evident from Fig. 3.1, as is the $\mathbb{Z}^2$ periodicity. Note that the two protocells together form a patch which may be taken as the repeating unit. This suggests the following definition:
Definition 14 The protomulticell of \( \mathcal{O}(W) \) is

\[
\mathcal{M} = \bigcup_{|J|=p} C_J
\] (3.17)

The multicells of \( \mathcal{O}(W) \) are the various integer vector translates \( \mathbf{n} + \mathcal{M} \). where \( \mathbf{n} \in \mathbb{Z}^d \). We say that \( \mathbf{n} + \mathcal{M} \) is based at the lattice point \( \mathbf{n} \).

In the case of oblique tilings of genus \((1, 2)\), we have

\[
A = \begin{bmatrix} 1 \\ u \\ v \end{bmatrix}
\] (3.18)

\[
P = \frac{1}{\Delta_{12}^2} \begin{bmatrix} 1 & u & v \\ u & u^2 & uv \\ v & uv & v^2 \end{bmatrix}
\] (3.19)

\[
Q = \frac{1}{\Delta_{12}^2} \begin{bmatrix} u^2 + v^2 & -u & -v \\ -uv & 1 + v^2 & -v \\ -v & -uv & 1 + u^2 \end{bmatrix}
\] (3.20)

where \( \Delta_{12} = \sqrt{1 + u^2 + v^2} \) and where for simplicity we assume \( 0 < u < v < 1 \).

In this case, there are three protocells, each the Cartesian product of a line segment with a rhombus:

\[
C_{(1)} = -P(F_{(1)}) + Q(F_{(2,3)})
\]

\[
C_{(2)} = -P(F_{(2)}) + Q(F_{(1,3)})
\]

\[
C_{(3)} = -P(F_{(3)}) + Q(F_{(1,2)})
\] (3.21)

See Figure 3.2 for an example, where cells of type \( C_{(1)} \) are colored white, cells of type \( C_{(2)} \) are colored green, and cells of type \( C_{(3)} \) are colored red\(^4\). Notice (see the top row of Fig 3.2) that the base of each multicell is a flat hexagon which viewed along \( W \) looks just like an ordinary cube.

To understand how the multicells fit together, see the middle row of Fig. 3.2. Note (middle left picture) how the red cell \( e_1 + C_{(3)} \) of the multicell \( e_1 + M \), the green cell

\(^4\)If this document has been printed on a black and white printer, red will appear as dark gray, green as medium gray, and white as light gray.
3.3. THE OBLIQUE TILING

Figure 3.2: The $(1, 2)$ oblique tiling $O(W)$, where $W$ is the indicated one dimensional subspace of $\mathbb{R}^3$. The left column depicts the situation from the viewpoint $-(19, 13, 7)$ and the right column uses the "opposite" viewpoint $(19, 13, 7)$. Top: a single multicell. Middle: showing how the multicells fit together. Bottom: part of the oblique tiling, showing the $\mathbb{Z}^3$ periodicity.
\( e_2 + C_{(2)} \) of the multicell \( e_2 + M \), and the white cell \( e_3 + C_{(1)} \) of the multicell \( e_3 + M \) fit under the flat hexagonal base of \( M \) (which is in front in the left middle picture, and almost hidden behind the other three multicells in the right middle picture).

In the case of oblique tilings of genus \((2, 1)\), we have

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ u & v \end{bmatrix} \tag{3.22}
\]

\[
P = \frac{1}{\Delta_{21}} \begin{bmatrix} 1 + v^2 & -uv & u \\ -uv & 1 + u^2 & v \\ u & v & u^2 + v^2 \end{bmatrix} \tag{3.23}
\]

\[
Q = \frac{1}{\Delta_{21}} \begin{bmatrix} u^2 & uv & -u \\ uv & v^2 & -v \\ -u & -v & 1 \end{bmatrix} \tag{3.24}
\]

where \( \Delta_{21} = \sqrt{1 + u^2 + v^2} \) and where for simplicity we assume \( 0 < u < v < 1 \).

Once again we have three protocells, each the Cartesian product of a line segment with a rhombus:

\[
C_{(1,2)} = -P(F_{(1,2)}) + Q(F_{(3)})
\]

\[
C_{(1,3)} = -P(F_{(1,3)}) + Q(F_{(2)}) \tag{3.25}
\]

\[
C_{(2,3)} = -P(F_{(2,3)}) + Q(F_{(1)})
\]

See Fig. 3.3, where cells of type \( C_{(1,2)} \) are colored white, cells of type \( C_{(1,3)} \) are colored green, and cells of type \( C_{(2,3)} \) are colored red. To understand how the multicells fit together, see the middle row of Fig. 3.3. Note (middle right picture) how the white cell \( C_{(1)} \) of the multicell \( M \) fits into grove formed by the green cell \( e_3 + C_{(2)} \) and red cell \( e_3 + C_{(3)} \) of the multicell \( e_3 + M \), forming a flat hexagonal plate (which partially penetrates the multicell \( e_3 + M \).

In the case of oblique tilings of genus \((2, 2)\), we have the rather frightening expressions:

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.26}
\]

\[
P = \frac{1}{\Delta_{22}} \begin{bmatrix} 1 + c^2 + b^2 & -ac + bd & -ad + bc & -ac - bd + ad^2 & b + bc^2 - acd \\ -ac - bd & 1 + b^2 + c^2 & c + b^2 c - abd & a^2 + c^2 + (ad - bc)^2 & ab + cd \\ a - bc + ad^2 & c + b^2 c - abd & a^2 + c^2 + (ad - bc)^2 & ab + cd & b^2 + d^2 + (ad - bc)^2 \\ b + bc^2 - acd & -abc + d + a^2 d & ab + cd & b^2 + d^2 + (ad - bc)^2 & \end{bmatrix} \tag{3.27}
\]

\[
Q = \frac{1}{\Delta_{22}} \begin{bmatrix} a^2 + b^2 + (ad - bc)^2 & -ac + bd & c^2 + a^2 + (ad - bc)^2 & -ac + bd - a^2 d & -b - bc^2 + acd \\ -ac + bd & c^2 + a^2 + (ad - bc)^2 & -ac - bc + abd & a + bc - d^2 & ab - cd \\ -a - bc + ad^2 & c^2 + a^2 + (ad - bc)^2 & -ac - bc + abd & 1 + b^2 + d^2 & -ab - cd \\ -a - bc + ad^2 & c^2 + a^2 + (ad - bc)^2 & -ac - bc + abd & 1 + b^2 + d^2 & -ab - cd \end{bmatrix} \tag{3.28}
\]
$W = \text{col} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{2} & \sqrt{3} \end{bmatrix}$

Figure 3.3: The $(2, 1)$ oblique tiling $\mathcal{O}(W)$, where $W$ is the indicated two dimensional subspace of $\mathbb{R}^3$. The left column depicts the situation from the viewpoint $(19, 13, 7)$ and the right column uses the viewpoint $(-19, -13, 3)$. Top: a single multicell. Middle: showing how the multicells fit together. Bottom: part of the oblique tiling, showing the $\mathbb{Z}^3$ periodicity.
where \( \Delta_{22} = \sqrt{1 + a^2 + b^2 + c^2 + d^2 + (ad - bc)^2} \).

Now there are six protocells

\[
\begin{align*}
C_{(1,2)} &= -P(F_{(1,2)}) + Q(F_{(3,4)}) \\
C_{(1,3)} &= -P(F_{(1,3)}) + Q(F_{(2,4)}) \\
C_{(1,4)} &= -P(F_{(1,4)}) + Q(F_{(2,3)}) \\
C_{(2,3)} &= -P(F_{(2,3)}) + Q(F_{(1,4)}) \\
C_{(2,4)} &= -P(F_{(2,4)}) + Q(F_{(1,3)}) \\
C_{(3,4)} &= -P(F_{(3,4)}) + Q(F_{(1,2)})
\end{align*}
\]

(3.29)

Each of these four dimensional protocells is the Cartesian product of two rhombs.

The multicell is four dimensional, but we can project it into \( W \) and \( W^\perp \). See Fig. 3.4 for an example; in this figure, each rhombus is the "base" of one of the six cells. The apparent overlaps are spurious, being caused by the projection; bear in mind that the cells stick out into \( \mathbb{R}^4 \) with two additional degrees of freedom, just enough to enable them to avoid intersections.

Look back at Fig. 3.1 and observe that \( \mathcal{O}(W) \) can be constructed from \( T \)'s: each \( T \) consists of a segment parallel to \( W \) (a step) and another orthogonal to \( W \) (a riser).

Recalling the notion of the \( p \)-skeleton, we can make the following definitions:

**Definition 15** The protoriser \( R \) of \( \mathcal{O}(W) \) is

\[
R = Q \left( \text{skel}^p I^d \right) = \bigcup_{|J|=p} \bigcup_{K \subseteq J} \left[ \left( \sum_{k \in K} q_k \right) + T_{J^c}^* \right]
\]

(3.30)

The protostep \( S \) of \( \mathcal{O}(W) \) is

\[
S = P \left( -\text{skel}^p I^d \right) = \bigcup_{|J|=q} \bigcup_{K \subseteq J} \left[ \left( -\sum_{k \in K} p_k \right) + T_J \right]
\]

(3.31)

The risers of \( \mathcal{O}(W) \) are the \( n + R, \ n \in \mathbb{Z}^d \), and the steps are the \( n + S, \ n \in \mathbb{Z}^d \).
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Figure 3.4: Top left: projection into $W$ of the protomulticell of $O(W)$, where $W$ is the indicated two dimensional subspace of $\mathbb{R}^4$. Top right: projection into $W^\perp$ of the protomulticell. Bottom left: the protostep of $O(W)$, depicted in $W$. Bottom right: the protoriser of $O(W)$, depicted in $W^\perp$.

\[ W = \text{col} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{3} & \sqrt{7} \\ \sqrt{5} & \sqrt{2} \end{bmatrix} \]
Compare the appearance of the multicell (projected into $W$ and $W^\perp$) with the appearance of steps and risers in Fig. 3.4.

It is crucially important to understand that, as Figs. 3.1 and Figs. 3.5 suggest, steps and risers are actually part of the $p$-skeleton and $q$-skeleton, respectively, of $\mathcal{O}(W)$. That is, we may think of steps as $p$-dimensional "plates" in $\mathcal{O}(W)$ and likewise for risers; see Fig. 3.6. Indeed, the oblique tiling arises from repeating T’s made from pairs of risers and steps.

We next wish to prove a fact about the volume of the multicells of $\mathcal{O}(W)$. We will need the following fact:

**Lemma 3.13** A positive semidefinite matrix has non-negative principal minors. In particular, if it has rank $r$, its principal minors of size $r$ are all positive.

See for instance [39].

**Proposition 3.14** The volume of each multicell of $\mathcal{O}(W)$ is unity:

$$\text{vol}_d M = \text{vol}_d \bigcup_{|J|=p} C_J = \sum_{|J|=1} \text{vol}_d C_J = 1$$

**Proof:** This is easier to understand if we perform the necessary computation in a special case. We will prove the claim for $(2,1)$ tilings. Recall that $p_j + q_j = e_j$. Then

$$e_1 \wedge e_2 \wedge e_3 = (p_1 + q_1) \wedge (p_2 + q_2) \wedge (p_3 + q_3)$$

$$= (p_1 + q_1) \wedge (p_2 \wedge p_3 + p_2 \wedge q_3 + q_2 \wedge p_3 + q_2 \wedge q_3)$$

$$= p_1 \wedge p_2 \wedge p_3 + p_1 \wedge p_2 \wedge q_3 + p_1 \wedge q_2 \wedge p_3 + p_1 \wedge q_2 \wedge q_3$$

$$+ q_1 \wedge p_2 \wedge p_3 + q_1 \wedge p_2 \wedge q_3 + q_1 \wedge q_2 \wedge p_3 + q_1 \wedge q_2 \wedge q_3$$

$$= p_1 \wedge p_2 \wedge q_3 + p_1 \wedge q_2 \wedge p_3 + q_1 \wedge p_2 \wedge p_3$$

(where any term containing more than two of the $p_j$ or more than one of the $q_j$ has dropped out). Now, these multivectors are all parallel since they have rank 3, and
3.3. THE OBLIQUE TILING

Figure 3.5: Top: steps and risers in the (1, 2) oblique tiling $\mathcal{O}(W)$ of Fig. 3.2. Top left: cells projecting under $P$ to the protostep of $\mathcal{O}(W)$; the line segment indicates the location of $W$. Top right: cells projecting under $Q$ to the protoriser (a hexagonal shaped plate). Bottom: steps and risers in the (2, 1) oblique tiling space $\mathcal{O}(W)$ of Fig. 3.3. Bottom left: cells projecting under $P$ to the protostep (a hexagonal shaped plate). Bottom right: cells projecting under $Q$ to a riser; the line segment indicates the location of $W^\perp$. 
Figure 3.6: Top: the protostep (pictured in $W$) for the oblique tiling $\mathcal{O}(W)$, where $W$ is the indicated two dimensional subspace of $\mathbb{R}^5$. Bottom: the proctor (pictured in $W^\perp$) for the same oblique tiling.
3.4. STURMIAN TILINGS

$\Lambda^3 \mathbb{R}^3$ is one dimensional. Moreover, since shears preserve volume, we can write

$$p_1 \wedge q_2 \wedge p_3 = p_1 \wedge e_2 \wedge p_3 = \det[p_1 p_2 p_3] e_1 \wedge e_2 \wedge e_3$$

(where the last equality follows from Lemma 3.6). Here the determinant is the minor $P(J, J)$ where $J = \{1, 3\}$, so it is positive by Lemma 3.13. This means that $p_1 \wedge p_2 \wedge q_3$ is not only parallel to $e_1 \wedge e_2 \wedge e_3$, but points in the same direction. Likewise for $p_1 \wedge q_2 \wedge q_3$ and $q_1 \wedge p_2 \wedge p_3$. Thus,

$$1 = \|e_1 \wedge e_2 \wedge e_3\|$$

$$= \|p_1 \wedge p_2 \wedge q_3\| + \|p_1 \wedge q_2 \wedge q_3\| + \|q_1 \wedge p_2 \wedge p_3\|$$

$$= \text{vol } C_{1,2} + \text{vol } C_{1,3} + \text{vol } C_{2,3}$$

which establishes the desired identity. Similarly for the general case.

Theorem 3.15 The oblique tiling is a tiling of $\mathbb{R}^d$ with $\mathbb{Z}^d$ periodicity.

Proof: Since this is proven very carefully in [103], we only provide a sketch here. The tricky point is to prove that the interiors of adjacent multicells do not overlap. Then, by the $\mathbb{Z}^d$ periodicity and the fact that the multicells have unit volume, the cells of $O(W)$ cover all of $\mathbb{R}^d$, with nonoverlapping interiors.

3.4 Sturmian Tilings

Recall that each cell $n + C_J$ of $O(W)$ is the Cartesian product of a zonotope $n + T_J \subset n + W$ with an orthogonal zonotope $n + T_J^\perp \subset n + W^\perp$. This means that if we take any affine subspace parallel to $W$; that is, any coset $x + W$, where $x \in \mathbb{R}^d$ (we can of course assume that $x \in W^\perp$), then $x + W$ will intersect certain of the cells $n + C_J$ of $O(W)$; moreover, because each such cell is the product of an $p$-dimensional convex set
parallel to \( W \) and an \( q \)-dimensional convex set parallel to \( W^\perp \), if \( x + W \) hits \( n + C_J \) it will—in the generic case—intersect it in a copy of \( T_J \). (Nongeneric intersections are discussed in Section 4.2.)

We can now obtain a tiling \( T(x + W) \) of \( \mathbb{R}^d \) by passing the cutplane \( x + W \) through the oblique tiling \( O(W) \) where \( x + W \) is identified first with \( W \), by translation, and then with \( \mathbb{R}^p \) in the obvious way. More formally, we make the following definition:

**Definition 16** The Sturmian tiling \( T(x + W) \) defined by the cutplane \( x + W \) is the collection:

\[ T(x + W) = \{ P(n) + T_J : (n + C_J) \cap (x + W) \neq \emptyset, |J| = p \} \]

The collection of all such tilings,

\[ S(W) = \{ T(x + W) : x \in W^\perp \} \]

is called the Sturmian system defined by \( W \). The \( C_J \) are called the prototiles of \( S(W) \) and the \( n + C_J \in T(x + W) \) are called the tiles of \( T(x + W) \).

See Figs. 3.7 and 3.8 for pictures of some typical cutplanes.

We stress that \( S(W) \) is not quite the same as the Sturmian tiling space \( \tau(W) \) with a continuous translation action; see Section 3.9. We also stress that for some “nongeneric” cutplanes, Def. 16 permits the possibility that for certain cutplanes, several overlapping tiles will try to co-exist at certain locations called singular patches; see Section 4.2. Thus, to some extent the word “tiling” is a misnomer. Nonetheless, we not only accept this deficiency, we embrace it; see Chapter 6. In any case, the generic cutplane will always yield a true tiling of \( W \).

In Chapter 1 we mentioned that Sturmian tilings can be regarded as arising from “digital approximations” to the cutplanes \( x + W \). We are now in a position to formalize this idea.
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\[ W = \text{col} \left[ \frac{1 + \sqrt{5}}{2} \right] \]

Figure 3.7: The cutline \(1.1 \mathbf{q}_1 + W\) for the Sturmian tiling space \(S(W)\), where \(W\) is the indicated one dimensional subspace of \(\mathbb{R}^2\). Left: the cells of \(O(W)\) which are cut by the cutline are shaded. Right: the corresponding digital approximation is shown as bold line segments.

**Definition 17** The digital approximation \(D(x + W)\) to \(x + W\) is the collection:

\[ D(x + W) = \{ n + F_J : (n + C_J) \cap (x + W) \neq \emptyset, \ |J| = p \} \]

The \(n + F_J\) are called the facets of \(D(x + W)\).

See Figs. 3.7 and 3.8. Note that since \(T_J = P(F_J)\), the tiles of \(T(x + W)\) may be obtained by simply projecting the \(p\)-dimensional facets \(n + F_J\) in the digital approximation \(D(x + W)\) to the cutplane \(x + W\).

**Theorem 3.16** Each digital approximation \(D(x + W)\) is a continuous surface homeomorphic to \(\mathbb{R}^p\).

This is pretty obvious from the pictures; for a careful proof see [103].
\[ W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{2} & \sqrt{3} \end{bmatrix} \]

Figure 3.8: The cutplane \( 0.1q_1 + 0.2q_2 + W \), where \( W \) is the indicated one dimensional subspace of \( \mathbb{R}^2 \). Left: the cells of \( O(W) \) cut by the cutplane. Right: the corresponding digital approximation.

The term “approximation” requires some qualification. It is evident from Fig. 3.7 that the facets (line segments) \( n + F_1, n + F_2 \) in \( D(x + W) \) are always “close” to \( x + W \), but they are not necessarily the closest such facets. It is possible to obtain the digital approximation directly from \( x + W \) and \( \mathbb{Z}^d \), without first constructing \( O(W) \) and the tiling \( T(x + W) \). The idea is to “fatten” \( x + W \) by forming the “cylinder” \( W + Q(-I^d) \), where \( -I^d \) is the unit cube obtained by taking the negatives of every vector in the standard unit cube \( I^d \). Then a facet \( n + F_j \) is in \( D(x + W) \) iff it is contained in the cylinder; this justifies our assertion that the facets included in the digital approximation to \( x + W \) are always “close” to \( x + W \), if not necessarily “the closest” facets. See Senechal [124] for more information on this “projection method” of obtaining Sturmian tilings. We also mention here in passing that the risers are also called atomic surfaces and that still another equivalent method of obtaining Sturmian tilings involves looking at the risers hit by the cutplane; see for
$$W = \text{col} \left[ \frac{1 + \sqrt{5}}{2} \right]$$

Figure 3.9: The oblique tiling $O(W)$, where $W$ is the indicated one dimensional subspace of $\mathbb{R}^2$. The pencil is shaded; the window is depicted as a bold line segment cutting the pencil into two halves.

instance [125].

If $W$ is totally irrational, the tilings $T(x + W)$ with one vertex at the origin of $W$ (or, if you prefer, of $\mathbb{R}^p$) are precisely those such that $Q(x)$ is in the riser $Q(\text{skel}^q I^d)$; see Fig. 3.9. This suggests the following definition.

**Definition 18** The set $\Pi = Q(\text{skel}^q I^d) \times W \subset \mathbb{R}^d$ is called the pencil of $S(W)$. In this context, the protoriser is called the window.

### 3.5 Degeneracy

It is possible that some of the prototiles of $S(W)$ have zero $p$ dimensional volume, in which case we call it degenerate. This happens iff $O(W)$ is similarly degenerate.

For example, consider the case of $(1, 2)$ tilings (see Eq. 3.23). The areas of the
prototiles are
\[
\begin{align*}
\text{vol}_2 T_{(1, 2)} &= \sqrt{\det A'_{12} A_{12}} = 1/\Delta_{12}, & A_{12} &= [p1 \ p2] \\
\text{vol}_2 T_{(1, 3)} &= \sqrt{\det A'_{13} A_{13}} = |b|/\Delta_{12}, & A_{13} &= [p1 \ p3] \\
\text{vol}_2 T_{(2, 3)} &= \sqrt{\det A'_{23} A_{23}} = |a|/\Delta_{12}, & A_{23} &= [p2 \ p3]
\end{align*}
\]
(see for instance [10]). We see that we need both \(a, b \neq 0\) in order that all the tiles be nondegenerate. The fact that \(\alpha_{12}\) is never zero in this example is a consequence of our assumption that by elementary column operations \(A\) may be brought into the given form; since \(A\) was assumed to be of full rank not all the areas can vanish, so by reordering the coordinates we can assume that \(\alpha_{12} \neq 0\); or, equivalently, that \(A\) has the form given in Eq. (3.4).

For a slightly more complex example, consider the case of \((2, 2)\) tilings (see Eq. 3.27). The areas of the tiles are
\[
\begin{align*}
\text{vol}_2 T_{(1, 2)} &= 1/\Delta_{22}, & \text{vol}_2 T_{(1, 3)} &= |c|/\Delta_{22}, & \text{vol}_2 T_{(1, 4)} &= |d|/\Delta_{22}, \\
\text{vol}_2 T_{(2, 3)} &= |a|/\Delta_{22}, & \text{vol}_2 T_{(2, 4)} &= |b|/\Delta_{22}, & \text{vol}_2 T_{(3, 4)} &= |ad - bc|/\Delta_{22}
\end{align*}
\]
These examples suggest the following:

**Proposition 3.17** Let \(W = \text{col } A\), where \(A\) is an \(p\) by \(d\) matrix. The \(p\)-dimensional volume of the prototile \(T_J\) of \(S(W)\) is
\[
\text{vol}_p T_J = \frac{|\det A(J, \cdot)|}{\sqrt{\sum_K (\det A(K, \cdot))^2}}
\]
where the sum is taken over all the minors \(\det A(K, \cdot)\) of \(A\). Moreover
\[
\sum_K (\text{vol}_p T_K)^2 = 1
\]

**Proof:** By Corollary 3.12, the volumes \(\text{vol}_p T_J = \|P_J\|\) are precisely the direction cosines \(\cos \theta_J\), where \(\theta_J\) is the angle between \(E_J\) and the Plücker line \(\mathcal{P}(W)\). Suppose \(A = [v_1 v_2 \ldots v_p]\). Then
\[
\text{vol}_p T_J = \cos \theta_J = \frac{\langle \wedge_{j=1}^p v_j, E_J \rangle}{\| \wedge_{j=1}^p v_j \|} = \frac{\det A(J, \cdot)}{\sqrt{\sum_{k=1}^p (\det A(K, \cdot))^2}}
\]
as claimed.

If \( p = 1 \), this is just the ordinary Pythagorean theorem in \( \mathbb{R}^{1+q} \), and then the numbers \( \text{vol}_1 T_f \) are the direction cosines of the line \( W \). Thus, in the general case, the numbers \( \text{vol}_p T_f \) can be regarded as "generalized direction cosines" of the subspace \( W \). We emphasize that this formula holds for any matrix \( A \) such that \( W = \text{col} \ A \).

Prop. 3.17 immediately implies the nondegeneracy condition we need:

**Theorem 3.18** The Sturmian system \( S(W) \), where \( W = \text{col} \ A \), is nondegenerate if and only if the \( p \) by \( p \) minors of \( A \) are all nonzero.

We say \( W \) has rank \( r \) if \( W = \text{col} \ A \), where \( A \) has rank \( r \). The point is that the rank of \( W \) can be less than \( \dim W \) if \( W \) is partially aligned with the coordinate planes of \( \mathbb{R}^d \).

**Corollary 3.19** Suppose \( W = \text{col} \ A \), where \( A \) has the form

\[
A = \begin{bmatrix} I \\ E \end{bmatrix}
\]

Then \( S(W) \) is nondegenerate only if each entry of \( E \) is nonzero.

That is, recalling that the entries of \( E \) are precisely the components of the affine coordinate patch discussed in Section 3.2, \( S(W) \) is nondegenerate iff \( W \) lies in the interior of one the standard affine coordinate patches of \( G(p, q) \).

**Proof:** If \( E \) is a row vector, it is easy to see that the \( p \) by \( p \) minors of \( A \) are all nonzero only iff each entry of \( E \) is nonzero. From this one can argue that this remains true if \( E \) has more than one row, and the result follows from Theorem 3.18.

The converse to Corollary 3.19 is untrue if \( E \) has more than one row.
3.6 Multigrids

Consider the tiling $T(x + W)$ of genus $(2, 1)$ depicted in Figure 3.10. If we think of each coordinate $x_j$ as defining a “height function”, the digital approximation $D(x + W)$ naturally decomposes into terraces (see the top of Fig. 3.10). These terraces are separated by short “retaining walls”, or just “walls” for short (see Figure 3.10, bottom).

Consider the terrace at top right of Fig. 3.10. It consists of sections of cells of form $n + C_K$ such that $n_3 = 0$, where $n_3$ is the third component of $n$. The corresponding wall (bottom right) consists of sections of those cells in this terrace which have the form $n + C_{(1,3)}$ or $n + C_{(2,3)}$. This suggests the following general definitions.
3.6. MULTIGRIDS

Definition 19 Let $S(W)$ be a Sturmian system of dimension $p > 1$. For each $n \in \mathbb{Z}$ and each integer $j$ with $1 \leq j \leq d$, define the terrace $E_j(n, x + W)$ to be the following collection of tiles drawn from $T(x + W)$:

$$E_j(n, x + W) = \{ P(m) + T_K \in T(x + W) : m_j = n \} \quad (3.32)$$

(where $m_j$ is the $j$-th component of $m$). The corresponding wall $W_j(n, x + W)$ is the subcollection

$$W_j(n, x + W) = \{ P(m) + T_K \in E_j(n) : j \in K \} \quad (3.33)$$

When a single tiling $T(x + W)$ is under discussion, it is convenient to shorten the notation $E_j(n, x + W)$ to $E_j(n)$ and likewise for $W_j(n)$. We will denote the family of terraces $\{ E_j(n) : n \in \mathbb{Z} \}$ by $E_j(\cdot)$ and likewise for $W_j(\cdot)$.

Returning to the example of the tiling $T(x + W)$ depicted in Fig. 3.10, consider the three families of evenly spaced coordinate planes $H_1(n), H_2(n), H_3(n)$, where $H_1(n)$ is the plane with Cartesian equation $x_1 = n_1$, and so on. Observe that they cut $x + W$ in three families of lines as shown at top in Figure 3.11. More generally, we can make the following definition.

Definition 20 Let $S(W)$ be a Sturmian system of dimension $p > 1$. For each $n \in \mathbb{Z}$ and each integer $j$ with $1 \leq j \leq d$, let $H_j(n)$ be the coordinate hyperplane in $\mathbb{R}^d$ with Cartesian equation $x_j = n$. The $p - 1$ dimensional affine subspaces of the cutplane $x + W$ given by the intersections with the $H_j(n)$, namely

$$G_j(n, x + W) = H_n(j) \cap (x + W) \quad (3.34)$$

are called gridplanes. The collection

$$G(x + W) = \{ G_j(n, x + W) : n \in \mathbb{Z}, 1 \leq j \leq d \} \quad (3.35)$$

is called a multigrid.
Figure 3.11: Top: the grid lines in $\mathbb{R}^3$ for a tiling of genus $(2,1)$, with the offset from the origin to $x + W$ shown as a short heavy line. Bottom left: the corresponding multigrid in $\mathbb{R}^2$ (identified with $W$); the left edge of this picture corresponds to the bottom edge of the tilted plane $W$ in the picture above. Bottom right: the corresponding tiling with the vertex neighborhood of the origin shaded; this vertex neighborhood corresponds to the (irregular) pentagon enclosing the origin in the multigrid at left.
3.6. MULTIGRIDS

Notice that the grid plane \( G_j(n, x + W) \), or \( G_j(n) \) for short, corresponds exactly to the wall \( H_j(n) \). Notice further that for each \( j \), the family of gridplanes \( G_j(\cdot) = \{G_j(n) : n \in \mathbb{Z}\} \) consists of evenly spaced hyperplanes in \( x + W \), but that this spacing is usually quite different for different \( j \). Also, if \( x = 0 \), the gridplane \( G_j(0) \subset W \) is orthogonal to the vector \( p_j \) (but is usually not a linear subspace, only an affine subspace).

Lemma 3.20 Let \( S(W) \) be a Sturmian system of dimension \( p > 1 \). Then for each selection \( J \) of \( p \) coordinates, and each \( n \in \mathbb{Z}^d \), we have

\[
P(n) + T_J = \bigcap_{j \in J} W_j(n_j)
\]

Proof: (Sketch) Roughly speaking, the intersection apparently leaves \( q \) of the components of \( n \) undetermined, but this is the codimension of \( W \), so in fact these are completely determined too.

In other words, each nonempty intersection of \( p \) gridplanes from different families corresponds to a unique tile in some tiling \( T(x + W) \), and vice versa. (Of course, most intersections will not give a cell in a particular tiling in \( S(W) \).) This shows how the multigrid \( G(x + W) \) completely defines the tiling \( T(x + W) \), up to translation; each intersection of \( p \) grid planes corresponds to a unique tiling in the Sturmian tiling. As one might expect from this discussion, it can be shown that the multigrid and oblique tiling viewpoints are equivalent [40].

Proposition 3.21 Let \( S(W) \) be a Sturmian system of dimension \( p > 1 \). Then each wall \( W_j(n, x + W) \) forms an unbroken barrier dividing the tiling \( T(x + W) \) into two halves.

Proof: This follows from the analogous property of the \( p - 1 \) dimensional gridplane \( G_j(n) \).
3.7 Robinson Shifts

Consider the tiling depicted in Fig. 3.12. On the left, the terraces \( E_1(\cdot) \) are colored alternately white and red, while the terraces \( E_2(\cdot) \) are colored alternately white and green. The places where red and green terraces intersect "inherit" the color brown (red plus green), and so forth. This gives a four coloring (white, red, green, brown) of a supertiling \( S_{(1,2)}(x + W) \) whose tiles are certain patches of tiles from the original tiling \( T(x + W) \). The colors are not part of the definition of \( S_{(1,2)}(x + W) \); they are simply used to explain its definition.

In the example depicted in Fig. 3.12, there are just three prototiles for the supertiling; the first is a protopatch consisting one tile each of types \( T_{(1,2)}, T_{(1,3)}, \) and \( T_{(2,3)} \), the second consists of one tile each of types \( T_{(1,2)}, T_{(1,3)}, \) and the third consists of a single tile of type \( T_{(1,2)} \). If we denote these three prototiles by \( A, B, C \) respectively, the upper left corner of the supertiling \( S_{(1,2)}(x + W) \) shown at left in Fig. 3.12 may be represented symbolically as

\[
\begin{array}{ccccc}
C & C & A & C & B \\
B & A & C & B & A \\
A & C & B & A & C \\
C & B & A & C & A \\
A & C & C & A & C \\
\end{array}
\]  

(3.36)

Thus, the supertiling \( S_{(1,2)}(x + W) \) may regarded as a symbolic sequence indexed by \( \mathbb{Z}^2 \). In this way, we obtain a two dimensional shift space \( S_{(1,2)}(W) \) whose "points" are functions \( f : \mathbb{Z}^2 \to \Sigma \), where \( \Sigma \) is the set of supertiles. This construction is due to E. A. Robinson [114].

Similarly, in the middle of Figure 3.12, the terraces \( E_1(\cdot) \) are again colored alternately white and red while the \( E_3(\cdot) \) terraces are colored alternately white and blue. The places where the red and blue terraces intersect "inherit" the color cyan (red plus blue), giving a four coloring (white, red, blue, cyan) of the supertile \( S_{(1,3)}(x + W) \). Similarly for the supertiling \( S_{(2,3)}(x + W) \) defined by the terraces \( E_2(\cdot) \) and \( E_3(\cdot) \) shown on the right in Figure 3.12.
Definition 21 Let $S(W)$ be a nondegenerate Sturmian system with $p > 1$ and let $J$ be a selection of $p$ coordinates. The tiles of the supertiling $S_J(x + W)$ are precisely the patches in $T(x + W)$ of form $\alpha = E_j(m) \cap E_k(n)$, where $m, n \in \mathbb{Z}$ and $j \neq k \in J$. The set of all such supertilings is the Robinson shift space

$$S_J(W) = \{S_J(x + W) : x \in W^\perp\}$$

Lemma 3.22 There are finitely many prototiles for each supertiling $S_J(x + W)$.

Proof: (Sketch) The plane $W$ rises with finite nonzero slope in each direction so the terraces have bounded width. This means that all the patches of form $E_j(m) \cap E_k(n)$ fit inside a disk of some finite radius. Plainly there are only finitely many patches which can fit inside a given disk.

The number of prototiles, however, grows rapidly with the codimension $q$. 
Figure 3.13: Two supertilings of \( T(x + W) \), where \( W \) is the indicated two dimensional subspace of \( \mathbb{R}^4 \). Top: the supertiling \( S_{(1,2)}(x + W) \). Bottom: the supertiling \( S_{(1,3)}(x + W) \). Due to the symmetries of this example, there are only two "essentially different" Robinson shift spaces associated with this tiling space.
3.8 Slide Rules

The multigrid construction works only if \( p > 1 \); happily, there is another construction, the slide rule construction, which we introduce in this section, which works only for \( p = 1 \). Thus, the slide rule construction very neatly complements the multigrid method, to which it is very closely related, indeed, multigrids can be thought of as the natural multidimensional generalization of slide rules.

The idea is a straightforward generalization of the sliderule construction for \((1,1)\) Sturmian systems described in Section 2.8. Recall that in the case of a nondegenerate line \( W \) in \( \mathbb{R}^2 \) spanned by \([\frac{1}{\alpha}]\) (where \( W \) nondegenerate means \( \alpha \neq 0 \)), or equivalently, by \([\frac{1}{\alpha}]\), the norms of these two matrices, namely \( \sqrt{1 + \alpha^2} \) and \( \sqrt{1 + 1/\alpha^2} \) respectively, gave the spacings for the two rulers. In the same way, in the case of a nondegenerate line \( W \) in \( \mathbb{R}^3 \), consider the three vectors

\[
\begin{bmatrix}
1 \\
\alpha \\
\beta
\end{bmatrix},
\begin{bmatrix}
1 \\
\frac{1}{\alpha} \\
\frac{1}{\beta}
\end{bmatrix},
\begin{bmatrix}
1 \\
\frac{1}{\alpha} \\
\frac{1}{\beta}
\end{bmatrix}
\]

The norms of these vectors, namely

\[
\sqrt{1 + \alpha^2 + \beta^2}, \quad \sqrt{\frac{1}{\alpha^2} + 1 + \frac{\beta^2}{\alpha^2}}, \quad \sqrt{\frac{1}{\beta^2} + \frac{\alpha^2}{\beta^2} + 1}
\]

give the spacing of the tick marks on the three slides. Once again, varying the relative positions of the slides corresponds to varying the the Sturmian sequence in this particular Sturmian shift; see Fig. 3.14. Note that “wavelength ratio” for
\( T_{(1)}, T_{(2)}, T_{(3)} \) (respectively) is \( 1 : \frac{1}{\alpha} : \frac{1}{\beta} \), corresponding to a “frequency ratio” of \( 1 : \alpha : \beta \). (See 4.8 for relative tile frequencies in \((p, q)\) Sturmian systems in general.)

### 3.9 Sturmian Tiling Dynamical Systems

We begin by modifying the definition of \( T(x + W) \) in a slight but significant way.

**Definition 22** Given \( x \in \mathbb{R}^d \), the Sturmian tiling \( T_W(x) \) is the collection

\[
T_W(x) = \{ P(-x + n + C_J) : (n + C_J) \cap (x + W) \neq \emptyset \}
\]

(3.37)

The Sturmian tiling space \( \mathcal{T}(W) \) is

\[
\mathcal{T}(W) = \{ T_W(x) : x \in \mathbb{R}^d \}
\]

(3.38)

(The alert and expert reader will observer that we are treating singular tilings (see 4.2) as if they were true tilings, at the expense of enlarging the prototile set.)

Note that \( P(-x + n + C_J) = -P(x) + \sum_{j=1}^{d} n_j p_j + T_J \) and that \( T_W(0) = T(x + W) \), so \( S(W) \subset \mathcal{T}(W) \); the latter contains all the translates in \( \mathbb{R}^d \) of the tilings in \( S(W) \).

The point is that now the additive group \( \mathbb{R}^p \) (identified with \( W \)) acts on \( \mathcal{T}(W) \); specifically, \( w \in W \) takes \( T_W(x) \) to \( T_W(x + w) \). The reason for the negative sign in Eq. 3.37 is that we are thinking of translating the origin of \( \mathbb{R}^p \) in \( W \) by \( w \in W \).

It is crucially important to understand that there is nothing like a one-to-one correspondence between points \( x \in \mathbb{R}^d \) and tilings \( T_W(x) \in \mathcal{T}(W) \). Indeed, we have the following.

**Lemma 3.23** The obvious action by \( \mathbb{R}^d \) on \( \mathcal{T}(W) \) is \( \mathbb{Z}^d \) periodic; that is, for any \( m \in \mathbb{Z}^d \),

\[
T_W(x + m) = T_W(x)
\]

(3.39)
3.10. **STEP SYMMETRIC STURMIAN TILINGS**

**Proof:** We compute as follows:

\[
T_W(x + m) = \{-P(x + m) + P(n) + T_J : (n + C_J) \cap (x + m + W) \neq \emptyset\}
\]

\[
= \{-P(x) + P(n - m) + T_J : (n + C_J) \cap (x + m + W) \neq \emptyset\}
\]

\[
= \{-P(x) + P(n - m) + T_J : (n - m + C_J) \cap (x + W) \neq \emptyset\}
\]

\[
= T_W(x)
\]

Proposition 3.24 Suppose \(W\) is totally irrational. The set \(\{T_W(x) : x \in \Pi\}\) is precisely the set of tilings in \(\mathcal{T}(W)\) with one vertex at the origin.

Every translation orbit in \(\mathcal{T}(W)\) has a representative with one vertex at the origin. On the other hand, by sliding \(W\) along \(W^\perp\), if \(W\) is totally irrational we will obtain slightly different patterns for every \(w \in \Pi\).

3.10 **Step Symmetric Sturmian Tilings**

In this section we explain the construction of Sturmian systems \(S(W)\) whose protostep \(P(\text{skel}^p(-I^d))\), considered as a convex polytope in \(W\), has some kind of geometrical symmetry.

Sturmian tiling spaces of genus \((p, q)\) are defined, as we have seen, by a pair of orthogonal subspaces \(W, W^\perp\) of dimension \(p, q\) respectively. Such pairs of subspaces arise naturally in several areas of mathematics. One of the most important is the classical representation theory of groups, which studies how the elements of an abstract group may be represented as linear operators, or if you prefer, as matrices. (For the elementary theory of groups and the correspondence between linear operators and matrices, see [10]; for the representation theory of finite groups, see [77].) Now real linear operators typically have **invariant subspaces** which are taken to themselves by the given operator.
This suggests that, for instance, to obtain a system $S(W)$ with a step having an $k$-fold rotational symmetry, we should examine linear representations of the cyclic group $C_k$. This group consists of $k$ elements which may be written $1, g, g^2, \ldots, g^{k-1}$: we say that the group is generated by $g$ because multiplying $g$ by itself over and over gives all the other elements of the group; because $g^k = 1$, where 1 is the identity element of the group. There is an obvious representation of $C_k$ with represents the generator $g$ by an operator which cyclically permutes the standard basis vectors of $\mathbb{R}^k$. For instance the generator of $C_3$ can be represented by the matrix

$$R_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (3.40)$$

This is an orthogonal matrix, i.e. an invertible matrix whose transpose agrees with its inverse. Indeed, it is a special orthogonal matrix because its determinant is one.

An important theorem states, roughly speaking, that every special orthogonal matrix can be decomposed into rotations in mutually orthogonal two dimensional planes; in odd dimensions only there is also a single fixed line, the rotation axis. In even dimensions there are no axes of rotation. (See for instance section 8.4 of [5].)

This theorem is actually a familiar fact of experience in the case of three by three orthogonal matrices: every orthogonal operator on $\mathbb{R}^3$ can be thought of as a rotation in some plane $W$ in $\mathbb{R}^3$: the operator fixes every point in $W^\perp$, which is of course the axis of rotation. In particular, $R_3$ fixes the line spanned by $(1,1,1)$ (this is an eigenvector for the eigenvalue 1 of $R_3$) and rotates by 120 degrees in the orthogonal plane. (The reader should try to verify this by using his thumb, index and middle fingers to represent the three mutually orthogonal basis vectors $e_1, e_2, e_3$.)

Similarly, the generator of $C_5$ may be represented by the matrix

$$R_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (3.41)$$
The characteristic equation (see [10]) of $R_5$ is $t^5 - 1 = 0$; thus the complex eigenvalues of $R_5$ are the complex fifth roots of unity. This equation factors rationally as $(t - 1)(t^4 + t^3 + t^2 + t + 1) = 0$ and over $\mathbb{R}$ as

$$(t - 1) (t^2 - 2 \cos (2\pi/5) + 1) (t^2 - 2 \cos (4\pi/5) + 1)$$

This factorization reflects that fact that $R_5$ decomposes into a $1/5$ turn in one invariant plane and a $2/5$ turn in the other. To see this, note that the complex eigenvectors may be written $(1, 1, 1, 1, 1), (1, \zeta, \zeta^2, \zeta^3, \zeta^4)$, and $(1, \zeta^2, \zeta^4, \zeta^6, \zeta^8)$, where $\zeta = \exp(2\pi i)/5$. Taking the real and imaginary parts and normalizing (to get unit norm vectors) gives the rows of the orthogonal matrix

$$U_5 = \sqrt{\frac{2}{5}} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & \cos (2\pi/5) & \cos (4\pi/5) & \cos (6\pi/5) & \cos (8\pi/5) \\ 0 & \sin (2\pi/5) & \sin (4\pi/5) & \sin (6\pi/5) & \sin (8\pi/5) \\ 1 & \cos (4\pi/5) & \cos (8\pi/5) & \cos (12\pi/5) & \cos (16\pi/5) \\ 0 & \sin (4\pi/5) & \sin (8\pi/5) & \sin (12\pi/5) & \sin (16\pi/5) \end{bmatrix} \quad (3.42)$$

Here $U_5$ changes to a new adapted basis bringing our operator into the simple form

$$U_5 R_5 U_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos (2\pi/5) & -\sin (2\pi/5) & 0 & 0 \\ 0 & 0 & \cos (2\pi/5) & 0 & 0 \\ 0 & 0 & 0 & \cos (4\pi/5) & -\sin (4\pi/5) \\ 0 & 0 & 0 & 0 & \cos (4\pi/5) \end{bmatrix}$$

which verifies directly the claim that $R_5$ decomposes into a $1/5$ turn in one invariant plane and a $2/5$ turn in the other.

Similarly, $R_7$ decomposes into a $1/7$ turn in its first invariant plane, a $2/7$ turn in its second invariant plane, and a $3/7$ turn in its third invariant plane. And so forth for general $R_k$, where $k$ is odd.

Now, take $W$ to be the invariant plane where $R_k$ effects a $1/k$-th turn, where $k$ is odd. This defines a whole sequence of tiling spaces of genus $(2, k - 2)$, where $k > 2$ is odd. In the case $k = 5$, we take $W$ to be plane spanned by the second and third columns of $U_5$, or equivalently (dropping the scalar factor $\sqrt{2/5}$), by the column
vectors of the matrix

\[
A_5 = \begin{bmatrix}
    1 & 0 \\
    \cos(2\pi/5) & \sin(2\pi/5) \\
    \cos(4\pi/5) & \sin(4\pi/5) \\
    \cos(6\pi/5) & \sin(6\pi/5) \\
    \cos(8\pi/5) & \sin(8\pi/5)
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    -1+\sqrt{5}/4 & \sqrt{\frac{5-\sqrt{5}}{8}} \\
    -1-\sqrt{5}/4 & \sqrt{\frac{5+\sqrt{5}}{8}} \\
    -1-\sqrt{5}/4 & -\sqrt{\frac{5+\sqrt{5}}{8}} \\
    -1+\sqrt{5}/4 & -\sqrt{\frac{5-\sqrt{5}}{8}}
\end{bmatrix}
\] (3.43)

This gives a space of Sturmian tilings which includes the *Penrose tilings* of the plane by thick and thin rhombs. These tilings were the first Sturmian tilings of the plane to be studied; they were discovered around 1970 in an entirely different way by Roger Penrose, and popularized in one of Martin Gardner’s “Mathematical Games” columns [41]. See Figure 3.15; the “local five-fold rotational symmetry” apparent in this picture reflects the fact that the definition of \(W\) ensures that the steps consists of a projection of the five cube into \(\mathbb{R}^2\) whose boundary is a regular decagon; see Fig. 3.17.

*(Warning! We have not claimed that every tiling in this particular \(T(W)\) is a Penrose tiling, for that is not true. What is true (as we shall discuss in more detail below) is that the Penrose tilings form a *minimal subsystem* of \(T(W)\); however, there are infinitely many additional minimal subsystems.)*

It is tempting to try to obtain step symmetrical tilings in even dimensions. However, the \(W\) defined by the obvious matrix representation of a four cycle in \(\mathbb{R}^4\) won’t work; this gives a degenerate tiling space, as the reader may verify. (The problem is that some of the \(p_j\) coincide.) However, a minor modification saves the situation; instead of \(C_4\) we represent \(C_8\). The generator of \(C_8\) can be represented by the matrix

\[
R_4 = \begin{bmatrix}
    0 & 0 & 0 & -1 \\
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0
\end{bmatrix}
\] (3.44)

The characteristic equation of \(R_4\) is \(t^4 + 1 = 0\), which can be factored over \(\mathbb{R}\) as

\[(t^2 - 2\cos(\pi/4) + 1)(t^2 - 2\cos(3\pi/4) + 1)\]
Figure 3.15: A typical Penrose tiling. The Penrose tilings comprise a minimal subsystem of the genus (2,3) Sturmian tiling space for the indicated two dimensional subspace $W$. This subspace is one of the irreducible invariant subspaces of the canonical five-cycle in $\mathbb{R}^5$, namely the one in which the five-cycle effects a one-fifth turn.
Figure 3.16: A typical tiling from another minimal subsystem of the same \((2,3)\) Sturmian tiling space. Each subsystem consists of tilings which "all look alike", but which differ (possibly only upon close examination) from those in any other subsystem.
Figure 3.17: Top: the step (pictured in $W$) for $O(W)$, where $W$ is the invariant subspace of the canonical five-cycle in $\mathbb{R}^5$ in which this five-cycle effects a one-fifth turn. Note the perfect ten fold rotational symmetry about the center. Bottom: the riser (pictured in $W^\perp$) for the same oblique tiling. In this picture, the origin is represented by the large ball at lower left; note the perfect five fold rotational symmetry about the axis from the origin through the point at upper right. Note that the vertices of the riser are distributed as follows: from left to right, the South Pole (the origin), a small pentagon at 60 degrees South, a large pentagon at 20 degrees South, another large pentagon at 20 degrees North, a small pentagon at 60 degrees North, and the North Pole. The meaning of these pentagons will be explained in Section 4.7.
which corresponds to the fact that $R_4$ decomposes into a 1/8 turn in its first invariant plane and a 3/8 turn in its second invariant plane. To see this, note that the complex eigenvectors can be written $(1, \zeta, \zeta^2, \zeta^3)$, $(1, \zeta^2, \zeta^4, \zeta^6)$, and $(1, \zeta^3, \zeta^5, \zeta^9)$, where $\zeta = e^{i\pi/4}$. Taking real and imaginary parts and normalizing gives the rows of the orthogonal matrix

$$U_4 = \sqrt{\frac{2}{4}} \begin{bmatrix}
1 & \cos(\pi/4) & \cos(2\pi/4) & \cos(3\pi/4) \\
0 & \sin(\pi/4) & \sin(2\pi/4) & \sin(3\pi/4) \\
1 & \cos(3\pi/4) & \cos(6\pi/4) & \cos(9\pi/4) \\
0 & \sin(3\pi/4) & \sin(6\pi/4) & \sin(9\pi/4)
\end{bmatrix}$$

(3.45)

$$= \sqrt{\frac{2}{4}} \begin{bmatrix}
1 & \sqrt{1/2} & 0 & -\sqrt{1/2} \\
0 & \sqrt{1/2} & 1 & \sqrt{1/2} \\
1 & -\sqrt{1/2} & 0 & \sqrt{1/2} \\
0 & \sqrt{1/2} & -1 & \sqrt{1/2}
\end{bmatrix}$$

(3.46)

Here $U_4$ changes to the adapted basis for $R_4$; bringing the matrix of our operator into the form

$$U_4 R_4 U_4^{-1} = \begin{bmatrix}
\cos(\pi/4) & -\sin(\pi/4) & 0 & 0 \\
\sin(\pi/4) & \cos(\pi/4) & 0 & 0 \\
0 & 0 & \cos(3\pi/4) & -\sin(3\pi/4) \\
0 & 0 & \sin(3\pi/4) & \cos(3\pi/4)
\end{bmatrix}$$

which verifies directly the claim that $R_4$ decomposes into a 1/8-th turn in its first invariant plane and a 3/8-th turn in its second invariant plane.

Similarly, $R_6$ decomposes into a 1/12 turn in its first invariant plane, a 3/12 = 1/4 turn in its second invariant plane, and a 5/12 turn in its third invariant plane. And so forth for $R_k$, where $k$ is even.

Now, take $W$ to be the invariant plane in which $R_k$ ($k$ even) effects a 1/(2k)-th turn. In the case $k = 4$, we take $W$ to be the plane spanned by the first two rows of $U_4$, or equivalently by the columns of

$$A_4 = \begin{bmatrix}
1 & 0 \\
\sqrt{1/2} & \sqrt{1/2} \\
0 & 1 \\
-\sqrt{1/2} & \sqrt{1/2}
\end{bmatrix}$$

(3.47)

This gives a Sturmian tiling space of genus $(2,2)$ which turns out to be a space of tilings first studied by Robert Ammann at about the same time as Penrose discovered
his tilings. See Figure 3.18; the local eight-fold rotational symmetry visible in this picture reflects the fact that the steps consist of a projection of the four cube into \( \mathbb{R}^2 \) whose boundary is a regular octagon.

Similarly, if \( k = 6 \), we find

\[
A_6 = \begin{bmatrix}
1 & 0 \\
\sqrt{3}/2 & 1/2 \\
1/2 & \sqrt{3}/2 \\
0 & 1 \\
-1/2 & \sqrt{3}/2 \\
-\sqrt{3}/2 & 1/2
\end{bmatrix}
\]

(3.48)

We have essentially proven:

**Proposition 3.25** For every \( q > 1 \), there is \((2, q)\) Sturmian system whose steps have \(2(q + 2)\)-fold rotational symmetry.

The foregoing discussion is largely based upon section 2.6.3 in [124].

So far we have considered only rotational symmetries of \((2, q)\) tiling spaces. More generally, we might consider representations of more interesting finite groups such as the sixty element alternating group (see [10]). Indeed, it turns out that this group can be represented in \( \mathbb{R}^6 \) and it then turns out to have a pair of invariant subspaces, each of dimension three. Taking \( W \) to be one of these subspaces, we obtain a Sturmian tiling in which the step is a projection of the five cube into \( \mathbb{R}^3 \) whose boundary is a semi-regular polyhedron having the same symmetries as the regular icosahedron. This Sturmian tiling space has been proposed as a model for a real three dimensional quasicrystal. (For regular polyhedra and their higher dimensional analogues, see [22].)

Here is a quite general way to obtain step symmetric Sturmian tilings of genus \((p, q)\). Start with a nonsingular non-negative integer matrix; this defines a linear operator on \( \mathbb{Z}^d \); that is, an endomorphism of the torus \( T^d \). If we choose this matrix to (for example) symmetric, it will have real eigenspaces, and we can often choose \( W \) to be a direct sum of these eigenspaces. For instance, this is just how the tiling depicted at the right of Figure 1.2 arises. In Chapter 5 we'll see how such toral
$W = \begin{bmatrix}
1 & 0 \\
\cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\
\cos \frac{3\pi}{4} & \sin \frac{3\pi}{4}
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
\sqrt{1/2} & \sqrt{1/2} \\
0 & 1 \\
-\sqrt{1/2} & \sqrt{1/2}
\end{bmatrix}$

Figure 3.18: A typical Ammann octagonal tiling. These tilings comprise precisely the genus (2, 2) Sturmian tiling space for the indicated two dimensional subspace $W$ of $\mathbb{R}^4$. This subspace is one of the irreducible invariant subspaces of the obvious four dimensional representation of the finite group $\mathbb{Z}_8$. 
Figure 3.19: Top: the step (pictured in \( W \)) for \( \mathcal{O}(W) \), where \( W \) is the invariant subspace of the canonical eight cycle in \( \mathbb{R}^4 \) in which this cycle effects a one-eighth turn. Note the perfect eight fold rotational symmetry. Bottom: the riser (pictured in \( W^\perp \)) for the same oblique tiling. Note the perfect eight fold rotational symmetry.
endomorphisms arise quite naturally in the study of Sturmian tiling spaces which possess compositions.
Chapter 4

Elementary Properties of Sturmian Systems

In this chapter we investigate some of the more elementary properties of Sturmian systems. Among other things, we will take a first look at several important concepts, include dual Sturmian systems (Section 4.1), singular tilings (Section 4.2, the hierarchical nature of Sturmian systems, including walls, subwalls, ribbons, and ribbon vectors (Section 4.3), periods and Smith form (Section 4.4), periodic approximations to aperiodic Sturmian systems (Section 4.5), almost periods for Sturmian tilings (Section 4.6), ergodic decomposition and orbit closures of Sturmian systems (Section 4.7), tile frequencies (Section 4.8), vertex neighborhoods (Section 4.9), and enumerating the alternatives (and the probability of each alternative) for the recurrence of tiles (Section 4.10).

4.1 Duality

As the reader may have already observed, the subspaces $W, W^\perp$ play (essentially) symmetric roles in the oblique tiling construction. That is, because each protocell of $\mathcal{O}(W)$ is the Cartesian product of two zonotopes, one lying in $W$ and the other in $W^\perp$, instead of taking cut planes parallel to $W$, we could just as well take cut planes parallel to $W^\perp$, as suggested in Fig. 4.1. (Note well: it is the tiling spaces which are dual, not individual tilings.)

Given the simplicity of this observation, it is rather annoying that $\mathcal{O}(W)$ does not quite agree with $\mathcal{O}(W^\perp)$; rather $\mathcal{O}(W^\perp) = -\mathcal{O}(W)$; that is, the protocells of $\mathcal{O}(W^\perp)$ have the form $-C_J$, where $C_J$ is a protocell of $\mathcal{O}(W)$; see Fig. 4.2. Nonetheless we shall say that the Sturmian systems $S(W)$ and $S(W^\perp)$ are dual, and likewise for $\mathcal{T}(W)$ and $\mathcal{T}(W^\perp)$. 
Figure 4.1: The notion of duality: the cells of $\mathcal{O}(W)$ cut by the line $0.1q_1 + 0.01q_2 + W$, together with the cells cut by the plane $0.1q_1 + 0.01q_2 + W^\perp$. Here $W$ is as usual the column space of the indicated matrix.
4.1. DUALITY

Figure 4.2: Comparison of $\mathcal{O}(W)$ and $\mathcal{O}(W^\perp)$ for a typical genus $(1,1)$ example.

Despite the fact that $\mathcal{S}(W)$ consists of tilings of $\mathbb{R}^p$ and $\mathcal{S}(W^\perp)$ consists of tilings of $\mathbb{R}^q$, where usually $p \neq q$, their properties are very closely related. As a first example of this theme, we have the following surprising fact:

**Proposition 4.1** The volumes of corresponding tiles in $\mathcal{S}(W)$ and $\mathcal{S}(W^\perp)$ agree:

$$\text{vol}_p T_J = \text{vol}_q T_J^\ast$$

**Proof:** Write $\text{vol}_p T_J = a_J$ and $\text{vol}_q T_J^\ast = b_J$. Then, since the $a_J$ are direction cosines we have $\sum a_J^2 = 1$ and likewise $\sum b_J^2 = 1$. Moreover $\text{vol}_d C_J = a_J b_J$ so $\sum a_J b_J = 1$. Therefore

$$\sum (a_J - b_J)^2 = \sum a_J^2 - 2 \sum a_J b_J + \sum b_J^2 = 0$$

which implies that each term in the sum must vanish. Thus $a_J = b_J$ for each $J$ and we are done.

Prop. 4.1 is remarkable because the tiles in question usually have different dimensions; yet their volumes agree numerically.
We immediately obtain the following consequence:

**Corollary 4.2** $S(W^\perp)$ is nondegenerate *iff* $S(W)$ is.

### 4.2 Singularities

As the reader may already have noticed, for certain values of $x \in W^\perp$, the construction of the Sturmian tiling $T(x + W)$ in Section 3.4) does not always define a true tiling, because whenever the cutline $x + W$ runs through a step (see Fig. 4.3) will will have an “ambiguous region” where two patches try to coexist. Following de Bruijn [26], we call such ambiguous regions **singular patches** and we call any tiling containing such a patch a **singular tiling**; those tilings containing no singular patches are called **regular**.

In particular, the tiling $T(0 + W) \in S(W)$ is always singular, because the cutline $W$ runs right through the step based at the origin (see Fig. 4.3). Following analogous usage in [7], we call $T(0 + W)$ the **Christoffel tiling**.

Singular tilings can have striking symmetries; see Fig. 4.4, which illustrates the two types of singular tilings which can occur in the space of Penrose tilings (recall this is only a small part of the space $S(W)$ where $W$ is an appropriate invariant subspace of the canonical five cycle $R_5$). (The **cartwheel tiling** shown at top is unique, up to translation; there are uncountably many **monoworm** tilings like the one shown at bottom, up to translation.) Notice how the cartwheel tiling consists of a single singular dodecagonal patch, which close inspection reveals to nothing other than the step pictured in Fig. 3.17, plus radiating “spokes” made up of “thick” and “thin” singular hexagonal patches (in ten different orientations), which close inspection reveals are certain parts of a step. This illustrates the idea, which we explore further in Chapter 6, that any singular tiling may be decomposed into **elementary singularities** which are compact singular patches drawn from a finite list (for any one Sturmian system). In the lower picture in Fig. 4.4, the characteristic pattern of long and short
Figure 4.3: The Christoffel tiling $T(0 + W) \in S(W)$, where $W$ is spanned by $(\tau, 1)$, where $\tau$ is the Golden Ratio. Top: the tiling is ambiguous where the cutline runs through a step. Bottom: the digital approximation is ambiguous at the bold square: should the approximation read $LS$ or $SL$ at the indicated place?
singular hexagons is called a worm (a term coined by J. H. Conway; see [41][53]);
this particular singular tiling has only one worm so it is called a monoworm tiling;
the cartwheel tiling contains five worms.

In this section, we focus on elementary singularities. In Chapter 6 we will explore
the striking patterns of recurring singularities seen in Fig. 4.4, which as we shall see
only hints at the very rich variety of possible patterns.

Let us recall how the oblique tiling cells determine what tile is placed at a particu-
lar location in the tiling $T(x + W)$. The generic cutline passes through the interior
of each oblique tiling cell it hits, in which case the corresponding tiling has no sin-
gularities. This “regular piercing” is illustrated in Figure 4.5. Here, we see that
the cutline $0.4q_1 - 0.3q_2 + W$ happens to pierce the interior of a green cell, namely
$0 + C_{(2)}$, and a white cell, namely $e_1 + 2e_2 + 4e_3 + C_{(1)}$. This means that the tiling
$T(0.4q_1 - 0.3q_2 + W) \in S(W)$ contains a green tile $T_{(2)}$ (it has length $\|p_w\|$ and right
hand endpoint at 0) and a white tile $e_1 + 2e_2 + 4e_3 + T_{(1)}$ (it has length $\|p_1\|$ and
right hand endpoint at $p_1 + 2p_2 + 4p_3$).

However, it is perfectly possible for a cutline to strike the projection of the inter-
face between two, or even three, oblique tiling cells; this is exactly how elementary
singularities arise. The first possibility is illustrated in Figure 4.6 and the second in
Figure 4.7.

In Figure 4.6, we see that the cutline $0.5q_1 + W$ pierces the face separating the
white and green cells based at 0, and also the face separating the green cell based
at $-e_2$ from the white cell based at $-e_3$. This is a first order singularity; its
occurrence means that we have a choice between having a green tile $T_{(2)}$ (with length
$\|p_2\|$ and right hand endpoint at 0) and a white tile $-p_2 + T_{(3)}$ (with length $\|p_3\|$ and
right hand endpoint at $-p_2$), or else a white tile $T_{(3)}$ (with right hand endpoint at
0) and a green tile $-p_3 + T_{(2)}$ (with right hand endpoint at $-p_3$). Thus, we have the
two resolutions “white-green” and “green-white”, and because these two resolutions
arise as the “front” and “back” faces of the projection to $W$ of the square $F_{(2,3)}$, we
Figure 4.4: Top: the “cartwheel” Penrose tiling; this is the Christoffel tiling for $S(W)$ where $W$ is the invariant subspace defined in Eq. 3.43. Bottom: a “monoworm” Penrose tiling, defined by the cutplane $\frac{1}{2}(q_1 + q_2) + W$
Figure 4.5: Left: this cutline pierces a green cell at 0 and a white cell at $e_1 + 2e_2 + 4e_3$. Right: In $W^\perp$, the cutline is represented by the indicated point, which lies in the interior of both the projection of the green cell at 0, namely $C_{(2)}$, and the projection of the white cell at $e_1 + 2e_2 + 4e_3$, namely $e_1 + 2e_2 + 4e_3 + C_{(3)}$. 
say this is a first-order singularity of type $P(F_{(2,3)})$. Similarly, we will have a first order singularities of type $P(F_{(1,2)})$ or $P(F_{(1,3)})$ whenever the cutline pierces the face separating some red cell from a green cell or a white cell, respectively.

In Figure 4.7, the cutline coincides with the edges joining several faces separating various cells of the oblique tiling. This is a second order singularity; just one type of second order singularity if possible in this situation, namely $P(F_{(1,2,3)})$. A careful analysis similar to the one just given shows that we have six resolutions, which we can briefly describe as "white-green-red", "white-red-green", and so forth for the other possible orderings of one white, one green, and one red tile.

If we start with the Christoffel tiling depicted in Fig. 4.7 and move the cutline very slightly along $q_1$, we "partially resolve" the second order singularity at the origin into a first order singularity. By moving the cutline very slightly off $q_1$, we can resolve this first order singularity, obtaining a regular tiling. This process, which we call unfolding the original singularity, is illustrated in Figure 4.9. In the second step, by perturbing to $0.01q_1$ we in effect chose the left front face of the cube seen in the left picture. If instead we had perturbed to $-0.01q_1 + W$, we would have chosen the right rear face of the cube. If we had perturbed to $0.01q_3 + W$, we would have chosen the upper face of the cube, and so forth. ( Needless to say, we are really thinking of taking the "limit" of $\varepsilon q_j + W$ as $\varepsilon \to 0$.)

Let us consider what happens as we move the cutline continuously along $q_3$. For the most part, we obtain tilings with an isolated "white/green" first order singularity at the origin, and no other singularities. However, we occasionally obtain a tiling with a single additional first order singularity, of type "white/red" or "green/red". This happens whenever the edge $q_3$ is crossed by some edge, belonging to some other window, of type $q_1$ or $q_2$, respectively. In fact, we actually have a dense set of $v$ lying along $q_3$ which yield such "anomalous" singular tilings. Similarly, if we move the cutline continuously along $q_1$, for the most part we obtain tilings with an isolated "green/red" first order singularity at the origin, but there is nonetheless a dense set
Figure 4.6: Left: this cutline pierces a pair of faces between two pairs of oblique tiling cells. Right: consequently, we have an ambiguous or singular patch; specifically, we have the choice between a green tile followed by a white tile or vice versa.

Figure 4.7: Left: the cutline $W$ hits a vertex of $Z^3$ and thus pierces a line segment which touches six oblique tiling cells. Right: consequently, we have the choice of a white tile followed a green tile followed by a red tile, or one of the other five orderings.
Figure 4.9: Unfolding the Christoffel tiling. Left: the digital approximation to $W$ has a second order singularity (skeleton of a cube) at the origin. Middle: the digital approximation to $0.01q_1 + W$ has only a first order singularity (skeleton of a square) at the origin. Right: the digital approximation to $0.01q_1 + 0.01q_2 + W$ is regular.
of $v$ in $q_1$ which yield tilings with one additional “white/red” singularity, and another
dense set of $v$ which yield tilings with an additional “white/green” singularity.

Since we have briefly indicated in Chapter 3 how to construct Sturmian tilings
using a multigrid, it may be of interest to see how singular tilings arise from the
multigrid point of view. In Fig. 4.10, we show how a multigrid of genus $(2, 1)$ has a
singularity whenever three lines meet at a point (generically, of course, at most two
lines meet at a given point).

The examples of genus $(1, 1)$ and $(2, 1)$ tilings are to some extent misleading since
it turns out that singular tilings can be much more complicated in general Sturmian
systems than they can be in codimension one systems. We will explore the possibilities
in more detail in Chapter 6.

4.3 Walls

Our main goal in this section will be to prove the remarkable fact, mentioned in
Section 1.1, that each wall of a $(p, q)$ Sturmian tiling is “combinatorially equivalent”
to a specific tiling in a specific $(p - 1, q)$ Sturmian system.

Recall from Section 3.4 the following definitions. The hyperplane $x_j = m$ is
denoted $H_j(m)$ and the digital approximation to the cut plane $x + W$ is

$$D(x + W) = \{n + F_K : (n + C_K) \cap (x + W) \neq \emptyset, \ |K| = p\}$$

Recall from Section 3.6 that the wall defined by $H_j(m)$ is

$$W_j(m, x + W) = \{P(n + F_K) \in E_j(m, x + W) : j \in K\}$$

(where we have used $P(n + C_K) = P(n + F_K) = P(n) + T_K$ to slightly reformulate
the definition given in Section 3.6).

Definition 23 Let $S(W)$ be a Sturmian system of dimension $p > 1$. For each $n \in Z$
and each integer $j$ with $1 \leq j \leq d$ (where as usual $d = p + q$, dim$W = p$ and
4.3. WALLS

Figure 4.10: A Christoffel tiling of genus \((2,1)\). Top: three grid lines intersect \(0 + W\) at the origin. Bottom left: the multigrid has a three-fold intersection at the origin. Bottom right: the corresponding tiling has a singular hexagon at the origin, corresponding to the place where \(0 + W\) runs into a step.
\( \dim W^\perp = q \), define the lane \( L_j(n, x + W) \) to be

\[
L_j(n, x + W) = \{ n + F_k \in D(x + W) : n_j = n, j \in K \} \tag{4.1}
\]

Note that \( L_j(n, x + W) \) is to \( D(x + W) \) as \( W_j(n, x + W) \) is to \( T(x + W) \).

It will also be convenient in this section to let \( \pi_j : \mathbb{R}^{p+q} \to \mathbb{R}^{p-1+q} \) denote the canonical projection deleting the \( j \)-th component of every vector in \( \mathbb{R}^{p+q} \).

Consider the \((2, 1)\) Sturmian system with

\[
W = \text{col} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -e & -\tau \end{bmatrix}
\]

where \( \tau = (1 + \sqrt{5})/2 \) is the Golden Ratio. The picture at the top of Fig. 4.11 suggests that each lane \( L_j(n, x + W) \) belonging to the digital approximation \( D(x + W) \) to a particular tiling in \( S(W) \) projects under \( \pi_j \) to the the digital approximation for a particular tiling in \( S(\pi_j(W)) \). In particular, assuming without loss of generality that the offset \( x \in \mathbb{R}^{p+q} \) is chosen to lie in \( W \cap H_j(0) \), it is easy to see that we have

\[
\pi_j(L_j(0, x + W)) = D(\pi_j(x + W))
\]

A little thought then reveals that, in the case \( j = 1 \) illustrated in Fig. 4.11,

\[
W \cap H_1(0) = \text{col} \begin{bmatrix} 0 \\ 1 \\ -\tau \end{bmatrix}
\]

and

\[
W \cap H_1(n) = n \begin{bmatrix} 1 \\ 0 \\ -e \end{bmatrix} + W \cap H_1(0)
\]

This means that if, for instance,

\[
x = \begin{bmatrix} 0 \\ 0.2 \\ 0.3 \end{bmatrix}
\]

(see Fig. 4.11), then the cutline in

\[
\pi_1(W) = \text{col} \begin{bmatrix} 1 \\ -\tau \end{bmatrix}
\]
4.3. **WALLS**

\[
x = \begin{bmatrix} 0 \\ 0.2 \\ 0.3 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -e & -r \end{bmatrix}
\]

Figure 4.11: Top: the lanes (left to right) \( L_1(1, x+W) \), \( L_1(0, x+W) \), \( L_1(-1, x+W) \) for the indicated tiling \( T(x+W) \) in the \( (2,1) \) Sturmian system \( S(W) \) are shaded. Note that \( x \) has been chosen to lie in \( W \cap H_1(0) \). Bottom: the digital approximations for the tilings in the \( (1,1) \) Sturmian system \( S(\pi_1(W)) \) which correspond (left to right) to the these lanes, as explained in the text. (In all three pictures, a vertical segment is cut off at the bottom of the picture; this is simply an artifact of the formalism.)
\[
x = \begin{bmatrix} 0 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}, \quad W = \col \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{7} & \sqrt{2} \\ \sqrt{5} & \sqrt{3} \end{bmatrix}
\]

Figure 4.12: Top: the lanes (left to right) \( L_1(-1, x + W), L_1(0, x + W) \) and \( L_1(1, x + W) \) from the indicated tiling \( T(x + W) \) in the (2, 2) Sturmian system \( S(W) \) are shaded. Note that \( x \) has been chosen to lie in \( W \cap H_1(0) \). Bottom: the digital approximations to the tilings in the (1, 2) Sturmian system \( S(\pi_1(W)) \) which correspond (left to right) to these lanes. (In all three pictures, a vertical segment is cut off at the bottom of the picture; this is simply an artifact of the formalism.)
corresponding to the lane $L_1(n, x + W)$ is given by

$$
\pi_1 \left( n \begin{bmatrix} 1 & 0 \\ -e & 0.2 \\ 0.3 & 1 \end{bmatrix} + \text{col } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = n \begin{bmatrix} 0 \\ -e \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} + \text{col } \begin{bmatrix} 1 \\ -\tau \end{bmatrix}
$$

In Fig. 4.11, this is illustrated for $n = -1, 0, 1$; the reader can verify that the illustrated lanes from the digital approximation for the $(2, 1)$ tiling correspond facet for facet with the digital approximation for the appropriate $(1, 1)$ tiling.

This gives the desired connection between the $x_1$ walls of the $(2, 1)$ Sturmian system $S(W)$ and the $(1, 1)$ Sturmian system $S(\pi_1(W))$.

Next, consider the $(2, 2)$ Sturmian system with

$$
W = \text{col } \begin{bmatrix} 1 & 0 \\ \sqrt{7} & \sqrt{2} \\ \sqrt{5} & \sqrt{3} \end{bmatrix}
$$

The picture at the top of Fig. 4.12 depicts the walls (left to right) $W_1(0, x + W)$, $W_1(0, x + W)$, $W_1(1, x + W)$ for the indicated tiling $T(x + W)$; these are pictured in lieu of corresponding lanes $D_1(-1, x + W)$, $D_1(0, x + W)$, $D_1(1, x + W)$ in $\mathbb{R}^4$. Now we have

$$
W \cap H_1(0) = \text{col } \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \\ \sqrt{5} \end{bmatrix}
$$

and

$$
W \cap H_1(n) = n \begin{bmatrix} 1 \\ 0 \\ 1 \\ \sqrt{5} \end{bmatrix} + W \cap H_1(0)
$$

This means that if, for instance,

$$
x = \begin{bmatrix} 0 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}
$$

(see Fig. 4.12), then the cutline in

$$
\pi_1(W) = \text{col } \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \end{bmatrix}
$$
corresponding to the lane \( L_1(n, x + W) \) is given by

\[
\pi_1 \left( n \begin{bmatrix} 1 \\ 0 \\ \frac{\sqrt{7}}{\sqrt{5}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \\ 0.4 \end{bmatrix} + \text{col} \begin{bmatrix} 0 \\ 1 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \right) = n \begin{bmatrix} 0 \\ \frac{\sqrt{7}}{\sqrt{5}} \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} + \text{col} \begin{bmatrix} 1 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}
\]

In Fig. 4.12, this is illustrated for \( n = -1, 0, 1 \); the reader can verify that the illustrated lanes from the digital approximation for the \((2, 2)\) tiling correspond facet for facet with the digital approximation for the appropriate \((1, 2)\) tiling.

This gives the desired connection between the \( x_1 \) walls of the \((2, 2)\) Sturmian system \( S(W) \) and the \((1, 2)\) Sturmian system \( S(\pi_1(W)) \).

Next, consider the \((3, 2)\) Sturmian system with

\[
W = \text{col} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{5}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}
\]

The picture at the top of Fig 4.13 illustrates the wall \( W_2(1, x + W) \), which is shown in lieu of the lane \( L_2(1, x + W) \) in \( \mathbb{R}^5 \). Now we have

\[
W \cap H_2(0) = \text{col} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}
\]

and

\[
W \cap H_2(n) = n \begin{bmatrix} 0 \\ 1 \\ \frac{\sqrt{5}}{\sqrt{2}} \end{bmatrix} + W \cap H_2(0)
\]

This means that if, for instance,

\[
x = \begin{bmatrix} 0.1 \\ 0 \\ 0.3 \\ 0.4 \\ 0.5 \end{bmatrix}
\]

(see Fig. 4.12), then the cutplane in

\[
\pi_2(W) = \text{col} \begin{bmatrix} 1 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}
\]
4.3. WALLS

\[ x = \begin{bmatrix} 0.1 \\ 0 \\ 0.3 \\ 0.4 \\ 0.5 \end{bmatrix}, \quad W = \text{col} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{11} \\ \sqrt{3} & \sqrt{2} & \sqrt{7} \end{bmatrix} \]

Figure 4.13: Top: two views of the wall \( W_2(1, x+W) \) for the indicated (3, 2) Sturmian tiling \( T(x+W) \) in \( S(W) \). Note that \( x \) has been chosen to lie \( W \cap H_2(0) \). Bottom: the corresponding (2, 2) Sturmian tiling in \( S(\pi_2(W)) \).
corresponding to the lane \( L_2(n, x + W) \) is given by

\[
\pi_2 \left( n \begin{bmatrix} 0 \\ 1 \\ 0 \\ \sqrt{5} \\ \sqrt{2} \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \\ 0.5 \end{bmatrix} + \text{col} \begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \\ \sqrt{3} \end{bmatrix} \right) = n \begin{bmatrix} 0 \\ \sqrt{5} \\ \sqrt{2} \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \end{bmatrix} + \text{col} \begin{bmatrix} 1 \\ 0 \\ \sqrt{5} \end{bmatrix}
\]

In Fig. 4.13, this is illustrated for \( n = 1 \); the reader can verify that the illustrated wall in the (3,2) tiling corresponds tile for tile with the appropriate (2,2) tiling.

The preceding discussion may be easily extended to prove the following result.

**Theorem 4.3** Given any Sturmian tiling \( x + W \) in \( S(W) \), where without loss of generality we may assume that \( x \in H_j(0) \), we have

\[
\pi_j (L_j(n, x + W)) = D(\pi_j(nw + x + W))
\]

where \( w \) is any vector in \( W \) with \( j \)-th component equal to one.

Note that in Eq. 4.2, we have on the left the projection (to the coordinate hyperplane \( H_j(0) \), identified with \( \mathbb{R}^{p-1+q} \)) of a lane for a \((p, q)\) tiling; on the right, we have the digital approximation for a \((p - 1, q)\) tiling.

Theorem 4.3 can be rephrased as follows: *every wall of any tiling in a given \((p, q)\) Sturmian system is combinatorially identical to a specific tiling in a particular \((p - 1, q)\) Sturmian system.* Of course, we can immediately apply this result to the walls themselves; the conclusion is that by considering first walls, then walls of walls, or subwalls of the original tiling, we can construct any \((p, q)\) tiling by stacking specific \((p - 1, q)\) tilings, each of which is constructed by stacking specific \((p - 2, q)\) tilings, and so forth, until we have reduced the whole structure to one dimensional subwalls, which are combinatorially equivalent to certain \((1, q)\) tilings of \( \mathbb{R} \); that is, to particular elements of specific one dimensional Sturmian shifts with an alphabet \( d \) letters. These one dimensional subwalls are called ribbons.

(A slight quibble: really, we should be stacking terraces, not just the walls. However, stacking walls actually determines the structure uniquely, as the multigrid construction shows. We should perhaps also note here that just as the walls of \( T(x + W) \)
4.3. WALLS

4.3. WALLS

Correspond to the grid planes in the multigrid representation of this tiling, so the smaller dimensional multigrid defined by the intersections of transverse grid planes with any one grid plane gives the network of subwalls in the corresponding wall, and so forth, until again reaches the ribbons, which are represented in the multigrid by the lines formed by the intersection of \( p \) grid planes.

It is worth illustrating a simple procedure for systematically determining the \((p - 1, q)\) Sturmian systems corresponding to each of the \(d\) walls of a \((p, q)\) Sturmian system, the \((p - 2, q)\) systems corresponding to the \((\frac{1}{d})\) subwalls of dimension \(p - 2\), and so forth\(^1\). Consider the \((3, 2)\) Sturmian system with

\[
W = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\sqrt{2} & \sqrt{5} & \sqrt{11} \\
\sqrt{3} & \sqrt{2} & \sqrt{7}
\end{bmatrix}
\]

(Recall from Section 3.5 that whenever \( S(W) \) is nondegenerate, \( W \) may be written in the form \( W = \text{col} \left[ \frac{I}{E} \right] \), where \( I \) is the \( p \) by \( p \) identity matrix and \( E \) is some \( q \) by \( p \) matrix.) In our example, we can already read off the subspaces associated with the \(x_1\) walls, namely

\[
W \cap H_1(0) = \text{col} \begin{bmatrix}
0 & 0 \\
0 & 1 \\
\sqrt{5} & \sqrt{11} \\
\sqrt{2} & \sqrt{7}
\end{bmatrix}, \text{ or } W_1 = \text{col} \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\sqrt{5} & \sqrt{11} \\
\sqrt{2} & \sqrt{7}
\end{bmatrix}
\]

with the \(x_2\) walls, namely

\[
W \cap H_2(0) = \text{col} \begin{bmatrix}
1 & 0 \\
0 & 0 \\
\sqrt{2} & \sqrt{11} \\
\sqrt{3} & \sqrt{7}
\end{bmatrix}, \text{ or } W_2 = \text{col} \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\sqrt{2} & \sqrt{11} \\
\sqrt{3} & \sqrt{7}
\end{bmatrix}
\]

and with the \(x_3\) walls, namely

\[
W \cap H_3(0) = \text{col} \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\sqrt{2} & \sqrt{5} \\
\sqrt{3} & \sqrt{2}
\end{bmatrix}, \text{ or } W_3 = \text{col} \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\sqrt{2} & \sqrt{5} \\
\sqrt{3} & \sqrt{2}
\end{bmatrix}
\]

\(^1\)Here, \((\frac{d}{r}) = \binom{d}{d-r}\) is the binomial coefficient "\(d\) choose \(r\)."
Here we have put \( W_j = \pi_j(W) \), so that each wall \( W_j(n, x + W) \) is combinatorially identical to a specific tiling in \( \mathcal{S}(W_j) \).

But we can find \( W \cap H_4(0) \) by taking linear combinations

\[
\begin{bmatrix}
\sqrt{5} \\
\sqrt{2} \\
\sqrt{3}
\end{bmatrix} - \sqrt{2} \begin{bmatrix}
0 \\
\sqrt{5} \\
\sqrt{2}
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} - \sqrt{2} \begin{bmatrix}
0 \\
\sqrt{11} \\
\sqrt{7}
\end{bmatrix}
\]

or

\[
W \cap H_4(0) = \text{col} \begin{bmatrix}
\sqrt{5} & \sqrt{11} \\
-\sqrt{2} & 0 \\
0 & -\sqrt{2}
\end{bmatrix}^{\top} \begin{bmatrix}
\sqrt{15} - \sqrt{4} \\
\sqrt{33} - \sqrt{14}
\end{bmatrix}
\]

and likewise for \( W \cap H_5(0) \).

Next, the intersection

\[
W \cap H_1(0) \cap H_2(0) = \text{col} \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

defines the \((1, 2)\) system \( \mathcal{S}(W_{(1,2)}) \), where

\[
W_{(1,2)} = \text{col} \begin{bmatrix}
\sqrt{11} \\
\sqrt{7}
\end{bmatrix}
\]

corresponding to the \( x_1, x_2 \) subwalls (or ribbons), the intersection

\[
W \cap H_2(0) \cap H_4(0) = \text{col} \begin{bmatrix}
\sqrt{11} \\
0 \\
-\sqrt{2}
\end{bmatrix}
\]

defines the \((1, 2)\) system \( \mathcal{S}(W_{(2,4)}) \), where

\[
W_{(2,4)} = \text{col} \begin{bmatrix}
\sqrt{11} \\
-\sqrt{2}
\end{bmatrix}
\]

corresponding to the \( x_2, x_4 \) subwalls (or ribbons), and so forth.

This procedure focuses attention on the one dimensional subspaces \( W_{(i,k)} \).
Definition 24 Let $S(W)$ be a $(p, q)$ Sturmian system. The $(p - 1)$ one dimensional subspaces of $\mathbb{R}^d$ (where $d = p + q$ as usual)

$$W_J = W \cap \left\{ \bigcap_{j \in J} H_j(0) \right\}$$

(4.3)

where $|J| = p - 1$, are called ribbon spaces. A vector in a ribbon space is called a ribbon vector.

This terminology is entirely consistent with eigenspace and eigenvector.

The point is that not only do any $p$ linearly independent ribbon vectors span $W$, but any $p - 1$ of them span some $W_j$, and so forth, so that the ribbon spaces completely determine the combinatorial equivalence between the subwalls of any Sturmian system and tilings from the appropriate smaller dimensional Sturmian systems.

As one might expect, the ribbon spaces for symmetric Sturmian tiling spaces are particularly simple. For future reference, we note here that four ribbon spaces of the Ammann tiling space, that is, the $C_8$ step symmetric $(2, 2)$ Sturmian system, are spanned by ribbon vectors

$$
\begin{bmatrix}
0 \\
1 \\
\sqrt{2} \\
1
\end{bmatrix}, \quad 
\begin{bmatrix}
1 \\
0 \\
-1 \\
-\sqrt{2}
\end{bmatrix}, \quad 
\begin{bmatrix}
\sqrt{2} \\
1 \\
0 \\
\sqrt{2}
\end{bmatrix}, \quad 
\begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix}
$$

(4.4)

The five ribbon spaces of the $C_{10}$ step symmetric $(2, 3)$ Sturmian system (which includes the Penrose tilings as a subsystem) are spanned by the ribbon vectors

$$
\begin{bmatrix}
0 \\
\tau \\
-1 \\
-\tau
\end{bmatrix}, \quad 
\begin{bmatrix}
\tau \\
0 \\
-1 \\
-\tau
\end{bmatrix}, \quad 
\begin{bmatrix}
1 \\
\tau \\
0 \\
-\tau
\end{bmatrix}, \quad 
\begin{bmatrix}
-1 \\
\tau \\
0 \\
-\tau
\end{bmatrix}, \quad 
\begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix}
$$

(4.5)

where $\tau = (1 + \sqrt{5})/2$ is the Golden Ratio. The six ribbon spaces of the $C_{12}$ step symmetric $(2, 4)$ Sturmian tiling system are spanned by ribbon vectors

$$
\begin{bmatrix}
0 \\
1 \\
\frac{\sqrt{3}}{2} \\
\sqrt{3}
\end{bmatrix}, \quad 
\begin{bmatrix}
-1 \\
0 \\
1 \\
\frac{\sqrt{3}}{2}
\end{bmatrix}, \quad 
\begin{bmatrix}
-\sqrt{3} \\
-1 \\
0 \\
\sqrt{3}
\end{bmatrix}, \quad 
\begin{bmatrix}
2 \\
\sqrt{3} \\
1 \\
0
\end{bmatrix}, \quad 
\begin{bmatrix}
\frac{\sqrt{3}}{2} \\
2 \\
1 \\
-\sqrt{3}
\end{bmatrix}, \quad 
\begin{bmatrix}
1 \\
2 \\
1 \\
-1
\end{bmatrix}
$$

(4.6)
In these cases, by an "accident of low dimensions", only one irrational ratio occurs in each ribbon vector; in higher dimensional cases, there will generally be several. as is already seen in the symmetric (2, 5) Sturmian tiling system. The fact that only the ratios \( \sqrt{2}, \tau, \) and \( \sqrt{3} \) occur in the rotationally step symmetric (2, 2), (2, 3) and (2, 4) Sturmian systems, respectively, will turn out to mean that the number theory of these particular irrationals dominates the theory of these particular Sturmian systems.

### 4.4 Periods

Recall that a subspace \( W \) of \( \mathbb{R}^d \) is called irrational if it contains no integer vectors. On the other hand, if \( W \) contains a nonzero integer vector \( n \neq 0 \), this will be a period for every tiling in \( S(W) \). In the most extreme case, if \( W \) is rational (the column space of an integer matrix) then \( S(W) \) will consist of periodic tilings.

Consider first the case of genus (1, 1) tilings. In Fig. 4.14, \( W \) has slope 2/3. We claim that the system defined by \( W \) is finite and consists entirely of tilings with least period \( \sqrt{13} \). To see this, start at the point \((x, y) = (-1, 0)\) and go right three steps and up two steps. This gives a period of the oblique tiling which is also a period of the pencil (lightly shaded). Indeed, this motion shifts the pencil along the \( p \) axis by the distance \( 3\|p_1\| + 2\|p_2\| = \sqrt{13} \), corresponding to three long tiles and two short ones. Arguing similarly, we can prove:

**Proposition 4.4** Let \( W \) be a one dimensional subspace of \( \mathbb{R}^2 \), with slope \( 0 < v < 1 \). Then the Sturmian tiling \( T(x + W) \) is periodic iff \( v \) is rational. If so, \( S(W) \) is finite.

Moreover, if \( v = m/n \) then

1. the geometric period of \( T(x + W) \) is \( \sqrt{m^2 + n^2} \);

2. the combinatorial period is \( n \) long tiles and \( m \) short tiles.

More generally, suppose \( W \cap \mathbb{Z}^d \neq 0 \). Then each (integer coordinate) vector in \( W \cap \mathbb{Z}^d \neq 0 \) gives a period for the tilings of species \( W \). Moreover, every period of a
Figure 4.14: The oblique tiling $\mathcal{O}(W)$, where $W$ is the column space of the indicated matrix. Each of the tilings in $\mathcal{S}(W)$ is periodic, with three long and two short tiles in each tiling.
Sturmian tiling must arise in exactly this way, i.e. have the form

$$\sum_{j=1}^{d} k_j p_j = \sum_{j=1}^{d} k_j e_j$$

where each $k_j \in \mathbb{Z}$. It is easy to see that this is equivalent to the identity

$$\sum_{j=1}^{d} k_j q_j = 0$$

For instance, in the case of the step symmetric (2, 3) Sturmian system $S(W)$ with steps having perfect ten-fold rotational symmetry, the additive symmetric $\sum_{j=1}^{5} p_j = 0$ corresponds to the period $\sum_{j=1}^{5} q_j = \sum_{j=1}^{5} e_j$ of the (3, 2) Sturmian system $S(W^\perp)$. We’ll return to this idea in Section 4.7.

**Definition 25** Let $A$ be an integer matrix with $d$ rows. We say that its columns give a set of fundamental periods for the additive abelian group $H = \mathbb{Z}^d \cap \text{col} \ A$ if the additive abelian group generated by the columns of $A$ is $H$.

The point of the definition is that in general, the columns of $A$ will generate some subgroup $K$ of $H$. We illustrate this with a simple example.

Consider the matrix

$$A_3 = \begin{bmatrix} 1 & 0 \\ \cos(2\pi/3) & \sin(2\pi/3) \\ \cos(4\pi/3) & \sin(4\pi/3) \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ 1 & 0 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix} \quad (4.7)$$

which arises in connection with the two dimensional step symmetric Sturmian tilings with three fold rotational symmetry. By multiplying each column by an appropriate real number we can obtain the integer matrix

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \quad (4.8)$$

which has the same column space. Let $H = \text{col} \ A \cap \mathbb{Z}^3$; we might expect that the columns of $A$ generate $H$ as an additive abelian group. In fact, they only generate a
subgroup $K$ of index two. To see this, replace the first column by the sum of the two columns:

$$
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
-2 & -1
\end{bmatrix}
$$

which has the same column space as

$$
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & -1
\end{bmatrix}
$$

From Fig. 4.15 it is obvious that the vectors

$$p_1 - p_3 = e_1 - e_3$$
$$p_2 - p_3 = e_2 - e_3$$

do generate the Dirichlet cell for each tiling $T(x + W)$ in $S(W)$; that is, they are fundamental periods for $\mathbb{Z}^3 \cap W$.

Our next task is to solve two problems: first, given $A$, determine whether it generates $H = \mathbb{Z}^d \cap A$, or only a subgroup $K \leq H$, and second, if not, find a matrix $B$ which does. We begin by observing that $K$ and $H$ are both free abelian subgroups of the additive abelian group $G = \mathbb{Z}^d$. Thus, $G/K$ is an additive abelian group, and if its torsion group is nontrivial, then $A$ does not give fundamental periods of $S(W)$. The reason is that there will be lattice points of $\mathbb{Z}^d$ inside the Dirichlet cell of $K$, considered as a subset of $W$. That is, in general $G/K \simeq G/H \oplus H/K$, where $G/H$ is free abelian and the torsion group $H/K$ measures how badly $A$ fails to generate all of $H$. In the example above, for instance $G/H$ is isomorphic to $\mathbb{Z}^2$ while $H/K$ is isomorphic to $\mathbb{Z}_2$.

Fortunately, there is a standard algorithm for determining $G/K$ up to homomorphisms of additive abelian groups. This theory involves finding the Smith normal form of $A$, and it implies the following result.

**Theorem 4.5** $A$ gives a set of fundamental periods for $S(W)$ iff the invariant factors are all zero or one; that is, iff $H/K$ is trivial.
Figure 4.15: The periods of the symmetric tiling of genus (2, 1) are $p_1 - p_2, p_2 - p_3$. 

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$
Corollary 4.6 Let \( W = \text{col } A \) where \( A \) is an integer matrix with the form

\[
A = \begin{bmatrix}
I \\
E
\end{bmatrix}
\]

Then the \( S(W) \) is fully periodic with a set of fundamental periods given by the columns of \( A \).

Proof: The Smith form of the given matrix \( A \) is obviously

\[
\begin{bmatrix}
I \\
0
\end{bmatrix}
\]

and the result follows from Thm. 4.5.

Suppose that \( H \) has nontrivial invariant factors, i.e. some of the columns of \( S \) contain a diagonal entry other than zero or one. Then dividing these columns of \( S \) by the appropriate positive integer and then reversing all the column operations (but ignoring the row operations) involved in computing the Smith form of \( A \) gives a matrix \( B \) whose columns do give a set of fundamental periods.

We illustrate this process with the example of Eq. 4.8,

\[
\begin{bmatrix}
2 & 0 \\
-1 & 1 \\
-1 & -1
\end{bmatrix}
\]

Swapping the first and third rows gives

\[
\begin{bmatrix}
-1 & -1 \\
-1 & 1 \\
2 & 0
\end{bmatrix}
\]

Inverting the new first row gives

\[
\begin{bmatrix}
-1 & 1 \\
-1 & 1 \\
2 & 0
\end{bmatrix}
\]

Adding the first row to the second and subtracting twice the first row from the third gives

\[
\begin{bmatrix}
1 & 1 \\
0 & 2 \\
0 & -2
\end{bmatrix}
\]
Subtracting the first column from the second gives

\[
\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -2 \end{bmatrix}
\]

Adding the second row to the third gives the desired Smith normal form:

\[
\hat{S} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}
\]

The invariant factor 2 shows that \( A \) generates (as an additive abelian subgroup) not \( H = \mathbb{Z}^3 \cap \text{col} \ A \) but rather an index two subgroup \( K \), the torsion group \( H/K \) being isomorphic to \( \mathbb{Z}_2 \). Therefore we replace \( S \) by

\[
\hat{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

and reverse all row operations (in reverse order). Specifically, subtracting the second row from the third gives

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}
\]

Subtracting the first row from the second and adding twice the first row to the third gives

\[
\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}
\]

Inverting the first row gives

\[
\begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}
\]

Finally, swapping the first and third rows gives

\[
\hat{A} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ -1 & 0 \end{bmatrix}
\]

(4.11)

(In this case, human inspection shows that this group has the even simpler basis given by Eq. 4.10, but Eq. 4.11 works just as well.)

Eq. 4.11 looks almost the same as Eq. 4.8, and the reader might doubt that the former generates \( G \) whereas latter generates only a subgroup \( H \) with \( [G : H] = 2 \).
4.5. **PERIODIC APPROXIMATIONS**

But notice that squared areas of the Dirichlet cells are $\det A' A = 12$ and $\det \hat{A}' \hat{A} = 3$, respectively; thus, $\hat{A}'$ really does generate an integer vector supergroup of the one generated by $A$. The fact that $\hat{A}$ has Smith form $\hat{S}$ means that its columns give a set of fundamental periods for $\col A \cap \mathbb{Z}^3$; the value of $\det \hat{A}' \hat{A}$ partially characterizes how "tilted" $W$ is in $\mathbb{Z}^3$.

This procedure of using column operations only on the Smith normal form is superficially reminiscent of the theory of the Hermite normal form; see for instance [88].

Often $S(W)$ will have periods, but these may not fill up $W$. In this case, we say that $S(W)$ is semiperiodic. See Fig. 4.16 for some simple examples.

**Definition 26** Suppose $W$ is a subspace of $\mathbb{R}^d$. Then $V = \text{span}(W \cap \mathbb{Z}^d)$ is called the rational part of $W$.

**Definition 27** If $W$ is a subspace of $\mathbb{R}^d$ and $V$ is its rational part, then the Sturmian system $S(V)$ is called the periodic part of $S(W)$.

Note that if $\dim V = \ell$, where necessarily $\ell < m$, then $S(V)$ consists of tilings of $\mathbb{R}^\ell$ whereas $S(W)$ consists of tilings of $\mathbb{R}^p$.

### 4.5 Periodic Approximations

Look again at Fig. 4.14, which depicts $S(W_1)$ where $W_1 = (3, 2)$. We may consider $S(W_1)$ a periodic approximation of the aperiodic Sturmian system $S(W)$ where $W = (\tau, 1)$, where $\tau$ is the golden ratio.

Given an irrational subspace $W$, how can we find periodic approximations to $S(W)$? In the case of $(1, 1)$ tilings, this question is easy to answer. Consider for example $S(W)$, where $W = (\tau, 1)$. The continued fraction expansion (see any of [55][80][102][104][113] for the basic theory of continued fractions) of $\tau$ yields a sequence of approximants $1/2, 2/3, 3/5 \ldots$ and

$$W_1 = \text{col} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ W_2 = \text{col} \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ W_3 = \text{col} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \ldots$$
Figure 4.16: Two semiperiodic tiling of genus (2,1). Left: $\tau$ satisfies $\tau^2 - \tau - 1 = 0$ and the period is $p_3 - p_2 - p_1$. Right: $\alpha$ satisfies $2\alpha^2 - 3\alpha - 1 = 0$ and the period is $2p_3 - 3p_2 - p_1$.

give the desired periodic approximations $S(W_1), S(W_2), S(W_3)\ldots$ to $S(W)$. Both the linguistic properties and geometric properties of $S(W)$ are better approximated by $S(W_n)$ and $n$ grows.

In the case of $(1, n)$ Sturmian systems, the same basic approach works; we find rational approximations to any vector in $W$ and obtain from these the desired approximations $S(W_n)$. Naturally, we still want to find “efficient” periodic approximations, in the sense of having small fundamental periods. There are a general number of methods for doing this.

In the case of higher dimensional Sturmian systems, the simplest approach is to find the ribbon spaces for $S(W)$, as discussed in Section 4.3, and then find rational approximations to representative ribbon vectors.

If each ribbon vector only involves a single irrational ratio, the problem is considerably simplified, since we can use the ordinary continued fraction algorithm to
4.5. PERIODIC APPROXIMATIONS

Figure 4.17: A "coarse" periodic approximation to an Ammann tiling, with every other Dirichlet cell shown in light gray. It is in the (2, 2)-Sturmian tiling space for W spanned by the indicated matrix A; this plane is a "rational approximation" in the Grassmannian manifold G(2, 2) of the Ammann plane.

\[
A = \begin{bmatrix}
0 & 3 \\
2 & 2 \\
3 & 0 \\
2 & -2
\end{bmatrix}
\]
find the best possible approximations by integer vectors with bounded entries. For example, we obtain the Ammann tiling system by choosing any two of the ribbon vectors from Eq. 4.4; noting that the first and third are mutually orthogonal we take

\[
W = \text{col} \begin{bmatrix} 0 & \sqrt{2} \\ 1 & 1 \\ \sqrt{2} & 0 \\ 1 & -1 \end{bmatrix}
\]

Now, the first two continued fraction approximants of \(\sqrt{2}\) are 3/2 and 7/5. Using the first of these we obtain

\[
\begin{bmatrix} 0 & 3/2 \\ 1 & 1 \\ 3/2 & 0 \\ 1 & -1 \end{bmatrix}
\]

or, clearing denominators

\[
B_1 = \begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 3 & 0 \\ 2 & -2 \end{bmatrix}
\]

Now, the Smith form of \(B_1\) is

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

so we know that

\[
2p_2 + 3p_3 + 2p_4, \quad 3p_1 + 2p_2 - 2p_4
\]

form a set of fundamental periods for the first periodic approximation \(S(\text{col } B_1)\). See Fig. 4.17 for a picture of the resulting periodic approximation to the Ammann system; notice that because we started with mutually orthogonal column vectors which turned out to give fundamental periods, the Dirichlet cells may be taken to be squares as shown in this figure.

Using the approximation 5/7 we obtain

\[
\begin{bmatrix} 0 & 7/5 \\ 1 & 1 \\ 7/5 & 0 \\ 1 & -1 \end{bmatrix}
\]
Figure 4.18: A “finer” periodic approximation. The Dirichlet cell shown in light gray is formed using the two column vectors of $A$, which are “relatively prime” in the Grassmannian $G(2,2)$.

$$A = \begin{bmatrix} 0 & 7 \\ 5 & 5 \\ 5 & -5 \end{bmatrix}$$
or, equivalently

\[ B_2 = \begin{bmatrix} 0 & 7 \\ 5 & 5 \\ 7 & 0 \\ 5 & -5 \end{bmatrix} \]

Once again the Smith normal form shows that

\[ 5p_2 + 7p_3 + 5p_4, 7p_1 + 5p_2 - 5p_4 \]

form a set of fundamental periods for the second periodic approximation \( S(\text{col } B_2) \).

See Fig. 4.18 for a picture of the resulting periodic approximation.

Clearly, this procedure will yield arbitrarily good rational approximations to the Ammann tilings. It follows that one possible procedure for studying them would be study instead the periodic approximations.

Two points are worth stressing here.

First, if \( W' \) is a rational approximation to \( W \), then the tiling space defined by \( W' \) may be considered a \emph{periodic approximation} to the tiling space defined by \( W \). However, the tilings of species \( W' \) do not have quite the same prototiles have the original tilings of species \( W \); for instance in Fig. 4.17, close examination reveals that the tilted squares are slightly smaller than the untilted squares, so that some symmetries of the original Ammann tiling have been lost. Nonetheless, the combinatorial pattern of tiles in the approximation yields a valid periodic tiling by the original prototiles (with the same \emph{combinatorial} period as the original rational approximations).

Second, given \( R > 0 \), we can obtain a rational approximation \( W^{(n)} \) to \( W \) such that the Dirichlet cells of the tilings of species \( W^{(n)} \) enclose a disk of radius \( R \). This means that the Ammann tilings come as close as possible to being periodic without actually being periodic, in the sense that the Ammann tiling space can just as well be regarded as approximations of this sequence of periodic tiling spaces as the other way around!
4.6 Almost Periods

If the reader has access to a photocopier which can make transparencies he or she may wish to try the following experiment. Make two photocopies of the Penrose tiling depicted in Fig. 3.15, one on paper and one on a transparency. Superimpose the two so that the two images overlap perfectly. Now slide the transparency upwards, keeping the edge directions aligned. At first you will observe a "mess" (disagreement between the two pictures all over the plane) but then suddenly something entirely unexpected will happen; you will suddenly see exact agreement between the transparency and the underlying image over a large percentage of the plane, but not all of the plane, as in Fig. 4.19. As you continue sliding, this agreement will disappear, but after a while a new pattern of agreement over a still larger percentage of the plane will appear, as in Fig. 4.20. Notice that the patches of disagreement (shown in grey) form a network of worms intersecting in decagons, whereas the patches of agreement (shown in white) are disjoint and simply connected. Essentially identical patterns arise in Onoda's "partially nondeterministic" algorithm [28][127] for growing Penrose tilings, starting from any small initial patch of tiles.

It turns out that you can obtain perfect agreement on as large a percentage of the plane as you wish, provided you are willing to take a sufficiently large displacement, as suggested the sequence of Figs. 4.19-4.22. This is what we mean by saying that Penrose tilings are almost periodic. The displacements giving such almost periods are anything but arbitrary, as the appearance of the Fibonacci numbers 2, 3, 5, 8 suggests. For this reason we prefer the more fanciful term magic shifts to "almost periods". The appearance here of the Fibonacci sequence is our first indication that number theory might play a role in the theory of Sturmian tilings.

For a magic shift for the Ammann tilings; see Fig. 4.23.

To understand how magic shifts arise, observe that the orthoprojection of the integer vector $\sum_j n_j e_j$ onto $W$ is $\sum_j n_j p_j$, while its projection onto $W^\perp$ is $\sum_j n_j q_j$.
Figure 4.19: A Penrose tiling shifted against itself by $2(\mathbf{p}_5 - \mathbf{p}_2) + (\mathbf{p}_4 - \mathbf{p}_3) \approx (0, -3.1495)$. Note the patches of disagreement form a network of gray "worms" intersecting in decagons.
Figure 4.20: The same Penrose tiling shifted against itself by $3(p_3 - p_2) + 2(p_4 - p_3)$. The patches of disagreement still form a network of worms, but now the patches of agreement are larger.
Figure 4.21: The same Penrose tiling shifted against itself by $5(p_5 - p_2) + 3(p_4 - p_3)$. Here the patches of agreement are larger still. The expert reader may recognize the shapes appearing as patches of agreement as the "deadsets" (here called "kingdoms") which appear in Onoda's partially non-deterministic algorithm for growing Penrose tilings.
Figure 4.22: The same Penrose tiling shifted against itself by $8(p_5 - p_1) + 5(p_4 - p_3)$. Sturmian tilings indeed deserve the appellation "almost periodic".
Figure 4.23: A Ammann tiling shifted against itself by $3p_3 + 2(p_2 + p_4)$. The patches of disagreement form a network of worms intersecting at singular octagons. Unlike the Penrose tilings, the worms can also cross at right angles “without interfering with one another other”.
4.6. ALMOST PERIODS

Here we naturally have

\[ \sum_j n_j e_j = \left( \sum_j n_j p_j \right) + \left( \sum_j n_j q_j \right) \]

By the \( \mathbb{Z}^d \) periodicity of \( \mathcal{O}(W) \), this immediately implies that

\[ \left( \sum_j n_j p_j \right) + \mathcal{O}(W) = -\left( \sum_j n_j q_j \right) + \mathcal{O}(W) \]  \hspace{1cm} (4.12)

Now suppose we can find an integer vector \( \sum_j n_j e_j \) which is "very nearly" in \( W \). Then \( \sum n_j e_j \) will be close to its orthoprojection into \( W \), namely \( \sum_j n_j p_j \); that is, \( \| \sum_j n_j q_j \| \) will be small. (See Fig. 4.6.) This means that if we shift \( \mathcal{O}(W) \) by \( \sum_j n_j p_j \), the shifted oblique tiling \( \left( \sum_j n_j p_j \right) + \mathcal{O}(W) \) will almost agree with \( \mathcal{O}(W) \), except that each cell will be displaced slightly orthogonally to \( W \), in fact by precisely \( -\sum_j n_j q_j \). The point is that we can understand the result of shifting a tiling \( T(x + W) \) against itself by \( \sum_j n_j p_j \) by considering \( x + W \) as a cut plane through \( \mathcal{O}(W) \) and through \( -\left( \sum_j n_j q_j \right) + \mathcal{O}(W) \), respectively. Wherever this cut plane cuts through cells of different types in the competing oblique tilings, shifting the tiling \( T(x + W) \) against itself by \( \sum_j n_j p_j \) will result in a disagreement in this particular location. Moreover, the \( q \) volume of overlap in \( W^\perp \) of tiles of the same type when comparing the multicell with the same multicell translated in \( W^\perp \), relative to the total \( q \) volume of the window, gives the frequency of vertices with a discrepancy.

The following theorems are well known (see for instance Section 2.6 of [124]):

**Theorem 4.7** Suppose \( W \) is a subspace of \( \mathbb{R}^d \). Then \( T(W) \) is almost aperiodic.

**Theorem 4.8** Suppose \( W \) is a subspace of \( \mathbb{R}^d \). Then the following are equivalent:

1. \( W \) is irrational (i.e. \( W \cap \mathbb{Z}^d = \emptyset \)),

2. \( P(\mathbb{Z}^d) \) is dense (and uniformly distributed) in \( W \),

3. \( Q|\mathbb{Z}^d \) is one-one,
4. \( T(W) \) is aperiodic.

Theorem 4.7 can be proven in a much more general context than we consider in this thesis using the theory of model sets; see [96] (the vertices of any Sturmian tiling \( T(x + W) \) form a model set). Indeed, roughly speaking every Sturmian system is almost periodic in any direction in which it is not periodic.

In Theorem 4.8, note that \( Q(\mathbb{Z}^d) \) is not in general dense, However, \( Q(\mathbb{Z}^d) \) will be dense (and uniformly distributed) in certain parallel \( k \)-flats in \( W^\perp \), where \( 0 < k \leq q \); in particular, note that because \( W \subset \text{rat} \ W \) we have \( (\text{rat} \ W)^\perp \subset W^\perp \). and then \( Q(\mathbb{Z}^d) \cap (\text{rat} \ W)^\perp \) is dense in \( (\text{rat} \ W)^\perp \) (see Section 4.7). Also, \( T(W) \) is not in general ergodic (again, see Section 4.7).

Note that Theorem 4.8 says that \( W \) is not irrational, i.e. \( W \) has a period \( n \in W \cap \mathbb{Z}^d \) iff \( Q(\mathbb{Z}^d) \) is not one-one; of course, the point is that in this situation \( Q(mn) = Q(n) \) for \( m \in \mathbb{Z} \).

In the remainder of this section, we concentrate on two problems:

1. How can we actually find magic shifts \( \mathbf{w} \)?

2. How large does \( ||\mathbf{w}|| \) need to be to obtain perfect agreement over a given percentage of \( \mathbb{R}^p \)?

Let us consider first the simplest case, the \((1, 1)\) Sturmian systems. As we have seen, the effect of displacing \( \mathcal{O}(W) \) by \( n_1 \mathbf{p}_1 + n_2 \mathbf{p}_2 \) is merely to shift each cell along \( W^\perp \) by the small displacement \( -(n_1 \mathbf{q}_1 + n_2 \mathbf{q}_2) \). This means of course (see Fig. 4.25) that along any one cut line, we obtain the same tile whether we look at \( \mathcal{O}(W) \) or the displaced oblique tiling \( n_1 \mathbf{p}_1 + n_2 \mathbf{p}_2 + \mathcal{O}(W) \). The disagreements come precisely where \( x + W \) cuts through one of the "rectangles of disagreement", each of which has height \( ||n_1 \mathbf{q}_1 + n_2 \mathbf{q}_2|| \).

Observe that there are two natural ways to measure the "amount of disagreement" between \( T(x + W) \) and \( T(x + \mathbf{w} + W) \). We can either count the number of tiles which
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Figure 4.24: The two multicells drawn with thin lines belong to the oblique tiling $\mathcal{O}(W)$ and are separated by the integer vector $n_1 e_1 + n_2 e_2$, which almost lies in $W$. The multicell drawn with bold lines belongs to the shifted oblique tiling $n_1 p_1 + n_2 p_2 + \mathcal{O}(W)$; the cells of this shifted tiling are each displaced along $W^\perp$ by $-(n_1 q_1 + n_2 q_2)$ from an identical cell in $\mathcal{O}(W)$.

\[
W = \col \begin{bmatrix} 7 \\ 1 \end{bmatrix}
\]

Figure 4.25: These pictures show the $(1, 1)$ oblique tiling $\mathcal{O}(W)$, where $W$ is the indicated line, before (bold lines) and after (thin lines) translations along $W$. Left: $3p_1 + 2p_2$; here $\|3q_1 + 2q_2\| = 0.124108$. Right: $5p_1 + 3p_2$; here $\|5q_1 + 3q_2\| = 0.0767031$. 
disagree inside some large interval around the origin, relative to the total number of tiles in that interval, and take the limit as the interval grows (by the homogeneity this won’t depend on the starting point), or we can measure the relative area of the disagreements within the interval, and take the limit as the interval grows. Let us call the first notion the **error frequency** and the second **error coverage**. From Fig. 4.25 it is evident that the error frequency is

\[
\frac{2\|n_1q_1 + n_2q_2\|}{\|q_1 + q_2\|}
\]

(4.13)

To find the error coverage, note that the coverage of long tiles in error is

\[
\frac{\|n_1q_1 + n_2q_2\| \|p_1\|}{\|q_1 + q_2\|}
\]

while the coverage of short tiles in error is

\[
\frac{\|n_1q_1 + n_2q_2\| \|p_2\|}{\|q_1 + q_2\|}
\]

But for (1, 1) tilings, we have \(\|p_1\| = \|q_2\|\) and \(\|p_2\| = \|q_1\|\), so that the total error coverage is just

\[
\varepsilon = \|n_1q_1 + n_2q_2\|
\]

(4.14)

**Theorem 4.9** Suppose \(S(W)\) is any aperiodic \((1, 1)\) Sturmian system. Then there exist magic shifts \(n_1p_1 + n_2p_2\) with an error coverage \(\varepsilon\) such that

\[
\max\{|n_1|, |n_2|\} < \frac{1}{\varepsilon}
\]

(4.15)

**Proof:** Consider first the case

\[
W = \text{col} \begin{bmatrix} 1 \\ v \end{bmatrix}
\]

where \(0 < v < 1\). The theory of simple continued fractions [55] shows that there are
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infinitely many positive integer solutions \((n_1, n_2)\) to the inequality\(^2\)

\[
\left| \frac{n_2}{n_1} - v \right| < \frac{1}{n_1^2}
\]  

(4.16)

In particular, we may take \(n_2/n_1\) to be any of the continued fraction approximants obtained by truncating the continued fraction

\[ v = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \]

From Eq. 3.15 we have

\[
n_1q_1 + n_2q_2 = \frac{1}{1 + v^2} \left( n_1 \begin{bmatrix} v^2 \\ -v \end{bmatrix} + n_2 \begin{bmatrix} -v \\ 1 \end{bmatrix} \right)
\]

\[
= \frac{-n_1v + n_2}{1 + v^2} \begin{bmatrix} -v \\ 1 \end{bmatrix}
\]

whence

\[
\|n_1q_1 + n_2q_2\| = \frac{|n_2 - n_1v|}{\sqrt{1 + v^2}} = |n_2 - n_1v| \cos \theta
\]

where \(W\) makes angle \(\theta\) with the \(x_1\)-axis. Now Eq. 4.16 gives

\[
\|n_1q_1 + n_2q_2\| < \frac{\cos \theta}{n_1}
\]

(4.17)

which is valid for all shifts \(n_1p_1 + n_2p_2\) satisfying Eq. 4.16. For sufficiently large \(n_j\), \(n_1 > n_2\) so we can write

\[
\varepsilon = \|n_1q_1 + n_2q_2\| < \frac{1}{\max\{|n_1|, |n_2|\}}
\]

where the dependence on the slope has disappeared. Indeed, by symmetry this is true for \(1 < v < \infty\) as well, so we are done.

\[\square\]

\(^2\) Eq. 4.16 can be improved to give the bound

\[
\|\frac{n_2}{n_1} - v\| < \frac{1}{\sqrt{5}n_1^2}
\]

and if \(v\) is not a unimodular image of the golden ratio, still further; see [55] for more information. If we agree to neglect null sets of slopes \(v\), we can get even stronger results; see [80].
Next, consider the case of $(1,q)$ Sturmian systems. Once again, we begin by observing that if $\sum n_j e_j$ "very nearly" lies in $W$, then displacing $O(W)$ by $\sum n_j p_j$ yields almost perfect agreement, with each cell of $\sum n_j p_j + O(W)$ being displaced along $W^\perp$ by the small vector $-\sum n_j q_j$. If $1 - \varepsilon_J$ is the fraction of overlap between $T^*_J$ and $-\sum n_j q_j + T^*_J$ (note that this is the area of a zonotope which is homothetic to $T^*_J$ itself), then the error coverage is

$$\frac{\sum_J \varepsilon_J \text{vol}_p T_J}{\sum_J \text{vol}_q T^*_J}$$
where the sum runs over selections \( J \) of a single coordinate, i.e. \( J = \{ j \} \), or

\[
\sum_{j=1}^{d} \varepsilon_j \frac{\cos \theta_j}{\sum_{j=1}^{d} \cos \theta_j}
\]

(4.18)

where \( \theta_j \) are the Euler angles of the line \( W \) in \( \mathbb{R}^d \).

**Lemma 4.10** Let \( S(W) \) be any nondegenerate \((1,q)\) Sturmian system. Given any \( \epsilon > 0 \), there exist vectors \( \sum_j n_j p_j \in W \) with

1. \( \max\{n_j : 1 \leq j \leq d\} < \epsilon^{-q} \).
2. \( \| \sum_j n_j q_j \| < \sqrt{q} \epsilon \).

**Proof:** Consider first the case

\[
W = \text{col} \begin{bmatrix} v_1 \\ \vdots \\ v_q \\ 1 \end{bmatrix}
\]

where \( 0 < v_1 < v_2 < \ldots < v_q < 1 \). A theorem of Dirichlet (see [55] Theorem 200) states that given any such \( v_j \) and \( \epsilon > 0 \), there exist \( n_1, n_2, \ldots, n_q, n_{q+1} \) such that

1. \( n_{q+1} < \epsilon^{-q} \).
2. \( |n_{q+1} v_j - n_j| < \epsilon \) for \( 1 \leq j \leq q \).

(Indeed, we may as well also require \( n_1 < n_2 < \ldots < n_q < n_{q+1} < \epsilon^{-q} \).) This implies that

\[
\left\| \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_q \\ n_{q+1} \end{bmatrix} - n_{q+1} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_q \\ 1 \end{bmatrix} \right\|^2 = \sum_{j=1}^{q} |n_{q+1} v_j - n_j|^2 < q \epsilon^2
\]

But because \( \sum_{j=1}^{q+1} n_j q_j \) is the orthoprojection of the integer vector \( \sum_{j=1}^{q+1} n_j e_j \) on \( W^\perp \), we have

\[
\left\| \sum_{j=1}^{q+1} n_j q_j \right\| < \left\| \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_q \\ n_{q+1} \end{bmatrix} - n_{q+1} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_q \\ 1 \end{bmatrix} \right\| < \sqrt{q} \epsilon
\]
The obvious symmetry arguments now allow us to extend this conclusion to any non-degenerate \((1, q)\) Sturmian system.

While the frequency of disagreements is not directly proportional to \(\| \sum_j n_j q_j \|\) if \(q > 1\), nonetheless it is clear that by choosing \(\epsilon\) sufficiently small and finding a shift \(\sum_j n_j p_j\) satisfying the two criteria of Lem 4.10, we can obtain perfect agreement over an arbitrarily large percentage of the line. Thus, we have shown the following.

**Theorem 4.11** Every nondegenerate \((1, q)\) Sturmian system \(S(W)\) is either periodic (rational \(W\)) or almost periodic.

More generally still, it follows immediately from Theorem 4.11 that every ribbon in any nondegenerate \((p, q)\) Sturmian system is either periodic or almost periodic, with all ribbons in a given family sharing the same almost periods or magic shifts.

See Fig. 4.27, where the lower row of pictures corresponds to the \((1, 2)\) magic shifts illustrated in Fig. 4.26.

**Lemma 4.12** Let \(S(W)\) be any nondegenerate \((p, q)\) Sturmian system. Given any \(\epsilon > 0\), there exist vectors \(\sum_j n_j p_j \in W\) with

1. \(\max\{ n_j : 1 \leq j \leq d \} < p \epsilon^{-q}\),

2. \(\| \sum_j n_j q_j \| < \sqrt{pq} \epsilon\).

**Proof:** Choose \(p\) distinct families of ribbons, with ribbon spaces \(W \cap_{k \in K} H_k(0)\), where \(K\) is some selection of \(p - 1\) coordinates. For the \(K\)-th family, by Lemma 4.10 we can find \(\sum_{j \notin K} m_j^K p_j \in W \cap H_k(0)\) such that:

1. \(\max\{ m_j^K : j \notin K \} < \epsilon^{-q}\),

2. \(\| \sum_{j \notin K} m_j^K q_j \|^2 < q \epsilon^2\).
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\begin{equation*}
W = \text{col} \begin{bmatrix}
1 & 0 \\
\sqrt{5} & \sqrt{3} \\
\sqrt{2} & \sqrt{7}
\end{bmatrix}
\end{equation*}

Figure 4.27: Some magic shifts for a typical (2, 2) Sturmian system \( S(W) \).
Figure 4.28: The magic shift $w = 3p_3 + 10p_4 + 8p_5$ for the $(3, 2)$ Sturmian system $S(W)$, where $W$ is the column space of the indicated matrix. This shift arises from a rational approximation to the third column of the matrix $A_1$, one of the ribbon vectors for $S(W)$. In the picture, the wire-frame cells indicate patches of disagreement between the illustrated tiling $T_W(x)$ and the shifted tiling $T_W(w + x)$ and the solid cells belong to a selection of one ribbon from each of the ten families of ribbons of $T_W(x)$, if such a cell does not belong to both tilings it is shown as a wire frame cell. The intent is to convey the sense that the two tilings agree perfectly over “most” of $\mathbb{R}^3$. 

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{11} \\ \sqrt{3} & \sqrt{2} & \sqrt{13} \end{bmatrix}$$
Adding over the $p$ families gives
\[
\left\| \sum_{j=1}^{d} n_j q_j \right\|^2 = \left\| \sum_{K \sum_{j=1}^{d} m_j^K q_j \right\|^2 < pq \varepsilon^2
\]
where
\[
\max\{n_j : 1 \leq j \leq d\} \leq p \varepsilon^{-q}
\]

In the case of the Penrose tilings, from the ribbon vector
\[
\begin{bmatrix}
0 \\
-1 \\
2 \\
-2
\end{bmatrix}
\]
we obtain the magic shifts
\[
\begin{bmatrix}
0 \\
3 \\
5 \\
8
\end{bmatrix},
\begin{bmatrix}
2 \\
2 \\
3 \\
5
\end{bmatrix},
\begin{bmatrix}
-1 \\
-2 \\
-3 \\
-5
\end{bmatrix}, \ldots
\]
used in Figs. 4.19 through 4.22.

4.7 Subsystems

Suppose
\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}
\]
The components $x_j$ are said to be rationally independent (see for instance p.33 of [79]) if
\[
\sum_{j=1}^{d} n_j x_j = 0
\]
only if all the $n_j = 0$; that is if the only integer vector $n$ which is orthogonal to $x$ is 0. This suggests the following definition.
Definition 28 An integer vector $n$ gives a rational relation for a linear subspace $W$ of $\mathbb{R}^d$ if

$$n \cdot w = 0$$

(4.19)

for all $w \in W$.

Proposition 4.13 Let $W$ be a subspace of $\mathbb{R}^d$, let $P$ be the orthoprojection onto $W$, and write $p_j = P(e_j)$ as usual. Suppose $n$ is an integer vector. Then the following are equivalent:

1. $n$ gives a period for $W$.

2. the $p_j$ satisfy the identity $\sum_{j=1}^{d} n_j p_j = 0$.

3. $n$ gives a rational relation for $W$.

Proof: $n$ is a period for $W$ iff $n \in W$ iff

$$0 = P(n) = P \left( \sum_{j=1}^{d} n_j e_j \right) = \sum_{j=1}^{d} n_j p_j$$

Moreover, $n \in W$ iff $n$ is a rational relation for $W$. 

Recall from Section 2.3 that the nature of the $W$-flow on the torus $T^d$ is determined by the rational closure of $W$; the orbit closures are precisely the cosets of the subtorus $\pi(\text{rat } W)$, and the orbits themselves, which are cosets of $\pi(W)$, are dense in these subtori.

Robinson [114] has shown that the $\mathbb{R}^p$ translation action on the Sturmian tiling space $\mathcal{T}(W)$ is an almost one-one, finite thickness extension of the $W$-flow on $T^d$. In particular, Haar measure on $T^d$ gives rise to a unique invariant measure on $\mathcal{T}(W)$, and the ergodic decomposition for the $W$-flow on $T^d$ immediately yields the following:
Corollary 4.14 Let $W$ be a subspace of $\mathbb{R}^d$. Then the following are equivalent:

1. $W^\perp$ is irrational,

2. $Q(\mathbb{Z}^d)$ is dense in $W^\perp$.

3. $P|\mathbb{Z}^d$ is one-one,

4. $W$ has no rational relations.

5. $T(W^\perp)$ is aperiodic.

6. the $W$-flow on $T^d$ is strictly ergodic (minimal and uniquely ergodic),

7. $T(W)$ is uniquely ergodic.

In general, if $W^\perp$ is not irrational, $Q(\mathbb{Z}^d) \cap (\text{rat } W)^\perp$ will be dense and uniformly distributed in $(\text{rat } W)^\perp$. Indeed, each ergodic component

$$S(y + V) = \{T(x + W) : x + W \subset y + V\}$$

corresponds to one of the subtori, and the $\mathbb{R}^p$ action on $S(y + V)$ is uniquely ergodic; that is, the tilings in $S(y + V)$ all belong to the same local isomorphism class (see [86]).

In fact, we can almost show that the orbit closure of $T(x + W)$ is

$$\{T(y + w + W) : w \in W, y + W \subset x + V\}$$

This is clearly true: the idea is that we can unfold the Christoffel tiling while remaining within $V$. In any case, this appears to be a "folk theorem": for instance Prop 3.5 of Thang Le [86] (for the proof, Le cites an unpublished work of Levitov) says that any two tilings $T(x + W), T(y + W)$ such that $x - y \in \text{rat } W$ belong to the same local isomorphism class. This would imply that $T(W)$ is minimal iff $T(W)$ is uniquely ergodic.
Indeed, the $W$-flow on $T^d$ (and thus $\mathcal{T}(W)$) has a $k$ parameter family of orbit closures, where

\[
k = \text{codim} \text{rat } W = \dim \text{span}(W^\perp \cap \mathbb{Z}^d) = \text{number of independent periods of } W^\perp = \text{number of independent rational relations for } W
\]

For example, suppose

\[
W = \text{col } \begin{bmatrix} \frac{1}{2} \\ \sqrt{3} \end{bmatrix}
\]

Then

\[
W^\perp = \text{col } \begin{bmatrix} 2 & \sqrt{3} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}
\]

so

\[
\text{rat } W = \text{col } \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

Here, we have the single rational relation $x_1 = 2x_2$ on $W$ and, correspondingly, $\text{codim} \text{rat } W = 1$. In this case, the orbit closures can be identified with $\pi(tq_1 + W)$ where $0 \leq t < 1/2$; see Fig. 4.29.

Here is another example. Let

\[
W = \text{col } \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}
\]

where $\alpha \approx 1.780776406$ is one root of $2\alpha^2 - 3\alpha - 1$. Then

\[
W^\perp = \text{col } \begin{bmatrix} \alpha & \alpha^2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \text{col } \begin{bmatrix} \alpha & 3\alpha + 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \text{col } \begin{bmatrix} \alpha & 1 \\ -1 & 3 \\ 0 & -2 \end{bmatrix}
\]

so $\mathcal{T}(W^\perp)$ has a single fundamental period which we may take to be $(1, 3, -2)$. This corresponds to the single integer relation $x_1 + 3x_2 - 2x_3 = 0$ on $W$. (A tiling in $\mathcal{T}(W^\perp)$ was illustrated at right in Fig. 4.16.) We have

\[
(\text{rat } W)^\perp = \text{col } \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \text{rat } W = \text{col } \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}
\]
Figure 4.29: Top: \( \pi(\text{rat } W) \) and \( \pi(0.25q_1 + \text{rat } W) \), where \( \pi : \mathbb{R}^3 \rightarrow T^3 \) is the canonical projection map (see Section 2.3). Bottom: the corresponding pictures (schematic) in the window in \( W^\perp \); the numbers are intended to help the reader understand the identifications made by \( \pi \). (Note that facets of type \( Q(F_{(1)}) \) appear here as horizontal segments.) In the bottom picture, cutlines \( x + W \) appear as points inside the window, and \( x + W \) lies on one of the bold segments iff \( T(x + W) \) belongs to the corresponding orbit closure in \( T(W) \).

Here, the single period of \( T(W^\perp) \) corresponds to the single dimension of \( (\text{rat } W)^\perp \), which parameterizes the orbit closures of \( T(W) \).

If \( W \) happens to be an invariant subspace of an integer matrix \( A \), the integer relations for \( W \) are intimately connected with the rational canonical form of \( A \). We summarize the relevant theory in the following lemma.

**Lemma 4.15** Let \( T \) be a linear operator on a vector space over a field \( F \). Recall the minimal polynomial \( \mu_T \) is the unique monic polynomial over \( F \) such that any polynomial \( p \) such that \( p(T) = 0 \) is divisible by \( \mu_T \). Suppose the minimal polynomial
Figure 4.30: Whenever $V \subset W$, where $\dim V = k$, the digital approximation to $y + V$ is part of the $k$-skeleton of the digital approximation of $y + W$. In case $V^\perp = \text{rat } W = \text{rat } W$, as here, $T(W)$ then decomposes into a $k$ parameter family of minimal subsystems.
Figure 4.31: The Dirichlet cells of the periodic tiling $T(x + V) \in S(V)$, where $V = \text{rat } W$, are indicated by “checkerboard” shading.
of $T$ factors into monic polynomials $\mu_j$, each irreducible over $\mathbb{Q}$:

$$\mu = \prod_{j=1}^{n} \mu_j^{e_j}$$

where each $e_j$ is a positive integer. Then our vector space is the direct sum of $T$-invariant subspaces

$$V_j = \text{null} \mu_j(T)^{e_j}$$

called the primary components with respect to $T$, where the minimal polynomial of

$T$ restricted to $V_j$, $\mu_{T|V_j}$, is $\mu_j^{e_j}$.

Recall that a $T$-cyclic subspace $V$ contains a vector $v$ such that $V$ is spanned by $v, T(v), T^2(v), \ldots$. Each primary component is a direct sum of $T$-cyclic subspaces, $V_j = \oplus V_{j,k}$, where the minimal polynomials $\mu_{T|V_{j,k}} = \mu_j^{e_j^{(k)}}$ are called the elementary divisors of $T$, and for each $j$ we have

$$e_j = e_{j,1} \geq e_{j,2} \geq \ldots$$

The elementary divisors form a complete set of invariants of $T$ up to similarity, where $S, T$ are similar if $T = U S U^{-1}$ where $U$ is a invertible matrix over $F$.

For proof see for instance [10].

Lemma 4.16 Suppose $A$ is a integer matrix. Then the primary components of $A$ are rational invariant subspaces of $A$.

Proof: Take $F = \mathbb{Q}$ in Lemma 4.15 and let $T$ be the linear operator defined by $A$ acting on $\mathbb{Q}^d$ (where $d$ is the number of columns of $A$). Recall that since each $\mu_k$ is monic and irreducible over $\mathbb{Q}$, they actually have integer coefficients. Therefore each primary component

$$V_j = \text{null} \mu_j(T)^{e_j} = \text{col} \prod_{k \neq j} \mu_k(T)^{e_k}$$

is a rational subspace.
Corollary 4.17 Suppose $A$ is an integer matrix. If $W$ is one of the $A$-cyclic subspaces contained in a primary component $V$ of $A$, then $\text{rat} W \subset V$.

Let us see how this works for the $(2, q)$ step symmetric tilings introduced in Section 3.10. Recall that these are constructed by taking $W$ to an invariant subspace of a $d$-cycle $R$ (if $d$ odd), or a $2d$-cycle (if $d$ is even). The resulting tilings have steps invariant under the dihedral groups $D_d$ (if $d$ is odd) or $D_{2d}$ (if $d$ is even). (Here, $D_n$ is the dihedral group with $2n$ elements, the semi-direct product of the $n$ element cyclic group $C_n$ with the two element cyclic group $C_2$.) Indeed, $W$ is a two-dimensional $R$-cyclic subspace generated (say) by $p_i$. To relate this to the above theory, note that the characteristic polynomial of $R$ is $t^d - 1$ (if $d$ odd) or $t^d + 1$ (if $d$ even).

Recall that the minimal polynomial $\mu_T$ divides the characteristic polynomial $\chi_T$, and $\mu_T = \chi_T$ iff $T$ is similar to the companion matrix of $\chi_T$. (See for instance [49]). But it easily seen that $R$ is the companion matrix. Thus, if $d$ is odd, the minimal polynomial of $R$ is

$$
\mu_R(t) = t^d - 1 = \prod_{k|d} \varphi_k(t) \quad (4.20)
$$

where $\varphi_k$ is the $k$-th cyclotomic polynomial

$$
\varphi_k(t) = \prod (t - \omega_j)
$$

where the product is taken over the primitive $k$-th roots of unity $\omega_j$. The cyclotomic polynomials are monic irreducible and have integer coefficients; the degree of $\varphi_k$ is given by the Euler totient function $\phi(k)$. (See for instance [57].) If $d$ is even, the minimal polynomial of $R$ is

$$
\mu_R(t) = t^d + 1 = \prod_{k|2d, k|d} \varphi_k(t) \quad (4.21)
$$

For example, if $d = 5$, we have

$$
\mu_R(t) = t^5 - 1 = (t - 1)(t^4 + t^3 + t^2 + t + 1) = \varphi_1(t)\varphi_5(t)
$$
Here, \( \phi_1 \) gives the primary component

\[
V_1 = \text{nul}(R - I) = \text{col} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

This contains and in fact equals a \( R \)-cyclic subspace, the fixed line of \( R \). Note that \( V_1 \cap \mathbb{Z}^5 = V_1 \) and that \( V_1 \) is up to homothety a copy of \( \mathbb{Z} \). Likewise, \( \varphi_5 \) gives the primary component

\[
V_2 = \text{nul}(R^4 + R^3 + R^2 + R + I) = \text{col} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\]

This contains two \( R \)-cyclic subspaces: \( R \) effects a one-fifth turn on one of these and a two-fifths turn on the other. To construct \((2, 3)\) tilings with a step symmetric under the group \( D_{10} \) we can choose \( W \) to be either of these \( R \)-cyclic subspaces. and then \( \text{rat} \, W = V_2 \), in which case

\[
\dim \text{rat} \, W = \deg \varphi_5 = \phi(5) = 4
\]

Thus, \( T(W) \) is aperiodic but non-minimal, with orbit closures forming a family with

\[
\text{codim} \, \text{rat} \, W = 5 - \phi(5) = 1
\]

real parameter, corresponding the single period of \( T(W^\perp) \). Readers familiar with Lie algebras will recognize that up to homothety, \( \mathbb{Z}^5 \cap V_2 \) is the root lattice often designated by the symbol \( A_4 \); see [19].

Fig. 4.32 depicts the riser for this Sturmian system, which is a three dimensional polyhedron with rotational symmetry about the indicated axis. The orbit closure \( \text{rat} \, W \), being four dimensional and containing \( W \), must have a two dimensional intersection with \( W^\perp \). Indeed, it appears in this picture as a family of 2-flats orthogonal to the axis of rotational symmetry of the riser. We can regard the riser as the window comprising those \( x \in W^\perp \) such that \( T(x + W) \) contains a vertex at the origin (that is,
4.7. SUBSYSTEMS

Figure 4.32: Two orbit closures depicted in the window, for the (2, 3) Sturmian system \( S(W) \) where \( W \) is the invariant plane where the canonical five cycle effects a one fifth turn. Top: the Penrose tilings. Bottom: another orbit closure. The easiest way to interpret these pictures is to think of \( x + \text{rat } W \) as a four dimensional subtorus of \( T^d \) (which is identified with the unit cube), and project via \( Q \) into \( W^\perp \); then the torus appears as the window and the subtorus as the pentagonal sections.
these points represent the different translation orbits of $\mathcal{T}(W)$, and then the $x$ lying in the orbit closure rat $W$ are precisely the South Pole, the North Pole, and the four pentagons shown in the left hand picture in Fig. 4.32. This orbit closure is precisely the set of Penrose tilings having one vertex at the origin. Other orbit closures (intersected with the set of tilings having one vertex at the origin) correspond to distinct family of evenly spaces slices of the window; see the right hand picture of Fig. 4.32 for an example.

If $d = 4$, we have

$$\mu_R(t) = t^4 + 1$$

which is already irreducible over $\mathbb{Q}$. Thus, the two invariant subspaces in which $R$ effects a $1/8$ or $3/8$ turn (respectively) are irrational. To obtain a tiling space with steps symmetric under the cyclic group $D_8$, we may choose $W$ to be either of these invariant subspaces, and then both $\mathcal{T}(W)$ and $\mathcal{T}(W^\perp)$ are aperiodic minimal.

If $d = 6$, we have

$$\mu_R(t) = t^6 + 1 = (t^2 + 1)(t^4 - t^2 + 1) = \varphi_4(t)\varphi_{12}(t)$$

Here $\varphi_4$ gives the primary component

$$V_1 = \text{nul}(R^2 + I) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Here $\mathbb{Z}^4 \cap V_1$ (up to homothety) is just $\mathbb{Z}^2$, and $V_1$ contains, and in fact agrees with, the cyclic subspace in which $R$ effects a one third turn. Likewise $\varphi_{12}$ gives the primary component

$$V_2 = \text{nul}(R^4 - R^2 + I) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, $\mathbb{Z}^4 \cap V_2$ is, up to homothety, a Cartesian product of root lattices, namely $A_2 \times A_2$, and $V_1$ contains two cyclic subspaces, in which $R$ effects $1/12$ and $5/12$ turns.
respectively. In order to obtain tilings with steps symmetric under $D_{12}$, we can take $W$ to be either of these cyclic subspaces. Then $\text{rat } W = V_2$, and
\[ \dim \text{rat } W = \deg \varphi_{12} = \phi(12) = 4 \]
Thus, $\mathcal{T}(W)$ is aperiodic but nonminimal, with orbit closures forming a family with
\[ \text{codim} \text{rat } W = 6 - \phi(12) = 2 \]
real parameters, corresponding to the two independent periods of $\mathcal{T}(W^\perp)$.

See Table 4.1 for a summary of the rational canonical form of the first few cycles.

We can sum up this discussion in the following Proposition, first proven by Nizecki [101], using the multigrid formalism.

**Proposition 4.18** Consider the $(2, q)$ Sturmian tiling spaces $\mathcal{T}(W)$ which have steps symmetric under the dihedral group $D_{2d}$, where $d = 2 + q$. Then $\mathcal{T}(W)$ has a
\[ \begin{cases} d - \phi(d) & d \text{ odd} \\ d - \phi(2d) & d \text{ even} \end{cases} \]
parameter family of orbit closures.

In particular, if $d$ is a power of two, $\mathcal{T}(W)$ and its dual are both aperiodic minimal; if $p$ is an odd prime, $\mathcal{T}(W^\perp)$ has a single period and $\mathcal{T}(W)$ is aperiodic, with a one parameter family of orbit closures.

### 4.8 Frequencies of Tiles

**Definition 29** The coverage in $T(x+W)$ of the $J$-th prototile $T+J$, written $\text{cov } T_J$, is that fraction of the volume of $\mathbb{R}^p$ which lies in the interior of some copy of $T_J$.

This fraction is well defined because of the unique ergodicity of the $W$-flow on each orbit closure $\pi(w + \text{rat } W)$ in $T_d$, which lifts to a unique ergodic measure on the corresponding orbit closure of $\mathcal{T}(W)$. 
<table>
<thead>
<tr>
<th>$d$</th>
<th>Primary component $V$</th>
<th>Lattice $V \cap \mathbb{Z}^d$ (up to homothety)</th>
<th>Cyclic Subspaces $W_j$</th>
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<td>3</td>
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<td>$A_2$</td>
<td>1/3 turn</td>
</tr>
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</tr>
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<td>$\mathbb{Z}$</td>
<td>fixed line</td>
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<td>$A_4$</td>
<td>1/5, 2/5 turns</td>
</tr>
<tr>
<td>6</td>
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<td>$\mathbb{Z}^2$</td>
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</tr>
<tr>
<td></td>
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<td>1/12, 5/12 turns</td>
</tr>
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<tr>
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<td>$A_2$</td>
<td>1/3 turn</td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
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<td>$A_2$</td>
<td>1/3 turn</td>
</tr>
<tr>
<td></td>
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<td>$A_4$</td>
<td>1/5, 2/5 turns</td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
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<td>$A_{18}$</td>
<td>1/19, 2/19, 3/19... 9/19 turns</td>
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</tbody>
</table>

Table 4.1: Summary of the rational canonical forms of $d$-cycles (if $d$ odd) and $2d$-cycles (if $d$ even) for small $d$. Notice that each $\varphi_k$ is associated with a particular lattice; $\varphi_{15}$ is the first cyclotomic polynomial which is not associated with a root lattice or Cartesian product of same.
Lemma 4.19 For each $J$ with $|J| = p$, we have the relation
\[ \text{cov} T_J = \text{vol}_d C_J \] (4.22)

Proof: Because $O(W)$ is periodic, we know that
\[ \text{cov} C_J = \frac{\text{vol}_d C_J}{\sum_{|K| = p} \text{vol}_d C_K} = \text{vol}_d C_J \]
(where we used the fact that $\sum_{|K| = p} \text{vol}_d C_K = 1$), so the point is to show that
$\text{cov} T_J = \text{cov} C_J$, independently of which cut line $x + W$ we use. But, given some sufficiently small $\delta > 0$, all but $\epsilon$ of $R^p$ of $x + W$ will correspond to places where small balls of diameter $\delta$ fit entirely inside some cell $C_J$. Within this “good” portion of $R^p$, the claim is clearly true. But as $\delta$ approaches zero, so does $\epsilon$.

Theorem 4.20 For any pair of prototiles $T_J, T_K$ we have
\[ \frac{\text{frq} T_J}{\text{frq} T_K} = \frac{\text{vol}_p T_J}{\text{vol}_p T_K} \]
That is, the relative frequencies of the prototiles are given by their relative volumes.

Proof: First, observe that (by the uniform distribution) $\text{cov} T_J = c \text{frq} T_J \text{ vol}_p T_J$, for some constant $c > 0$ not depending on $J$. Then by Proposition 4.1 and Lemma 4.19,
\[
\left( \frac{\text{vol}_p T_J}{\text{vol}_p T_K} \right)^2 = \frac{\text{vol}_p T_J}{\text{vol}_p T_K} \frac{\text{vol}_p T_J}{\text{vol}_p T_K} \frac{\text{vol}_d C_J}{\text{vol}_d C_K} \frac{\text{cov} T_J}{\text{cov} T_K} = \frac{\text{frq} T_J}{\text{frq} T_K} \frac{\text{vol}_p T_J}{\text{vol}_p T_K}
\]
Cancelling a factor of $\frac{\text{vol}_p T_J}{\text{vol}_p T_K}$ now gives the desired result.
Figure 4.33: The prototiles of the symmetric (2, 3) tiling and its dual. Top left: thick rhombus $T_{1,2}$ (area 0.380423) Middle and bottom left: two views of the full rhombohedron $T^*_{3,4,5}$ (volume 0.380423; its faces consist of four thin rhombs and two thick ones). Top right: thin rhombus $T_{1,3}$ (area 0.235114) Middle and bottom right: two views of the squat rhombohedron $T^*_{2,4,5}$ (volume 0.235114; its faces consist of six thick rhombs). The relative frequencies of thick to thin rhombs in $S(W)$, and of full to squat rhombohedrons in $S(W^L)$, is $\tau : 1$, where $\tau$ is the Golden Mean.
4.9. VERTEX NEIGHBORHOODS

Thus, in any Sturmian tiling $T(x+W)$, copies of the $J$-th prototile $T_J$, where $|J| = p = \dim W$, occur with frequency equal (not just proportional) to its $p$-dimensional volume. Moreover, this frequency does not vary from tiling to tiling in $\mathcal{S}(W)$; this is remarkable [124] because the frequency of a given patch generally varies from one subsystem to the next within $\mathcal{S}(W)$.

4.9 Vertex Neighborhoods

The set of tiles sharing a given vertex in a tiling is called the vertex neighborhood of that vertex. In this section we study the problem of enumerating the possible vertex neighborhoods of a given Sturmian tiling. Our plan of attack is easily explained in the simplest case, the genus $(1, 1)$ tilings. Consider again the window (bold line segment) shown in Figure 3.9. The cut lines $y + W$ passing through this window, where $W$ is the line spanned by $(1 + \sqrt{5}, 2)$, correspond to the tilings of species $W$ which have one vertex at the origin of $\mathbb{R}$. Starting from the top of the window and moving down we see that the possible vertex neighborhoods at this vertex are $XY, XX, YX$ respectively (where $X$ denotes a long tile and $Y$ a short tile). It is easy to see that by the $\mathbb{Z}^2$ periodicity of the oblique tiling that the same statement holds for any other vertex in this tiling or any other of the same species. Indeed, apart from uninteresting degenerate cases, this statement holds for any Sturmian tiling of genus $(1, 1)$.

Consider next the case of Sturmian tilings of genus $(2, 1)$. For example, take first $W^\bot$ to be spanned by $(1, \tau, \pi)$; the reader should consult Figure 4.34 for the following discussion. At top left we have indicated the location of $W^\bot$ in the Grassmannian $G(1, 2)$; the meaning of the partition into trapezoids will be explained in a later section. At top right we have drawn part of a typical tiling of species $W$. The oblique tiling cells surrounding the window of the pencil consisting of the tilings with one vertex at the origin of $\mathbb{R}^2$ are shown at bottom left. Specifically, these cells are listed in the following table:
<table>
<thead>
<tr>
<th>Base Point</th>
<th>Cell Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$C_{(1,2)}$, $C_{(1,3)}$, $C_{(2,3)}$</td>
</tr>
<tr>
<td>$e_1$</td>
<td>$C_{(1,2)}$, $C_{(1,3)}$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>$C_{(1,2)}$, $C_{(2,3)}$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>$C_{(1,3)}$, $C_{(2,3)}$</td>
</tr>
<tr>
<td>$e_1 + e_2$</td>
<td>$C_{(1,2)}$</td>
</tr>
<tr>
<td>$e_1 + e_3$</td>
<td>$C_{(1,3)}$</td>
</tr>
<tr>
<td>$e_2 + e_3$</td>
<td>$C_{(2,3)}$</td>
</tr>
</tbody>
</table>

Starting at the top and moving the cut plane down through this configuration "enumerates" the seven vertex neighborhoods shown at bottom right. The reader should check that all of these neighborhoods, and no others, appear in the tiling shown at top right. Note too that the relative thicknesses of the "slabs" corresponding to the tilings having a given vertex neighborhood at the origin gives the relative frequencies with which these neighborhoods appear in any tiling in this tiling space.

As we vary $W$, the relative proportions of the oblique tiling cells change, which means that the catalog of vertex neighborhoods will undergo "bifurcations" at certain critical positions of $W$. In the case of tilings of genus $(2, 1)$, it turns out there is essentially only one other catalog, which is depicted in Figure 4.35. (This claim corresponds to the fact that there are essentially one two types of cell in the partition of $G(1, 1)$ into trapezoids in Figure 4.34: the edges of the trapezoids are the "critical positions" of $W$.)

However, the simplest truly nontrivial cases occur with genus $(2, 2)$ tilings. Now the window is a convex polygon; for an example see Figure 4.36. This polygon is nothing other than $Q(K)$ where $K$ is the unit cube and $Q$ is as usual the orthoprojection onto $W^\perp$. The edges of $K$ project to form a network of line segments which partition the window into smaller convex polygons. Each of these polygons corresponds to a different vertex neighborhood. We should think of each such convex polygon as the intersection of $q$-dimensional zonotopes, the projections onto $W^\perp$ of various faces of the cube $K$; each of these zonotopes contains all $y$ such that the tiling with cut plane $y + W$ contains a given tile in the vertex neighborhood of the the origin. The relative area of these polygons determines the relative frequency of the various vertex
Figure 4.34: The catalog of vertex neighborhoods of a tiling of genus (2,1). Top left: the location of the line $W^1$ in the Grassmannian manifold $G(1,2)$. Top right: a typical tiling of species $W$. Bottom left: the oblique tilings cells surrounding the window (a line segment) of the pencil of all tilings having one vertex at the origin of $\mathbb{R}^2$. Bottom right: schematic picture of the seven types of vertex neighborhoods.
Figure 4.35: The catalog of vertex neighborhoods of another tiling of genus (2, 1). Top left: the location of the line $W^\perp$ in the Grassmannian manifold $G(1, 2)$. Top right: a typical tiling of species $W$. Bottom left: the oblique tilings cells surrounding the window (a line segment) of the pencil of all tilings having one vertex at the origin of $\mathbb{R}^2$. Bottom right: schematic picture of the seven types of vertex neighborhoods. These are essentially the only two catalogs possible for genus (2, 1) tilings.

$$A = \begin{bmatrix} e & \pi \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$
4.9. **VERTEX NEIGHBORHOODS**

neighborhoods.

To see how this works, consider the irregular quadrilateral at the top of the window shown at upper left of Figure 4.36. This quadrilateral is the intersection of the projection of three two dimensional facets of \( K \), each corresponding to a different tile. Put another way, it is the intersection of the projection of \( W \perp \) of three oblique tiling cells, namely:

<table>
<thead>
<tr>
<th>Base Point</th>
<th>Cell Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_2 )</td>
<td>( C_{(2,3)} )</td>
</tr>
<tr>
<td>( e_4 )</td>
<td>( C_{(3,4)} )</td>
</tr>
<tr>
<td>( e_2 + e_4 )</td>
<td>( C_{(2,4)} )</td>
</tr>
</tbody>
</table>

See the top of Figure 4.37.

For a second example, consider the irregular pentagon at upper left of the window. This pentagon is the intersection of the projection of five two dimensional facets of \( K \), each corresponding to a different tile. Put another way, it is the intersection of the projection on \( W \perp \) of five oblique tiling cells, namely:

<table>
<thead>
<tr>
<th>Base Point</th>
<th>Cell Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( C_{(1,3)}, C_{(3,4)} )</td>
</tr>
<tr>
<td>( e_2 )</td>
<td>( C_{(2,4)} )</td>
</tr>
<tr>
<td>( e_4 )</td>
<td>( C_{(1,4)} )</td>
</tr>
<tr>
<td>( e_2 + e_4 )</td>
<td>( C_{(2,4)} )</td>
</tr>
</tbody>
</table>

See Figure 4.37.

In this example, \( W \) was chosen to be generic, i.e. to enjoy no special symmetries. However, the projection \( Q(K) \) is always centrally symmetric in \( W \perp \), and the catalog of vertex neighborhoods for a \( (2, q) \) tiling will always reflect this central symmetry; except for the small centrally located quadrilateral in the window depicted in Figure 4.36, each vertex neighborhood has a partner obtained by “central inversion”.

In many cases, of course, \( W \) has additional symmetries. In particular, the symmetric tiling space of genus \( (2, 2) \) has a window which is a regular octagon, and the internal configuration reflects this eight fold symmetry. This greatly simplifies the task of enumerating the vertex neighborhoods, since we can now say that “up to symmetry” (or “up to rotation in \( \mathbb{R}^2 \)”) there are only six neighborhoods, which we have
Figure 4.36: The vertex neighborhoods of the Sturmian tiling space, of genus \((2, 2)\), defined by taking \(W\) to be the column space of the indicated matrix \(A\). Top left: the window is a convex polygon in \(W^A\). Top right: a typical tiling in this space. Bottom left: schematic picture of the vertex neighborhood corresponding to the irregular quadrilateral at the top of the window; this quadrilateral is the intersection of three rhombic cells, and the corresponding vertex neighborhood has three tiles. Bottom center: schematic picture of the vertex neighborhood corresponding to the irregular pentagon at upper left in the the window; this pentagon is the intersection of five rhombic cells, and the corresponding vertex neighborhood has five tiles. Both of these regions within the window are relatively large, corresponding to the fact that these are relatively common vertex neighborhoods for this particular tiling space. Bottom right: the four \(p_j\).
Figure 4.37: Two vertex neighborhoods, depicted in $W^\perp$ (left) and in $W$ (right). Top left: three rhombic regions intersect to give an irregular quadrilateral cell in the window. Top right: the corresponding vertex neighborhood contains three rhombic; this picture shows the tiling with cutplane $-0.9q_3 - 0.25q_1 + W$, which is shown as a point at left. Bottom left: five rhombic regions intersect to give an irregular pentagonal cell in the window. Bottom right: the corresponding vertex neighborhood contains five rhombics; this picture shows the tiling with cutplane $-0.5(q_3 + q_1) + W$, which is shown as a point at left.
named respectively the peg (three tiles), ray, vamp (five tiles each), squid (six tiles), cactus (seven tiles), and flower (eight tiles). The symmetry of the window ensures that (for instance) the eight possible orientations of the peg occur with equal frequency. The reader will find it instructive to label each region in an enlarged picture of the window with a schematic picture of the corresponding vertex neighborhood. Notice that as we go clockwise around the octagon, the neighborhoods rotate clockwise “three times as fast”; the reason for this should be obvious from our discussion of the representation of $C_8$ in $\mathbb{R}^4$.

Incidentally, notice that in Figure 4.17, the “squid” in the lower right corner of each Dirichlet cell occurs nowhere else in its cell, showing that the stated periods are indeed fundamental. Similarly for the upside down “cactus” at bottom center of each Dirichlet cell in Figure 4.18.

An additional caveat is necessary when $T(W)$ is non-minimal; or equivalently, when $T(W^\perp)$ is semiperiodic. In that case, it makes sense to ask about the catalog of vertex neighborhoods (and their relative frequencies) in each orbit closure. Interestingly enough, these will in general vary as we vary the orbit closure.

We will explain this using the example of the symmetric $(2, 3)$ tilings. The window for this tiling space (see Figure 4.39) is the union of the intersections of the following oblique tiling cells with $W^\perp$: 
Figure 4.38: Top: the window for the symmetric (2, 2) tiling space (Ammann tiling space) is a regular octagon. Bottom: schematic pictures of the six vertex neighborhoods (up to rotation) of the Ammann tiling space. Clockwise from top left: the peg (think “square peg in round vertical hole”), the ray (he is swimming toward the lower right part of the page), the vamp (his arms are raised and he’s wearing a long cloak which brushes the ground), the squid (he’s got three arms visible and he’s swimming toward the lower left part of the page), flower, and cactus (its growing toward the upper right part of the page).
<table>
<thead>
<tr>
<th>Base Point</th>
<th>Cell Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$C_{1,2}$, $C_{1,3}$, $C_{1,4}$, $C_{1,5}$, $C_{2,3}$, $C_{2,4}$, $C_{2,5}$, $C_{3,4}$, $C_{3,5}$, $C_{4,5}$</td>
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<tr>
<td>$e_1$</td>
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<tr>
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<td>$C_{1,2}$, $C_{2,3}$, $C_{2,4}$, $C_{2,5}$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>$C_{1,3}$, $C_{2,3}$, $C_{3,4}$, $C_{3,5}$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>$C_{1,4}$, $C_{2,4}$, $C_{3,4}$, $C_{4,5}$</td>
</tr>
<tr>
<td>$e_5$</td>
<td>$C_{1,5}$, $C_{2,5}$, $C_{3,5}$, $C_{4,5}$</td>
</tr>
<tr>
<td>$e_1 + e_2$</td>
<td>$C_{1,2}$</td>
</tr>
<tr>
<td>$e_1 + e_3$</td>
<td>$C_{1,3}$</td>
</tr>
<tr>
<td>$e_1 + e_4$</td>
<td>$C_{1,4}$</td>
</tr>
<tr>
<td>$e_1 + e_5$</td>
<td>$C_{1,5}$</td>
</tr>
<tr>
<td>$e_2 + e_3$</td>
<td>$C_{2,3}$</td>
</tr>
<tr>
<td>$e_2 + e_4$</td>
<td>$C_{2,4}$</td>
</tr>
<tr>
<td>$e_2 + e_5$</td>
<td>$C_{2,5}$</td>
</tr>
<tr>
<td>$e_3 + e_4$</td>
<td>$C_{3,4}$</td>
</tr>
<tr>
<td>$e_3 + e_5$</td>
<td>$C_{3,5}$</td>
</tr>
<tr>
<td>$e_4 + e_5$</td>
<td>$C_{4,5}$</td>
</tr>
</tbody>
</table>

(These are of course precisely those cells whose projection to $W$ includes a vertex at the origin of $\mathbb{R}^2$, which is identified with $W$.)

Each vertex neighborhood of the origin corresponds to a convex polyhedral cell inside this window. For instance, the peg neighborhood shown at top in Figure 4.40 arises from the three oblique tiling cells

<table>
<thead>
<tr>
<th>Base Point</th>
<th>Cell Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1 + e_3$</td>
<td>$C_{1,3}$</td>
</tr>
<tr>
<td>$e_1 + e_4$</td>
<td>$C_{1,4}$</td>
</tr>
<tr>
<td>$e_3 + e_4$</td>
<td>$C_{2,4}$</td>
</tr>
</tbody>
</table>

whereas the jellyfish neighborhood shown at bottom in Figure 4.40 arises from the five oblique tiling cells

<table>
<thead>
<tr>
<th>Base Point</th>
<th>Cell Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_3$</td>
<td>$C_{1,3}$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>$C_{1,4}$</td>
</tr>
<tr>
<td>$e_2 + e_5$</td>
<td>$C_{2,5}$</td>
</tr>
<tr>
<td>$e_2 + e_3$</td>
<td>$C_{2,3}$</td>
</tr>
<tr>
<td>$e_4 + e_5$</td>
<td>$C_{4,5}$</td>
</tr>
</tbody>
</table>

Note that both lists are sublists of the list of the cells touching the window.

The relative volume of the corresponding polyhedral cells gives the relative frequency in $T(W)$ of these two neighborhoods. However, it also makes sense to ask for the catalog of vertex neighborhoods and their relative frequencies within a given
subsystem of $T(W)$, and this question in general has a quite different answer. In particular, although there are 17 possible vertex neighborhoods for Sturmian tilings of species $W$ (according to one way of counting them), only seven of these can occur in Penrose tilings!

To see how this works, recall that the window can also be thought of as the orthoprojection on $W^\perp$ of the unit five dimensional cube. Note that the $2^5 = 32$ vertices fall into planes containing respectively 1, 5, 10, 10, 5, 1 vertices. These form a point, a regular pentagon, two nested regular pentagons, and so forth. Recall further that the axis of rotational symmetry within $W^\perp$ of the window, namely $q_1 + q_2 + q_3 + q_4 + q_5$, gives a line parameterizing the subsystems of $T(W)$. In particular, within the window itself, apart from the top and bottom points there are only four pentagons of Penrose tilings having a vertex at the origin. (See Figure 4.39.) Now the point is that the polyhedral cell defining the peg neighborhood shown at the top of Figure 4.40 happens to intersect on of the four pentagons, so this neighborhood can occur in a Penrose tiling. On the other hand, the polyhedral cell defining the jellyfish neighborhood shown at the bottom of Figure 4.40 does not hit any of the pentagons, so this neighborhood cannot occur in a Penrose tiling. However, it can and does occur in other subsystems of $T(W)$; see for instance Figure 3.16.

Within the subsystem of Penrose tilings, the relative areas of the intersections of the relevant polyhedral cells with the various pentagons of course determines the relative frequencies within this subsystem of the allowed vertex neighborhoods, and similarly for the other subsystems.

4.10 Recurrence of Tiles

In this section we study the recurrence of tiles in Sturmian tiling of genus $(p, q)$. We begin by observing that tiles recur along ribbons (see Figure 3.10 for an example of this). Our claim is that (in the generic case) each $p$-dimensional tile $T$ is associated
Figure 4.39: Top: the window for the symmetric (2,3) tiling space (defined by taking $W$ to be the column space of the indicated matrix) is a semiregular polyhedron called a rhombic icosahedron. Bottom: the interior of the window is partitioned into convex polyhedral cells (tiles of $\mathcal{S}(W^\perp)$), each corresponding to one possible vertex neighborhood for tilings of species $W$. In particular, the cell $Q(C_{(2,5)}) = T_{(2,5)}$ is appears in the lower left of the window. Points in the pentagons are Penrose tilings (the set of Penrose tilings is the orbit closure rat $W$) with one vertex at the origin; that is, these points represent translation orbits of Penrose tilings.
Figure 4.40: Top: the intersection (left) of three oblique tiling cells gives the convex polyhedron corresponding to the peg neighborhood (right). This vertex neighborhood occurs in Penrose tilings; see Figure 3.15. Bottom: the intersection (left) of five oblique tiling cells gives the convex polyhedron corresponding to the jellyfish neighborhood (right). This vertex neighborhood does not occur in any Penrose tiling, but does occur in other subsystems of $T(W)$; see Figure 3.16.
with $2^2$ recurrence alternatives along each type of ribbon.

We will explain this by considering the example of the Ammann tiling space. Consider the square tile

$$T_{(2,4)} = \{-t_2p_2 - t_4p_4 : 0 \leq t_2, t_4 \leq 1\}$$

of $T(W)$, which is dual to the square tile

$$T^*_{(1,3)} = \{t_1q_2 + t_3q_3 : 0 \leq t_1, t_3 \leq 1\}$$

of $T^*(W^\perp)$. (as has already been observed, $T^*(W^\perp)$ is in this case another copy of $T(W)$, so this tiling space is self dual). The recurrence of this tile along a given ribbon is determined as follows. For certain integer displacements $z \in \mathbb{Z}^4$, the projection of $W^\perp$ of $z + C_{(2,4)}$ will overlap with that of $C_{(2,4)}$, i.e. with $T^*_{(1,3)}$. In particular, there are precisely four alternatives for the first recurrence along the ribbon orthogonal to $p_2$ as shown in Figure 4.41.

The relative frequencies of these alternatives of course agree with the relative frequencies of the four regions in the square at top right of Figure 4.41; we can readily compute these areas, with the following result:

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_2$</td>
<td>$\frac{3-2\sqrt{2}}{2} \approx 0.0857864$</td>
</tr>
<tr>
<td>$p_2 + p_1$</td>
<td>$-\frac{1+\sqrt{2}}{2} \approx 0.207107$</td>
</tr>
<tr>
<td>$p_2 + p_3$</td>
<td>$-\frac{1+\sqrt{2}}{2} \approx 0.207107$</td>
</tr>
<tr>
<td>$p_2 + p_1 + p_3$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

Next, consider the rhombic tile

$$T_{(1,2)} = \{-t_1p_1 - t_2p_2 : 0 \leq t_1, t_2 \leq 1\}$$

of $T(W)$, which is dual to the rhombic tile

$$T^*_{(3,4)} = \{t_3q_3 + t_4q_4 : 0 \leq t_3, t_4 \leq 1\}$$

of $T^*(W^\perp)$. Because of the algebraic symmetries of the $q_i$, it so happens that there are in fact only two recurrence alternatives along each ribbon; see Figure 4.42. We obtain the following frequencies of the two alternatives along the two ribbons:
Figure 4.41: Recurrence alternatives for the tile $T_{(2,4)}$ along the ribbon orthogonal to $p_2$ in the Ammann tiling space. Top: the four alternatives as they arise in $W^\perp$. Middle: the four alternatives as they appear in $W$. Bottom left: determination of the first order structure in $W^\perp$. Bottom right: the resulting diagram showing the first order constraints on recurrences along the ribbon orthogonal to $p_2$. A similar analysis holds for recurrence along the ribbon orthogonal to $p_4$. 
Figure 4.42: The recurrence of the rhombic tile $T_{(1,2)}$ in the Ammann tiling space, along the ribbon orthogonal to $p_1$ (left) and $p_2$ (right).

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^2 + p_3 + p_4$</td>
<td>$2 - \sqrt{2} \approx 0.585786$</td>
</tr>
<tr>
<td>$p^2 + 2p_3 + p_4$</td>
<td>$\sqrt{2} - 1 \approx 0.414214$</td>
</tr>
<tr>
<td>$p_1 - p_3 - p_4$</td>
<td>$2 - \sqrt{2} \approx 0.585786$</td>
</tr>
<tr>
<td>$p_1 - 2p_3 - p_4$</td>
<td>$\sqrt{2} - 1 \approx 0.414214$</td>
</tr>
</tbody>
</table>

In the same way, we may determine the zeroth order recurrence structure of tiles in any Sturmian tiling. The reader may wish to check our work against the picture of typical Ammann tiling shown in Figure 3.18.

In the same way, we can determine the first order structure as shown at the bottom of Figure 4.41. However, this quickly grows tedious, so we will not pursue this example any further.

In the special case of genus (1,1) tilings, our result says that there are precisely two recurrence alternatives for any prototile; indeed, we will see later that there are precisely two recurrence alternatives for any protopatch whatsoever, and we will see how these may be enumerated with great convenience.
Chapter 5

Compositions of Oblique Tilings

We saw in Section 2.4 that integer coefficient matrices take $\mathbb{Z}^d$ into itself and therefore define toral endomorphisms. In particular, symmetric matrices, i.e. matrices $L$ such that $L' = L$, always have real and mutually orthogonal eigenspaces and this suggests choosing $W$ to be spanned by such eigenspaces. It is natural to expect that $L$ will then act "nicely" on $\mathcal{O}(W)$ and therefore on the Sturmian tiling space $\mathcal{T}(W)$, and this is in fact the case. As usual the simplest case is the tiling spaces of genus $(1,1)$ and we begin by studying this case in some detail. In the two succeeding sections, we offer a sketchy account of some of ways in which the reasonably satisfactory theory in $\mathbb{R}^2$ breaks down in $\mathbb{R}^d$.

5.1 Compositions of Oblique Tilings in $\mathbb{R}^2$

Consider the symmetric integer matrix

$$L = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Note $\det L = -1$, so $L$ defines a toral automorphism. The eigenspaces of $L$ are easily determined; there is of course an expanding eigenspace

$$W = \text{col} \begin{bmatrix} 1 + \sqrt{5} \\ 2 \end{bmatrix}$$

and a contracting eigenspace $W^\perp$ (the two eigenspaces are orthogonal because $L$ is symmetric). Applying $L$ to $\mathcal{O}(W)$, we obtain the picture shown in Figure 5.1; we see that each large square is expanded and contracted so as to give a long rectangle which contains precisely one small square and part of a large square, whereas the small squares are expanded and contracted so as to give short rectangles which cover precisely the part of the large squares left uncovered by the long rectangles. This
means that any cut line $y + W$ cuts through both the original and the new oblique tilings, and that the new tiling can be obtained by composition from the original tiling using the substitution rules $X \leftarrow XY$, $Y \leftarrow X$. The fact that this substitution is applied uniformly means that the composition defines a continuous map $\mathcal{T}(W) \to \mathcal{T}(W)$; the fact that it is invertible means that we have a new $\mathbb{Z}$ action on $\mathcal{T}(W)$.

For two more examples see Figures 5.2 and 5.3.

**Theorem 5.1** Let $W$ be the expanding eigenspace of a symmetric two by two integer matrix $L$. Then the multicell of $\mathcal{O}(W)$ gives a 2 element Markov partition for the endomorphism induced on $T^2$ by $L$.

This is immediate from the proof of Theorem 8.4 in [1], upon comparing Fig. 3.1 with Fig. 29 of [1]. (Note that Adler's figure depicts the situation for a non-symmetric matrix; in our case the matrix is symmetric and the eigenspaces are orthogonal.)

Let us consider for a moment an integer matrix whose determinant is not one (see Figure 5.4). In this case, we will in general have at least one "internal vertex" and consequently the composition will be discontinuous (in Figure 5.4, the choice between $Y \leftarrow YXXY$ and $Y \leftarrow XYXY$ cannot be made by looking only within a bounded neighborhood).

It is easy to enumerate the composition matrices with "small" non-negative entries. Here and in the sequel it will be convenient to treat the cases $\det L = 1$ and $\det L = -1$ separately. Every symmetric matrix of determinant one may be written

$$L = \begin{bmatrix} a & b \\ b & \frac{b^2 + 1}{a} \end{bmatrix}$$

where $b^2 = -1 \mod a$ and where $a > 0$. In the case $a = 1$ we see that any $b$ gives a composition matrix. In the case $a = 2$ any odd $b$ works. However, no such matrix can have $a = 3$ because no choice of $b$ satisfies $b^2 = -1 \mod 3$. Similarly for $a = 4$. For $a = 5$ we find that any value of $b = 3 \mod 5$ works. And so forth. A similar
5.1. COMPOSITIONS OF OBLIQUE TILINGS IN $\mathbb{R}^2$

$L = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad W = \text{col} \begin{bmatrix} 1 + \sqrt{5} \\ 2 \end{bmatrix}$

$X \leftarrow XY, \quad Y \leftarrow X$

Figure 5.1: The Fibonacci composition of $\mathcal{O}(W)$; the corresponding toral automorphism is defined by the composition matrix $L$. The corresponding substitution is also indicated, where $X$ stands for the long tile and $Y$ for the short one.
\[ L = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}, \quad W = \col \left[ 3 + \frac{\sqrt{13}}{2} \right] \]

\[ X \leftarrow X.X.X.Y, \quad Y \leftarrow X \]

Figure 5.2: The composition of \( O(W) \); the corresponding toral automorphism is defined by the composition matrix \( L \). The corresponding substitution is also indicated; note that the first column of \( L \) shows that three long tiles (\( X \)) and one short tile (\( Y \)) compose to one large long tile, while the second column shows that one long tile composes to one large short tile.
5.1. COMPOSITIONS OF OBLIQUE TILINGS IN $\mathbb{R}^2$

\[ L = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \quad W = \text{col} \left[ \begin{array}{c} 1 + \sqrt{2} \\ 1 \end{array} \right] \]

\[ X \leftarrow XXYXXYYX, \quad Y \leftarrow XXY \]

Figure 5.3: The composition of $\mathcal{O}(W)$. 
$L = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 + \sqrt{17} \\ 4 \end{bmatrix}$

$X \leftarrow XYXYX, \quad Y \leftarrow \begin{cases} YXXY \\ XYXY \end{cases}$

Figure 5.4: A toral endomorphism (onto but not one-one) gives a discontinuous composition of the corresponding Sturmian tilings.
analysis may be carried out in the case $\det L = -1$. We shall return a bit later to the question of enumerating composition matrices for $(1, 1)$ oblique tilings.

An obvious question is: which $W$ permit compositions arising from a toral automorphism?

**Lemma 5.2** Suppose $O(W)$ is a $(1, 1)$ oblique tiling permitting a nontrivial composition arising from a toral automorphism. Then the slope $v$ of $W$ satisfies a quadratic equation of form

$$v^2 + qv \pm 1 = 0 \quad (5.1)$$

where $q \in \mathbb{Q}$.

**Proof:** Once again taking first the case $\det L = 1$, write

$$L = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where $b^2 = -1 \mod a$, $a \neq 0$, and $c = (b^2 + 1)/a$. The characteristic equation of $L$ is now

$$\alpha^2 - (a + c)\alpha + 1 = 0$$

from which we find that the eigenvectors of $L$ are given by $(1, v)$ and $(-v, 1)$ where

$$v^2 + \frac{a - c}{b}v - 1 = 0 \quad (5.2)$$

Similarly if $\det L = -1$, the characteristic equation is

$$\alpha^2 - (a + c)\alpha - 1 = 0$$

and the eigenvectors are $(1, v)$ and $(-v, 1)$ where $v$ must again satisfy (5.2). \[\blacksquare\]

Thus, the only $W$ possessing oblique tiling compositions arising in the manner considered here are lines with slopes which are quadratic integers of a rather special form.
Definition 30 Let $\mathcal{O}(W)$ be an oblique tiling. The group of symmetric integer matrices having $W$ as an expanding eigenspace and $W^\perp$ as a contracting eigenspace is called the isotropic composition group of $\mathcal{O}(W)$.

Theorem 5.3 Let $\mathcal{O}(W)$ be a $(1,1)$ oblique tiling. Then the composition group is either trivial or isomorphic to $\mathbb{Z}$, according to whether the slope $v$ of $W$ satisfies a quadratic equation of form

$$v^2 + qv \pm 1 = 0$$

where $q \in \mathbb{Q}$.

To prove this, as usual we consider first the case that $\det L = 1$. Write $L$ as above and note that

$$a(-c) = -(1 + b^2), \ a + (-c) = br$$

where $r$ is some rational number. Thus, $a, -c$ are the roots of the equation

$$t^2 - rbt - (1 + b^2) = 0$$

Write $r = M/N$, where $M, N$ are relatively prime. Let $Y$ be the integer such that $b = YN, br = YM$. Then since $a, -c$ are integers

$$(rb)^2 + 4(1 + b^2) = X^2$$

is an integer. We conclude that there are integers $X, Y$ such that

$$Y^2M^2 + 4(1 + Y^2N^2) = X^2$$

or

$$X^2 - (M^2 + 4N^2)Y^2 = 4 \quad (5.3)$$

which the reader may recognize as a minor variant of a certain equation (see [31][55]).

Arguing similarly in the case $\det = -1$ we are led to the equation

$$X^2 - (M^2 + 4N^2)Y^2 = -4 \quad (5.4)$$
5.1. COMPOSITIONS OF OBLIQUE TILINGS IN $\mathbb{R}^2$

In either case, any solution $X, Y$ leads to a composition matrix $L$ having the desired eigenspace, to wit, $L = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where

$$a = \frac{X + YM}{2}, \quad b = YN, \quad c = \frac{X - YM}{2}$$

It is well known [55] that we can enumerate all solutions of Pell’s equation $X^2 + KY^2 = 1$ (for a specific integer $K$) using the continued fraction expansion of $\sqrt{K}$; this idea is easily modified to enumerate all the composition matrices for a given $W$. The method is best explained by example.

Consider $W$ spanned by $(v, 1)$ where $v$ satisfies $v^2 + 5/7v - 1 = 0$. Now we have $M = 5, N = 7$, so $M^2 + 4N^2 = 221$ and from any solution $X, Y$ to (5.3) we obtain the composition matrix $L = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where

$$a = \frac{X + 5Y}{2}, \quad b = 7Y, \quad c = \frac{X - 5Y}{2}$$

Now the continued fraction expansion of $\sqrt{221}$ is

$$[14, 1, 6, 2, 6, 1, 28, 1, 6, 2, 6, 1, 28]$$

which has period six (the continued fraction expansion of any quadratic irrational with can arise in this context is guaranteed to be periodic; see [55]). Writing the $n$-th convergent as $h_n/k_n$ we select those convergents such that $h_n^2 - 221k_n^2$ is 4 or 1. In the first case we may take $X_n = h_n, Y_n = k_n$; in the second we may take $X_n = 2h_n, Y_n = 2k_n$. In tabular form, the computations look like this:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$</th>
<th>$h_n$</th>
<th>$k_n$</th>
<th>$h_n^2 - 221k_n^2$</th>
<th>$X_n$</th>
<th>$Y_n$</th>
<th>$L_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>15</td>
<td>1</td>
<td>4</td>
<td>15</td>
<td>1</td>
<td>$[10/7]$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>104</td>
<td>7</td>
<td>-13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>223</td>
<td>15</td>
<td>4</td>
<td>223</td>
<td>15</td>
<td>$[150/105]$</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>1442</td>
<td>97</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1665</td>
<td>112</td>
<td>1</td>
<td>3330</td>
<td>224</td>
<td>$[1568/1105]$</td>
</tr>
<tr>
<td>6</td>
<td>28</td>
<td>48062</td>
<td>3233</td>
<td>-25</td>
<td>49727</td>
<td>3345</td>
<td>$[3345/1850]$</td>
</tr>
</tbody>
</table>

However, in practice a much faster way to obtain the solutions is to note that

$$\frac{X_n + DY_n}{2} = \left(\frac{X_1 + DY_1}{2}\right)^n$$
where \( D = M^2 + 4N^2 \), from which we can obtain the recurrence

\[
(X_n, Y_n) = X_1(X_{n-1}, Y_{n-1}) + (X_{n-2}, Y_{n-2})
\]

(5.5)

where \((X_1, Y_1)\) is the smallest positive solution (corresponding to the generator of the composition group).

A related approach to solving \( X^2 - DY^2 = E \) (where \( D, E \in \mathbb{Z} \)) is the “cyclic” method explained in Section 1.9 and 7.4 of [31].

This concludes the proof of Theorem 5.3.

Let us return to the question of enumerating the composition matrices. As we have said, a (continuous) composition matrix for a \((1, 1)\) oblique tiling is a two by two symmetric invertible integer matrix. But all the determinant one integer matrices occur in the Farey tree (see [66]), so the proper composition matrices are exactly the symmetric matrices appearing in this tree. Moreover, if \( L \) is such a matrix, corresponding to some finite downward path in the Farey tree, i.e. some word in the generators, then \( L^2 \) is obtained by concatenating two copies of this word, i.e. following the original downward path by a copy of itself. So the generators of groups of proper compositions are the symmetric matrices such that the Farey tree contains no strictly higher symmetric matrices (except for the identity). The “square roots” of these generators are improper generators of the full composition group associated with a given slope one dimensional subspace \( W \).

We close this section with one final remark. We have of course used euclidean inner products throughout this dissertation. However, the author has observed that one can construct oblique tilings in \textit{Minkowski plane} for a given one dimensional subspace \( W \) of slope \( v = \tanh \alpha \) (rather than \( v = \tan \alpha \)); instead of two squares of lengths \( \cos \alpha, \sin \alpha \) respectively, the protocells are two rectangles, the large one of length \( \cosh \alpha \) and height \( \cosh \alpha - \sinh \alpha = \exp(-\theta) \), and the small one of length \( \cosh \theta \) and height \( \sinh \theta \). These oblique tilings possess a theory of compositions closely analogous to the euclidean theory developed in this section.
5.2 Compositions of Oblique Tilings in $\mathbb{R}^3$

Despite the sketchy nature of the preceding section, it should be clear that the theory of $(1, 1)$ compositions is complete and satisfactory. Unfortunately, as soon as we look at the next most complicated case, the $(2, 1)$ oblique tilings (recall these are pretty much the same as the $(1, 2)$ oblique tilings!), the theory becomes substantially more involved. The first complication we must grapple with is that, unlike the case of $(1, 1)$ tilings, where because the characteristic looks like $t^2 - nt \pm 1$, it was clear that there is exactly one expanding and one contracting eigenspace, in higher dimensions there can very well be several distinct real expanding eigenspaces, as well as several distinct contracting eigenspaces. Thus, in general, there is no reason to expect that a given symmetric integer matrix $L$ will permit us to choose a pair of subspaces $W, W^\perp$ such that $L$ is expanding by $\lambda$ in $W$ and contracting by $1/\lambda$ in $W^\perp$. In other words, we must now distinguish between isotropic compositions and nonisotropic compositions, which might act differently in different directions. In the latter case, the composed tiling will clearly not have prototiles which are homothetic to (expanded or contracted copies of) the original prototiles, but they will be affine images of them, so nonisotropic compositions might also be called affine compositions to distinguish them from self similar compositions.

Indeed, in the case of $(2, 1)$ or $(1, 2)$ oblique tilings, it is not hard to see that we cannot have any nontrivial invertible isotropic compositions!

**Theorem 5.4** There are no nontrivial invertible isotropic compositions of oblique tilings in $\mathbb{R}^3$.

**Proof:** We will consider the case of proper isotropic compositions; improper compositions may be treated similarly.

Suppose $L$ is a proper isotropic composition. Let $W$ be a two dimensional eigenspace with eigenvalue $\alpha > 0$. Then $W^\perp$ must be a one dimensional eigenspace with eigenvalue $1/\alpha^2$, since the product of the eigenvalues must be one. Thus, the characteristic
polynomial must have the form
\[
\chi_L(t) = (t - \alpha)^2 (t - \alpha^{-2})
= t^3 - (2\alpha + 1/\alpha^2)t^2 + (\alpha^2 + 2/\alpha)t - 1
= t^3 - Mt^2 + Nt - 1
\]

where
\[
M = 2\alpha + \frac{1}{\alpha^2}
\quad (5.7)
\]
\[
N = \alpha^2 + \frac{2}{\alpha}
\quad (5.8)
\]

are integers. Multiplying through (5.7) by \(\alpha^2\) and (5.8) by \(\alpha\) and transposing terms gives the following two cubic polynomial identities satisfied by \(\alpha\):
\[
2\alpha^3 - M\alpha^2 + 1 = 0
\quad (5.9)
\]
\[
\alpha^3 - N\alpha + 2 = 0
\quad (5.10)
\]

On the one hand, subtracting (5.9) from twice (5.10) gives
\[
M\alpha^2 - 2N\alpha + 3 = 0
\quad (5.11)
\]

On the other hand, subtracting (5.10) from twice (5.9) gives
\[
3\alpha^3 - 2M\alpha^2 + N\alpha = 0
\]

or (factoring out the common factor of \(\alpha\) on the left hand side)
\[
3\alpha^2 - 2M\alpha + N = 0
\quad (5.12)
\]

Subtracting three times (5.12) from (5.11) gives
\[
(6N - 2M^2)\alpha + MN - 9 = 0
\quad (5.13)
\]
5.2. COMPOSITIONS OF OBLIQUE TILINGS IN $\mathbb{R}^3$

If the coefficient of $\alpha$ in (5.13) vanishes, then $3N = M^2$ and also $MN = 9$, so $27 = 3MN = M^3$ or $M = 3$, whence $N = 3$ and, plugging these values back into (5.6), we see that $\alpha = 1$. Otherwise, (5.13) implies that $\alpha$ is rational:

$$\alpha = \frac{9 - MN}{6N - 2M^2}$$

But then, because $\alpha$ is both rational and an algebraic integer, it must be an integer (see Problem 11 in Section 5.1 of [57]). But now subtracting (5.10) from (5.9) gives

$$\alpha^3 - M\alpha^2 + N\alpha - 1 = 0$$

or $\alpha(\alpha^2 - M\alpha + N) = 1$, which shows that $\alpha$ divides one; since $\alpha > 0$ and is an integer, we conclude again that $\alpha = 1$. \hfill \Box

(The author's first two attempts to prove this apparently straightforward result were unsuccessful, and he is grateful to Mr. Erik Scott, Prof. Robert Phelps, and Prof. Ralph Greenberg for various observations which have been incorporated into the above argument.)

For an example of a nonisotropic composition in $\mathbb{R}^3$, see Figures 5.5 and 5.6. Note that $L$ takes the oblique tiling to a new oblique tiling whose periodicity is no longer $\mathbb{Z}^3$, but rather is given by the columns of $L$. Cut planes through this new oblique tiling give an "affine distortion" of a unique tiling in $\mathcal{T}(W)$, so we can consider $L$ to define an operator on $\mathcal{T}(W)$.

A more subtle phenomenon first encountered in three dimensional oblique tilings is that invertibility of $L$ is no longer sufficient to guarantee the continuity of the composition.

Problem 5.5 Find criteria for the continuity of the induced composition on $S(W)$.

Of course, even if the composition is not continuous, one can still investigate ergodic theory phenomena.
Figure 5.5: A genus (2, 1) oblique tiling with a composition having characteristic \( t^3 - t^2 - 2t + 1 \). We choose \( W \) to be the direct sum of the eigenspaces for the two expanding eigenvalues, which are approximately 1.80194 and -1.24698; this implies that \( W^\perp \) is the eigenspace for the contracting eigenvalue, which is approximately 0.445042. Top: the multicell before (left) and after (right) applying \( L \). Bottom: a small portion of the oblique tiling before (left) and after (right) applying \( L \).
5.2. *COMPOSITIONS OF OBLIQUE TILINGS IN* $\mathbb{R}^3$

\[
L = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & -1 \\
1 & -1 & 0
\end{bmatrix}
\]

Figure 5.6: A typical tiling from the space shown in Figure 5.5 before (faint lines) and after (bold lines) composition.
Figure 5.7: A typical tiling from a genus (2, 1) tiling with a composition having characteristic $t^3 - 5t + 1$, before (faint lines) and after (bold lines) composition. $W$ is the direct sum of the eigenspaces for the two expanding eigenvalues, which are approximately $-2.33006$ and $2.12842$; this implies that $W^\perp$ is the eigenspace for the contracting eigenvalue, which is approximately $0.20164$.
5.2. COMPOSITIONS OF OBLIQUE TILINGS IN $\mathbb{R}^3$

$L = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Figure 5.8: A typical tiling from a genus $(2,1)$ tiling with a composition having characteristic $t^3 - 5t^2 + 6t - 1$ before (faint lines) and after (bold lines) composition. $W$ is the direct sum of the eigenspaces for the two expanding eigenvalues, which are approximately $3.24698$ and $1.55496$; this implies that $W^\perp$ is the eigenspace for the contracting eigenvalue, which is approximately $0.198062$. 
Problem 5.6 If $L$ induces a nonisotropic composition, formulate a way to associate the affine image with a particular tiling in the original space, and thus to define a measure-preserving map on $S(W)$.

One phenomenon of interest is that the composition might map orbit closures onto new orbit closures.

Problem 5.7 Investigate the ergodic theory of the induced composition of a non-isotropic oblique tiling composition.

As a start, compute the entropy of the induced composition. Another point of entry: look for periodic points; in this connection see the connections with classical number theory found by Vivaldi [136].

We should mention one more new phenomenon: the multicell no longer gives a Markov partition of the underlying toral automorphism. Indeed, Bowen [79] proved that the tiles of a Markov partition must be fractals in the case of $T^d$, where $d > 2$.

Problem 5.8 Show that iterating the composition and rescaling yields in the limit the fractal "tiles" of a Markov partition for the torus.

5.3 Compositions of Higher Dimensional Oblique Tilings

In this section we briefly explore some further ramifications which can appear in still higher dimensions.

We begin with a discussion of invertible isotropic compositions of $(2, 2)$ oblique tilings. In this case, the characteristic polynomial of $L$ evidently has the form

$$\chi_L(t) = (t \pm \alpha) (t \pm \alpha) \left( t \pm \frac{1}{\alpha} \right) \left( t \pm \frac{1}{\alpha} \right)$$

(5.14)

where we assume $\alpha > 1$ is the absolute value of the expanding eigenvalue. (We can rule out $(t \pm \alpha)^2 (t \pm \frac{1}{\alpha}) (t \pm 1)$ by a modification of the argument in Theorem 5.4.) As one might expect, there are rather severe restrictions on the algebraic nature of the eigenvalues.
Theorem 5.9 Let $L$ be an invertible isotropic proper nontrivial composition of a $(2,2)$ oblique tiling, and let $\alpha > 1$ be the absolute value of the expanding eigenvalue. Put

$$M = \alpha + \frac{1}{\alpha}, \quad N = \alpha - \frac{1}{\alpha}$$

Then the characteristic of $L$ has one of the following forms:

1. $(t^2 \pm Mt + 1)^2$, where $M \in \mathbb{Z}$,

2. $(t^2 \pm Nt - 1)^2$, where $N \in \mathbb{Z}$,

3. $(t^2 - Mt + 1)(t^2 + Mt + 1)$, where $M \in \mathbb{Z}$,

4. $t^4 + (M^2 - 2)t^2 + 1$, where $M^2 \in \mathbb{Z}$.

In the first two cases, $L$ effects reflections in neither $W$ nor $W^\perp$. (In the first case $L$ effects inversions in neither subspace or in both of them; in the second case it effects an inversion in precisely one of them.) In the last two cases, $L$ effects reflections in both subspaces, and thus does not induce a proper composition on $S(W)$.

In the first three cases, $\alpha$ is a quadratic irrational, and in the last case, it is the root of an irreducible quartic.

Proof: We proceed by case by case analysis of each possible choice of signs in (5.14), such that the product of all the eigenvalues is unity. For instance, if all signs are chosen to be negative, we have a characteristic of form

$$(t - \alpha)(t - \alpha) \left( t - \frac{1}{\alpha} \right) \left( t - \frac{1}{\alpha} \right) = \left( (t - \alpha) \left( t - \frac{1}{\alpha} \right) \right)^2 = (t^2 - Mt + 1)^2 = t^4 - 2Mt^3 + (M^2 + 2)t^2 - 2Mt + 1$$

where $2M, M^2 + 2$ are integers, so that $M$ is a half integer whose square is an integer; that is, $M$ is an integer and $\alpha$ is a quadratic irrational. Moreover, $L$ effects no
inversions or reflections in either $W$ or $W^\perp$. The analysis of the sign choices $;++--$, $;--++$, $;+++$ is similar, except that $L$ effects an inversion in one or both of $W,W^\perp$. This disposes of the first two alternatives listed in the statement of the theorem.

The remaining choices of signs are $++-+$, $+-+-$, $+-+-+-+-$, and $+-+-+$. These are obviously all equivalent, and in each case $L$ effects a reflection in both $W$ and $W^\perp$. The first of these yields a characteristic of form

$$(t - \alpha)(t + \alpha) \left( t - \frac{1}{\alpha} \right) \left( t + \frac{1}{\alpha} \right) = (t - \alpha) \left( t - \frac{1}{\alpha} \right) \left( t + \alpha \right) \left( t + \frac{1}{\alpha} \right)$$

$$= (t^2 - Mt + 1)(t^2 + Mt + 1)$$

$$= t^4 + (M^2 - 2)t^2 + 1$$

If $M$ is an integer, we have the third alternative listed in the statement; if not, $M^2$ is an integer and we have the fourth alternative.

A similar statement can be proven for improper invertible isotropic compositions of $(2,2)$ oblique tilings; we leave this analysis to the reader and turn our attention to illustrating some of the possibilities listed in Theorem 5.9.

If we search for small integer matrices with characteristic

$$\chi(t) = (t^2 - 3t + 1)(t^2 + 3t + 1)$$

(this illustrates the third alternative listed in Theorem 5.9), we find that one such matrix is

$$L = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

We note that this matrix has primary components

$$\text{nul}(L^2 - 3L + I) = \text{col} \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

and

$$\text{nul}(L^2 + 3L + I) = \text{col} \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$
By construction, \( L \) has eigenvalues
\[
\frac{3 + \sqrt{5}}{2}, \quad \frac{-(3 + \sqrt{5})}{2}, \quad \frac{3 - \sqrt{5}}{2}, \quad \frac{-(3 - \sqrt{5})}{2}
\]
The eigenvector for \( \frac{3 + \sqrt{5}}{2} \) is
\[
(3 + \sqrt{5}) \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}
\]
(where we have written in the given form to show that it is in the primary component \( \text{nul}(L^2 + 3L + I) \) and that for \( \frac{-(3 + \sqrt{5})}{2} \) is
\[
(3 + \sqrt{5}) \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}
\]
(which is in the primary component \( \text{nul} L^2 - 3L + I \)). We choose \( W \) to be spanned by these two eigenvectors, so that \( L \) effects a reflection across a one dimensional subspace of \( W \) and a dilation by \( \frac{3 + \sqrt{5}}{2} \), as claimed; see Fig. 5.9. Computing the ribbon vectors, we easily find a more convenient expression for \( W \), namely
\[
W = \text{col} \begin{bmatrix} 5 + 3\sqrt{5} & 0 \\ 4 & 2 \\ 2 & -4 \\ 0 & 5 + 3\sqrt{5} \end{bmatrix}
\]
This yields an aperiodic and minimal Sturmian system \( S(W) \).

If we search for small integer matrices with the characteristic
\[
\chi(t) = (t^2 - 2t - 1)^2
\]
(corresponding to the second alternative listed in Theorem 5.9), we find that one such matrix is
\[
L = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}
\]
This has minimal polynomial \( \mu_L(t) = t^2 - 2t - 1 \) and trivial primary decomposition. Choosing \( W \) to be the eigenspace of the expanding eigenvalue \( 1 + \sqrt{2} \), we recover the Ammann tilings \( S(W) \). However, the induced composition, illustrated in Fig. 5.10, is evidently not the same as the composition discussed in [53][124].
Figure 5.9: A \((2, 2)\) Sturmian system \(S(W)\) with an improper inflation \(L\) with characteristic \(x_L(t) = (t^2 - 3t + 1)(t^2 + 3t + 1)\). The effect of \(L\) on \(W\) is to reflect across a one dimensional subspace and to expand by \(1 + \tau \approx 2.61803\), where \(\tau\) is the golden ratio.
Figure 5.10: This oblique tiling composition $L$, with characteristic $\chi_L(t) = (t^2 - 2t - 1)^2$, induces a composition on the space of Ammann octagonal tilings.
Problem 5.10 Determine whether the "standard" inflation for the Ammann tilings discussed in [53][124] can arise from a composition of the oblique tiling. Determine whether the induced composition illustrated in Fig. 5.10 is continuous and determine its relationship with the "standard" inflation.

If we search for small integer matrices with the irreducible characteristic

\[ \chi(t) = t^4 - 4t^2 + 1 \]

(corresponding to the fourth alternative listed in Theorem 5.9), we find that one such matrix is

\[ L = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \]

The eigenvalues are

\[ -\sqrt{2} - \sqrt{3}, \sqrt{2} - \sqrt{3}, -\sqrt{2} + \sqrt{3}, \sqrt{2} + \sqrt{3} \]

Here, note that \( \sqrt{2} + \sqrt{3} = \frac{1+\sqrt{3}}{\sqrt{2}} \) and so forth. The eigenvectors for the last two eigenvalues are

\[ \begin{bmatrix} (1 + \sqrt{2})(1 + \sqrt{3}) \\ 2 + \sqrt{2} \\ \sqrt{2} \\ 1 + \sqrt{3} \end{bmatrix}, \begin{bmatrix} (1 - \sqrt{2})(1 + \sqrt{3}) \\ 2 - \sqrt{2} \\ -\sqrt{2} \\ 1 + \sqrt{3} \end{bmatrix} \]

If we choose \( W \) to be the span of these eigenvectors, \( L \) will induce an improper but isotropic composition on \( S(W) \). Taking ribbon vectors, we easily find the simplified expression

\[ W = \text{col} \begin{bmatrix} 1 + \sqrt{3} & 0 \\ 1 & 1 \\ 0 & 1 + \sqrt{3} \end{bmatrix} \]

See Fig. 5.11.

We next turn our attention to compositions of (2, 3) oblique tilings (again, this is pretty much the same thing as compositions of (3, 2) oblique tilings). Begin by observing that any isotropic invertible composition must have characteristic polynomial

\[ (t \pm \alpha)(t \pm \alpha)(t \pm \beta)(t \pm \beta)(t \pm \beta) \]  \hspace{1cm} (5.15)
5.3. COMPOSITIONS OF HIGHER DIMENSIONAL OBLIQUE TILINGS

\[
W = \begin{bmatrix} 1 + \sqrt{3} & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 1 + \sqrt{3} \end{bmatrix} \\
L = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}
\]

Figure 5.11: A (2, 2) Sturmian system $S(W)$ with an improper inflation $L$ with irreducible characteristic $\chi_L(t) = t^4 - 4t^2 + 1$. The effect of $L$ on $W$ is to reflect across a one dimensional subspace and to expand by $\sqrt{2 + \sqrt{3}} = (1 + \sqrt{3})/\sqrt{2} \approx 1.93185$. 
where \( \alpha^2 \beta^3 = 1 \), and where certain other polynomials in \( \alpha, \beta \) must yield integers. For instance, choosing all the signs to be negative, we can write
\[
(t - \alpha)^2 (t - \alpha^{-2/3})^3 = t^5 - (2\alpha + 3\alpha^{-2/3}) t^4 \\
+ (3\alpha^{-4/3} + 6\alpha^{1/3} + \alpha^2) t^3 \\
- (3\alpha^{4/3} + 6\alpha^{-1/3} + \alpha^{-2}) t^2 \\
+ (3\alpha^{2/3} + 2\alpha^{-1}) t - 1
\]
(5.16)
where the coefficients must all be integers. This evidently imposes very severe restrictions on \( \alpha \).

Faced with this situation, it seems easier to relax our insistence upon isotropic compositions, and to insist only that the induced composition on \( S(W) \) be isotropic. Then we can consider operators which characteristics of the form
\[
(t \pm \alpha)(t \pm \alpha) \left( t \pm \frac{1}{\alpha} \right) \left( t \pm \frac{1}{\alpha} \right) (t \pm 1)
\]
which will obviously involve restrictions similar to those in Theorem 5.9. For example, if we search for small integer matrices with characteristic
\[
\chi(t) = (t^2 - 3t - 1)^2(t - 1)
\]
we find that one such matrix is
\[
L = \begin{bmatrix}
-1 & 0 & 1 & 1 & 0 \\
0 & -1 & 0 & 1 & 1 \\
1 & 0 & -1 & 0 & 1 \\
1 & 1 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & -1
\end{bmatrix}
\]

Proceeding as before, one can verify that the eigenspace \( W \) of the eigenvalue \( \frac{-3 + \sqrt{5}}{2} \) yields the Sturmian system \( S(W) \) whose Christoffel component is the space of Penrose tilings. The induced composition is the square of the inflation map discovered by Penrose; see Fig. 5.12. The effect of \( L \) on \( W^\perp \) is to fix the invariant line (the period of \( S(W^\perp) \)), but to contract within the orthogonal complement in \( W^\perp \) of this line, and to expand in \( W \). No reflections or inversions are involved; see Fig. 5.13. Because \( L \)
Figure 5.12: The expanding eigenspace for the eigenvalue $\tau$ (the golden ratio) of this composition, which has characteristic $\chi_L(t) = (t^2 - 3t - 1)^2(t - 1)$, gives the Penrose tilings, and the induced composition is the square of the inflation map discovered by Penrose.
\[ W = \begin{bmatrix} 0 & \tau \\ \tau & -1 \\ 1 & 1 \\ -1 & -\tau \\ -\tau & 0 \end{bmatrix} \quad L = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} \]

Figure 5.13: This picture shows the effect of \( L \) on another orbit closure in \( S(W) \).
5.3. **COMPOSITIONS OF HIGHER DIMENSIONAL OBLIQUE TILINGS**

fixes the invariant line, regarded as a map on $S(W)$, the induced composition takes each orbit closure to itself.

**Problem 5.11** Investigate the continuity and ergodic theory of this inflation restricted to a given orbit closure of $\mathcal{T}(W)$.

If we drop our insistence that a composition must be *invertible* on the entire system $S(W)$, we can recover the original Penrose composition as the induced composition restricted to the Christoffel orbit closure. In particular, if we search for small integer matrices with characteristic $(t^2 - t - 1)^2(t - 3)$, we find the matrix

$$L = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1
\end{bmatrix}$$

has a two dimensional expanding eigenspace $W$ with eigenvalue $\tau$, and $\mathcal{T}(W)$ is precisely the Sturmian tiling space whose Christoffel component is the space of Penrose tilings; see Fig. 5.14. The effect of $L$ on $W^\perp$ is to stretch by a factor of 3 along the invariant line, and contract by $1/\tau$ orthogonally to the period, which means that two components other than the Christoffel component are mapped onto this one. (The effect on $W$ is to stretch isotropically by $\tau$; no reflections or inversions occur in either $W$ or $W^\perp$.) In other words, this *noninvertible* composition maps certain components onto other components.

**Problem 5.12** Investigate the continuity properties and ergodic theory of this composition as a map on the one parameter family of components of $S(W)$.

(The reader can verify that there are matrices with characteristic $(t^2 - t - 1)^2(t - 2)$ which also give the original Penrose inflation on the Christoffel component, etc.)

In a similar way, one can discuss the compositions of the $(2,q)$ step symmetric tilings considered at the end of Section 3.10.
Figure 5.14: The expanding eigenspace for the eigenvalue \( \tau \) (the golden ratio) of this noninvertible composition, which has characteristic \( \chi_L(t) = (t^2 - t - 1)^2(t - 3) \), gives the Penrose tilings, and the induced composition is precisely the inflation map discovered by Penrose.

\[
W = \text{col} \begin{bmatrix}
0 & \tau \\
\tau & -1 \\
1 & 1 \\
-1 & -\tau \\
-\tau & 0
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]
Recall that in the case of oblique tilings in $\mathbb{R}^2$, we could show that the composition group of any $(1,1)$ Sturmian system is either trivial or isomorphic to $\mathbb{Z}$. It is natural to ask whether this remains true in higher dimensions. The answer is no, at least if one requires that the oblique tiling composition be isotropic in $W$ but permits it to be nonisotropic in $W^\perp$, as the following example shows.

Consider the $(2,6)$ Sturmian system $\mathcal{S}(W)$ whose steps have $D_{16}$ symmetry. It can be shown that the matrix

$$L_1 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}$$

expands $W$ by $1 + \cos \pi/8 \approx 2.84776$, and expands the invariant subspaces where the canonical 16 cycle effects a $3/16$ turn by $1 + \cos 3\pi/8 \approx 1.76537$, contracts the invariant subspace of the $5/16$ turn by $1 + \cos 5\pi/8 \approx 0.234633$, and contracts the invariant subspace of the $7/16$ turn by $-(1 + \cos 7\pi/8) \approx 0.847759$ and also effects a half turn there. On the other hand, the matrix

$$L_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}$$

expands $W$ by

$$1 + 2\cos \pi/8 + 2\cos \pi/4 + 2\cos 3\pi/8 \approx 5.02734$$

and effects a nonisotropic expansion and contraction on $W^\perp$. Both of these matrices give invertible compositions, but neither is a power of the other; that is, they give distinct cyclic subgroups of the composition group.
Problem 5.13 Determine the composition group in the previous example. More generally, determine those groups which can arise as the composition group of some Sturmian system.
Chapter 6

Singularities of Sturmian Systems

In this chapter we continue our investigation into the singular tilings of Sturmian systems. In Section 6.1 we completely classify the elementary singularities introduced in Section 4.2. In Section 6.2 we study the recurrence of singular patches in linear arrays called faults, and explain how to classify them using Young-Ferrers diagrams. In Section 6.3 we also characterize of family of ergodic components and the Christoffel tilings of one dimensional Sturmian systems and their duals. We also discuss the correspondence between the faults of a one dimensional Sturmian system and periodic ribbons in its dual. In Section 6.4 we study faults which coincide with ribbons; these are called worms. We study in detail the worms of the Sturmian system which contains the Penrose tilings. In Section 6.5 we study a new phenomenon, fault planes, in which a given type of singular patch occurs in a quasiperiodic planar array inside a singular tiling, and show how to construct Sturmian systems with “arbitrarily complicated” singular tilings.

6.1 Elementary Singularities

Recall from Section 4.2 that in the case of (1, 2) Sturmian tilings, there are essentially three ways a cut line \( x + W \) (where we take \( x \in W^\perp \)) can strike a particular cell \( n + C_J \) of the oblique tiling \( O(W) \). These are exemplified by the following examples:

1. \( x \) lies in the interior of \( Q(n + F_{(1,2)}) \), a tile in the window (see Fig. 4.5), which means that \( x + W \) pierces \( n + C_{(3)} \), so \( T(x + W) \) contains the tile \( P(n + F_{(3)}) = P(n) + T_{(3)} \).

2. \( x \) lies in the interior of \( Q(n + F_{(1)}) \), an edge in the window, which means that
\( x + W \) grazes both \( n + C_{(2)} \) and \( n + C_{(3)} \) (see Fig. 4.6). Consequently, two tiles are trying to coexist at \( P(n) \); the result is that \( T(x + W) \) contains a singular patch \( P(n + F_{(2,3)}) \), a first order elementary singularity.

3. \( x = Q(n) \), a vertex in the window, which means that \( x + W \) grazes the cells \( n + C_{(1)}, n + C_{(2)}, \) and \( n + C_{(3)} \) (see Fig. 4.7). Consequently, three tiles are trying to coexist at \( P(n) \); the result is that \( T(x + W) \) contains a singular patch \( P(n + I^3) \), a second order elementary singularity.

Thus, there are \( \binom{3}{2} = 3 \) types of tiles, there are \( \binom{3}{1} = 3 \) types of first order elementary singularities, and there are \( \binom{3}{3} = 1 \) type of second order elementary singularity.

In the case of (2, 2) Sturmian systems there are again essentially three ways a cut line \( x + W \) (where \( x \in W^+ \)) can strike a particular cell \( n + C_J \) of \( \mathcal{O}(W) \). These are exemplified the following examples (see Fig. 6.1):

1. \( x \) lies in the interior of \( Q(n + F_{(1,2)}) \), a tile in the window, which means that \( x + W \) pierces the cell \( n + C_{(3,4)} \), so \( T(x + W) \) contains the tile \( P(n + F_{(3,4)}) = P(n) + T_{(3,4)} \).

2. \( x \) lies in the interior of \( Q(n + F_{(1)}) \), an edge in the window, which means that \( x + W \) grazes \( n + C_{(2,3)}, n + C_{(2,4)}, \) and \( n + C_{(3,4)} \) (see Fig. 4.6). Consequently, three tiles are trying to coexist at \( P(n) \); the result is that \( T(x + W) \) contains a singular patch \( P(n + F_{(2,3,4)}) \), a first order elementary singularity.

3. \( x = Q(n) \), a vertex in the window, which means that \( x + W \) grazes the cells \( n + C_{(1,2)}, n + C_{(1,3)}, n + C_{(1,4)}, n + C_{(2,3)}, n + C_{(2,4)}, \) and \( n + C_{(3,4)} \) (see Fig. 4.7). Consequently, six tiles are trying to coexist at \( P(n) \); the result is that \( T(x + W) \) contains a singular patch \( P(n + I^4) \), a second order elementary singularity.

Thus, there are \( \binom{6}{2} = 6 \) types of tile, there are \( \binom{6}{1} = 4 \) types of first order elementary singularities, and there is \( \binom{6}{3} = 1 \) type of second order elementary singularity.
Figure 6.1: Elementary singularities in tilings $T(x + W)$ of a typical $(2, 2)$ Sturmian system. Top left: $x = Q(n) + 0.5q_1$, which lies in the interior of $Q(n + F(1))$, an edge in the window. Top right: the corresponding singular patch is $P(n + F(2, 2, 4))$, a first order elementary singularity. Middle left: $x = Q(n) + 0.5q_3$, which lies in the interior of $Q(n + F(3))$, another edge. Middle right: the corresponding singular patch is $P(n + F(1, 2, 4))$, another first order elementary singularity. Bottom left: $x = Q(n)$, a vertex in the window. Bottom right: the corresponding singular patch is $P(n + F(4))$, a second order elementary singularity. (In these pictures we have taken $n = 0$.)
In the case of $(2, 3)$ Sturmian systems, there are essentially four ways a cut plane $x + W$ (where $x \in W^\perp$) can strike a cell $n + C_J$ of $\mathcal{O}(W)$. These are exemplified by the following examples:

1. $x$ lies in the interior of $Q(n + F_{(1,2,3)})$, a tile in the window, which means that $x + W$ pierces the cell $n + C_{\{4,5\}}$, so $T(x + W)$ contains the tile $P(n + F_{\{4,5\}}) = P(n) + T_{\{4,5\}}$.

2. $x$ lies in the interior of $Q(n + F_{(1,2)})$, a edge in the window, which means that $x + W$ grazes the cells $n + C_{\{3,4\}}, n + C_{\{3,5\}},$ and $n + C_{\{4,5\}}$. Consequently, three tiles are trying to coexist at $P(n)$; the result is that $T(x + W)$ contains the singular patch $P(n + F_{\{3,4,5\}})$, a first order elementary singularity.

3. $x$ lies in the interior of $Q(n + F_{(1)})$, an edge in the window, which means that $x + W$ grazes the cells $n + C_{\{2,3\}}, n + C_{\{2,4\}}, n + C_{\{2,5\}}, n + C_{\{3,4\}}, n + C_{\{3,5\}},$ and $n + C_{\{4,5\}}$. Consequently, six tiles are trying to coexist at $P(n)$; the result is that $T(x + W)$ contains the singular patch $P(n + F_{\{2,3,4,5\}})$, a second order elementary singularity.

4. $x = Q(n)$, a vertex in the window, which means that $x + W$ grazes ten cells based at $n$. Consequently, ten tiles are trying to coexist at $P(n)$; the result is that $T(x + W)$ contains the singular patch $P(n + I^5)$, a third order elementary singularity.

There are $\binom{5}{3} = 10$ types of tiles, $\binom{5}{2} = 10$ types of first order elementary singularities, $\binom{5}{1} = 5$ types of second order elementary singularities, and $\binom{5}{0} = 1$ type of third order elementary singularity.

If we define $F_q = 0 \in \mathbb{R}^d$, we can summarizes this discussion in Table 6.1 or, more formally, in the following theorem.
Figure 6.2: Elementary singularities in tilings \(T(x + W)\) in a typical \((2,3)\) Sturmian system. Top left: \(x = Q(n) + 0.2(q_1 + q_2)\), which lies in the interior of \(Q(n + F_{(1,2)})\), a two-dimensional face in the window. Top right: the corresponding singular patch is \(P(n + F_{(3,4,8)})\), a first order elementary singularity. Middle left: \(x = Q(n) + 0.5q_1\), which lies in the interior of \(Q(n + F_{(1)})\), an edge in the window. Middle right: the corresponding singular patch is \(P(n + F_{(2,3,4,5)})\), a second order elementary singularity. Bottom left: \(x = Q(n)\), a vertex in the window. Bottom right: the corresponding singular patch is \(P(n + I^5)\), a third order elementary singularity. (In these pictures we have taken \(n = 0\).)
| p | q | \( Q(\mathbf{x}) \) lies in interior of | \( \mathbf{x} + W \) hits these cells: | \( T(\mathbf{x} + W) \) contains this patch: | order of singularity |
|---|---|---|---|---|
| 1 | 1 | \( Q(\mathbf{n} + F_{(1)}) \) | \( \mathbf{n} + C_{(2)} \), \( \mathbf{n} + C_{(1)}, \mathbf{n} + C_{(2)} \) | \( P(\mathbf{n} + F_{(1)}) \), \( P(\mathbf{n} + F_{(1,1,2)}) \) | regular, first |
| 2 | 1 | \( Q(\mathbf{n} + F_{(1,2)}) \) | \( \mathbf{n} + C_{(3)} \), \( \mathbf{n} + C_{(2)}, \mathbf{n} + C_{(3)} \), \( \mathbf{n} + C_{(1)}, \mathbf{n} + C_{(2)}, \mathbf{n} + C_{(3)} \) | \( P(\mathbf{n} + F_{(1,2)}), P(\mathbf{n} + F_{(2,3)}), P(\mathbf{n} + F_{(1,2,3)}) \) | regular, first, second |
| 2 | 2 | \( Q(\mathbf{n} + F_{(1,2)}) \) | \( \mathbf{n} + C_{(3,4)} \), \( \mathbf{n} + C_{(2,3)}, \mathbf{n} + C_{(2,4)}, \mathbf{n} + C_{(3,4)} \), \( \mathbf{n} + C_{(1,2)}, \ldots \mathbf{n} + C_{(3,4)} \) | \( P(\mathbf{n} + F_{(1,2)}), P(\mathbf{n} + F_{(2,3)}), P(\mathbf{n} + F_{(1,2,3)}) \) | regular, first, second |
| 2 | 3 | \( Q(\mathbf{n} + F_{(1,2,3)}) \) | \( \mathbf{n} + C_{(4,5)} \), \( \mathbf{n} + C_{(3,4)}, \mathbf{n} + C_{(3,5)}, \mathbf{n} + C_{(4,5)} \), \( \mathbf{n} + C_{(2,4)}, \ldots \mathbf{n} + C_{(4,5)} \), \( \mathbf{n} + C_{(1,2)}, \ldots \mathbf{n} + C_{(4,5)} \) | \( P(\mathbf{n} + F_{(1,2,3)}), P(\mathbf{n} + F_{(3,4,5)}), P(\mathbf{n} + F_{(2,3,4,5)}) \) | regular, first, second |

Table 6.1: The catalog of elementary singularities for some small values of \( p \) and \( q \). Notice that the order of each singular patch \( P(F) \) is the codimension of the corresponding zonotope \( Q(F_{(p)}) \) in \( W^\bot \).

(Note: for notational convenience, we take "\( \mathbf{x} \) lies in the interior of \( Q(\mathbf{n}) \)" to mean just \( \mathbf{x} = Q(\mathbf{n}) \).)
6.2. LINEAR RECURRENCE OF SINGULARITIES

**Theorem 6.1** Let \( S(W) \) be a \((p, q)\) Sturmian tiling space. Suppose \( F_J \) is some facet of the unit cube \( I^d \), say with \(|J| = p + m\) where \(1 \leq m \leq q\). Then given \( x \in W^\perp\), \( T(x + W) \) has a \( m \)-th order singular patch \( P(n + F_J) \) at \( P(n) \in W \) iff \( x \) lies in the interior of \( Q(n + F_{J\circ}) \). Moreover, for each \( m \) with \(1 \leq m \leq q\), each \( J \) with \(|J| = p + m\) gives a different type of elementary singularity of order \( m \), so there are \( d \) choose \( p + m \) types of \( m \)-th order elementary singularities.

This gives the desired classification of elementary singularities. Depending on the properties of \( W \), a tiling \( T(x + W) \) may contain more than one elementary singularity, as we shall see.

Note that Theorem 6.1 implies that the Christoffel tiling \( T(0 + W) \) of \( S(W) \) always contains a \( q \)-th order singularity (the maximal order) at the origin, so singular tilings are a ubiquitous feature of Sturmian systems.

Notice that our remarks in Section 4.2 on the unfolding of singularities generalizes to the case of \((p, q)\) Sturmian systems. In particular, by appropriately perturbing the cut plane \( 0 + W \) of the Christoffel tiling \( T(0 + W) \) we can systematically unfold the \( q \)-th order elementary singularity at the origin; this corresponds to choosing some hyperface of the singular patch \( P(I^d) \), obtaining a singularity of order \( q - 1 \), then choosing a face of this hyperface, obtaining a singularity of order \( q - 2 \), and so forth.

This suggests (correctly) that most of our effort should be directed toward understanding how the appearance of the Christoffel tiling of \( S(W) \) changes as (for fixed \( p, q \)) we vary \( W \).

6.2 Linear Recurrence of Singularities

In Section 4.2 we discussed in some detail the singular tilings of \( S(W) \) in the case

\[
W = \text{col} \begin{bmatrix} 1 \\ r \\ e \end{bmatrix}
\]
Figure 6.3: Top: the Christoffel tiling of this tiling space has an isolated second order singularity at the origin (skeleton of cube).

where $\tau = (1 + \sqrt{5})/2 \approx 1.61803$ is the Golden Ratio. (Note that $W$ is irrational, so $S(W)$ contains only aperiodic tilings.) We found that the Christoffel tiling has an isolated second order singularity at the origin (see Fig. 6.3), and, apparently, no other singularities.

Let us now consider $(1, 2)$ Sturmian systems which exhibits some striking new
behavior; for instance take

\[ W = \text{col} \begin{bmatrix} 1 \\ 2 \\ \sqrt{3} \end{bmatrix} \]

Note that \( W \) is irrational, so that \( \mathcal{S}(W) \) contains only aperiodic tilings. However, the Christoffel tiling looks quite different from the previous example: it has not only a second order singularity at the origin but also an infinite array of first order singularities, which are all projections of the same square, up to translation (see Fig. 6.4). This quasiperiodic array of first order singular patches, all of the same type, is called a \textbf{first order fault}.

It is not hard to understand how such a fault can arise. Because the vector \((1, 2, \sqrt{3})\) is in \( W \), we obtain the relation

\[ q_1 + 2q_2 + \sqrt{3}q_3 = 0 \]

This forces the \( W^\perp \) edge

\[ q_1 + 2q_2 + [\sqrt{3}]q_3 + Q(F_{(3)}) = q_1 + 2q_2 + q_3 + Q(F_{(3)}) \]

and the parallel edge

\[ q_1 + 2q_2 + [\sqrt{3}]q_3 + Q(F_{(3)}) = q_1 + 2q_2 + 2q_3 + Q(F_{(3)}) \]

to cover the edge \( Q(F_{(3)}) \), as shown at the bottom of Fig. 6.4. This means that whenever we have a first order singularity of type \( P(F_{(1,2)}) \) at the origin, we must have another such at \emph{either} \( p_1 + 2p_2 + p_3 \) or \( p_1 + 2p_2 + 2p_3 \). Moreover, these alternatives are chosen with frequencies \( [\sqrt{3}] - \sqrt{3} \approx 0.267949 \) and \( \sqrt{3} - [\sqrt{3}] \approx 0.732051 \) respectively, as we range over all the tilings with a “white/green” first order singularity at the origin. These frequencies are simply the relative proportion of overlap by the edges with basepoints at \( q_1 + 2q_2 + q_3, q_1 + 2q_2 + 2q_3 \) respectively of the edge with basepoint at the origin. Indeed, we must have the same two recurrence alternatives (occurring with the same frequencies) at \emph{any} place where a “white/green” singularity occurs, so
Figure 6.4: Top: the Christoffel tiling of this tiling space has recurrent first order singularities (skeletons of horizontal squares) in addition to a single second order singularity at the origin (skeleton of cube). Bottom left: the windows for $0$, $p_1 + 2p_2 + p_3$, and $p_1 + 2p_2 + 2p_3$. The endpoints of the $q_3$ edge are shown as dots. Bottom right: a schematic diagram showing how the relation $q_1 + 2q_2 + \sqrt{3}q_3 = 0$ forces the “head” of the $W^+$ edge $q_1 + 2q_2 + q_3 + Q(F_{(3)})$ and the “tail” of the edge $q_1 + 2q_2 + 2q_3 + Q(F_{(3)})$ to overlap the edge $Q(F_{(3)})$, which means that every occurrence of a first order singular patch of type $P(n + F_{(1,3)})$ is always accompanied at either $n + p_1 + 2p_2 + p_3$ or $n + p_1 + 2p_2 + 2p_3$ by an identical patch (with the second alternative being chosen more frequently).
we must have a quasiperiodic array of first order "white/green" singularities in any tiling which contains one such singularity. 

On the other hand, in our first example, because \( W \) is irrational \( Q|Z^3 \) is one-one, and because \( W^\perp \) is also irrational \( Q(Z^4) \) is dense and uniformly distributed in \( W^\perp \). Thus, translates of Christoffel tilings (and their unfoldings) are dense in \( T(W) \), but while edges of type \( Q(F_{(1)}) \) are dense and in fact uniformly distributed in \( W^\perp \), they never overlap; that is, at most one first order singularity of type \( P(F_{(1)}) \) can occur in each tiling; likewise for the other two types of first order singularity. Thus, singular tilings in this Sturmian system always look like the Christoffel tiling (with a single isolated second order singularity) or like one of the three tilings (each with a different isolated first order singularity) obtained by unfolding the Christoffel tiling, or else is a tiling \( T(x + W) \) such that \( Q(x) \) is not the projection of an integer vector but does lies on the intersection of two edges of different types; in this case, \( T(x + W) \) will contain precisely one first order singularity of each of two different types. Each of these types of singular tilings is dense in \( T(W) \), as is required by the fact that \( T(W) \) is uniquely ergodic (since \( W^\perp \) is irrational). See Fig. 6.5 to get some idea of the way the each type of edge \( Q(F_{(j)}) \) forms a disjoint family dense in \( W^\perp \), with each such edge intersecting infinitely many others from the other two families of edges (in a single point each).

Another way to think about this to fix an edge in the window, say \( Q(n + F_{(1)}) \), and consider all the \( Q(x) \) lying on that edge. These are precisely the tilings having a singularity of type \( P(F_{(2,3)}) \) at \( P(n) \). As we move along \( Q(n + F_{(1)}) \), we encounter translates of the Christoffel tiling only at either endpoint; we also encounter a dense set of intersections with edges of type \( Q(F_{(2)}) \) and also a dense set of intersections with edges of type \( Q(F_{(3)}) \); these represent tilings with precisely one type \( P(F_{(2,3)}) \) and one type \( P(F_{(1,3)}) \) singularity each, or precisely one type \( P(F_{(2,3)}) \) and one type \( P(F_{(1,2)}) \) singularity each, respectively. These are not unfoldings of the Christoffel tiling as we defined unfolding, but nonetheless all these types of singular tilings are
\[ W = \col \begin{bmatrix} 1 \\ \tau \\ e \end{bmatrix} \]

\[ W = \col \begin{bmatrix} 1 \\ \frac{2}{\sqrt{3}} \end{bmatrix} \]

Figure 6.5: Left: here both \( W \) and \( W^\perp \) are irrational, and none of the edges overlap and there are no recurrent singularities. Each family of edges \( Q(F_{(1)}) \) is a disjoint collection dense in \( W^\perp \). Right: here \( W \) is irrational but \( W^\perp \) is not. The edges of type \( Q(F_{(3)}) \) lie densely along parallel lines in \( W^\perp \); these lines are cosets of \((\text{rat } W) \cap W^\perp \). The edges of type \( Q(F_{(2)}) \) and \( Q(F_{(1)}) \).

dense in \( \mathcal{T}(W) \). Similarly for moving along \( Q(n + F_{(2)}) \) or \( Q(n + F_{(3)}) \).

The situation is quite different for the second example. Here, if we move along a fixed edge \( Q(n + F_{(3)}) \), we encounter translates of the Christoffel tiling only at the endpoints, and dense sets of transverse intersections with edges of type \( Q(F_{(1)}) \) or \( Q(F_{(2)}) \). However, if we move along a fixed edge \( Q(n + F_{(1)}) \), we encounter a translate of the Christoffel tiling at the midpoint. And of course if we move along a fixed edge \( Q(n + F_{(3)}) \) (thus remaining within the orbit closure of the Christoffel tiling), we are always overlapping infinitely many other such edges, and also encounter a dense set of translates of the Christoffel tiling, and a dense set of transverse intersections with edges of form \( Q(F_{(1)}) \). (But we do not encounter the interior of any type \( Q(F_{(2)}) \) edges). Note that each orbit closure (corresponding to a coset of the two dimensional subtorus \( \pi(\text{rat } W) \) of \( T^3 \)) has its own characteristic singular tilings.
In Fig. 6.6 we have shown the result of “unfolding” the Christoffel tiling (which has a second order singularity at the origin; see Fig. 6.4) along the \( q_1, q_2 \) and \( q_3 \) directions.

Notice that the appearance of the resulting singular tilings (all of which contain only first order singularities) may be summarized as follows:

1. \( T(\varepsilon q_1) \) has a \( P(F_{1,2}) \) type singularity at the origin, and no further singularities,

2. \( T(\varepsilon q_2) \) has a \( P(F_{1,3}) \) type singularity at the origin, and no further singularities,

3. \( T(\varepsilon q_3) \) has a first order fault with repeated singularities of type \( P(F_{1,2}) \),

4. \( T(\frac{1}{2} q_1) \) has a first order fault as before, but also an isolated first order singularity of type \( P(F_{1,3}) \) at the origin.

Obviously, the integer identity \( x_1 = 2x_2 \) obeyed by points on \( W \) in the second example lies behind the existence of recurrent singularities in this example. This suggests making the following simple observation.

**Lemma 6.2** Let \( W \) be a one dimensional nondegenerate subspace of \( \mathbb{R}^3 \). Then \( W \) is spanned by a vector of precisely one of the following types:

1. an integer vector,

2. a vector having two integer and one irrational components,

3. a vector having one integer and two incommensurate irrational components.

In each case, the vector has no vanishing components.
$W = \text{col } \begin{bmatrix} 1 \\ 2 \\ \sqrt{3} \end{bmatrix}$

Figure 6.6: Clockwise from top left: the perturbations of the Christoffel tiling $T(\varepsilon q_1)$, $T(\varepsilon q_2)$, $T(\varepsilon q_3)$ respectively, where $\varepsilon > 0$ is very small, and the anomalous tiling $T(0.5 q_1 + W)$. 
6.2. LINEAR RECURRENTNESS OF SINGULARITIES

Proof: Let \( W \) be spanned by \( v \neq 0 \). Because \( W \) is nondegenerate, no component of \( v \) vanishes. By scalar multiplying we can obtain \( v_2 \) with first component equal to one. If the remaining components are rationals, we can clear denominators to obtain a vector of the first form. If the remaining components are incommensurate irrationals, we have a vector of the third form. If the remaining components are commensurate irrationals, a further scalar multiplication gives \( v_3 \) with two integer components and one irrational component, and we have a vector of the second form. Otherwise, one component of \( v_2 \) must be rational and the other irrational, and by a further scalar multiplication we obtain a vector \( v_3 \) of the second form.

It follows immediately that the Christoffel tiling of any nondegenerate \((1,2)\)-Sturmian tiling space must look like one of the following:

1. periodically repeated second order singularities, and no other singularities,

2. a second order singularity at the origin together with a first order fault, and no other singularities,

3. a second order singularity at the origin, and no other singularities.

This discussion can be readily generalized.

**Definition 31** Let \( S(W) \) be a \((p,q)\) Sturmian system. Suppose that for some selection \( J \) of coordinate indices, \( |J| = p + m \), the vector

\[
    w = \sum_{j \in J} n_j e_j + \sum_{j \notin J} \alpha_j e_j
\]

(\( \text{where the } n_j \in \mathbb{Z} \text{ and the } \alpha_j \text{ are irrational} \)) lies in \( W \). Then \( w \) is called a \( m \)-th order recurrence vector for \( S(W) \).

Notice this generalizes the notion of a period for \( S(W) \). We can immediately generalize Prop. 4.13 as follows.
Proposition 6.3 Let $W$ be a $p$-dimensional subspace of $\mathbb{R}^d$, let $P, Q$ be the orthoprojections on $W, W^\perp$, and let $p_j = P(e_j), q_j = Q(e_j)$ as usual. Then the following are equivalent:

1. $v = \sum_{j \in J} n_j e_j + \sum_{j \notin J} \alpha_j e_j$ lies in $W$, i.e. is a recurrence vector for $S(W)$,

2. the $q_j$ satisfy the identity

$$\sum_{j \in J} n_j q_j + \sum_{j \notin J} \alpha_j q_j = 0$$

Proof: $v$ lies in $W$ iff $Q(v) = \sum_{j \in J} n_j q_j + \sum_{j \notin J} \alpha_j q_j = 0$.

The point is of course that whenever we have a $m$-th order recurrence vector in $W$, we will have edges of codimension $m$ in the window which exactly overlap, giving recurrent $m$-th order singularities in the Christoffel tiling for $S(W)$.

For example, in the $(1, 3)$ Sturmian system $S(W)$ with

$$W = \text{col} \begin{bmatrix} 1 \\ 2/3 \\ \sqrt{2} \end{bmatrix}$$

the second order recurrence vector

$$w = e_1 + 2e_2 + 3e_3 + \sqrt{2}e_4$$

gives the identity

$$q_1 + 2q_2 + 3q_3 + \sqrt{2}q_4 = 0$$

which means that the $W^\perp$ edge $Q(F_{\langle 4 \rangle})$ overlaps

$$q_1 + 2q_2 + 3q_3 + [\sqrt{2}]q_4 + Q(F_{\langle 4 \rangle}) = q_1 + 2q_2 + 3q_3 + q_4 + Q(F_{\langle 4 \rangle})$$

and

$$q_1 + 2q_2 + 3q_3 + [\sqrt{2}]q_4 + Q(F_{\langle 4 \rangle}) = q_1 + 2q_2 + 3q_3 + 2q_4 + Q(F_{\langle 4 \rangle})$$
6.2. \textit{LINEAR RECURRENT OF SINGULARITIES} \quad 225

giving a quasiperiodic array of second order singularities of type $P(F_{(1,2,3)})$ with the $2^{3-2}$ first recurrence alternatives

$$ p_1 + 2p_2 + 3p_3 + p_4 $$

$$ p_1 + 2p_2 + 3p_3 + 2p_4 $$

which are chosen according to the corresponding relative overlap.

In the $(1,3)$ Sturmian system $S(W)$ with

$$ W = \begin{bmatrix} 1 \\ 2 \\ \sqrt{2} \\ \sqrt{3} \end{bmatrix} $$

the first order recurrence vector

$$ w = e_1 + 2e_2 + \sqrt{2}e_3 + \sqrt{3}e_4 $$

gives the identity

$$ q_1 + 2q_2 + \sqrt{2}q_3 + \sqrt{3}q_4 = 0 $$

which means that the edge $Q(F_{(3,4)})$ in the window overlaps

$$ q_1 + 2q_2 + [\sqrt{2}]q_3 + [\sqrt{3}]q_4 + Q(F_{(3,4)}) = q_1 + 2q_2 + q_3 + q_4 + Q(F_{(3,4)}) $$

and

$$ q_1 + 2q_2 + [\sqrt{2}]q_3 + [\sqrt{3}]q_4 + Q(F_{(3,4)}) = q_1 + 2q_2 + q_3 + 2q_4 + Q(F_{(3,4)}) $$

and

$$ q_1 + 2q_2 + [\sqrt{2}]q_3 + [\sqrt{3}]q_4 + Q(F_{(3,4)}) = q_1 + 2q_2 + 2q_3 + q_4 + Q(F_{(3,4)}) $$

and

$$ q_1 + 2q_2 + [\sqrt{2}]q_3 + [\sqrt{3}]q_4 + Q(F_{(3,4)}) = q_1 + 2q_2 + q_3 + q_4 + Q(F_{(3,4)}) $$
giving a quasiperiodic array of first order singularities $P(F_{1,2})$ with the $2^{3-1} = 4$
first recurrence alternatives

\[ p_1 + 2p_2 + p_3 + p_4 \]
\[ p_1 + 2p_2 + p_3 + 2p_4 \]
\[ p_1 + 2p_2 + 2p_3 + p_4 \]
\[ p_1 + 2p_2 + 2p_3 + 2p_4 \]

which are chosen according to the corresponding relative overlap.

In the (2, 2) Sturmian system $S(W)$ with

\[ W = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & \sqrt{5} \\ \sqrt{2} & \sqrt{3} \end{bmatrix} \]

the first order recurrence vector

\[ w = e_1 + 2e_2 + 3e_3 + \sqrt{2}e_4 \]

gives the identity

\[ q_1 + 2q_2 + 3q_3 + \sqrt{2}q_4 = 0 \]

which means that the edge $Q(F_{4})$ overlaps

\[ q_1 + 2q_2 + 3q_3 + [\sqrt{2}]q_4 + Q(F_{4}) = q_1 + 2q_2 + 3q_3 + q_4 + Q(F_{4}) \]

and

\[ q_1 + 2q_2 + 3q_3 + [\sqrt{2}]q_4 + Q(F_{4}) = q_1 + 2q_2 + 3q_3 + 2q_4 + Q(F_{4}) \]

giving a quasiperiodic array of second order singularities of type $P(F_{1,2,3})$ with the
$2^{2-1} = 2$ first recurrence alternatives

\[ p_1 + 2p_2 + 3p_3 + p_4 \]
\[ p_1 + 2p_2 + 3p_3 + 2p_4 \]
which are chosen according to the corresponding relative overlap. Note that the second column, $3e_1 + e_2 + \sqrt{5}e_3 + \sqrt{3}e_4$ does not give a recurrence vector because it doesn't have enough integer components. Note too that the same vectors can play different roles for different subspaces.

We can sum up this discussion in the following theorem (in which we put $F_j = 0$).

**Theorem 6.4** Let $W$ be a $p$-dimensional subspace in $\mathbb{R}^d$. Let $|J| = p+m$ and suppose

$$w = \sum_{j \in J} n_j e_j + \sum_{j \in J} \alpha_j e_j$$

is a $m$-th order recurrence vector for $S(W)$. Then

1. whenever one singularity of type $P(F_j)$ occurs in some $T(x + W)$, so must infinitely many translates of this patch,

2. these singular patches of type $P(F_j)$ occur with the $2^{q-m}$ first recurrence alternatives

$$\sum_{j \in J} n_j p_j + \sum_{j \in J} m_j e_j$$

(where $m_j = [\alpha_j]$ or $[\alpha_j]$),

3. these alternatives are chosen (as we vary over the set of all tilings having a particular such singular patch) with frequencies given by the relative overlap in $W^\perp$ of the edge

$$\sum_{j \in J} n_j q_j + \sum_{j \in J} m_j q_j + Q(F_{j^c})$$

with $Q(F_{j^c})$,

4. Any tiling in $S(W)$ which contains one singularity of type $P(F_j)$ contains an entire quasiperiodic array of them, with the same first recurrence alternatives and the same frequencies.
5. In particular, the Christoffel tiling for $S(W)$ contains a quasiperiodic array of $m$-th order singularities of type $P(F)$.

Two caveats: first, we shall see in Section 6.4 that the frequencies with which the recurrence alternatives are chosen, as we vary over tilings in a particular orbit closure which contain a given singular patch, is usually not the same as the frequencies found above. Indeed some recurrence alternatives may be forbidden altogether (the situation here is analogous to our remarks about patch frequencies; the frequency of occurrence in a particular orbit closure is generally different from the frequency of occurrence in $S(W)$ as a whole, and indeed some patches might be entirely forbidden in some orbit closures). Second, while we focus here on linear quasiperiodic arrays, we note that the above does not rule out multidimensional arrays of singularities of a given type in higher dimensional Sturmian tilings (see Section 6.5).

Recurrence vectors span one dimensional subspaces; a fundamental observation is that these can very well contain new recurrence vectors. For example, consider

$$\begin{bmatrix}
1 \\
2 \\
3 \\
\sqrt{2} \\
3\sqrt{2} \\
\sqrt{2} \\
\sqrt{5}
\end{bmatrix}$$

If this lies in some two dimensional subspace $W$, it gives a first order recurrence vector for $S(W)$. Scalar multiplying by $1/\sqrt{2}$ gives the new first order recurrence vector

$$\begin{bmatrix}
1/\sqrt{2} \\
2/\sqrt{2} \\
3/\sqrt{2} \\
1 \\
3 \\
1 \\
\sqrt{5/2}
\end{bmatrix}$$

This suggests the following definition.
6.2. LINEAR RECURRENCE OF SINGULARITIES

Definition 32 A vector $x = \sum_j x_j e_j$ in $\mathbb{R}^d$ is said to be nondegenerate if all its components are nonzero. If $x$ is nondegenerate, we say that two components $x_j, x_k$ are in the same rational class if they are rational multiples of each other.

Consequently, we can assign to each nondegenerate vector a Young-Ferrers diagram (like those employed to classify partitions of a set of $d$ elements, among other uses). For example, the vector

$$x = \begin{bmatrix} 7 \\ \pi \\ 3 \\ -5 \\ \sqrt{2} \\ e \\ 3\sqrt{2} \end{bmatrix}$$

may be assigned the Young diagram

```
 1 3 4
5 7
2
6
```

Here, the first, third and fourth components of $x$ are rational, and the fifth and seventh are multiples of $\sqrt{2}$, while the second and sixth are incommensurate with all the other components. This gives four rational classes of sizes three, two, one and one, respectively, corresponding to the four rows of the diagram. It is sometimes convenient to indicate the indices of the members of each rational class by small numbers written in the appropriate box of the Young diagram, as shown. Note that each row of the Young diagram corresponds to a basis vector of the rational closure of the one-dimensional subspace spanned by $x$

$$\text{rat span } x = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$
Here, the first column corresponds to the first row of the Young diagram (the rational class of size three), the third column to the second row (the rational class of size two), and the second and fourth columns to the remaining rows (rational classes of size one).

Note that recurrence vectors for a \((p, q)\) system have at least one class of size greater than \(p\), and perhaps more. Needless to say, most vectors have only singleton classes (corresponding to the case \(\text{rat span } x = \mathbb{R}^d\)).

The reason for the restriction to nondegenerate vectors above is clear: zero components “want” to belong to all the rational classes at once! By passing to the subwalls of \(S(W)\) if necessary, we can ignore this possibility.

**Theorem 6.5** The possible types of nondegenerate recurrence vectors for \((p, q)\) Sturmian systems correspond to the partitions of the \(d\) components into rational classes, such that at least one class has more than \(p\) elements.

### 6.3 Singularities of One Dimensional Systems and their Duals

In this section we focus on nondegenerate codimension and dimension one Sturmian systems. Our first task is to characterize the family of orbit closures of Sturmian systems and the appearance of Christoffel tilings of codimension one.

**Theorem 6.6** Suppose \(S(W)\) is a nondegenerate Sturmian system of codimension one. Then every tiling of \(S(W)\) contains at worst a first order singularity, and these are isolated unless \(S(W)\) has a period. Moreover, there is only type of first order singularity, so there is a unique type of singular patch. If \(S(W)\) is aperiodic, the Christoffel tiling contains an isolated first order singularity at the origin and no other singularities, and in this case \(T(W)\) is minimal.

**Proof:** Suppose \(T(x + W)\) contains a singular patch. By Theorem 6.1, the only elementary singularities which can occur in \(S(W)\) are first order singularities of form
6.3. SINGULARITIES OF ONE DIMENSIONAL SYSTEMS AND THEIR DUALS

\( P(n + K) \), where \( K \) is the unit cube in \( \mathbb{R}^d \) and where \( n \in \mathbb{Z}^d \). Suppose \( T(x + W) \) contains two such singular patches, say \( P(m + K) \) and \( P(n + K) \). By Theorem 6.4, \( T(x + W) \) contains \( P(n + K) \) iff \( x = Q(n) \), so \( x = Q(m) = Q(n) \) or \( Q(m - n) = 0 \); that is, \( m - n \in W \), so the recurrence arises from the period \( m - n \).

We saw in Section 4.7 that the orbit closures of \( T(W) \) are parameterized by \( (\text{rat } W)^\perp \); but if \( W \) has codimension one, \( \text{rat } W \) is either all of \( \mathbb{R}^d \) or else \( W \) is rational. Thus, if \( T(W) \) is aperiodic, it must be minimal.

At the other extreme, if \( S(W) \) is a dimension one Sturmian system, there are more possibilities, but we can still characterize both the family of orbit closures and the appearance of the Christoffel tilings using the Young diagrams.

**Theorem 6.7** Suppose \( W \) is a one dimensional subspace of \( \mathbb{R}^d \), and suppose \( W \) is associated with \( r \) rational classes of sizes \( k_1 \geq k_2 \geq \ldots k_r > 0 \), where \( k_1 + k_2 + \ldots k_r = d \). Then \( \dim \text{rat } W = r \), the number of rows of the Young diagram (the number of rational classes), and each row (rational class) of of size \( k_j = 1 + m \), where \( m > 0 \), gives an \( m \)-th order fault.

**Proof:** The assertion concerning the parameterization of the orbit closures follows immediately from our remarks after Definition 32. For the second assertion, observer that each rational class of \( k_j = 1 + m \), where \( m > 0 \) permits any vector in \( W \), after a suitable scalar multiplication, to serve as a recurrence vector giving an \( m \)-th order fault.

By Cor. 4.14, \( T(W) \) is uniquely ergodic iff \( T(W^\perp) \) is aperiodic. In particular, if \( W \) is a one dimensional subspace, \( T(W) \) is uniquely ergodic unless \( W \) has a period. This period need not be a periodic ribbon, as the examples in Fig. 6.7 show. The importance of periodic ribbons is indicated by the following fact.
\[
W = \text{col} \begin{bmatrix}
2\sqrt{3} \\
4\sqrt{3} - 3
\end{bmatrix}
\]

\[
W^\perp = \text{col} \begin{bmatrix}
1 \\
\frac{1}{\sqrt{3}}
\end{bmatrix}
\]

\[
W^\perp = \text{col} \begin{bmatrix}
-2 & -\sqrt{3} \\
1 & 0 \\
1 & 2
\end{bmatrix}
\]

\[
W^\perp = \text{col} \begin{bmatrix}
-2 & -\sqrt{3} \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Figure 6.7: Two aperiodic (1, 2) tiling spaces, each having a one parameter family of orbit closures, with their dual (2,1) tiling spaces, each having a single fundamental period. Left: the period of the dual is not aligned with any ribbon, and the Christoffel tiling has no faults. Right: the $z_3$-ribbons of the dual (shaded) are periodic, and the Christoffel tiling has a corresponding first order fault.
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Proposition 6.8 The Christoffel tiling of $S(W)$ contains recurrent singularities iff $S(W^\perp)$ has periodic ribbons.

Proof: It suffices to consider for instance the case of $(1,4)$ Sturmian systems. We can write $W$ and the ribbon spaces of $S(W^\perp)$ like this:

$$
\begin{bmatrix}
1 & \alpha & \beta & \gamma & \delta & 0 & 0 & 0 & 0 & 0 \\
\alpha & -1 & 0 & 0 & 0 & \beta & \delta & 0 & 0 & 0 \\
\beta & 0 & -1 & 0 & 0 & -\alpha & 0 & 0 & \gamma & \delta \\
\gamma & 0 & 0 & -1 & 0 & 0 & -\alpha & 0 & -\beta & 0 \\
\delta & 0 & 0 & 0 & -1 & 0 & 0 & -\alpha & 0 & -\beta & -\gamma \\
\end{bmatrix}
$$

Here, the first, second, third, and fourth ribbon vectors can be chosen to be rational iff $x_2, x_3, x_4, x_5$ respectively are in the rational class of $x_1$. Likewise, the next three ribbon vectors can be chosen to be rational iff $x_3, x_4, x_5$ respectively are in the rational class of $x_2$. The next two can be chosen to rational iff $x_4, x_5$ respectively are in the rational class of $x_3$, and the last can be chosen to be rational iff $x_5$ is in the rational class of $x_4$. On the other hand, since $W$ is one dimensional, any rational class of size $m + 1$, where $m > 0$, corresponds to an $m$-th order fault.

Indeed, we can say much more. Suppose $W$ is associated with the Young diagram:

```
+ + + +
+ + +
+ +
+.
```

Then

- $S(W)$ has a fourth order fault contributed by the first row, of length $1 + 4 = 5$; this fault is associated with $(\frac{5}{2}) = 10$ periodic ribbons in $S(W^\perp)$,

- $S(W)$ has a second order fault contributed by the second row, of length $1 + 2 = 3$; this fault is associated with $(\frac{3}{2}) = 3$ periodic ribbons in $S(W^\perp)$,

- $S(W)$ has two first order faults contributed by the next two rows, of length $1 + 1 = 2$; these are each associated with one periodic ribbon in $S(W^\perp)$. 


<table>
<thead>
<tr>
<th>Young Diagram</th>
<th>Example: $W$ and ribbon spaces of $S(W)$</th>
<th>Number of rational ribbons of $T(0 + W)$</th>
<th>Singularities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; \sqrt{2} &amp; \sqrt{3} &amp; 0 \ \sqrt{2} &amp; -1 &amp; 0 &amp; \sqrt{3} \ \sqrt{3} &amp; 0 &amp; -1 &amp; -\sqrt{2} \end{bmatrix}$</td>
<td>0</td>
<td>isolated only</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 &amp; \sqrt{3} &amp; 0 \ 2 &amp; -1 &amp; 0 &amp; \sqrt{3} \ \sqrt{3} &amp; 0 &amp; -1 &amp; -2 \end{bmatrix}$</td>
<td>1</td>
<td>first order fault</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 &amp; 3 &amp; 0 \ 2 &amp; -1 &amp; 0 &amp; 3 \ 3 &amp; 0 &amp; -1 &amp; -2 \end{bmatrix}$</td>
<td>3</td>
<td>periodic</td>
</tr>
</tbody>
</table>

Table 6.2: Examples of the three possible types of Christoffel tilings in $(1,2)$ Sturmian systems and the corresponding periodic ribbons in the dual system.

Thus, in this example, $S(W^\perp)$ has a total of $10 + 3 + 1 + 1 = 15$ periodic ribbons, out of a total of $\binom{15}{2} = 105$ ribbons.

See Tables 6.2 to 6.4 for some concrete examples of how this works.

### 6.4 Worms

In this section we study a special kind of fault.

**Definition 33** A worm is a ribbon whose ribbon space (recall this is a one dimensional subspace in $\mathbb{R}^d$) is spanned by a degenerate recurrence vector.

In one dimensional systems, every fault is a worm (notice that we consider two distinct recurrence vectors which are scalar multiples of each other to define distinct worms), but in higher dimensional systems, this need not be true.
<table>
<thead>
<tr>
<th>Young Diagram</th>
<th>Example: $W$ and ribbon spaces of $S(W^\perp)$</th>
<th>Number of rational ribbons</th>
<th>Singularities of $T(0 + W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; \sqrt{2} &amp; \sqrt{3} &amp; \sqrt{5} &amp; 0 &amp; 0 &amp; 0 \ \sqrt{2} &amp; -1 &amp; 0 &amp; 0 &amp; \sqrt{3} &amp; \sqrt{5} &amp; 0 \ \sqrt{3} &amp; 0 &amp; -1 &amp; 0 &amp; -\sqrt{2} &amp; 0 &amp; \sqrt{5} \ \sqrt{5} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -\sqrt{2} &amp; -\sqrt{3} \end{bmatrix}$</td>
<td>0</td>
<td>isolated only</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 &amp; \sqrt{2} &amp; \sqrt{3} &amp; 0 &amp; 0 &amp; 0 \ 2 &amp; -1 &amp; 0 &amp; 0 &amp; \sqrt{2} &amp; \sqrt{3} &amp; 0 \ \sqrt{2} &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; 0 &amp; \sqrt{3} \ \sqrt{3} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; -\sqrt{2} \end{bmatrix}$</td>
<td>1</td>
<td>first order fault</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 &amp; 2\sqrt{5} &amp; 3\sqrt{5} &amp; 0 &amp; 0 &amp; 0 \ 2 &amp; -1 &amp; 0 &amp; 0 &amp; 2\sqrt{5} &amp; 3\sqrt{5} &amp; 0 \ 2\sqrt{5} &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; 0 &amp; 3 \ 3\sqrt{5} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; -2 \end{bmatrix}$</td>
<td>2</td>
<td>two first order faults</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 &amp; 3 &amp; \sqrt{5} &amp; 0 &amp; 0 &amp; 0 \ 2 &amp; -1 &amp; 0 &amp; 0 &amp; 3 &amp; \sqrt{5} &amp; 0 \ 3 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; 0 &amp; \sqrt{5} \ \sqrt{5} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; -3 \end{bmatrix}$</td>
<td>3</td>
<td>second order fault</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 &amp; 3 &amp; 5 &amp; 0 &amp; 0 &amp; 0 \ 2 &amp; -1 &amp; 0 &amp; 0 &amp; 3 &amp; 5 &amp; 0 \ 3 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; 0 &amp; 5 \ 5 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; -3 \end{bmatrix}$</td>
<td>6</td>
<td>periodic</td>
</tr>
</tbody>
</table>

Table 6.3: Examples of the five possible types of Christoffel tilings in $(1, 3)$ Sturmian systems and the corresponding periodic ribbons in the dual system.

For example, Fig. 6.8 shows a system whose Christoffel tiling contains both a worm and a fault which is not a worm. Here,

$$W = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & 2 \\ 1 & \sqrt{3} \end{bmatrix}$$

The worm arises from the first column, a recurrence vector which happens to lie in the $x_2$-ribbon space. That is, it arises from the relation $q_1 + q_4 = -\sqrt{2}q_3$, and the associated recurrence alternatives $p_1 + p_4 + p_3, p_1 + p_4 + 2p_3$ run along the $x_2$-ribbon.
<table>
<thead>
<tr>
<th>Young Diagram</th>
<th>Example: $W$ and ribbon spaces of $S(W^\perp)$</th>
<th>Number of rational ribbons of $T(0 + W)$</th>
<th>Singularities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{bmatrix} \sqrt{2} &amp; \sqrt{3} &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ \sqrt{3} &amp; -1 &amp; 0 &amp; 0 &amp; -\sqrt{2} &amp; 0 &amp; 0 &amp; 0 \ \sqrt{2} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -\sqrt{2} &amp; 0 &amp; -\sqrt{2} \ \sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; -\sqrt{2} \end{bmatrix}$</td>
<td>0</td>
<td>isolated only</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 \sqrt{2} &amp; \sqrt{3} &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ \sqrt{2} &amp; -1 &amp; 0 &amp; 0 &amp; \sqrt{3} &amp; 0 &amp; 0 &amp; 0 \ \sqrt{3} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; \sqrt{3} &amp; 0 \ \sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; -\sqrt{3} \end{bmatrix}$</td>
<td>1</td>
<td>first order fault</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 \sqrt{2} &amp; \sqrt{3} &amp; 2\sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ \sqrt{2} &amp; -1 &amp; 0 &amp; 0 &amp; \sqrt{3} &amp; 0 &amp; 0 &amp; 0 \ \sqrt{3} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; \sqrt{3} &amp; 0 \ \sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; -\sqrt{3} \end{bmatrix}$</td>
<td>2</td>
<td>two first order faults</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 \sqrt{2} &amp; \sqrt{3} &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ \sqrt{2} &amp; -1 &amp; 0 &amp; 0 &amp; 3 &amp; 0 &amp; 0 &amp; 0 \ \sqrt{3} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; 0 &amp; -3 \ \sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; 0 \end{bmatrix}$</td>
<td>3</td>
<td>second order fault</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 \sqrt{2} &amp; \sqrt{3} &amp; 2\sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ \sqrt{2} &amp; -1 &amp; 0 &amp; 0 &amp; 3 &amp; 0 &amp; 0 &amp; 0 \ \sqrt{3} &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; 0 &amp; -3 \ \sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; -2 &amp; 0 \end{bmatrix}$</td>
<td>4</td>
<td>second order fault, first order fault</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 &amp; 3 &amp; 5 \sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 2 &amp; -1 &amp; 0 &amp; 0 &amp; 3 &amp; 5 \sqrt{2} &amp; 0 &amp; 0 \ 3 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; -2 &amp; 0 &amp; -3 \ 5 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; -2 &amp; 0 \end{bmatrix}$</td>
<td>6</td>
<td>third order fault</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 2 &amp; 3 &amp; 5 \sqrt{2} &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 2 &amp; -1 &amp; 0 &amp; 0 &amp; 3 &amp; 5 \sqrt{2} &amp; 0 &amp; 0 \ 3 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; -2 &amp; 0 &amp; -3 \ 5 &amp; 0 &amp; 0 &amp; -1 &amp; 0 &amp; 0 &amp; -2 &amp; 0 \end{bmatrix}$</td>
<td>10</td>
<td>periodic</td>
</tr>
</tbody>
</table>

Table 6.4: Examples of the seven possible types of Christoffel tilings in $(1, 4)$ Sturmian tiling spaces and the corresponding periodic ribbons in the dual space.
The fault arises from the second column, a recurrence vector which does not lie in any of the four ribbon spaces. That is, it arises from the relation $q_1 + q_2 + 2q_3 = -\sqrt{3}q_4,$ and the associated recurrence alternatives $p_1 + p_2 + 2p_3 + p_4, p_1 + p_2 + 2p_3 + 2p_4$ do not run along any ribbon.

The classification of Christoffel tilings in higher dimensional Sturmian systems is greatly complicated by the possible presence of transverse faults, i.e. faults which do not, like worms, run along some ribbon.
In principle, it is clear that we can hope to characterize the possible worms for specific $p, q$. For instance, consider nondegenerate $(2, 2)$ Sturmian systems. We can write the ribbons of $S(W)$ and $S(W^\perp)$ like this:

\[
\begin{bmatrix}
1 & 0 & -\gamma & \delta \\
0 & 1 & -\beta & \Delta \\
\gamma & \alpha & 0 & \Delta \\
\delta & \beta & \Delta & 0
\end{bmatrix}
\begin{bmatrix}
0 & \Delta & \beta & \alpha \\
\Delta & 0 & \delta & \gamma \\
\beta & \delta & 0 & -\gamma \\
-\alpha & -\gamma & -1 & 0
\end{bmatrix}
\]

where $\Delta = \alpha\delta - \beta\gamma$ is nonzero by the assumption that $S(W)$ is nondegenerate. From this we can make the following observations:

- If $1, \alpha, \beta, \gamma, \delta, \Delta$ are mutually incommensurate, neither $T(0 + W)$ nor $T(0 + W^\perp)$ will have any worms,

- If $1, \alpha, \beta, \gamma, \delta$ are mutually incommensurate, but $\Delta$ is commensurate with one of $\alpha, \beta, \gamma, \delta$, then $T(0 + W)$ and $T(0 + W^\perp)$ each will have precisely one worm,

- If precisely one of $\alpha, \beta, \gamma, \delta$ is rational, but the other three mutually incommensurate, then $T(0 + W)$ and $T(0 + W^\perp)$ each will have precisely one or two worms, depending on whether these three are incommensurate with $\Delta$,

- A full classification of even the possible worms in the Christoffel tiling of even the $(2, 2)$ Sturmian systems would seem to be rather involved (or else this author simply hasn't hit upon the right way of thinking about the problem),

Rather than pursuing an (apparently) ugly theorem, let us illustrate some the possibilities, which may give a better idea of the variety possible even in the singularities of $(2, 2)$ Sturmian systems. At the same time, we will illustrate the principle that what we might call the "number theoretic properties" of the subspace $W$ exert complete control over the character of $S(W)$ in general, and the appearance of its singular tilings in particular.
In Fig. 6.9, at top left we have

\[
W = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\sqrt{2} & \sqrt{2} \\
\sqrt{3} & \sqrt{5}
\end{bmatrix}
\]

which has one recurrence vector which is also a ribbon vector and thus gives a worm, namely

\[
\begin{bmatrix}
0 \\
1/2 \\
\sqrt{5}
\end{bmatrix}
\]

At top right we have

\[
W = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\sqrt{2} & \sqrt{2} \\
\sqrt{3} & 2\sqrt{2}
\end{bmatrix}
\]

which has two recurrence vectors which are also ribbon vectors and thus give worms, namely

\[
\begin{bmatrix}
0 \\
\sqrt{1/2} \\
1/2
\end{bmatrix}, \begin{bmatrix}
-1 \\
1 \\
2\sqrt{2} - \sqrt{3}
\end{bmatrix}
\]

At bottom left we have

\[
W = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
3 & 2\sqrt{2} \\
\sqrt{2} & 3\sqrt{2}
\end{bmatrix}
\]

which has three recurrence vectors which are also ribbon vectors, namely

\[
\begin{bmatrix}
0 \\
\sqrt{1/2} \\
2/3
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
3
\end{bmatrix}, \begin{bmatrix}
3 \\
-2\sqrt{2} + 9
\end{bmatrix}
\]

At bottom right we have

\[
W = \begin{bmatrix}
1 & 2 \\
2 & -1 \\
0 & \sqrt{2} \\
\sqrt{2} & 0
\end{bmatrix}
\]

for which all four ribbon spaces contain recurrence vectors, namely

\[
\begin{bmatrix}
0 \\
5/\sqrt{2} \\
-1 \\
2
\end{bmatrix}, \begin{bmatrix}
5/\sqrt{2} \\
0 \\
2 \\
1/\sqrt{2}
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
0 \\
1/\sqrt{2}
\end{bmatrix}, \begin{bmatrix}
2 \\
-1 \\
0 \\
0
\end{bmatrix}
\]
Figure 6.9: Top left: a Christoffel tiling with one worm. Top right: one with two worms. Bottom left: one with three worms. Bottom right: one with four worms. The first three only have inversion symmetry; the last has $C_4$ symmetry.
In Fig. 6.10 we give an example of a (2, 3) Christoffel tiling with a first and second order worm.

Not very surprisingly, invertible integer square matrices (thought of as composition matrices for oblique tilings) tend to give step symmetrical tiling spaces. These often exhibit attractive patterns of worms; see Figs. 6.11 through 6.13.

We devote the rest of this section to a fairly detailed study of the worms of the (2, 3) system $S(W)$ where $W$ is the invariant plane where the five cycle effects a one fifth turn:

$$W = \begin{bmatrix} 0 & \tau \\ \tau & 0 \\ 1 & -\tau \\ -1 & -1 \\ -\tau & 1 \end{bmatrix}$$

Recall that for this subspace

$$\text{rat } W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and that the Penrose tilings correspond to the orbit closure $\pi(\text{rat } W)$ in the $W$-flow on $T^5$; a tiling from this orbit closure and one from the orbit closure $(1/2)(q_1 + q_2 + q_3 + q_4 + q_5 + \text{rat } W)$, where were illustrated in Fig. 3.15 and Fig. 3.16 respectively. Recall too that the ribbon spaces are spanned by the vectors

$$\begin{bmatrix} 0 \\ \tau \\ 1 \\ -1 \\ -\tau \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \tau \\ 1 \\ -\tau \end{bmatrix}, \begin{bmatrix} 1 \\ -\tau \\ 0 \\ 0 \\ \tau \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since these are evidently all recurrence vectors (in two ways each) we expect to obtain ten worms in the Christoffel tiling. However, the picture in Fig. 4.4 appears to show five worms, not ten. A second look shows that each worm consists of two types of singular patch, so that each worm is really two overlapping worms. One of these is comprised of thin singular hexagons (the male worm) and the other of fat singular hexagons (the female worm). However, close examination shows something far more
$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{2} & 1 \\ 2 & \sqrt{5} \\ \sqrt{3} & -1 \end{bmatrix}$

Figure 6.10: A (2, 3) Christoffel tiling featuring an isolated fifth order singularity at the origin, with a second order worm running vertically and a first order worm running diagonally.
6.4. WORMS

\[ W = \text{col} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \quad \quad L = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \]

Figure 6.11: The Christoffel tiling of a minimal aperiodic \( S(W) \) such that \( \mathcal{O}(W) \) has the composition matrix \( L \); the minimal polynomial is \( \mu_L(z) = z^2 - 2z - 1 \). This is the Ammann octagonal tiling space; the Christoffel tiling has \( D_8 \) symmetry.
Figure 6.12: The Christoffel tiling of a minimal aperiodic \( S(W) \) such that \( \sigma(W) \) has the composition matrix \( L \); the minimal polynomial is \( \mu_L(x) = x^2 - 4x - 1 \). The Christoffel tiling has \( D_4 \) symmetry.

\[
W = \begin{bmatrix}
2 & \sqrt{5} \\
\sqrt{5} & 2 \\
0 & 1 \\
1 & 0
\end{bmatrix}
\quad
L = \begin{bmatrix}
2 & 2 & 1 & 0 \\
2 & 0 & 2 & -2 \\
1 & 0 & 2 & -2 \\
0 & 1 & -2 & 2
\end{bmatrix}
\]
Figure 6.13: The Christoffel tiling of a minimal aperiodic \( S(W) \) such that \( \mathcal{O}(W) \) has the composition matrix \( L \); the minimal polynomial is \( \mu_L(x) = x^4 - 4x^2 + 1 \). The Christoffel tiling has \( D_4 \) symmetry.
puzzling; since each first order patch \( P(F_J) \) corresponds to a zonotope in the window \( Q(F_J) \) of codimension one, we expect four recurrence alternatives. Yet each type of first order singular patch appears to have only two. Also, it is fair to ask why, if the overlapping male and female worms are "independent", their singular patches never overlap; indeed, in the singular Penrose tilings they abut one another without any gaps; we might say they are conjoined in an extremely intimate embrace. Our geometric explanation of this behavior is somewhat involved, but it does provide a good example of how the theory works in a situation which is nontrivial but still fairly simple (we'll illustrate much more complex systems in the next section).

Because of the rotational symmetry of \( S(W) \), up to rotation and translation, there are only two types of singular patches, which we call thin and fat singular hexagons. (See the monoworm tiling in Fig. 4.4 for examples of each type.) Accordingly, we will examine in detail the behavior of "vertical" \( x_1 \) worms.

Let us first focus our attention on understanding the recurrence of singular patches of type \( P(F_{(1,3,4)}) \) (giving the thin singular hexagons in a vertical male worm). Since \( P(F_{(1,3,4)}) \) corresponds to \( Q(F_{(2,5)}) \), we see that the relevant recurrence vector is

\[
\begin{bmatrix}
0 \\
\tau \\
1 \\
-1 \\
-\tau
\end{bmatrix}
\]

According to Theorem 6.4, this gives four recurrence alternatives:

- \( p_2 - p_5 + p_3 - p_4 \)
- \( 2p_2 - p_5 + p_3 - p_4 \)
- \( 2p_2 - 2p_5 + p_3 - p_4 \)
- \( p_2 - 2p_5 + p_3 - p_4 \)

See Fig. 6.14, where each recurrence alternative \( \sum n_j p_j \) corresponds to one of the translates \( Q(F_{(2,5)}) + \sum n_j q_j \) of \( T^*_{(2,5)} \) = \( Q(F_{(2,5)}) \). In this picture, the set of all tilings
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$T(x+W)$ containing the thin hexagonal singular patch $0+P(F_{(1,3,4)})$ is represented by the set of $x$ lying in $0+Q(F_{(2,5)})$ (where it is understood that for convenience we have identified the various resolutions of each singular tiling). As we vary $x$ over this set, we obtain tilings exhibiting each of the four recurrence alternatives to the next highest singular patch in the $x_1$-worm. Because the overlap of $2q_2 - 2q_5 + q_3 - q_4 + Q(F_{(2,5)})$ with $0 + Q(F_{(2,5)})$ is larger than the overlap with the other three translates, the corresponding alternative $2p_2 - 2p_5 + p_3 - p_4$ is chosen more frequently, in $S(W)$ as a whole. However, this frequency is not the same as the frequency in a given tiling $T(y+W)$, or which is the same thing, the frequency as we vary $x$ over the translation orbits of the orbit closure of $T(y+W)$.

Indeed, examining Fig. 6.14 more closely, we can see that the set of *Penrose tilings* containing the singular patch $0 + P(F_{(1,3,4)})$ corresponds to the line segment where the lowermost pentagon intersects $0 + Q(F_{(2,5)})$, and indeed, this happens to only hit two of the four overlaps, namely $q_2 - q_5 + q_3 - q_4 + Q(F_{(2,5)})$ and $2q_2 - 2q_5 + q_3 - q_4 + Q(F_{(2,5)})$. Consequently, only the two alternatives $p_2 - p_5 + p_3 - p_4$ and $2p_2 - 2p_5 + p_3 - p_4$ appear in $x_1$-worms in tilings from this orbit closure (i.e., in Penrose tilings). Indeed, it is not hard to see that in this orbit closure, the short alternative is chosen with frequency $1 - 1/\tau \approx 0.38196$ and the long alternative with frequency $1/\tau \approx 0.61803$, these being the relative lengths of the two line segments concerned.

Compare Fig. 6.15, where we depict the situation for the orbit closure $(1/2)(q_1 + q_2 + q_3 + q_4 + q_5) + \text{rat } W$. This orbit closure does intersect the overlaps missed by the Penrose tilings, but it misses $q_2 - q_5 + q_3 - q_4 + Q(F_{(2,5)})$. Consequently, only the three recurrence alternatives $2p_2 - p_5 + p_3 - p_4$, $p_2 - 2p_5 + p_3 - p_4$, and $2p_2 - 2p_5 + p_3 - p_4$ can occur in tilings in this orbit closure which have an $x_1$-worm.

Indeed, one can show that distinct orbit closures have characteristic frequencies for the recurrence alternatives of thin hexagons. Fig. 6.16 illustrates how the frequencies of the alternatives can vary drastically from tiling to tiling in this system, depending
\[ q_2 - q_5 + q_3 - q_4 + Q(F_{(2,5)}) \]
\[ 2q_2 - q_5 + q_3 - q_4 + Q(F_{(2,5)}) \]
\[ 2q_2 - 2q_5 + q_3 - q_4 + Q(F_{(2,5)}) \]
\[ q_2 - 2q_5 + q_3 - q_4 + Q(F_{(2,5)}) \]

Figure 6.14: The window for the (2,3) Sturmian system \( S(W) \), where \( W \) is the invariant subspace in which the canonical five cycle effects a one fifth turn. At the left side, we have the zonotope \( Q(F_{(2,5)}) \), with the four translates listed above shown as bold face outlines. The first of these alternatives gives the short alternative in the Penrose worm and corresponds to the upper leftmost of the four translates; the third gives the long alternative and corresponds to the lower rightmost of the four translates. The parallel pentagons consist of \( \mathbf{x} \) which are in both rat \( W \) and the window, i.e. \( \mathbf{x} \) such that \( T(\mathbf{x} + W) \) is a Penrose tiling in which the singular patch \( 0 + P(F_{(1,3,4)}) \) appears.
\[ q_2 - q_5 + q_3 - q_4 + Q(F_{(2,5)}) \]
\[ 2q_2 - q_5 + q_3 - q_4 + Q(F_{(2,5)}) \]
\[ 2q_2 - 2q_5 + q_3 - q_4 + Q(F_{(2,5)}) \]
\[ q_2 - 2q_5 + q_3 - q_4 + Q(F_{(2,5)}) \]

Figure 6.15: This picture shows same situation as in Fig. 6.14, except that this time we consider the orbit closure \((1/2)(q_1 + q_2 + q_3 + q_4 + q_5) + \text{rat} \ W\).
on which orbit closure it belongs to.

Let us now turn our attention to singular patches of type \( P(F_{1,2,5}) \) (giving the fat singular hexagons in a vertical female worm). Since \( P(F_{1,2,5}) \) corresponds to \( Q(F_{3,4}) \), we see that the relevant recurrence vector is a rescaling of the recurrence vector we have just been discussing, namely

\[
\begin{bmatrix}
    0 \\
    1 \\
    1/\tau \\
    -1/\tau \\
   -1
\end{bmatrix}
\]

According to Theorem 6.4, this gives four recurrence alternatives:

\[ P_2 - P_5 \]

\[ P_2 - P_5 + P_3 \]

\[ P_2 - P_5 + P_3 - P_4 \]

\[ P_2 - P_5 - P_4 \]

See Fig. 6.17, where each recurrence alternative \( \sum n_j p_j \) corresponds to a translate \( Q(F_{1,2,5}) + \sum n_j q_j \). The top leftmost translate is \( q_2 - q_5 + q_3 - q_4 + Q(F_{3,4}) \), and the bottom rightmost translate is \( q_2 - q_5 + Q(F_{3,4}) \). Once again, we can contrast the frequencies with which these alternatives are chosen, as we vary over the tilings containing the singular patch \( P(F_{1,2,5}) \), with the frequencies within a given tiling, which is the same thing as the frequencies in an orbit closure (ergodic component). In particular, we can once again see at a glance that in the case of the orbit closure \( \text{rat} W \) (the Penrose tilings) only two of the four alternatives are possible, and these are chosen with relative frequencies 0.61803 for \( P_2 - P_5 + P_3 - P_4 \) and 0.38196 for \( P_2 - P_5 \). The frequencies for distinct orbit closures can be very different; see Fig. 6.18, where we can see that in the orbit closure \( (1/2)(q_1 + q_2 + q_3 + q_4 + q_5) + \text{rat} W \), the alternative \( P_2 - P_5 \) doesn't occur at all. However, the frequency of a "left jig" agrees
Figure 6.16: Top left: $x = 0.2(q_1 + q_5)$ pictured in the window. Top right: the corresponding tiling. Bottom left: $x = 0.4(q_1 + q_5)$ pictured in the window. Bottom right: the corresponding tiling.
\[ q_2 - q_5 + Q(F_{(3,4)}) \]
\[ q_2 - q_5 + q_3 + Q(F_{(3,4)}) \]
\[ q_2 - q_5 + q_3 - q_4 + Q(F_{(3,4)}) \]
\[ q_2 - q_5 - q_4 + Q(F_{(3,4)}) \]

Figure 6.17: The window for the (2, 3) Sturmian system \( S(W) \), where \( W \) is the invariant subspace in which the canonical five cycle effects a one fifth turn. At the left side, we have the zonotope \( Q(F_{(3,4)}) \), with the four translates listed above shown as bold face outlines. The parallel pentagons indicate the Penrose orbit closure.
Figure 6.18: This picture shows same situation as in Fig. 6.17, except that this time we consider the orbit closure \((1/2)(q_1 + q_2 + q_3 + q_4 + q_5) + \text{rat } W\).
with the frequency of a "right jig", as must happen if the worm is to remain close to a vertical line in the cutplane.

Once again, the frequencies which the alternatives for fat hexagons are chosen can vary drastically from tiling to tiling, depending on which orbit closure it is in. See Fig. 6.19.

Next, let us consider again the picture of the Penrose tiling $x_1$-worm in Fig. 4.4. Observe that whenever the long alternative $p_2 + p_3 - p_4 - p_5$ is chosen for a fat hexagon, a thin hexagon fits right into the gap. Fig. 6.20 explains why: $q_2 + q_5 + Q(F_{2,5})$ intersects $Q(F_{3,4})$ in exactly the line segment corresponding to the choice of the long alternative. Thus, whenever the long alternative for a fat hex is chosen, a thin hex must fit in the gap. Similarly, $2(q_2 + q_5 + Q(F_{2,5}))$ intersects $Q(F_{3,4})$ exactly in the line segment corresponding to the choice of the short alternative. Thus, whenever the short alternative for a fat hex is chosen, there is a thin hex at $2(p_2 + p_5)$; in other words, two adjacent fat hexes must be succeeded by a thin hex.

Finally, let us consider second order singularities in $S(W)$. Recall that for instance whenever $x \in Q(n + F_{\{1\}})$, then $T(x + W)$ contains the second order singular octagon $P(n + F_{\{3,4,5\}})$. Such a singularity can occur at most once in a given tiling. The Penrose orbit closure is distinguished by having singular tilings with isolated third order singularities (namely, translates of the Christoffel tiling); other orbit closures have singular tilings with an isolated second order singularity, accompanied by four first order worms. For instance, any tiling of the form $T(tq_1 + W)$, where $0 < t < 1$, will have an isolated second order singularity at the origin, namely $P(F_{\{2,3,4,5\}})$, which corresponds to the edge $Q(F_{\{1\}})$ in the window, and four worms corresponding to the four zonotopes $Q(F_{\{1,2\}})$, $Q(F_{\{1,3\}})$, $Q(F_{\{1,4\}})$, and $Q(F_{\{1,5\}})$ which meet the edge $Q(F_{\{1\}})$. An example is shown in Fig. 6.21.
Figure 6.19: Top left: $x = 0.2(q_1 + q_5)$ pictured in the window. Top right: the corresponding tiling. Bottom left: $x = 0.4(q_1 + q_5)$ pictured in the window. Bottom right: the corresponding tiling.
Figure 6.20: This picture shows both why Conway's "worms" have both fat and thin hexagons and also why the overlapping worms, consisting of fat hexagons and thin hexagons respectively, interlock to fit together without gaps.
Figure 6.21: This picture shows the tiling $T(0.5q_1 + W)$, which lies in the orbit closure $(1/2)(q_1 + q_2 + q_3 + q_4 + q_5) + \text{rat } W$, and which has an isolated second order singularity at the origin and four worms (two male and two female). Note the $D_2$ symmetry of this tiling. The two male and two female worms in this tiling take on three of the four recurrence alternatives with frequencies which are characteristic of this particular orbit closure; other orbit closures in this system (except the Penrose orbit closure) have similar tilings, but take on the various recurrence alternatives with different frequencies.
6.5 Fault Planes and Constructing Singular Tilings to Order

So far we have considered only one-dimensional faults; in this section we study higher dimensional faults. In our rather casual survey of worms in (2, 2) Sturmian systems in the previous section, we deliberately stopped just at the point where we would have stumbled over a higher dimensional quasiperiodic array of first order singularities. Consider the Christoffel tiling $T(0 + W)$, where

$$W = \text{col} \begin{bmatrix} 1 & 5 \\ 1 & 3 \\ 1 & 2 \\ \sqrt{2} & \sqrt{3} \end{bmatrix}$$

This has two first order recurrence vectors, given by the columns of the matrix above.

The ribbon spaces are spanned by the columns of the matrix

$$\begin{bmatrix} 0 & 2 & 3 & 5\sqrt{2/3} - 1 \\ 2 & 0 & 1 & 3\sqrt{2/3} - 1 \\ 3 & -1 & 0 & 2\sqrt{2/3} - 1 \\ 5\sqrt{2} - \sqrt{3} & -3\sqrt{2} + \sqrt{3} & -2\sqrt{2} + \sqrt{3} & 0 \end{bmatrix}$$

The first three of these give three separate worms involving singularities of type $P(F_{1,2,3})$; if a tiling in this system has even one occurrence of a patch of type $P(F_{1,2,3})$, it must have a whole quasiperiodic planar array of them, with characteristic recurrence alternatives along the $x_1$, $x_2$, and $x_3$ ribbons; the $x_4$ ribbon, however, threads between all these singularities. See Fig. 6.22.

Incidentally, with respect to the duality between fault lines of one dimensional Sturmian systems and periodic ribbons in the dual, it is worth mentioning that the ribbon spaces of the dual tiling are spanned by the columns of the matrix

$$\begin{bmatrix} 0 & -2\sqrt{2} + \sqrt{3} & 3\sqrt{2} - \sqrt{3} & -1 \\ 2\sqrt{2} - \sqrt{3} & 0 & -5\sqrt{2} + \sqrt{3} & 3 \\ -3\sqrt{2} + \sqrt{3} & 5\sqrt{2} - \sqrt{3} & 0 & -2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$$

Thus, the dual tiling has a single periodic ribbon and no worms.
6.5. FAULT PLANES AND CONSTRUCTING SINGULAR TILINGS TO ORDER

\[
W = \begin{bmatrix}
1 & 5 \\
1 & 3 \\
\sqrt{2} & \sqrt{3}
\end{bmatrix}
\]

Figure 6.22: This picture shows a (2, 2) Christoffel tiling with a first order fault plane and an isolated second order singularity at the origin.
More generally, consider a \((p, q)\) Sturmian system \(S(W)\). Let \(J, K\) be two selections of coordinates, with

\[|J| = p + m, \ |K| = p + n, \ |J \cap K| = p + r, \ 0 < m, n, r < q\]

and suppose \(W\) contains two recurrence vectors of the form

\[u = \sum_{j \in J} n_j e_j + \sum_{j \notin J} \alpha_j e_j\]
\[v = \sum_{k \in K} n_k e_k + \sum_{k \notin K} \alpha_k e_k\]

Here, \(u\) is associated with overlapping codimension \(m\) zonotopes of type \(Q(F_j)\) in the window, i.e. recurring \(m\)-th order singular patches of type \(P(F_j)\), whereas \(v\) is associated with overlapping codimension \(n\) zonotopes of type \(Q(F_k)\) in the window, i.e. recurring \(n\)-th order singular patches of type \(P(F_k)\). Now, \(u + v, u - v\) evidently give (in general) new \(r\)-th order recurrence vectors, in independent directions in \(W\), both involving \(r\)-th order singular patches of type \(P(F_{J \cap K})\). In short, singular patches of type \(P(F_{J \cap K})\) recur in a quasiperiodic pattern of dimension (at least) two. In this situation, this \(r\)-th order fault plane is in general accompanied by linear faults of larger orders \(m, n\). A simplified example is illustrated in Fig. 6.23.

We can sum up this discussion as follows.

**Theorem 6.9** Suppose there is some subspace \(V \subset W\) such that \(V\) is spanned by column vectors of the form

\[
\begin{bmatrix}
  n_{1,1} & \cdots & n_{1,r} \\
  n_{2,1} & \cdots & n_{2,r} \\
  \vdots & \cdots & \vdots \\
  n_{s,1} & \cdots & n_{s,r} \\
  \alpha_{s+1,1} & \cdots & \alpha_{s+1,r} \\
  \alpha_{s+2,1} & \cdots & \alpha_{s+2,r} \\
  \vdots & \cdots & \vdots \\
  \alpha_{d,1} & \cdots & \alpha_{d,r}
\end{bmatrix}
\]

where \(m + n = d\) and \(r \geq n\). Here the \(n_{j,k}\) are integers and the \(\alpha_{j,k}\) are irrational.

Then the Christoffel tiling for \(X(W)\) contains an \(r\)-dimensional fault plane containing isometric \(p\)-th order elementary singularities, where \(p = 1 + s - n\).
Figure 6.23: This picture shows a $(2,3)$ Christoffel tiling with a first fault fault plane of type $P(F_{1,2,3})$, a second order fault of type $P(F_{1,3,4})$, and an isolated third order singularity at the origin.
Theorem 6.9 allows us to construct arbitrarily complicated Christoffel tilings simply by going into higher dimensions. We can for instance have a fifth-order, three dimensional fault plane, a fourth order worm, a third order fault, and several second and first order faults. The "interactions" of the various recurrence vectors can be quite tricky to analyze, since it may not be evident from looking at the ribbon vectors that a recurrence vector not associated with a worm is present (for that matter, it may not be evident that a period is present).
Chapter 7

The Language and Grammar of Sturmian Tilings

In this chapter, we extend some of the themes first seen in our remarks about vertex neighborhoods (Section 4.9), tile frequencies (Sections 4.8), recurrence alternatives for tiles (Section 4.10) and singular patches (Section 6.4). Most of our discussion will focus on the case of $(1,1)$ systems, where we obtain a very satisfactory theory. As we shall see, the difficulties in passing to higher dimensional Sturmian systems are closely analogous to the well known difficulties (see for instance the monograph by Brentjes [12]) of extending the theory of one dimensional simple continued fractions to higher dimensions. (See [55][80][102][104] for some classical accounts; see [66][90][113] for more geometric presentations of the theory of one dimensional simple continued fractions.)

For various reasons (some to become apparent later in this chapter) it will be convenient to switch our conventions when discussing $(1,1)$ systems in this chapter; instead of (3.11) we use

$$C_J = T_J - T_{J^*} = \left\{ \sum_{j \in J} t_j p_j - \sum_{j \in J} t_j q_j : t_j \in [0,1], \ 1 \leq j \leq d \right\} \quad (7.1)$$

While this change may seem disconcerting, note that it doesn’t change anything essential, since our new $(1,1)$ oblique tilings are obtained by inversion $x \to -x$ from the old ones. Moreover, our discussion will focus so much on the case of $(1,1)$ tilings that in the first few sections, unless stated otherwise, the reader may assume that $S(W)$ means a $(1,1)$ Sturmian system where $W$ is a line with slope $0 \leq v \leq 1$. Of course, this is precisely the case which corresponds to the classical Sturmian shifts$^1$.

In Section 7.1, we introduce a geometric construction analogous to the higher block

$^1$The $(1,1)$ systems where $W$ is a line with slope in $\{-v, \pm 1/v\}$ give, for fixed $v > 0$, essentially the same shift.
shift (see for instance [81][87]) and show how this construction permits us to compute the allowed words of length two and their frequencies, which is the same as computing the first order Markov approximation. In Section 7.2 we explore the bifurcations in the Markov approximations of a given order (see [126] for a very clear explanation of the notion of Markov approximations to a shift space), which occur at rational $W$. We show how forgetting the information about frequencies leads to bifurcations in the underlying SFT approximations. In Section 7.3 we show how the classical Stern-Brocot-Farey tree (see for instance [24][48][66][82][90]) organizes these bifurcations, in the case of $(1,1)$ systems, and we discuss a generalization which plays a similar role in higher dimensions (see [47] for a different generalization). In Section 7.4 we turn our attention to the recurrence of words, and show that in the case of $(1,1)$ systems, the recurrence of arbitrary words is controlled by the recurrence of special words called kingdoms. We show how to systematically enumerate the kingdoms and their recurrence alternatives, in the case of $(1,1)$ Sturmian system. This gives an essentially complete picture of how the languages of the classical Sturmian shifts vary as we vary $W$; for instance, we will see how to read off the slope of the line $W$ to any desired degree of accuracy by examining the words of a given Sturmian shift. We also remark on a certain natural parsing into long and short words which generalizes some observations of de Bruijn and of Conway. In Section 7.5, we consider the fundamental concept of empires and determine the empires of given word in the case the classical Sturmian shifts. In Section 7.6 we briefly discuss how this theory generalizes to $(p,q)$ Sturmian systems. The sketchier sections of this chapter will be fleshed out in the on-line version of this document.
7.1 Higher Block Oblique Tilings

Let $S(W)$ be a Sturmian system. Recall from Section 4.8 that for each pair of prototiles,

$$\frac{\text{frq } T_J}{\text{frq } T_K} = \frac{\text{vol } T_J}{\text{vol } T_K}$$

In particular, if $W$ is a line in $\mathbb{R}^2$, passing through the origin with slope $v$, then assuming without loss of generality that $0 < v < 1$, according to (3.15) we have a long prototile of length $\frac{1}{\sqrt{1+v^2}} = \cos \arctan v$ and a short tile $S$ of length $\frac{v}{\sqrt{1+v^2}} = \sin \arctan v$. It follows that the frequency of occurrence of the long tile is $\frac{1}{1+v}$ and the frequency of occurrence of the short tile is $\frac{v}{1+v}$. Recall that these two prototiles correspond to the alphabet of the corresponding classical Sturmian shift. The frequencies just computed give the zeroth order Markov approximation to the shift corresponding to $S(W)$; this is the Bernoulli shift having the same alphabet and letter frequencies as the given shift.

The first order Markov approximation is the first Markov shift having the same alphabet, letter frequencies, and transition frequencies between letters as the given shift. By dissecting and recombining the tiles of the original oblique tiling $O(W)$ we obtain the first higher block oblique tiling $O^{(1)}(W)$ as shown in Fig. 7.1. This is a periodic tiling of $\mathbb{R}^2$ which has three prototiles; reading from top to bottom in the shaded multicell in the top picture shown in Fig. 7.1, we have

1. a small square followed (to its right) by the lower two thirds or so of a large square,

2. the upper two thirds or so of a large square followed by a small square,

3. the lower third or so of a large square followed by the upper third or so of another large square.
The point of this construction is that taking cut lines through \( O(1)(W) \) gives tilings constructed using three prototiles, correspond to the three words \( SL, LS, \) and \( LL \) respectively. This reflects that the fact that the word \( SS \) cannot occur in any \((1,1, \) system \( S(W), \) where \( W \) is a line with slope \( 0 < v < 1. \) That is, the first order SFT approximation to any such classical Sturmian shift is the familiar Golden Mean shift (see [81][87]).

Moreover, recall that the relative heights of the prototiles of \( O(W) \) give the relative frequency of the letters \( L \) and \( S, \) namely \( 1, v \) respectively. In the same way, the relative heights of the three prototiles of \( O(1)(W) \) give the relative frequencies of the words \( SL, LS, LL, \) namely \( 1, 1, 1 - v \) (see Fig. 7.2). We may then readily compute the transition frequencies needed to define the first order Markov approximation by the diagram at the bottom of Fig. 7.2. For instance,

\[
\text{frq}(LS) = \text{frq}(L) \text{frq}(S \text{ follows } L) \\
\text{frq}(LL) = \text{frq}(L) \text{frq}(L \text{ follows } L)
\]

from which we obtain the fact that the relative transition frequencies out of \( L \) are \( 1, 1 - v; \) these happen to be already normalized as just given, and we obtain the diagram at the bottom of Fig. 7.2.

Note that this diagram completely encodes the information more usually represented by a stochastic matrix \( A \) and a Perron eigenvector \( v \)

\[
A = \begin{bmatrix} 1 - v & 1 \\ v & 0 \end{bmatrix}, \quad v = \frac{1}{1 + v} \begin{bmatrix} 1 \\ v \end{bmatrix} \tag{7.2}
\]

where the first component corresponds to the letter \( L \) and the second to the letter \( S. \) For instance, the entry \( A(1,1) = 1 - v \) records the fact that \( L \) follows \( L \) with frequency \( 1 - v, \) whereas the entry \( A(2,1) = v \) records the fact that \( S \) follows \( L \) with frequency \( v. \)

Note that we read compose from right to left, and therefore read indices in the transition matrix from right to left (that is, \( A(i,j) \) records the frequency that state
7.1. **HIGHER BLOCK OBLIQUE TILINGS**

Figure 7.1: The construction of the first higher block oblique tiling $O^{(1)}(W)$ for a (1, 1) Sturmian system $S(W)$. Top: the three protocells correspond to the three words of length two, namely $SL$, $LS$, $LL$. Bottom: the unit cell of the higher block tiling has the twice the volume of the unit cell of the original oblique tiling.
Figure 7.2: Top: the relative heights of the three prototiles of $O^{(1)}(W)$, where $x = \cos \arctan v = 1/\sqrt{1+v^2}$ and $y = \sin \arctan v = v/\sqrt{1+v^2}$, give the relative frequencies of the three words $SL, LS, LL$. Bottom left: From the relative frequencies of $SL, LS, LL$ we may compute the first order transition frequencies between the letters $L, S$ to obtain the first order Markov approximation defined by the diagram. Bottom right: By forgetting the probabilistic structure we obtain the first order SFT approximation defined by this diagram; note that independent of $v$ we obtain the diagram defining the familiar Golden Mean shift.

$j$ is followed by state $i$, and we are use column vectors and right eigenvectors (i.e. column vectors $v$ such that $Av = \lambda v$); this agrees with the conventions in a standard linear algebra course. Most presentations of Markov chains on finite alphabets (see for instance [6][81]) make the opposite convention of reading composition from left to right, reading indices of the transition matrix from left to right, and use of row vectors and left eigenvectors (i.e. row vectors $v'$ such that $v'A = \lambda v'$).

The reader should verify that in (7.2), $Av = v$, corresponding to the fact that $v$ represents an (ergodic) invariant measure on the state space of the Markov chain.
7.2 Bifurcations in Patch Transitions

At the reader may already have guessed, there is an infinite hierarchy of further higher block oblique tilings obtained by dissecting and recombing the tiles of $O(W)$. They are all periodic, but the $n$-th higher block oblique tiling $O^{(n)}(W)$ has a multicell of volume $n$. However, an interesting and extremely significant complication arises as soon as we pass to the next highest tiling in this sequence, $O^{(2)}(W)$. In the previous section, the transition frequencies made sense for any $0 < v < 1$, and we saw that this enabled us to define the first order Markov approximation in terms of $v$, and then by forgetting the probabilistic structure, the underlying first order SFT approximation. Now, however we find that we have a bifurcation in the nature of second order SFT approximation at $v = 1/2$ (see Fig. 7.3).

If $0 < v < 1/2$, we obtain the following second order Markov approximation (see Fig. 7.4):

$$A = \begin{bmatrix} \frac{1-2v}{1-v} & 0 & 1 \\ \frac{1-v}{1-v} & 0 & 0 \\ \frac{1-v}{0} & 1 & 0 \end{bmatrix}, \quad v = \frac{1}{1+v} \begin{bmatrix} 1 - v \\ v \\ v \end{bmatrix} \quad (7.3)$$

where the first, second and third components respectively refer to the words $LL, LS, SL$. In (7.3) the Perron eigenvector $v$ is obtained from the relative heights of the prototiles of $O^{(1)}(W)$, and the transition frequencies are obtained from the relative heights of the prototiles of $O^{(2)}(W)$, as shown in Fig. 7.4.

On the other hand, if $1/2 < v < 1$, we obtain the following second order Markov approximation (see Fig. 7.5):

$$A = \begin{bmatrix} 0 & 0 & \frac{1-v}{2v-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad v = \frac{1}{1+v} \begin{bmatrix} 1 - v \\ v \\ v \end{bmatrix} \quad (7.4)$$

where once again the first, second and third components respectively refer to the words $LL, LS, SL$. In (7.4) the Perron eigenvector $v$ is the same as in (7.3); the transition frequencies are obtained from the relative heights of the prototiles of $O^{(2)}(W)$, as shown in Fig. 7.5.
Figure 7.3: The bifurcation in $\mathcal{O}^{(2)}(W)$ at $v = 1/2$. Top: the appearance of $\mathcal{O}^{(2)}(W)$ for $0 < v < 1/2$ (schematic). Middle: $v = 1/2$ (schematic). Bottom: $1/2 < v < 1$ (schematic). As $v$ increases, a pair of critical steps (shaded) move down and to the right until they simultaneously touch the boundaries of the unit cell, then continue down and to the right. Below the critical value, LLL is permitted while SLS is forbidden. Above this value, LLL is forbidden but SLS is permitted.
Figure 7.4: The second order Markov approximation for slopes $0 < v < 1/2$. Top: the appearance of $O^{(2)}(W)$, where $x = \cos \arctan v, y = \sin \arctan v$ as before. Bottom: the diagram defining the Markov approximation; the diagram defining the underlying SFT is obtained by erasing the transition frequencies.
Figure 7.5: The second order Markov approximation for slopes $1/2 < v < 1$. Top: the appearance of $\mathcal{O}^{(2)}(W)$. Bottom: the diagram defining the Markov approximation; the diagram defining the underlying SFT is obtained by erasing the transition frequencies.
Next, consider the third order Markov approximations for the case $0 < \nu < 1/2$. We find that there is a bifurcation at $\nu = 1/3$. If $0 < \nu < 1/3$, we obtain the following third order Markov approximation (see Fig. 7.6):

$$A = \begin{bmatrix}
\frac{1-3\nu}{1-2\nu} & 0 & 0 & 1 \\
\frac{1-2\nu}{1-2\nu} & 0 & 1 & 0 \\
\frac{1-2\nu}{1-2\nu} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}, \quad \nu = \frac{1}{1+\nu} \begin{bmatrix}
1 - 2\nu \\
\nu \\
\nu \\
\nu \\
\end{bmatrix} \quad (7.5)$$

where the first through fourth components correspond respectively to the states $LLL, LLS, LSL, SLL$. In (7.5) the Perron eigenvector $v$ is obtained from the relative heights of the prototiles of $\mathcal{O}^{(2)}(W)$, and the transition frequencies are obtained from the relative heights of the prototiles of $\mathcal{O}^{(3)}(W)$, as shown in Fig. 7.6.

On the other hand, if $1/3 < \nu < 1/2$, we obtain the following third order Markov approximation (see Fig. 7.7):

$$A = \begin{bmatrix}
0 & 0 & 0 & \frac{1-2\nu}{3\nu-1} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}, \quad \nu = \frac{1}{1+\nu} \begin{bmatrix}
1 - 2\nu \\
\nu \\
\nu \\
\nu \\
\end{bmatrix} \quad (7.6)$$

where the first through fourth components correspond respectively to the states $LLL, LLS, LSL, SLL$. In (7.6) the Perron eigenvector $v$ is the same as in (7.5); the transition frequencies are obtained from the relative heights of the prototiles of $\mathcal{O}^{(3)}(W)$, as shown in Fig. 7.7.

At this point, the reader might well expect a similar bifurcation in the third order Markov approximations for the case $1/2 < \nu < 1$ at $\nu = 2/3$, but this does not occur. Rather, for $1/2 < \nu < 1$, the third order Markov approximation is given by (see Fig. 7.8):

$$A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & \frac{1-\nu}{2\nu-1} & 0 & 0 \\
0 & \frac{2\nu-1}{\nu} & 0 & 0 \\
\end{bmatrix}, \quad \nu = \frac{1}{1+\nu} \begin{bmatrix}
1 - \nu \\
\nu \\
1 - \nu \\
2\nu - 1 \\
\end{bmatrix} \quad (7.7)$$

where the first through fourth components correspond respectively to the states $LLS, LSL, SLL, SLS$. In (7.7) the Perron eigenvector $v$ is obtained from the relative heights of the prototiles of $\mathcal{O}^{(2)}(W)$, and the transition frequencies are obtained from the relative heights of the prototiles of $\mathcal{O}^{(3)}(W)$, as shown in Fig. 7.8.
Figure 7.6: The third order Markov approximation for slopes $0 < v < 1/3$. Top: the appearance of $O(3)(W)$. Bottom: the diagram defining the Markov approximation; the diagram defining the underlying SFT is obtained by erasing the transition frequencies.
Figure 7.7: The third order Markov approximation for slopes $1/3 < v < 1/2$. Top: the appearance of $C^{(3)}(W)$. Bottom: the diagram defining the Markov approximation; the diagram defining the underlying SFT is obtained by erasing the transition frequencies.
Figure 7.8: The third order Markov approximation for slopes $1/2 < v < 1$. Top: the appearance of $O(3)(W)$. Bottom: the diagram defining the Markov approximation; the diagram defining the underlying SFT is obtained by erasing the transition frequencies.
7.3. **The Stern-Brocot-Farey Tree**

In the same way, we can determine the fourth order Markov approximations (see Fig. 7.9) and the underlying fourth order SFT approximations, and still higher order approximations.

7.3 **The Stern-Brocot-Farey Tree**

Plainly we need some way of organizing the bifurcations in the SFT approximations discussed in Section 7.2. It is clear why these bifurcations occur at rational slopes, and the crucial clue is that they occur first at 1/2, then at 1/3, 2/3, and so forth. This suggests, particularly if the reader is familiar with the role of the Stern-Brocot-Farey tree in organizing the simple continued fraction expansions of the slopes $0 < v < 1$ (see [66]), that this binary tree yields the organizing principle we seek.

The author has introduced a coding of the Farey intervals by unimodular matrices which turns out to be very useful for our further work. The idea to interpret the columns of the matrix as the left and right endpoint of a Farey interval. This is closely related to an interesting algebraic-geometric "divide and conquer" reinterpretation of the classical simple continued fraction algorithm. The essential points are these:

1. There is a bijection between unimodular matrices, leaves of the Stern-Brocot-Farey tree, Farey intervals, and SFT approximations to classical Sturmian shifts.

2. Each downward path in the tree corresponds to a line $W$ with a particular slope; each slope $0 < v < 1$ yields a different language.

3. The endpoints of each Farey interval encountered as one descends this path give the "best rational approximations" to $W$; these approximations converge quadratically (or faster) to $W$.

4. The simple continued fraction expansion of $v$ can be obtained by repeated "even division" of each Farey interval and choosing the right or left half according to
Figure 7.9: The fourth order Markov approximations for slopes $0 < v < 1$. Top left: $0 < v < 1/4$. Top right: $1/4 < v < 1/3$. Middle left: $1/3 < v < 1/2$. Middle right: $1/2 < v < 2/3$. Bottom: $2/3 < v < 1$. 
Figure 7.10: The bifurcations in the SFT approximations are organized by the Stern-Brocot-Farey tree, the same binary tree which organizes the simple continued fractions expansions of the slopes $0 < v < 1$. At top, we have the Golden Mean shift, the first order SFT approximation common to all classical Sturmian shifts, which bifurcates at $v = 1/2$ into two second order SFT approximations. The left child bifurcates at $v = 1/3$ into two third order SFT approximations; right child bifurcates at $v = 2/3$ into two fourth order SFT approximations (there is only one third order SFT approximation compatible with this second order approximation, as the reader may verify). (In this picture, we have replaced the letter $L$ by 0 and the letter $S$ by 1.)
which of the contains \( v \); this corresponds dynamically to applying the inverse of one of the two generators

\[
L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}
\]

5. The simple continued fraction expansion

\[
v = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}
\]

corresponds to the matrix product \( L^{a_1} R^{a_2} L^{a_3} \ldots \) or \( R^{a_1} L^{a_2} R^{a_3} \ldots \).

6. Applying this algorithm to a rational slope line recovers the euclidean algorithm which computes the greatest common divisor of two integers (the components of a vector spanning \( W \)).

7. The “even division” of each Farey interval is made by taking the Farey mediant of the endpoints.

See [62][66] for details.

### 7.4 Kingdoms and Recurrence Alternatives

The key observation about the SFT graphs (sometimes called Rauzy graphs [9]) which appear in Fig. 7.10 is that they all have the form of two loops with a common stretch; see Fig. 7.11. This common stretch represents a word which can be extended in two ways at each end. Such a word is called a kingdom. It turns out that the kingdoms are the key to understanding Sturmian languages.

The main points here are as follows:

1. Each leaf in the Stern-Brocot-Farey tree is associated with a unique kingdom.
2. For each classical Sturmian shift, the protokingdoms which appear in its language form a linear sequence in which each protokingdom is the "head" of its successor.

3. Each protokingdom is a palindrome.

4. The limit of the sequence of protokingdoms is (the right half of) the Christoffel word, corresponding to (the right half of) the tiling \( T(0 + W) \).

5. Each slope \( v \) determines and is determined by a unique sequence of protokingdoms; that is, the sequence of protokingdoms completely determines the language and indeed the shift itself.

6. Each kingdom recurs with precisely two first recurrence alternatives; we can either go around the short loop (short alternative) or the long loop (long alternative).

7. We can read these alternatives off the unimodular matrix encoding the leaf associated with a particular kingdom; for example, in Fig. 7.11, the long alternative
is four zeros and three ones (read off the right column) and the short alternative
is three zeros and two ones (read off the left column).

8. The kingdom corresponding to the left child is the new word created by taking
the short alternative and likewise for the right child. (There is never a gap
between the kingdom and its next occurrence, so this procedure, amazingly
enough, does in fact make sense.)

9. Every word $\alpha$ is contained in a unique smallest kingdom $K(\alpha)$, and has the
same recurrence properties as this kingdom.

10. The upshot is an algorithm for systematically reading off the Rauzy diagrams
defining the SFT approximations to $S(W)$, from the continued fraction expa-
sion of the slope of $W$. (This algorithm is described using two examples in [62].)

11. $O(W)$ has a composition iff the continued fraction expansion of the slope of $W$
is periodic, and then the substitution induced by the oblique tiling composition
can be read off from the kingdom at the first repeat.

See [62][66] for some more details.

7.5 Empires of Classical Sturmian Shifts

The empire $E(\alpha)$ of a word $\alpha$ is the set of all words which appear in any sequence in
the shift $X$ in which $\alpha$ appears; see Appendix A. In [62] the author has explained this
concept in more detail, with examples using Toeplitz shifts (e.g. the Adding Machine
shifts) and Sturmian shifts. In the latter case, it turns out that the empire of $\alpha$ consists
of $K(\alpha)$ flanked by strictly smaller outlying kingdoms; see Figs. 7.12 and 7.13. The
imperial terminology introduced by Conway [41] has of course inspired the author's
coinage of "kingdom" to describe the components of an empire.
Figure 7.12: The empire of the word $S$ in the Fibonacci shift consists of the kingdom $LSL$ flanked by a quasiperiodic array of kingdoms $L$. 
Figure 7.13: The empire of the word $LL$ in the Fibonacci shift consists of the kingdom $LSLLSL$ flanked by a quasiperiodic array of kingdoms $LSL$ and $L$. 
On another occasion the author will describe an algorithm for obtaining the complete empire of a given word $\alpha$ by shifting the Christoffel word (or rather the upper and lower resolutions of the Christoffel tiling) against itself; the appropriate shifts are again read off the unimodular matrix associated with the leaf of the Stern-Brocot-Farey tree which corresponds to the kingdom $K(\alpha)$.

7.6 Generalizations to Higher Dimensional Sturmian Systems

We conclude with a few brief remarks on how this picture generalizes to higher dimensional Sturmian systems. In Fig. 7.14 we illustrate the fundamental empire-cylinder duality; in this picture, the dark shaded vertex neighborhood forces all the light shaded tiles. Every tiling containing this patch occurs as a global section through a sheaf in the manner suggested by the remaining part of the picture; see [62] for some more details on this idea, which the author intends to return to on another occasion [64].

The main points of difference with the $(1,1)$ case are these:

1. The kingdoms do not form a linear sequence,

2. The smallest kingdom containing a given patch may not be finite.

In Figs. 7.15 to 7.17 we have depicted some typical empires in another the $(2,2)$ Sturmian system, the octagonal Ammann tilings. These pictures are striking visual testament to the “rigidity” of this system.

In Fig. 7.18 we illustrate an empire in the space of Penrose tilings. When $S(W)$ is not minimal, one may speak of empires within $S(W)$ or within an orbit closure; these will in general be quite different. Ammann is said to have determined the empires of the vertex neighborhoods of the Penrose tilings [53]. However, his results, as illustrated in [53], appear to conflict with the authors; specifically, certain tiles which are not in the empire according the figures in [53] are in fact forced, according
Figure 7.14: The empire (light shaded patch) of a typical patch (dark shaded tiles) in a (2, 2) Sturmian system $S(W)$. 
Figure 7.15: The empire (light shaded tiles) of the "ray" vertex neighborhood (dark shaded tiles) in the Ammann tiling system.
Figure 7.16: The empire (light shaded tiles) of the "squid" vertex neighborhood (dark shaded tiles) in the Ammann tiling system.
Figure 7.17: Because a "ray" occurs in the empire of the "squid", the entire empire of the "ray" (dark shaded tiles) must be included in the larger empire (light shaded tiles).
to the author. (These additional tiles do appear, as they should, in the pictures in [53].)
Figure 7.18: The empire (light shaded) of the "nun" vertex neighborhood (dark shaded tiles) in the Penrose component of the (2, 3) step symmetric Sturmian system.
Chapter 8

Projective Homotopy of Sturmian Tilings

The projective fundamental group is an attractive new topological conjugacy invariant which was defined for $\mathbb{Z}^d$ symbolic shift spaces and tiling spaces by Geller and Propp [43]. In this chapter, we provide some insights into what appears to be a difficult problem, namely to compute these groups for an arbitrary Sturmian system.

Before proceeding, we should point out that projective homotopy groups are not the only group theoretic conjugacy invariants which may be defined for Sturmian systems. In particular, the dimension group of certain Sturmian systems (see for instance [87]) can be computed by elementary means, and much work as been done recently on defining and more sophisticated K-theoretic invariants of symbolic dynamical systems; see the review paper by Putnam [110] and the paper by A. Hof [72]. Indeed, in his seminal book [17], Alain Connes used Penrose tilings as a central motivating example for the notion of a noncommutative geometry\(^1\) and Anderson and Putnam [2] have been able to compute such invariants for the space of Penrose tilings by taking advantage of the composition map.

In the sequel, due to lack of time and space, we assume that the reader is familiar with [43].

8.1 Projective Homotopy Groups

In [43], Geller and Propp were able to compute their projective homotopy groups for a number of examples derived from statistical mechanics and other sources. These examples share certain dynamical characteristics: they have positive entropy and have

\(^1\)Be warned that Connes' discussion of Penrose tilings has been characterized by another expert as "fundamentally misleading"; the author is insufficiently expert to comment on this controversy.
8.1. PROJECTIVE HOMOTOPY GROUPS

a certain "flexibility" which, in particular allows Geller and Propp to "modulate" a given symbolic patch into a special one within a bounded region of the plane—this allows them to establish projective connectivity, the analog of path connectedness for projective homotopy, and also the triviality of the group in question. (Geller and Propp also considered some examples of shifts which are the quotients of such shifts, which serve as something analogous to a universal cover; this allows them to compute some nontrivial projective fundamental groups as well.)

More recently, Schmidt [123] has shown how the well known tiling groups introduced by Conway and Lagarias [18] are related to the projective homotopy groups, and has used sophisticated cohomological techniques to compute the projective homotopy groups for some additional examples; however, his approach has apparently not yet permitted the computation of these groups for "rigid" dynamical systems.

Our goal in this chapter is very limited— to report briefly on the author's inconclusive attempts to extend the "hands-on" methods of Geller and Propp, involving the concrete manipulation of paths in scene space, to dynamical situations which are unlike the examples considered in [43][123]. For technical reasons having to do with the "hands on" nature of this approach, the goal was to compute the projective homotopy groups of the Robinson shifts rather than the Sturmian tiling spaces themselves. We stress at the outset that no attempt has been made here to take advantage of the composition maps defined on certain Sturmian tiling spaces (and their Christoffel orbit closure, such as the space of Penrose tilings). The plan was to approach the extreme rigidity of Robinson shifts via some intermediate stepping stones; see Table 8.1. This approach was partially successful at best. in particular, we will not be able to compute any groups for examples drastically different from those already treated in [43][123]. Nonetheless, we can describe here preliminary (and inconclusive) evidence for the existence of two opposing trends as one approaches the dynamical domain of extreme rigidity (as exemplified by Robinson shifts): apparently, the num-
<table>
<thead>
<tr>
<th>Property</th>
<th>Shifts in [43][123]</th>
<th>Ledrappier Shift</th>
<th>Chair Shift</th>
<th>Sturmian Shifts</th>
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</thead>
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<tr>
<td>Mixing?</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Entropy</td>
<td>positive</td>
<td>zero</td>
<td>zero</td>
<td>zero</td>
</tr>
<tr>
<td>Number of n x n scenes</td>
<td>$2^{cn^2}, 0 &lt; c &lt; 1$</td>
<td>$2^{2n-1}$</td>
<td>?</td>
<td>$cn$</td>
</tr>
<tr>
<td>Periodic Points</td>
<td>lots</td>
<td>some</td>
<td>none</td>
<td>none (generic)</td>
</tr>
<tr>
<td>Substitution Shift?</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>some (nongeneric)</td>
</tr>
<tr>
<td>Primitive Cmpts.</td>
<td>one</td>
<td>one</td>
<td>many</td>
<td>?</td>
</tr>
<tr>
<td>Projective Path Cmpts.</td>
<td>finitely gen.</td>
<td>inf. gen.?</td>
<td>nontrivial</td>
<td>nontrivial (sometimes?)</td>
</tr>
<tr>
<td>Projective Fnd. Grps.</td>
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<td></td>
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</table>

Table 8.1: Comparing the dynamical properties of some $\mathbb{Z}^2$-shifts.

The number of generators (associated with "rod-like" obstructions in the scene spaces\(^2\)) goes to infinity, but so does the number of path components. Indeed, it may be that translates of a given tiling are in different path components. It is possible that the group actually varies from place to place in (say) the Penrose tiling space, with one value associated with the tilings in the same projective path component as a cartwheel, another with those in the projective path component of a monoworm singular tiling, and other base points having perhaps trivial fundamental group.

### 8.2 Homotopically Projective Paths and Prohomotopy

It is useful to reformulate the definitions found in [43] in a different way. First consider paths in an inverse system of path connected topological spaces,

$$X = \lim_{E \in \mathcal{E}} X_E$$

where $X_E$ are spaces indexed by a directed set $\mathcal{E}$, and where whenever $E \leq F$ we have a continuous map $E \xrightarrow{\psi_E^{EF}} F$.

A point is $X$ is a sequence of points $x_E$, where each $x_E \in X_E$, such that whenever $E \leq F$, $x_E = \psi_E^{EF}(x_F)$. Similarly, a path from $x$ to $y$, where $x, y, \in X$, is a sequence of paths $(\gamma_E)$, where each $\gamma_E$ is a path in $X_E$ from $x_E$ to $y_E$, such that whenever

\(^2\)That is, obstructions to contracting loops, like a cylindrical "bubble" in scene space that you can loop around.
$E \leq F$, $\gamma_E = \psi_*^{EF}(\gamma_F)$. (Here the $\psi_*^{EF}$ are the usual induced maps on the homotopy groupoids of the space $X_E$.) In particular, if $\mathcal{X}$ is a $\mathbb{Z}^d$ shift space, for each bounded window $E$ we can take $X_E = S_E(\mathcal{X})$, the scene space as defined in [43], and then $X$ is the pattern-position space $\mathbb{R}^d \times \mathcal{X}$, which has the property that for each $x \in \mathcal{X}$ and each $s, t \in \mathbb{R}^d$, the point $(s, x)$ is path connected to $(t, x)$. However, since $\mathcal{X}$ is in general homeomorphic to a Cantor set, if $x \neq y$, then $(t, x)$ is never connected by a projective path to $(t, y)$.

Now define a homotopically projective path from $x$ to $y$ to be a sequence of paths $\gamma_E$ from $x_E$ to $y_E$, such that whenever $E \leq F$, $\gamma_E$ is homotopic relative to its basepoints to $\psi_*^{EF}(\gamma_F)$. Then we can define the projective path components to the set of points in $X$ reachable from a given point by homotopically projective paths.

This now gives a notion of homotopically projective path connectedness which agrees with the notion of projective connectivity defined in [43], and it is possible for distinct $x, y$ in pattern-position space to be connected by a homotopically projective path.

Given two homotopically projective paths, say $\gamma$ and $\omega$, we say that they are pro-homotopic if for each $E$, $\gamma_E$ is homotopic to $\omega_E$ relative to their common basepoints. We can now compose homotopically projective loops based at $x$ to obtain a group, the prohomotopy group, which agrees with the projective fundamental group defined in [43]. This group is constant as we vary the base point $(t, x)$ by changing $t$ while keeping $x$ fixed, and if we vary $x$ within a single homotopically projective path component, but may change if move to another component.

It would be natural to continue this line of thought by attempting to develop a theory of procovering spaces which would be analogous to the homotopy theory of ordinary covering spaces; however, it turns out that while many proofs go through without change, this theory appears not to be as useful as the author had hoped.

Examining the methods of Geller and Propp shows that their "hands-on" computations may be viewed as pushing around local sections, where we regard each scene space as a non-Hausdorff sheaf over $\mathbb{R}^d$. According to this point of view, pattern-
position space should also be viewed as a non-Hausdorff sheaf over \( \mathbb{R}^d \). However, this viewpoint seems to break down for \( \mathbb{R}^d \) actions (as opposed to \( \mathbb{Z}^d \) actions), where one can deform local sections "vertically" (as in a fiber-bundle) as well as "horizontally"; this appears to make the homotopically projective paths of \( \mathbb{R}^d \) actions more difficult to visualize geometrically. For this reason, we shall focus on trying to compute the projective homotopy groups of the Robinson shifts, rather than the projective homotopy groups of the Sturmian tiling spaces themselves.

### 8.3 The Ledrappier Shift

The Ledrappier shift \( \mathcal{L} \) is the \( \mathbb{Z}^2 \) symbolic shift defined over the alphabet \( \mathbb{Z}_2 \) by the rule

\[
x(m, n) + x(m + 1, n) + x(m, n + 1) \equiv 0 \mod 3
\]

where \( x : \mathbb{Z}^2 \to \mathbb{Z}_2 \). This shift was originally introduced by Ledrappier as a simple example of a dynamical system which is 2-mixing but not 3-mixing.

It is indeed topologically mixing, is a group shift (i.e. an additive group as a set of binary sequences), has a dense set of periodic points, is in fact a shift of finite type (see Figure 8.1), is not (evidently) a substitution shift, but has zero topological entropy; indeed, the number of \( n \) by \( n \) scenes is \( 2^{2n-1} \); clearly this number must grow like \( 2^{cn^2} \) for some \( 0 < c < 1 \) in order to have positive entropy.

The Ledrappier shift thus exhibits dynamical behavior intermediate between the "extreme flexibility" of the shifts of finite type considered in [43][123], which have positive entropy, are transitive, projectively connected, and often mixing, and the Robinson shifts (which are (usually) minimal, have zero entropy, in certain cases are substitution shifts, and have extremely slow growth of the number of patches of a given area, as the area increases).

In the case of the Ledrappier shift, the empires of (most) finite patches have a characteristic triangular shape; see Figure 8.2. The reason for this is simple. Suppose
8.3. THE LEDRAPPIER SHIFT

Figure 8.1: The eight forbidden two by two scenes defining the Ledrappier shift as a shift of finite type. The remaining eight scenes are all permitted.

Figure 8.2: The empire of a patch (grey) in the Ledrappier shift is (with some exceptions) the smallest “triangle” covering that patch; by definition, the symbols in the empire are completely determined by the original patch.
we fix \( x(m, n) = s \), where \( s \in \mathbb{Z}_2 \), and try to extend \( x \) to a legal element of the Ledrappier shift. If we choose a value for \( x(m, n - 1) \), this forces a particular value of \( x(m + 1, n - 1) \).

**Lemma 8.1** Let \( E \) denote a \( n \) by \( n \) square of "symbolic places". Every "\( E \)-shaped patch" or \( E \)-scene \( x|(E + n) \), where \( n \in \mathbb{Z}^2 \), can be extended to an element \( x \) of the Ledrappier shift \( \mathcal{L} \) having all zeros below a diagonal line running from upper left to lower right.

**Proof:** It suffices to consider the case \( x|E \). "Seed" the rows below \( E \) with diagonal ones and likewise "seed" the columns to the left of \( E \) with diagonal ones. (See Figure 8.3.) This forces all zeros below the diagonal row of ones, and gives an element \( x \) of the Ledrappier shift which contains the required patch \( E \). (Note that we have no control over the elements lying below the diagonal running through the upper right corner of \( E \); these are completely determined by \( x|E \) and our choice of a diagonal row of ones running just below the lower left corner of \( E \).)

**Theorem 8.2** The Ledrappier shift is projectively connected.
8.3. THE LEDRAPPIER SHIFT

Proof: Let \( x \in \mathcal{L} \) and let \( z \) be the all zeros element of \( \mathcal{L} \). We will construct a projective path from \(((0,0), x)\) to \(((0,0), z)\). Naturally this will imply that if \( y \in \mathcal{L} \) then by composition we can obtain a projective path from \(((0,0), x)\) to \(((0,0), y)\).

Let \( F \) be a square window. We can easily define a path \( \gamma_F \) in \( S_F(\mathcal{L}) \) from \(((0,0), x|F)\) to \(((0,0), z|F)\) as suggested by Figure 8.4.

Now suppose \( E \subset F \) are two square windows, with \( \Psi^{FE} \) the canonical projection from \( S_F(\mathcal{L}) \) to \( S_E(\mathcal{L}) \). We must show that \( \gamma'_E = \Psi^{FE}_*(\gamma_F) \) is homotopic to \( \gamma_E \). “Seed” the columns to the left of \( E \) with diagonal ones as suggested by Figure 8.5. (Note that everything shown in this figure is completely determined by this seeding plus \( \gamma'_E \).) Now we can homotope \( \gamma'_E \) to \( \gamma''_E \) as suggested by the Figure, and \( \gamma''_E \) is plainly homotopic to \( \gamma_E \) as suggested by Figure 8.6. Combining these gives the desired homotopy from \( \gamma'_E \) to \( \gamma_E \).

This means that the projective fundamental groups of \( \mathcal{L} \) at different base points \( x \in \mathcal{L} \) are always conjugate to one another, so we may as well speak of “the projective fundamental group of \( \mathcal{L} \)”.

Let \( z \) be the all zeros section and let \( p \) denote the section with all zeros except for a row of diagonal ones running through the origin from upper left to lower right. It is easy to construct a nontrivial loop (of infinite order) in each scene space \( S_F(\mathcal{L}) \) for square windows \( F \), as suggested in Figure 8.7, and the sequence of these loops is easily seen to form a projective loop in \( \mathcal{L} \).

Another way to see this is to consider “holed tilings”. The top right “hole” shown at right in Figure 8.8 can be enlarged and or moved diagonally by Conway moves [41][53], which consist of adding or removing square tiles (each marked with a zero or a one), such that each square neighborhood agrees with one of the legal Ledrappier neighborhoods shown in Figure 8.1. The hole can also be split into two, each of which can be independently moved along diagonally, but clearly the hole cannot be eliminated altogether by Conway moves. On the other hand, the hole at
Figure 8.4: The first half of a path from \((0, 0), z|F\) to \((0, 0), z|F\). The second half of the path lies entirely on the \(z\) section and runs back underneath the local section (shown here) which is defined by the first half of the path.
8.3. THE LEDRAPPIER SHIFT

Figure 8.5: The first half of the paths $\gamma_E'$ (vertical, moving down) and $\gamma''_E$ (horizontal, moving left). The second halves of both paths lie entirely on the $z$ section. The local section shown here is sufficiently "large" that we can obtain a linear homotopy from $\gamma_E'$ to $\gamma''_E$. The dotted rectangles suggest the "shape" of the path $\gamma_F$ in the higher scene space $S_F(\Omega)$. 

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Figure 6.6: The first half of the paths $\gamma_1$ (horizontal, moving left) and $\gamma_2$ (vertical, moving down) are sufficiently large that we can obtain a linear homotopy from $\gamma_1$ to $\gamma_2$.

<table>
<thead>
<tr>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 1</td>
<td>1 0 1</td>
</tr>
<tr>
<td>0 0 1</td>
<td>1 0 0</td>
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<tr>
<td>0 0 0</td>
<td>0 0 1</td>
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<tr>
<td>1 0 1</td>
<td>0 0 0</td>
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</table>

The second halves of both paths lie entirely on the $z$ section. The local section shown here is...
8.4. **THE FREE PARTICLE SHIFT**

bottom left (in the lefthand picture) can be moved diagonally but not vertically or horizontally, whereas the one at top left can be moved arbitrarily, and both can be completely eliminated as shown on the right.

The holed tiling shown in Figure 8.9 corresponds clearly to the cyclic group generated by the loop shown in Figure 8.7. Moreover, shifting the diagonal row of ones up one place apparently gives a distinct generator.

**Conjecture 8.3** *The projective fundamental group of \( \mathcal{L} \) is not only infinite but in fact infinitely generated.*

### 8.4 The Free Particle Shift

The **Free Particle Shift** \( \mathfrak{F} \) is obtained from the Ledrappier shift by extracting the "diagonal rows of ones (free particles) against a field of zeros" which appear in the Ledrappier shift. That is, define a shift of finite type \( \mathfrak{F} \) by stipulating that each two by two scene agree with one of the four shown in Figure 8.10. Note that \( \mathfrak{F} \) is essentially a *one dimensional* shift of finite type; taking the action to be generated by \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) gives the full two shift whereas taking the action to be generated by \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) gives the Golden Mean shift.

It is easy to see that the Free Particle shift is projectively connected; see Figure 8.12. However, \( F \) is not topologically mixing, because there is obviously no way the two by two scene shown at far left in Figure 8.10 can appear together with the one shown third from left with relative separation of form \( \begin{pmatrix} n \\ -n \end{pmatrix} \). Thus, projective connectivity does not imply topological mixing. On the other hand, as Geller and Propp remarked in [43], topological mixing implies that all the scene spaces are path connected (this condition does not imply projective connectivity). Geller and Propp further remarked in [43] that either topological mixing or projective path connectedness implies primitivity (that is, the shift space is a transitive \( \mathbb{Z}^2 \)-set without finite quotients).
Figure 8.7: A nontrivial loop in $S_E(\mathcal{L})$, $E$ a square window. Top: the first leg of the loop. Middle: the middle leg of the loop. Bottom: the last leg of the loop.
8.4. THE FREE PARTICLE SHIFT

Figure 8.8: A holed Ledrappier tiling with two nonessential holes and an one essential hole; the black diagonal bands schematically indicate "ones against a field of zeros". On the right, the essential hole has been separated into two parts. These can be recombined or moved diagonally, but never entirely eliminated.

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Figure 8.9: The nontrivial holed tiling for $\mathcal{L}$ corresponding to the cyclic subgroup generated by the homotopically projective loop shown in Figure 8.7.

Figure 8.10: The four permitted two by two scenes in the Free Particle shift.
Figure 8.11: The combinatorial constraints on words in the one dimensional version of the Free Particle shift, under horizontal translation.

Figure 8.12: The first half of a path $\gamma_F$ from $((0,0), z|F)$ to $((0,0), z|F)$ in the Free Particle shift. We simply continue the lowest diagonal row of ones and place a vertical translate of $z|F$ immediately below the row where this diagonal row runs off the picture to the right. The necessary homotopy from $\Psi^F_E(\gamma_F)$ to $\gamma_E$, where $E \subseteq F$, may now be easily constructed.
8.5. THE CHAIR SHIFT

Figure 8.13: The four prototiles of a chair tiling. We obtain a dynamically equivalent shift, the Chair Shift, by coding the square pieces making up each prototile by digits in \( \mathbb{Z}_8 \) as shown. Note that even digits are used for “corners” and that only eight digits are needed, not twelve.

```
0  1  5  2
3  0  1  3
7  3  0  1  5
4  1  3  0  1
0  1  7  3  0
3  7  4  1  3
7  4  1  7  7  3  0
4  1  5  6  4  1  3
```

Figure 8.14: From each Chair tiling (thick edged tiles) we obtain a unique element of the Chair Shift (thin edged squares labeled with a digit in \( \mathbb{Z}_8 \)) and vice versa; thus, the space of Chair Tilings and the Chair Shift are completely equivalent dynamical systems.

The same argument as used for \( \mathcal{L} \) suggests the following conjecture.

**Conjecture 8.4** The projective fundamental group of \( \mathcal{F} \) is infinitely generated.

8.5 The Chair Shift

The Chair Shift is the Wang tiling shift obtained from the space of Chair Tilings \([111],[124]\) by marking “interior edges” in the chair shaped prototiles to partition them into three squares each; see Figure 8.13 and Figure 8.14.

The Chair Shift \( \mathcal{C} \) is a substitution minimal shift; its dynamical properties are closer to those of the Robinson shifts than the the Ledrappier Shift. The scenes of
the Chair shift tend to recur in a “two-adic” pattern; this is closely related to the fact that its dynamical spectrum is $\mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$.

Now color the underlying square tiles of the Chair shift in checkerboard fashion as in Figure 8.15. It is not hard to see that among the even digits (associated with “corners” of chair tiles) the digits 0, 6 occur only on white squares whereas the digits 2, 4 only on black ones. This shows that for a two by two square window $E$, the scene space $S_E(\mathcal{C})$ has two primitive components with respect to the $\mathbb{Z}^2$ action.

A similar coloring with eight colors reveals an eightfold imprimitivity in $S_F(\mathcal{C})$ where $F$ is the four by four square window, and similarly for larger squares. (It suffices to consider only $2^n$ by $2^n$ square windows, since these yield a cofinal sequence in the inverse system of scene spaces.) We conclude that $\mathcal{C}$ has infinitely many homotopically projective path components; indeed, each element $x$ is in a different component form each of its translates.

Now consider pairs of “monochevron” tilings; see 8.16. These plainly give “rod-shaped obstructions” in the scene spaces which give rise to nontrivial homotopically projective loops based at any point in each monochevron tiling.

Thus, although—as in the case of the Ledrappier and Free Particle Shifts—
Figure 8.16: A pair of partial monochevron tilings. These can be extended to tilings of the entire plane by repeated inflation, excision, and translation. In a complete monochevron tiling, the grey "chevrons" form a diagonal running from upper left to lower right through the entire plane, and the chevron can be "flipped" to run in the opposite direction. Thus we obtain "rod-shaped obstructions" similar to those found in the Ledrappier and Free Particle Shifts.
there are “rod shaped obstructions” giving rise to countably many generators in the
groupoid, in the case of the Chair Shift none of these generators are in the same
component, suggesting a considerable simplification in the prohomotopy groups.

8.6 Robinson Shifts

In [26], N. G. de Bruijn showed that there are uncountably many distinct “monoworm”
Penrose tilings (up to translation) containing a single worm which can be “flipped” [41][53];
see Figure 8.17. Each such monoworm is evidently associated with a nontrivial gen-
erator of the prohomotopy group based at any point in the homotopically projective
path component of the monoworm tiling. Equivalently, removing a patch of tiles in-
terrupting the worm and flipping one semi-worm gives an essentially holed tiling; just
as in the Ledrappier and Free Particle Shifts, this hole can be moved along the worm,
which creates a “rod-shaped obstruction” in the scene spaces.

In addition there is the “cartwheel tiling”, which contains ten worms meeting in a
central decagon; see Fig. 4.4. These worms cannot be flipped independently; however,
if one removes the decagon to form a holed Penrose tiling, the semi-worms can be
flipped independently, yielding 62 patterns \(^3\) up to the obvious \(D_{10}\) symmetry; only
one of these patterns is an inessential hole corresponding to the original cartwheel
tiling. Of the remaining 61 worm patterns, one is associated with an essential hole
which can be translated along either of two worms; the other holes can be enlarged
but cannot be moved away from the center. See [53] for the worm patterns and the
translation of holes.

In one respect, however, we must correct the discussion in [53] of the “61 essential
holes” in Penrose tilings which were identified by Conway in unpublished (and even
unwritten) work described orally to Shephard. This classification of “holes” does not

\(^3\)The number 62 follows from an easy computation using the Polya counting formula for an
appropriate action on the “signs” of ten semi-worms by the group \(D_{10}\). For the Polya counting
formula, see Appendix B and references therein.
Figure 8.17: A Penrose monoworm tiling (middle; the worm is shaded) with its two resolutions (top and bottom).
in fact refer to “holed Penrose tilings” up to Conway equivalence in the sense used earlier, but rather to the shape of the boundaries themselves, up to Conway moves where we completely ignore the partial tiling outside the “hole”. The relation between holed tilings (up to Conway moves and the $D_{10}$ geometric symmetry) and boundaries (up to Conway moves and the $D_{10}$ geometric symmetry) is as follows. There are 60 “essential boundaries” which correspond to 60 of the 62 patterns obtained by flipping the semi-worms independently; each of these can serve only as the boundary of a unique essentially holed tiling, namely one of the obtained from the cartwheel by removing the central octagon and flipping the semiworms in an appropriate way. There is one removable boundary, called “Batman” by Conway, which corresponds to the original pattern of semiworms and which can serve only as the boundary of the inessentially holed tiling obtained by taking out any patch from the cartwheel. Finally, there is one remaining boundary shape, called “Asterisk” by Conway, which can serve either as the boundary of one of the essentially holed tilings obtained from the cartwheel, or as the boundary of any of an uncountable number of essentially holed tilings obtained from one of the monoworm tilings. This (hopefully) clarifies the remark in [53] that the “Asterisk” boundary class “does not force a unique tiling”! Note too that the “Asterisk” shape can be “translated” by Conway moves (if the exterior tiling corresponds to a monoworm, we see that it can be “translated” only along the worm), whereas the other sixty “essential” boundaries cannot be “translated” away from the center because this would cause two incompatible halves of some worm to try to join up, which they cannot do if one semiworm has been “flipped” relative to the other.

The generic Sturmian tiling lacks the subtle algebraic number theoretic symmetries which produce worms and other singularities. However, there are a huge number of such tilings which do have worms associated with “rod-shaped obstructions”, in addition to more complicated singular patterns. (These are associated, like the Penrose tilings, which a “composition”; the asymmetric Sturmian tilings lack such composi-
8.6. ROBINSON SHIFTS

In particular, the Ammann octagonal tiling [53][124] has a cartwheel in which eight worms meet in a central octagon (see Fig. 6.11); these cannot be "flipped" independently, but if the central octagon is removed to make a holed Ammann tiling, the semi-worms can be flipped independently to give 21 patterns⁴ up the $D_8$ symmetry.

There are many geometric symmetries other than $D_8$ allowed even for Sturmian tilings which, like the Ammann tiling, are produced from four dimensional cubical lattices [124]. The general situation is complex enough to be interesting, while presumably not being too complex to admit a nice description of the prohomotopy groups.

Problem 8.5 Clarify the relationship between the projective homotopy of a Robinson shift and that of the parent Sturmian tiling space. Does each Robinson shift associated with a given Sturmian tiling space have the same projective homotopy?

Conjecture 8.6 The projective homotopy group of a Sturmian tiling space (and its Robinson shifts) is controlled by the nature of its singular tilings; in particular, each fault is associated with a generator.

The "generic" Sturmian tiling space has no faults, so this would imply that the "most" such spaces actually have trivial projective homotopy (but are probably not projectively connected). This would tend to show that projective homotopy is not a very good invariant for an extremely rigid yet rich class of symbolic dynamical systems such as the Sturmian systems.

Problem 8.7 Characterize the projective components of a Sturmian tiling space (and its Robinson shifts). Clarify the relationship between projective components and orbit closures.

⁴Needless to say, the number 21 follows from another Polya counting computation.
Conjecture 8.8 The projective homotopy group of a Sturmian tiling space can differ from one projective component to the next, but only a finite number of groups arise in this way (for a fixed tiling space).

Problem 8.9 Find all the generators of the projective homotopy group of each projective component of a Sturmian tiling space (and its Robinson shifts).

Problem 8.10 In the special cases of a Sturmian system with matching rules or composition map, investigate how one may take advantage of this extra structure to compute the projective homotopy. Does the existence of a composition imply anything about the projective components?
Chapter 9

Micropolars of Sturmian Tilings

In this chapter, we introduce the notion of the micropolar of a parallelogram tiling, and discuss some problems concerning the nature of these tilings in the case of Sturmian systems.

9.1 Micropolars of Tilings by Polygons

![Diagram](image)

Figure 9.1: Left: a typical patch (solid lines) in a tiling by parallelograms and the corresponding patch of the micropolar tiling (dotted lines). Right: the micropolar (heavy solid lines) of the micropolar of the original tiling.

Because every tile in a \((2, q)\) Sturmian tiling is a convex polygon, we can take advantage of a construction (apparently due to this author) which works for any two dimensional tiling by convex polygons. Recall that every convex polytope possesses a polar which is also a convex polytope and which after suitable rescaling, just fits inside its parent; indeed, the vertices of the polar will be precisely the midpoints of the faces of the parent. In the case of a polygon, the rescaled polar is obtained simply by taking the midpoints of the edges and connecting these to form a new polygon in
the obvious way. See also Appendix A; see also [5] for more details on convex sets in general and polars in particular. The micropolar of tiling by polygons is obtained by taking the polar of each tile (rescaled as above); the “gaps” between the polygons so obtained will be new tiles; see Fig. 9.1. Thus, the operation of taking the micropolar can be repeated.

Problem 9.1 Formulate and investigate a notion of the “dynamics” of the micropolar map; that is, identify a class of rhomb tilings such that the micropolar of one tiling in the class is another such, make this into a topological space, and investigate the topological dynamics and ergodic theory of the micropolar map in this setting.

One particularly interesting feature of the micropolar of a tiling by parallelograms is that the prototiles of the micropolar are “graded” into a set of parallelograms corresponding to the old prototiles, together with set of polygons corresponding to the vertex neighborhoods of the original tiling. A polygon in the second set will be an n-gon exactly in case n edges meet at the corresponding vertex. See the lefthand picture in Fig. 9.1. Moreover, the micropolar of the micropolar will have three types of prototiles; smaller copies of the original parallelograms, parallelograms corresponding each type of edge in the original tiling, and polygonal tiles corresponding to the vertex neighborhoods of the original; see the righthand picture in Fig. 9.1.

Problem 9.2 Additional ideas will be needed to generalize this construction to tilings of \( \mathbb{R}^p \) by zonotopes (the problems come in “filling in the gaps” between the polars of the original tiles). Find them and determine a workable construction.

9.2 Micropolars of \((2, q)\) Sturmian Tilings

The result of applying the micropolar operation to a typical \((2, q)\) Sturmian tiling is typically quite attractive, and surprisingly “organic” in appearance; see Fig. 9.2. The
Figure 9.2: Top: a typical $(2, 2)$ Sturmian tiling. Bottom: the micropolar of this tiling.
\[ W = \text{col} \begin{bmatrix} 2 & \sqrt{5} \\ \sqrt{5} & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ W = \text{col} \begin{bmatrix} 1 + \sqrt{3} & 1 \\ 1 & -1 + \sqrt{3} \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \]

Figure 9.3: Two step symmetrical (2, 2) Sturmian tilings (left) and their micropolars (right).
local symmetries, if any, of particular vertex neighborhoods are often more transparent in the micropolar tiling than in the original; see Fig. 9.3.

There is an amusing way to interpret the micropolar of a Sturmian tiling, which the author first discovered in a completely different context a decade ago. Consider the (2,1) step symmetric Sturmian Systems $S(W)$ obtained from the invariant plane of the canonical three cycle in $\mathbb{R}^3$. “Lift” each tiling $T(x + W)$ to the corresponding digital approximation $D(x + W)$ and put the indefinite Lorentzian metric with line element

$$ds^2 = u^2 - v^2$$

on each square, where $u = x_1, v = x_2$ for $F_{1,2}$ and so forth. Then line segments connecting the midpoints are null geodesics (representing the world lines of photons). Moreover, the metric can be smoothly continued from one square to the next in the digital approximation, provided one deletes out the vertices, where all the angular deficits (for vertices of valence three) and excesses (for vertices of valence five or more) are concentrated. The result is a metrically flat, multiply connected spacetime with boundary (the boundary being the union of deleted points). In this spacetime, there is no global distinction between timelike and spacelike geodesics.

Now, the interesting thing about this interpretation is that the null geodesics are often homeomorphic to circles. This is particularly easy to see in the case of the (2,1) symmetric tiling, which happens to be periodic; see Fig. 9.4. At first glance, it may seem we have shown this only for the special null geodesics which arise by connecting midpoints of squares, but in fact, a little thought shows this is generic for null geodesics; the “midpoint” geodesics control the others, in the sense that only those null geodesics which run into a vertex (and hence “stop”) are not determined by the special null geodesics appearing in the micropolar. In particular, in this tiling, every null geodesic is homeomorphic to a circle, except for a negligible set of exceptional geodesics.
Problem 9.3 Formulate a reasonable notion of “negligible” and verify the above claim.

The tiling shown in Fig. 9.4 happens to be periodic, but close examination of the aperiodic (2, 2) micropolars shown in Fig. 9.5 and Fig. 9.6 shows the presence of many small closed null geodesics, together with what appear to be “wandering”, nonclosed null geodesics.

Note: in drawing the null geodesics by hand, it is helpful to observe that each vertex of the micropolar has valence four; the rule for continuing a null geodesic through the micropolar tiling vertex is simple: continue along the “opposite edge”, turning neither left nor right.

In the most extreme case, the step symmetric (2, q) Sturmian tilings, “small” null geodesics form very striking closed curves; see Figs. 9.7 and 9.9.

Conjecture 9.4 With a “negligible” set of exceptions, the null geodesics in the step symmetric (2, q) tilings arising from canonical cycles on $\mathbb{R}^d$, where $d = 2 + q$, are homeomorphic to circles.

At first blush, this appears similar to a result of Conway described in [53]. However, unlike the decorations discussed there, the null geodesics are not globally well behaved under the composition map of the Ammann or Penrose tiling spaces, and consequently Conway’s method does not apply. Despite this, it is possible to organize a selection of null geodesics of the Ammann and Penrose tilings into a striking hierarchical design; see Figs. 9.10 and 9.11.

We might mention here a computer experiment which suggests another interesting problem. Choose an “interesting” (2, q) Sturmian tiling space. Take a very good periodic approximation, and fix one ribbon vector. “Tilt” the remaining ribbon vector to progressively perturb away from the original tiling. Observe that the previously closed null geodesics alter their topology in an intricate but strikingly regular manner, as illustrated in Figs. 9.12 to 9.14.
Figure 9.4: A tiling from the (periodic) \((2,1)\) Sturmian systems \(S(W)\), where \(W\) is the invariant plane of the canonical three cycle on \(\mathbb{R}^3\) (top), and its micropolar (bottom).
Figure 9.5: This is the same micropolar tiling shown at top right in Fig. 9.3.

\[ W = \text{col} \begin{bmatrix} 2 & \sqrt{5} \\ \sqrt{5} & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \]
Figure 9.6: This is the same micropolar tiling shown at bottom right in Fig. 9.3.
Figure 9.7: The micropolar tiling of a typical Ammann tiling.
9.2. MICROPOLARS OF \((2, Q)\) STURMIAN TILINGS

Figure 9.8: The micropolar tiling of a typical Penrose tiling.
Figure 9.9: The micropolar tiling of another orbit closure in the same Sturmian system as in Fig. 9.8.
Figure 9.10: A hierarchy of null geodesics in the micropolar of an Ammann tiling.
Figure 9.11: A hierarchy of null geodesics in the micropolar of a Penrose tiling.
$W = \begin{bmatrix} 2 & 0 \\ -1 & 10 \\ -1 & -11 \end{bmatrix}$

Figure 9.12: A periodic approximation to the tiling in Fig. 9.4; note the widely separated "bands" of non-closed geodesics.
$W = \text{col} \begin{bmatrix} 2 & 0 \\ -1 & 6 \\ -1 & -7 \end{bmatrix}$

Figure 9.13: A coarser approximation to the same tiling; note the "bands" have moved closer.
9.2. MICROPOLARS OF $(2, Q)$ STURMIAN TILINGS

\[ W = \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ -1 & -4 \end{bmatrix} \]

Figure 9.14: A still coarser approximation; note the "bands" have moved closer yet.
Problem 9.5 Characterize the topological changes in the null geodesics on various scales as one moves about in the generalized Farey tree organizing the Sturmian systems of fixed genus. In particular, investigate what happens during a one parameter motion like the "tilting" experiment described above.
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Appendix A

What is a Concept?

A.1 Introduction

Pairs of dual lattices arise throughout mathematics. One of the most familiar examples involves the lattice of subspaces of a finite dimensional euclidean inner product space $X$ and its dual space $X^*$ (the space of all linear functionals $f : X \to \mathbb{R}$; see [10]). Here the subspace $A \subseteq X$ corresponds to its “annihilator”

$$\text{ann } A = \{ f \in X^* : f(x) = 0, \forall x \in A \}$$

which is itself a subspace of $X^*$. Indeed, every subspace of $X^*$ is the annihilator of a unique subspace of $X$, so we obtain a bijection between the subspaces of $X$ and the subspaces of $X^*$. The bijection is order reversing, in the sense that $A$ is a subspace of $B$ exactly when $\text{ann } B$ is a subspace of $\text{ann } A$. Moreover,

$$\text{ann}(A + B) = \text{ann } A \cap \text{ann } B$$
$$\text{ann}(A \cap B) = \text{ann } A + \text{ann } B$$

which says that taking meets (sums) in lattice of subspaces of $X$ corresponds to taking joins (intersections) in lattice of subspaces of $X^*$, and vice versa. This is what is meant by saying that the lattice of subspaces of $X$ is in galois duality with the lattice of subspaces of $X^*$.

Such galois dualities arise throughout mathematics, and are in fact fundamental in subjects such as field theory, operator theory, algebraic geometry, and dynamical systems. As a rule such dualities are best pictured by using the order reversing bijection to combine two dual lattices into one. Indeed, such a combined lattice is
conveniently constructed as a certain complete lattice, the lattice of concepts, which can be constructed from any relation from X to Y; that is, from any subset \( R \subset X \times Y \). Indeed, the most important galois dualities in mathematics arise in just this way, as we shall see by example below.

Here, a concept is a set of the form \((A, B) \subset R\), where \((A, B)\) are "dual" subsets. More precisely,

\[
B = ^\circ A = \{ y \in Y : (x, y) \in R, \forall x \in A \} \\
A = ^\bullet B = \{ x \in X : (x, y) \in R, \forall y \in B \}
\]

This notion was introduced by R. Wille [25], who was inspired by Aristotle's notion of epistemology ("study of knowledge"). According to Aristotle, every concept has a dual nature; its extent is the set of all objects which exhibit the concept, while its intent ("meaning") is the set of all predicates ("properties, attributes") these objects share in common. This idea has been made rigorous in modern work in topos theory (the theory of "categories which can serve as the foundation of all mathematics"), but this will not concern us here. (Interested readers are referred to [61] for a concise introduction to topos theory.)

The third thesis is that certain general facts concerning the lattice of concepts constitute the "trivial" part of the theory of any one galois duality; the "nontrivial" part often involves giving a separate characterization of the concepts. This will be illustrated in the examples, where such characterizations often turn out to be fundamental theorems of the subjects in question. Unfortunately, the trivial part is that most often omitted when these subjects are introduced in graduate courses! Probably this is because establishing a galois duality can be tedious and boring unless the machinery of abstract concept theory is available. As a consequence, the reader may learn here for the first time that lattices and duality are involved in an essential way in certain notions he has known about since his mathematical infancy.
A.2. THE LATTICE OF CONCEPTS

The organization of this paper is as follows. In the next section I discuss Aristotle-Wille theory of abstract concepts. In the third section I discuss examples of concepts drawn from analysis, operator theory, algebra, dynamical systems, and even elementary number theory.

This is an expository paper with minimal prerequisites. However, at certain points in the paper some familiarity with the usual topics of a first year graduate course in real analysis and a senior year modern algebra course is assumed. In addition, in the interests of space I assume that the reader has encountered the definition of a lattice at the level of [10][35] and the definition of a group action at the level of Appendix B.

A.2 The Lattice of Concepts

Suppose that $X, Y$ are two sets and that $\rho$ is a relation from $X$ to $Y$ given in the usual way by some subset of $X \times Y$, namely

$$R = \{(x, y) : x \rho y\} \quad (A.1)$$

Given $A \subseteq X$. define

$$\triangleright A = \{y : x \rho y \text{ for all } x \in A\} \quad (A.2)$$

Given $B \subseteq Y$, define

$$\triangleleft B = \{x : x \rho y \text{ for all } y \in B\} \quad (A.3)$$

These are "dual" definitions because they are symmetric under simultaneous interchange of $x \in X$ with $y \in Y$, $A \subseteq X$ with $B \subseteq Y$, and $\triangleleft$ with $\triangleright$. It follows that any true statement concerning $\triangleright$ and $\triangleleft$ yields another true statement under simultaneous interchange of the symbols $x \in X$ with $y \in Y$ and $\triangleleft$ with $\triangleright$. This duality principle cuts in half the work we must perform in setting up the theory.

It is convenient to observe that by definition the following are equivalent:
1. $x \rho y,$

2. $y \in \triangleright x,$

3. $x \in \triangleleft y.$

It follows at once that

$$\triangleright A = \cap_{x \in A} \triangleright x \quad (A.4)$$

$$\triangleleft B = \cap_{y \in B} \triangleleft y \quad (A.5)$$

Furthermore,

$$\triangleright \triangleleft A = \{x \in X : \triangleright x \supset \triangleright A\} \quad (A.6)$$

$$\triangleright \triangleright B = \{y \in Y : \triangleright y \supset \triangleright B\} \quad (A.7)$$

To see this, observe that

$$\triangleright \triangleleft A = \{x : x \rho y \text{ for all } y \in \triangleright A\} = \{x : y \in \triangleright x \text{ for all } y \in \triangleright A\} = \{x : \triangleright x \supset \triangleright A\}$$

and similarly for $\triangleright \triangleright B.$

**Lemma A.1** *The operators $\triangleright$ and $\triangleleft$ are order reversing; that is,*

- *If $A \subset A' \subset X,$ then $Y \supset \triangleright A \supset \triangleright A',$

- *If $B \subset B' \subset Y,$ then $X \supset \triangleleft B \supset \triangleleft B'.$

**Proof:** If $A \subset A',$ then $\triangleright A' = \cap_{x \in A'} \triangleright x$ is obtained by intersecting the $\triangleright x$ over a larger set of indices $x$ than is $\triangleright A = \cap_{x \in A} \triangleright x.$ This shows the first implication; the second implication follows by duality.
Lemma A.2  The operators \( \leadsto \) and \( \triangleright \) are order increasing; that is,

- For all \( A \subseteq X \), \( A \subseteq \leadsto A \),

- For all \( B \subseteq Y \), \( B \subseteq \triangleright B \).

Proof: If \( x \in A \) then \( \triangleright x \supset A \) by Lemma A.1, whence \( x \in \leadsto A = \{ x' \in X : \triangleright x' \supset \triangleright A \} \). This shows the first inclusion; the second follows by duality.

Lemma A.3  By further iterations of the operators \( \triangleright \), \( \triangleleft \) we obtain no new operators other than \( \triangleright \triangleright \) and \( \triangleleft \triangleleft \). That is,

- For all \( A \subseteq X \), \( \triangleright A = \triangleright \leadsto A \),

- For all \( B \subseteq Y \), \( \triangleleft B = \triangleleft \triangleleft B \).

Proof: On the one hand, \( A \subseteq \leadsto A \) implies \( \triangleright A \supset \triangleright \leadsto A \) by Lemma A.1. On the other hand, \( y \in \triangleright A \) implies \( \triangleleft y \supset \triangleleft \leadsto A \) by Lemma A.1, whence \( y \in \triangleleft \triangleleft A \); thus \( \triangleright A \subseteq \triangleright \triangleleft A \).

Combining these observations gives the first identity; the second follows by duality.

An operator \( \langle \cdot \rangle \) on subsets of a set \( X \) which satisfies

1. for all \( A \subseteq X \), \( A \subseteq \langle A \rangle \),

2. if \( A \subseteq A' \subseteq X \), then \( \langle A \rangle \subseteq \langle A' \rangle \subseteq X \),

3. for all \( A \subseteq X \), \( \langle \langle A \rangle \rangle = \langle A \rangle \)

is called an algebraic closure operator. Familiar examples include the topological closure of a subset of a topological space, the linear span of a subset of a vector space, and the subgroup generated by a subset of a group.
Lemma A.4 The operators $\mathcal{a}$ and $\mathcal{b}$ are algebraic closure operators.

Proof: We have already seen (Lemma A.2) that $A \subseteq \mathcal{a} A$ holds for all $A \subseteq X$. If $A \subseteq A' \subseteq X$, by two applications of Lemma A.1, we have $\mathcal{b} A \supseteq \mathcal{b} A'$ and thus $\mathcal{a} A \subseteq \mathcal{a} A'$. Next, $\mathcal{a}(\mathcal{b} \mathcal{a} A) = \mathcal{b} A$ by Lemma A.3. The other half of the Lemma follows by duality.

Lemma A.5 The operators $\mathcal{b}$ and $\mathcal{a}$ take unions to intersections: specifically

- If $\{A_j : j \in J\}$ is a collection of subsets $A_j \subseteq X$, then
  $$\mathcal{b} \bigcup_{j \in J} A_j = \bigcap_{j \in J} \mathcal{b} A_j$$

- If $\{B_k : k \in K\}$ is a collection of subset $B_k \subseteq Y$, then
  $$\mathcal{a} \bigcup_{k \in K} B_k = \bigcap_{k \in K} \mathcal{a} B_k$$

Proof: We compute as follows

$$\mathcal{b} \bigcup_{j \in J} A_j = \bigcap_{x \in \bigcup_{j \in J} A_j} \mathcal{b} x = \bigcap_{j \in J} \bigcap_{x \in A_j} \mathcal{b} x = \bigcap_{j \in J} \mathcal{b} A_j$$

The second claim now follows by duality.

A concept of $\rho$ is a set of the form $(A, B)$ where $A = \mathcal{a} B$ and $B = \mathcal{b} A$. The fundamental theorem concerning concepts is the following.

Theorem A.6 (Wille) The set of concepts forms a complete lattice; specifically, if $\{(A_j, B_j) : j \in J\}$ is a collection of concepts, then the meet and join taken over the collection are respectively

$$\bigwedge_{j \in J} (A_j, B_j) = (\bigcap_{j \in J} A_j, \mathcal{b} \bigcup_{j \in J} B_j)$$
\[ \bigvee_{j \in J} (A_j, B_j) = (\Rightarrow \bigcup_{j \in J} A_j, \cap_{j \in J} B_j) \]

The unique maximal concept is \((X, \triangleright X)\) and the unique minimal concept is \((\triangleleft Y, Y)\).

**Proof:** Define a relation on the set of concepts by stipulating that \((A, B) \leq (A', B')\) iff \(A \subseteq A'\); equivalently, iff \(B \supseteq B'\). It is easy to see that \(\leq\) is a partial order with unique maximal element \((X, \triangleright X)\) and unique minimal element \((\triangleleft Y, Y)\).

Given a collection of concepts \(S = \{(A_j, B_j) : j \in J\}\), let \(A = \cap_{j \in J} A_j\) and \(B = \Rightarrow \cup_{j \in J} B_j\). We need to show that \((A, B)\) is the greatest lower bound for \(S\). (The corresponding claim about least upper bounds will then follow by duality.)

**CLAIM:** \((A, B)\) is a concept of \(\rho\).

**Reason:** We first need to show that \(A = \cap_j A_j\) is closed in \(X\). But on the one hand (by Lemma A.2) \(A \subseteq \Rightarrow A\). On the other hand, for each \(A_k\) we have \(A_k \supseteq A\) whence (by Lemma A.2) \(A_k = \Rightarrow A_k \supseteq \Rightarrow A\), whence \(A = \cap_k A_k \supseteq \Rightarrow A\). Therefore \(A = \Rightarrow A\).

Moreover (by Lemma A.3 and Lemma A.5) we have

\[ \triangleleft B = \triangleleft \Rightarrow \cup_j B_j = \triangleleft \cup_j B_j = \cap_j \triangleleft B_j = \cap_j A_j = A \]

Thus, \((A, B)\) is a concept.

**CLAIM:** \((A, B)\) is a greatest lower bound for \(S\).

**Reason:** For each \(A_j\), we have \(A_j \supseteq A\), whence \((A_j, B_j) \leq (A, B)\), so \((A, B)\) is a lower bound. Suppose that \((A', B')\) is another lower bound. Then for each \(A_j\), we have \(A_j \subseteq A'\), whence (taking the intersection over all \(j\)), \(A \subseteq A'\). Therefore \((A, B) \leq (A', B')\).

An alternative way to state Theorem A.6 would be to say that the closed sets in \(X\) form a complete lattice in **galois duality** with the closed sets in \(Y\). Under this viewpoint, the maps \(\triangleright\) and \(\triangleleft\) are called a **galois connection** (see Exc. 11.3 in [25]).

See Figure A.1 for a schematic picture of a typical concept lattice.
Wille also showed that if we start with a complete lattice $L$ and take $\rho$ to be the induced partial ordering on $L$, we obtain a concept lattice isomorphic to $L$ (see Thm. 11.5 in [25]).

\[
\begin{align*}
(X, \triangleright X) \\
(\forall \{A \cup A', B \cap B'\}) \\
(A, B) \\
(A', B') \\
(A \cap A', \triangleright \forall \{B \cup B'\}) \\
(\forall Y, Y)
\end{align*}
\]

Figure A.1: A schematic diagram of a concept lattice, where $A = \triangleright B$, $B = \forall A$ and $A' = \triangleright B'$, $B' = \forall A'$. In this diagram, "smaller" concepts (with respect to the partial order $\leq$ defined in the proof of Theorem A.6) lie below "larger" ones.

### A.3 Concepts Galore!

In this section, we obtain the payoff for the rather dry theory of the previous section. I want to now introduce you to some of my favorite examples of concepts. They include some of the most important examples of "duality" in mathematics; a few are merely personal favorites of the author. For each example, I first give the relation which gives rise to a concept lattice (or, equivalently, a pair of lattices in galois duality), and then briefly discuss some of the more important features of the concept lattice.

Due to limitations of energy, time, and space, this section is extremely sketchy, but I have attempted to provide copious references accessible to beginning graduate students.

We first discuss a collection of symmetric relations from a set to itself. Under these
circumstances, there is no distinction between $\to$ and $\leadsto$, so we may as well replace both by a single symbol. From the galois connection point of view, such relations provide a self-duality on a complete lattice.

Example 1 Let $X$ be a Hilbert space and define a relation from $X$ to itself by stipulating that $x \rho y$ if $(x, y) = 0$.

We may as well replace both $\triangleleft$ and $\triangleright$ by the symbol $\perp$. Given $A \subseteq X$, its orthogonal complement is

$$A^\perp = \{x \in X : (x, a) = 0, \forall a \in A\}$$

Given $A \subseteq X$, $(A^\perp)^\perp$ is the smallest closed subspace containing $A$. The concepts are precisely the pairs of orthogonal complements $(V, V^\perp)$ of closed subspaces of $X$; see Figure A.2. The orthogonal complement of a unidimensional subspace is particularly interesting: it is a hyperplane. (See Chapter 3 of [105] for an excellent introduction to Hilbert spaces.)

![Figure A.2: A schematic diagram of the concept lattice for orthogonal complements in a Hilbert space $X$ (Example 1), where $U, V$ are closed subspaces of $X.$](image-url)
Example 2 Let $X$ be a Hilbert space and define a relation from $X$ to itself by stipulating that $x \rho y$ if $(x, y) \leq 1$.

We may as well replace both $\triangleright$ and $<$ by $\triangleright$. Then, given $A \subset X$, its polar $A^0$ is the topological closure of the convex hull of $A \cup \{0\}$; that is, the smallest closed convex set containing both $A$ and the origin of $X$. The concepts consist of pairs $(A, A^0)$, where $A$ is a closed convex set containing $0 \in X$ and $A^0$ is its polar. The polar of a ray from the origin is particularly interesting: it is a closed half space (see Dfn. 30.1 of [4]).

![Diagram](image)

Figure A.3: A schematic diagram of the concept lattice for polars in a Hilbert space $X$ (Example 2), where $A, B$ are closed convex sets containing $0 \in X$.

In the very special case that $X$ is ordinary three dimensional euclidean space, the polar of a cube (with centroid at the origin) is an octahedron, and the polar of a dodecahedron (with centroid at the origin) is an icosahedron.

Our next example was considered extremely important in the last century but now has almost been forgotten.

Example 3 Let $\mathbb{R}^{n+1}$ be equipped with a nondegenerate bilinear form $L : X \times X \to \mathbb{R}$ and let $Q = \{v \in \mathbb{R}^{n+1} : L(v, v) = 0\}$. (The classical name for $Q$ is “the Absolute”.)
Let $X = \mathbb{R}P^n$ be the set of lines through the origin of $\mathbb{R}^{n+1}$. Then $L$ induces a bilinear form on $X \times X$ and we can define a symmetric relation by stipulating that $x \rho y$ if for any representative $u$ of $x$ and any representative $v$ of $y$, we have $L(u, v) = 0$.

We may as well replace both $\triangleright$ and $\triangleleft$ by $\circ$. Here the concepts are precisely the pairs $(P, P^0)$ where $P$ is a projective subspace of dimension $k$ and $P^0$ is a certain projective subspace of codimension $k$. For instance, suppose $n = 2$ and let

$$L(u, v) = u_1v_1 + u_2v_2 - u_3v_3$$

The absolute is the double cone

$$Q = \{ u \in \mathbb{R}^3 : u_1^2 + u_2^2 - u_3^2 = 0 \}$$

We can identify $\mathbb{R}P^2$ with the plane $u_3 = 1$; now we see that the absolute appears in $\mathbb{R}P^2$ as an ellipse (in fact, a circle, in this example). Take a point $x \in \mathbb{R}P^2$ not on this ellipse; then $x$ defines a line spanned through the origin which intersects the absolute only at $0 \in \mathbb{R}^3$. Take the orthogonal plane according to the pseudoeuclidean "inner product" given by $L$. This plane intersects the plane $u_3 = 1$ in a line, which is the projective line dual to the original projective point $x \in \mathbb{R}P^2$. This establishes the classical duality between lines and points in $\mathbb{R}P^2$.

For instance, if the projective point $x$ actually lies on $Q$, then $x^0$ is the projective line tangent to $Q$ at $x$. If $Q$ is elliptical and $x$ lies outside $Q$, then $x^0$ is obtained geometrically by drawing tangents to $Q$ from $x$ and taking $x^0$ to be the line through the points where these tangents touch $Q$. The reader can work out similar geometric constructions for $x^0$ when $x$ lies inside $Q$, or when $Q$ is a hyperbola. Note that the lattice of concepts may be identified with the lattice of polars with respect to the projective metric on $\mathbb{R}P^2$ induced by $L$ (or equivalently, by giving a projective conic $Q$ in $\mathbb{R}P^2$). (See [10][140] for very brief introductions to projective geometry.)

Our next example is fundamental to operator theory.
Example 4 Let $X$ be a Banach algebra (i.e. a linear associative algebra with a norm such that $\|xy\| \leq \|x\| \cdot \|y\|$ for all $x, y \in X$ and such that $X$ is complete as a metric space). Define a symmetric relation on $X$ by stipulating that $x \rho y$ if $xy = yx$.

We may as well replace both $\triangleleft$ and $\triangleright$ by $\cdot$. Given $A \subset X$, $A'$ is the commutant

$$A' = \{x \in X : xy = yx \forall y \in A\}$$

which is a closed subalgebra of $X$. The closure of $A$ is the bicommutant $A''$, another closed subalgebra. The concepts are the pairs of subalgebras $(A, B)$ where $A = B'$ and $A' = B$. The maximal concept is $(X, Z)$, where $Z$ is the center of $X$, and the minimal concept is $(Z, X)$.

If $A$ is a commutative subalgebra, then $A''$ is a commutative Banach algebra. (See Exercise 49.11 of [4] and Thms. 11.22-23 of [118].) In particular, when $X = B(H)$ is the set of bounded (or equivalently, continuous) linear operators on a Hilbert space $H$, the concepts of form $(A, B)$ where either $A$ or $B$ is commutative, are (by definition) dual pairs of von Neumann algebras.

Our next example is undoubtedly one of most important constructions in all of mathematics.

Example 5 Define a relation on $\mathbb{Q}$ to itself by stipulating that $x \rho y$ if $x \leq y$.

Given $A \subset \mathbb{Q}$,

$$\triangleleft A = \{x \in \mathbb{Q} : x \leq a, \forall a \in A\} = \mathbb{Q} \cap (-\infty, \inf A]$$

is the set of lower bounds for $A$; dually,

$$\triangleright A = \{x \in \mathbb{Q} : a \leq x, \forall a \in A\} = [\sup A, \infty) \cap \mathbb{Q}$$

is the set of upper bounds for $A$. Moreover,

$$\triangledown A = \{x \in \mathbb{Q} : \triangleright x \supset \triangleright A\} = \mathbb{Q} \cap (-\infty, \sup A]$$
and

$$\vartriangleright A = \{ x \in \mathbb{Q} : \triangleleft x \supset \triangleleft A \} = [\inf A, \infty) \cap \mathbb{Q}$$

The concepts are precisely the Dedekind cuts \((I, J)\) where

$$I = (-\infty, x) \cap \mathbb{Q}, \ J = [x, \infty) \cap \mathbb{Q}, \ x \in \mathbb{R}$$

together with a unique maximal element, namely \((\mathbb{Q}, \emptyset)\), and a unique minimal element, namely \((\emptyset, \mathbb{Q})\). The lattice of concepts is therefore nothing other than the extended reals \(\mathbb{R} \cup \{-\infty, \infty\}\) equipped with the usual ordering and with meet and join operations given by the operators \(\inf\) and \(\sup\). (See Section P.5 of [36])

Figure A.4: A schematic diagram of the concept lattice for Dedekind cuts (Example 5), where \(A \subset \mathbb{Q}\).

**Example 6** Define a relation from \(\mathbb{R}^2\) to itself by stipulating that \(x \rho y\) iff

$$(y_0 - x_0)^2 - (x_1 - y_1)^2 \geq 0$$

$$y_0 - x_0 \geq 0$$

where \(x = (x_0, x_1)\) and \(y = (y_0, y_1)\).

Physics students will recognize the meaning of \(x \rho y\) as saying that “\(x\) is in the absolute past (backward light cone) of \(y\)”; or equivalently, “\(y\) is in the absolute future
(forward light cone) of \( x \). Put another way, “the event \( x \) can causally affect the event \( y \) by signals propagated at the speed of light (or less)”. Here \( \triangleleft x \) is the backwards light cone of \( x \) and \( \triangleright x \) is the forward light cone of \( x \). Furthermore,

\[
\triangleleft x = \{ y : \triangleright y \supset \triangleright x \} = \triangleleft x
\]

\[
\triangleright \triangleleft x = \{ y : \triangleleft y \supset \triangleleft x \} = \triangleright x
\]

This implies that the concept lattice contains a copy of \( \mathbb{R}^2 \); specifically, \( x \in \mathbb{R}^2 \) corresponds to the concept \((\triangleleft x, \triangleright x)\) representing the light cones (both forward and backward) based at \( x \). Indeed, the only additional concepts are \((\emptyset, \mathbb{R}^2)\) and \((\mathbb{R}^2, \emptyset)\), corresponding to the “first” and “last” events respectively.

![Diagram of concept lattice](image.png)

Figure A.5: Left: a schematic diagram of the concept lattice of light cones in a two dimensional model of flat spacetime (Example 6). Right: the meaning of the “latest common cause” (LCC) and “earliest common effect” (ECE) of two events \( x, y \), where \( \triangleright \{x, y\} \) (top) and \( \triangleleft \{x, y\} \) are shaded.

(It is instructive to consider how this picture is complicated when we consider the analogous partial order on \( \mathbb{R}^4 \): in this case it is no longer true that \( \triangleleft \{x, y\} = z \) for some \( z \).)

**Example 7** Define a relation from \( \mathbb{N} \) to itself by stipulating that \( d \rho n \) iff \( d \mid n \).
Given \( n \in \mathbb{N} \), \( \cdot n \) is the set of positive integer multiples of \( n \); dually, \( \cdot n \) is the set of positive divisors of \( n \). Thus, in contrast to the previous example, \( \cdot n \) is always \textit{infinite} whereas \( \cdot n \) is \textit{finite}. Nonetheless, in analogy with the previous example, we have

\[
\Phi n = \{ m : \cdot m \supset \cdot n \} = \cdot n \\
\cdot \Phi n = \{ m : \cdot m \supset \cdot n \} = \cdot n
\]

This implies that the concept lattice contains a copy of \( \mathbb{N} \); specifically, \( n \) corresponds to the concept

\[
(\cdot n, \cdot \cdot n) = (\cdot n, \cdot n) = (\Phi n, \cdot n)
\]

The maximal concept is \((1, \mathbb{N})\) and the minimal concept is \((\mathbb{N}, \emptyset)\). Also, \((\cdot m) \cap (\cdot n) = 1 \text{ iff } m, n \text{ are relatively prime (have no common divisors)}, \text{ and minimal nontrivial concepts have the form } (\cdot p, \cdot p) \text{ where } p \text{ is prime.} \text{ On the other hand, } (\cdot m) \cap (\cdot n) = \mathbb{N} \text{ only if } m = n = 1, \text{ and there are no maximal proper concepts.}

To every \textit{finite} set \( A \) of positive integers (containing at least two distinct elements), there corresponds precisely \textit{two} concepts. For on the one hand, the concept \((\cdot A, \cdot \cdot A)\) corresponds to the \textbf{common divisors} of \( A \) (that is, the divisors of GCD\((A)\)), on the left, and the multiples of GCD\((A)\), on the right. On the other hand, the concept \((\Phi A, \cdot A)\) corresponds to the \textbf{common multiples} of \( A \) (that is, the multiples of LCM\((A)\)), on the right, and the divisors of LCM\((A)\), on the left.

However, if \( A \) is infinite, \((\Phi A, \cdot A) = (\mathbb{N}, \emptyset)\) (the unique maximal concept), whereas \((\cdot A, \cdot \cdot A)\) may well be a proper nontrivial concept (for example, let \( A \) be the even integers). Thus, in analogy with the previous example, the only concept not of form \((\cdot m, \cdot m)\) is \((\mathbb{N}, \emptyset)\), which behaves like "the (nonexistent) integer which is the product of all the positive integers".

As the reader may have already realized, Examples 5-7 are special cases of a general construction (called Macneille-Dedekind completion) which can be applied to
any poset to obtain a complete lattice.

**Example 8** Let \((X, \leq)\) be a poset and define a relation on \(X\) by stipulating that \(x \prec y\) if \(x \leq y\).

Here, if \(A \subseteq X\),

\[ \triangleleft A = \{ x : x \leq y \text{ for all } y \in A \} \]

is the set of **lower bounds** for \(A\): dually

\[ \triangleright A = \{ y : x \leq y \text{ for all } x \in A \} \]

is the set of **upper bounds** for \(A\). Moreover,

\[ \triangleleft \triangleright A = \{ x : \triangleright x \supseteq \triangleright A \} \]

is the set of elements which are lower bounds of every upper bound of \(A\); dually

\[ \triangleright \triangleleft A = \{ x : \triangleleft x \supseteq \triangleleft A \} \]

is the set of elements which are upper bounds of every lower bound of \(A\).
A.3. CONCEPTS GALORE!

Observe that for every $x \in X$, $(\triangleleft x, \triangleright x) = (\triangleright x, \triangleright x)$, so that the concept lattice (as a poset) contains an isomorphic copy of $X$. More generally, if $x \neq y$ are elements of $X$, then $\triangleright \{x, y\} = \triangleright x \cap \triangleright y$ is the set of common upper bounds; if $x, y$ happen to have a least upper bound $u \in X$ then $\triangleright \{x, y\} = \triangleright u$; otherwise, the completion has provided a new “element” $x \lor y$. Similarly, $\triangleleft \{x, y\} = \triangleleft x \cap \triangleleft y$ is the set of common lower bounds; if $x, y$ happen to have a greatest lower bound $\ell \in X$, then $\triangleleft x \cap \triangleleft y = \triangleleft \ell$; otherwise, the completion has provided a new “element” $x \land y$. In general, the concepts of Example 8 are the pairs of form $(\triangleleft A, \triangleright A)$, which can be identified with the “element” $\inf A = \land_{x \in A} x$ (which usually does not already exist in $X$) or $(\triangleright A, \triangleright A)$, which can be identified with the “element” $\sup A = \lor_{x \in A} x$ (which usually does not already exist in $X$). Also, observe that $\triangleleft X = \emptyset$ unless $X$ already had a unique minimal element, and $\triangleright X = \emptyset$ unless $X$ already had a unique maximal element. The upshot is that the lattice of concepts is the smallest complete lattice containing a copy of $X$ (hence the term “completion”).

![Diagram]

Figure A.7: A schematic diagram of the concept lattice for the Macneille-Dedekind completion of a poset $X$ (Example 8).

---

1We should perhaps note that the completion of $\mathbb{R}^2$ in Example 6 is not the only reasonable notion of completion; see the conformal model on p. 919 of [95].
Example 9 Let $G$ be a locally compact abelian group. A character of $G$ is a continuous group homomorphism $\chi : G \to S^1$. The characters of $G$ form a group $G^\dagger$ which becomes a locally compact abelian group when given the topology of compact convergence. Define a relation from $G$ to $G^\dagger$ by stipulating that $g \rho \chi$ if $\chi(g) = 1$. (This relation is known as Pontryagin duality).

Here the closure operator on $G$ takes a subset $H \subset G$ to the smallest closed subgroup swallowing $H$, and likewise for $G^\dagger$. The concepts are the pairs $(H, \text{ann } H)$ where $H$ is a closed subgroup of $G$ and

$$\text{ann } H = \{ \chi : \chi(h) = 1 \text{ for all } h \in H \}$$

is the annihilator subgroup of $H$: note that $\text{ann } H$ is a closed subgroup of $G^\dagger$. See Figure A.8.

It turns out (see Theorem 27 of [97]) that

$$(G/H)^\dagger \simeq \text{ann } H \quad H^\dagger \simeq G^\dagger / \text{ann } H$$

The famous Pontryagin duality theorem (see Chapter 4 of [97]) says that $(G^\dagger)^\dagger$ is isomorphic as a topological group to $G$ itself. Thus, every topological or group theoretical characteristic of $G$ is reflected in some characteristic of $G^\dagger$. In particular, $G^\dagger$ is respectively compact (discrete) (torsion-free) whenever $G$ is discrete (compact) (connected). (See [137].) For instance, the dual of $G = S^1$ (compact) is $G^\dagger = \mathbb{Z}$ (discrete). Furthermore, the space of square integrable functions on $G$, $L^2(G, m)$, (where $m$ is Haar measure on $S^1$) is the closure of the span of the elementary characters $\chi_n(x) = e^{inx}$, so we can decompose $f \in L^2(G, m)$ as a sum of characters:

$$f(x) = \sum_{n \in \mathbb{Z}} f^\dagger(n) \chi_n(x)$$

where

$$f^\dagger(n) = \int_G f(x) \chi_n(x) dm(x)$$
A.3. CONCEPTS GALORE!

Of course $f^\dagger$ is just the usual discrete Fourier transform of $f$. This connection between Fourier analysis and characters is central to the subject of harmonic analysis [37].

![Diagram]

Figure A.8: A schematic diagram of the concept lattice for Pontryagin duality (Example 9).

Our next four examples are fundamental in symbolic dynamics (see [87]).

Let $\mathcal{B} = \{0, 1\}$. Then $\mathcal{B}^\mathbb{Z}$ with the product topology is called the full two-shift. (Note that this is a compact Hausdorff space homeomorphic to the standard Cantor set.) Alternatively, given $x, y \in \mathcal{B}^\mathbb{Z}$, define $d(x, y) = 2^{-n}$ where $n$ is the smallest integer (in absolute value) such that $x(n) \neq y(n)$. That is, we define $x, y$ to be “close” if they agree on a large “interval” around the origin $0 \in \mathbb{Z}$. Then the metric topology agrees with the product topology.

The shift map $\mathcal{B}^\mathbb{Z} \xrightarrow{S} \mathcal{B}^\mathbb{Z}$ is defined by setting $S(x)(n) = x(n + 1)$ for all $n \in \mathbb{Z}$. It is a homeomorphism, so $S$ generates an action by $\mathbb{Z}$ on $\mathcal{B}^\mathbb{Z}$. If $I$ is some finite sequence of consecutive integers, then $\alpha = x|I$ (where $x$ is some element of $\mathcal{B}^\mathbb{Z}$) is called a block; the “interval” $I$ is the carrier of $\alpha$. The shift map now induces an action on blocks, where $n \in \mathbb{Z}$ takes $x|I$ to $S^n(x)|(I + n)$. The orbit of $\alpha = x|I$ under this action is known as a protoblock, written $[\alpha]$. Suppose $X$ is a closed (hence compact) shift invariant subspace of $\mathcal{B}^\mathbb{Z}$. Then
\((X, S|X)\) is called a shift dynamical system. We say that \(x \in X\) threads a block \(\alpha\) if \(\alpha = x|I\) for some "interval" \(I\). We say that a protoblock \([\alpha]\) occurs in \(x \in X\) if \(\alpha = S^n(x)|I\) for some "interval" \(I\) and some \(n \in \mathbb{Z}\). (Note that this is well defined!) The collection of all protoblocks occurring in \(X\) is its language \(\mathcal{L}(X)\).

Given a block \(\alpha\) carried by \(I\), the cylinder over \(\alpha\) is the set of all \(x\) which represent \(\alpha\); that is,

\[
Z(\alpha) = \{x \in \mathbb{B}^\mathbb{Z} : x|I = \alpha\}
\]

Note that every ball of \(\mathbb{B}^\mathbb{Z}\) is a cylinder, and given any cylinder \(Z\), every \(x \in Z\) has an open neighborhood included in \(Z\), so \(Z\) is open. On the other hand, given \(\alpha\) carried by \(I\), the cylinder \(Z(\alpha)^c\) can be decomposed into a finite union of balls (consider the blocks \(\beta \neq \alpha\) which are also carried by \(I\)), so cylinders are both open and closed.

A homomorphism between two shift dynamical systems is a continuous map \(\varphi : X \to Y\) which respects the shift map; that is, the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{S} & X \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
Y & \xrightarrow{S} & Y
\end{array}
\]

commutes. The Curtis-Hedlund Theorem (see Theorem 6.2.9 of [87]) says that the homomorphisms are exactly the sliding block codes determined by listing the "images" in \(Y\) of a finite list of protoblocks. The image \(\varphi(X)\) is compact and shift invariant, and thus \((\varphi(X), S|\varphi(X))\) defines a quotient of \((X, S|X)\).

Let \(X\) be a compact shift invariant subspace. Then \(X^c\) is open, so given any \(y \in X^c\), we can find a ball \(B\) with \(y \in B \subset X^c\). But \(B = Z(\alpha)\) for some block \(\alpha\) which does not occur in \(X\). Moreover (by the shift invariance of \(X\)) \(S^n(y)\) is never in \(X\), so we can describe \(X^c\), and thus \(X\), by giving a collection of protoblocks which do not occur in \(X\). Conversely, every list of forbidden protoblocks determines a unique
compact shift invariant subspace (possibly empty) and thus a unique shift dynamical system.

**Example 10** Define a relation from protoblocks to compact shift invariant subspaces of $B^Z$ by stipulating that $[\alpha] \rho X$ iff $[\alpha]$ is a forbidden protoblock for $X$.

Given a collection $\mathcal{A}$ of forbidden protoblocks, $X = \triangleright \mathcal{A}$ is corresponding compact shift invariant subspace of $B^Z$; dually, given $A \subseteq X$, $\mathcal{A} = \triangleleft A$ is the set of all protoblocks forbidden in $X$. The concepts are pairs $(\mathcal{A}, X)$, where $X$ is a compact shift invariant subspace of $B^Z$ and $\mathcal{A}$ is the corresponding set of forbidden protoblocks. Maximal proper concepts correspond to pairs $(\mathcal{A}, X)$ where $X$ is a minimal shift system; such shift systems are extremely "exclusive" in the sense that a great many blocks are forbidden. At the opposite extreme, shift systems $(X, S|X)$, where $X = \triangleright A$ where $\mathcal{A}$ is finite are called finite type shift systems: they are much more "inclusive" than minimal shift systems, in the sense that they permit a great many blocks. If $x \in B^Z$, $\triangleright x$ is the orbit closure of $x$; this generally corresponds to a "small" shift system but may not be minimal.

$$
\begin{array}{c}
(\triangleleft B^Z, \emptyset) \\
(\triangleleft (\mathcal{A} \cup B), X \cap Y) \\
(\mathcal{A}, X) \\
(\mathcal{A} \cap B, X \cup Y) \\
(\emptyset, B^Z)
\end{array}
$$

Figure A.9: A schematic diagram of the concept lattice for shift dynamical systems (Example 10). Here $\mathcal{A} = \triangleleft X$ is the complement in $\triangleleft B^Z$ of the language of the shift system $X$; likewise for $B = \triangleleft Y$. 
Example 11 Define a relation from $\mathcal{L}(X)$ to itself by stipulating that $[\alpha] \rho [\beta]$ if (for appropriately chosen representative blocks) $\beta = x|I$ and $\alpha = x|J$, where $I \supset J$.

Of course the concept lattice here is a special case of the Macneille-Dedekind completion. In analogy with Example 7, $\prec[\alpha]$ is always finite whereas $\triangleright[\alpha]$ is always infinite.

Fix a compact shift invariant subspace $X$. It may happen that whenever $x \in X$ threads $\alpha = x|I$, it also threads some larger block $\beta = x|J$, where $J \supset I$. If so, we say that $\alpha$ forces $\beta$ in $X$. The collection of all blocks forced by $\alpha$ is its empire $E(\alpha)$. If $(X, S|X)$ is a finite type shift system, $E(\alpha)$ will consist of a single, usually somewhat larger block $\beta$. However, if $(X, S|X)$ is a minimal shift system, $E(\alpha)$ is often an infinite collection of disjoint blocks (see Chapter 7). Thus, minimal shift systems are in a very real sense “maximally rigid” whereas finite type shift systems are “maximally flexible”.

Example 12 Let $X$ be a compact shift invariant subspace. Define a relation from the blocks of $X$ to $X$ by stipulating that $\alpha \rho x$ iff $x$ threads $\alpha$.

Given $\alpha$, $\triangleright \alpha = Z(\alpha)$; dually, given $x \in X$, $\triangleright x = \{x\}$ (since no $y \neq x$ can thread every block threaded by $x$). Furthermore, $\preccurlyeq \alpha = E(\alpha)$. The concepts are pairs $(E, Z)$, where $E$ is a generalized empire consisting of the blocks forced by a (possibly infinite) collection of blocks, and $Z$ is a generalized cylinder consisting of the elements threading a (possibly infinite) collection of blocks. The maximal proper concepts correspond to pairs $(E, Z)$ where $Z$ is the cylinder over a single index $n \in \mathbb{Z}$. Note that $Z(\alpha) \cap Z(\beta) = \emptyset$ iff $\alpha, \beta$ are “incompatible”, whereas $E(\alpha) \cap E(\beta) = \emptyset$ iff $\alpha, \beta$ are “supercompatible” in the sense that they force no common blocks.

The reader familiar with sheaves may recognize that the relation between blocks and sequences is analogous to the relation between local and global sections of any
sheaf. Furthermore, there is an obvious analogy between the notion of forcing and logical implication. These analogies are developed in [64].

\[
\begin{align*}
&(\vartriangleleft X, \emptyset) \\
&(E(\{\alpha, \beta\}), Z(\alpha) \cap Z(\beta)) \\
&(E(\alpha), Z(\alpha)) \\
&(E(\beta), Z(\beta)) \\
&(E(\alpha) \cap E(\beta), Z(\alpha) \cup Z(\beta)) \\
&(\emptyset, X)
\end{align*}
\]

Figure A.10: A schematic diagram of the concept lattice for empires and cylinders (Example 12) in a fixed compact shift invariant subspace \(X\).

If \([\alpha], [\beta]\) are protoblocks, their concatenation \([\alpha] \ast [\beta]\) is the protoblock obtained by appending \(\beta\) to \(\alpha\).

**Example 13** Fix a compact shift invariant subspace \(X\). Define a partial ordering on its language \(\mathcal{L}(X)\) by saying that \([\alpha] \rho [\beta]\) iff \([\alpha] \ast [\beta]\) occurs in \(X\).

Given \([\alpha]\), \(\triangleright[\alpha]\) is known as the follower set for \([\alpha]\) and \(\triangleleft[\alpha]\) is the leader set.

A quotient of a finite type shift system is called a sofic system. Equivalently (see Thm. 3.2.1 and Thm. 3.2.10 of [87]), a shift system is sofic iff it has only a finite number of distinct follower sets. Equivalently (see Exc. 3.2.6 in [87]), a shift system is sofic iff it has only a finite number of distinct leader sets; in other words, iff the concept lattice in Example 13 is finite. Quotients, intersections, and (finite) unions of sofic shift systems are sofic.

In the case of a sofic system, we can construct a labeled digraph whose vertices are the follower sets of individual protoblocks, as follows. Given \(\triangleright[\alpha]\), draw an edge
labeled "0" from $\triangleright [\alpha]$ to $\triangleright [\alpha] \ast 0$ and draw an edge labeled "1" from $\triangleright \alpha$ to $\triangleright [\alpha] \ast 1$.

This gives a presentation of the sofic system. For example, let $X$ be the set of all $x \in \mathbb{B}^\mathbb{Z}$ which contain an even number of 0's between every occurrence of 1: the shift system $(X, S|X)$ is called the even shift system. It is readily verified that the leader and follower sets are as shown in Figure A.12; compare Fig. 3.2.2 of [87].

As a special case of Example 2, we have defined convex hulls in $E^d$ (a finite dimen-
sional Hilbert space). For many reasons it is desirable to have an analogous notion of convexity of sets, not of points, but of lines and more generally, \(k\)-dimensional affine subspaces of \(E^d\). That is, instead of convex sets in \(E^d\) we wish to study convex sets in the "extended Grassmannian" manifold \(G'(d, k)\), consisting of \(k\) dimensional affine subspaces, or \(k\)-flats, of \(E^d\), where \(0 < k < d\). It turns out that the "correct" definition [46] is yet another example of concept lattice; interestingly enough, this is very closely related to the empire-cylinder concepts of Example 12.

**Example 14** Define a relation from the extended Grassmannian manifold \(G'(d, k)\) of all \(k\)-flats \(W\) of \(E^d\), \(0 < k < d\), to \(C(E^d)\), the collection of all convex sets \(S\) in \(E^d\), as follows. Stipulate that \(W \rho S\) iff \(W \cap S \neq \emptyset\); that is, iff \(W\) meets \(S\).

Now, if \(S\) is a collection of convex sets, then \(\rho S\) is the set of all \(k\)-transversals of \(S\); that is, all \(k\)-flats meeting each member of \(S\). If \(\mathcal{W}\) is a collection of \(k\)-flats, then \(\rho \mathcal{W}\) is the \(k\)-dual of \(\mathcal{W}\); that is, the collection of all convex subsets in \(E^d\) which meet each member of \(\mathcal{W}\). Finally, \(\mathcal{W}\) is the convex hull in \(G'(d, k)\) of \(\mathcal{W}\).

Here, we have in effect defined a subset \(\mathcal{W} \subset G'(d, k)\) to be convex iff it is its own convex hull. A few examples taken from [46] may help clarify this notion of convexity:

- the rulings on the variety \(x^2 + y^2 - z^2 = 1\) in \(E^3\), plus all lines fitting 'inside' this variety, form a convex set in \(G'(3, 1)\),

- if \(S\) is a convex set in \(E^d\) and \(W\) is a \(k\) dimensional affine subspace passing through \(S\), then the set of all \(k\)-flats parallel to \(W\) and meeting \(K\) is convex in \(G'(d, k)\),

- any set of three mutually skew lines in \(E^3\) is convex in \(G'(3, 1)\); this example shows that convex sets in \(G'(d, k)\) are not necessarily connected!
Figure A.13: A schematic diagram of the lattice of convex sets in the extended Grassmannian $G'(m, k)$ (Example 14), where $U, V$ are convex sets in $G'(m, k)$.

Now, I claim that in this construction, $\mathcal{S}$, the set of $k$-transversals of a collection of convex sets $\mathcal{S}$, is analogous to the cylinder of a collection of patches, and likewise $\mathcal{D}\mathcal{S}$ is analogous to the empire of a collection of patches. One way to make sense of this claim is to exhibit a situation where the notion of cylinder set and the set of transversals agrees. Such a situation almost obtains in the construction of Sturmian tilings by the method of oblique tilings introduced in [103], but instead of taking all transversals, we only take those parallel to a particular flat. See Fig. A.14: the oblique tiling is the periodic tiling of $\mathbb{R}^2$ by small and large "tilted" squares, and the corresponding Sturmian tilings of $\mathbb{R}$ are obtained by passing cut lines (parallel to the bases of the squares) through the oblique tiling; the resulting tilings consist of long and short tiles (line segments).

Our next two examples concern two fundamental constructions of functional analysis.

**Example 15** Let $X$ be a Banach space (i.e. a complete normed linear space) and let $X^*$ (with the weak-* topology, which agrees with the topology of pointwise convergence; that is, $f_n \to f$ iff $f_n(x) \to x$ for all $x \in X$; see [36] p.161) be the dual space
Figure A.14: The empire of the patch consisting of a short tile followed by two long tiles in a certain Sturmian tiling space. Here the corresponding cylinder set consists of all tilings obtained by passing transversals through the three oblique tiling cells corresponding to the sequence SLL at the center of this picture (underlined in the schematic sequence above).
consisting of all continuous linear functionals \( f : X \to \mathbb{R} \). Define a relation from \( X^* \) to \( X \) by stipulating that \( f \rho x \) if \( f(x) = 0 \).

Given \( A \subseteq X \), \( \triangleright A = \mathrm{cl} \, \mathrm{span} \, A \) is the closure of the subspace spanned by \( A \); that is, the smallest closed subspace swallowing \( A \). Dually, given \( E \subseteq X^* \), \( \triangleleft E = \mathrm{cl} \, \mathrm{span} \, E \) is the smallest closed subspace swallowing \( E \). (See Thm. 38.14 in [4] or Thm. 47 in [118].) The concepts are pairs of the form \((\text{ann} A, A)\), where \( A \) is a closed subspace of \( X \) and

\[
\text{ann} A = \{ f \in X^* : f(x) = 0, \forall x \in A \}
\]

is its annihilator. See Figure A.15.

\[
\begin{array}{c}
0 \\
(X^*, 0) \\
(\mathrm{cl} (\text{ann} A + \text{ann} B), A \cap B) \\
(\text{ann} A, A) \\
(\text{ann} B, B) \\
(\text{ann} A \cap \text{ann} B, \mathrm{cl} (A + B)) \\
(0, X)
\end{array}
\]

Figure A.15: A schematic diagram of the concept lattice for annihilators in a Banach space \( X \) and its dual \( X^* \) (Example 15), where \( A, B \) are closed subspaces of \( X \).

This construction can be considerably generalized. The evaluation map taking \((f, x)\) to \( f(x) \) is a continuous bilinear form on \( X^* \times X \), and in general any such form gives rise to weak topologies and results analogous to those stated above; see Chapter 38 of [4]. (This also generalizes Example 3.)

There is another way to understand these relations. Given Banach spaces \( X, Y \) and a continuous linear map \( X \overset{\varphi}{\to} Y \), define its adjoint map \( X^* \overset{\varphi^*}{\to} Y^* \) by setting
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$\varphi^*g = g \circ \varphi$ for all $g \in Y^*$. This defines a contravariant functor. Then in the concept lattice for $X$ and $X^*$ we have

$$\ker \varphi = \triangleright \text{im} \varphi^*$$
$$\text{cl im} \varphi = \triangleright \ker \varphi^*$$

while in the concept lattice for $Y$ and $Y^*$ we have

$$\ker \varphi^* = \triangleleft \text{im} \varphi$$
$$\text{cl im} \varphi^* = \triangleleft \ker \varphi$$

Moreover, $\varphi$ is one-one iff $\text{im} \varphi^*$ is dense in $X^*$ and $\varphi^*$ is one-one iff $\text{im} \varphi$ is dense in $X$. (See Thm. 4.12 in [118].)

In particular, if $A$ is a closed subspace of $X$, then the inclusion map $A \rightarrow X$ has the adjoint $A^* \hookrightarrow X^*$, where $\iota^*$ restricts $f \in X^*$ to $A$. Here, $\iota^*$ is onto with $\ker \iota^* = \triangleleft A$. If $X^*$ is given the operator norm (the induced topology is finer than the weak-* topology, so $\triangleleft A$ is still closed) then there is a natural linear isometric homeomorphism from $X^*/\triangleleft A$ onto $A^*$. (See Thm. 44.3 in [4] or Thm. 4.9a in [118].)

Furthermore, the projection $X \twoheadrightarrow X/A$ has adjoint $X^* \hookrightarrow (X/A)^*$. Here, $\pi^*$ is one-one with $\text{im} \pi^* = \triangleright A$. If $(X/A)^*$ is given the operator norm, then there is a natural linear isometric homeomorphism from $(X/A)^*$ onto $\triangleleft A$ (See Thm. 44.4 in [4] or Thm. 4.9b in [118].)

**Example 16** Let $X$ be any topological vector space and let $X^*$ be its dual space (with the weak-* topology). Define a relation from $X^*$ to $X$ by stipulating that $f \rho x$ whenever $|f(x)| \leq 1$.

Given $A \subset X$ with $0 \in A$, its polar $A^0 = \triangleleft A$ is closed convex in $X^*$ and contains $0 \in X^*$. Dually, given $E \subset X^*$ such that $0 \in E$, its prepolar $\triangleright E$ is closed.
convex in $X$ and contains $0 \in \text{r} E$. (See Thms. 3.12 [118].) If $A$ has nonempty interior, the Banach-Alaoglu Theorem (see Thm. 3.15 of [118]) states that $A^0$ is in fact compact convex in $X^*$. Thus, concepts have the form $(E, A)$ where $E = \text{r} A$ and $A, E$ are closed convex sets containing $0 \in X^*$ and $0 \in X$, respectively, and where $E$ is compact whenever $A$ has nonempty interior. See Figure A.16.

![Diagram](image)

Figure A.16: A schematic diagram of the concept lattice of prepolars and polars for a topological vector space $X$ (Example 16), where $A, B$ are closed subspaces of $X$.

If $X$ is a locally convex topological vector space and $K$ is a nonempty compact convex subset, the Krein-Millman Theorem (see Thm. 36.9 of [4] or Thm. 2.5.4 of [105] or Thm. 3.23 of [118]) states that $\text{cl conv ext } K = K$, where $\text{ext } K$ are the extreme points of $K$. This allows one to generalize the idea that every point in a convex polytope can be obtained by averaging over the vertices.

**Example 17** Let $X$ be a compact Hausdorff space and let $C(X)$ be the space of continuous functions on $X$. Define a relation from $C(X)$ to $X$ by stipulating that $f \rho x$ if $f(x) = 0$.

Given $A \subseteq X$, $\text{r} A = \text{cl} A$ is just the topological closure of $A$; dually, given $E \subseteq C(X)$, $\text{r} E$ is the smallest closed ideal of $C(X)$ which swallows $E$. The concepts
are the pairs \((\text{ann } A, A)\) where \(A\) is a closed set in \(X\) and

\[\text{ann } A = \{f \in C(X) : f(x) = 0 \text{ for all } x \in A\}\]

is the corresponding ideal of functions vanishing on \(A\); see Figure A.17. We have the isometric isomorphism \(C(A) \cong C(X)/\triangleleft A\). The maximal proper concepts have the form \((M, \{x\})\) where \(M\) is the maximal ideal of all functions vanishing at \(x\). The minimal concept is \((C(X), \emptyset)\) and the maximal concept is \((0, X)\).

![Concept lattice diagram](image)

Figure A.17: A schematic diagram of the concept lattice for closed sets and ideals of continuous functions on the compact Hausdorff space \(X\) (Example 17), where \(A, B\) are closed sets in \(X\).

**Example 18** Let \(K\) be any field, and let \(R = K[x_1, x_2, \ldots x_n]\) be the ring of polynomials in \(n\) variables with coefficients in \(K\). Let \(X = K^n\) be the \(n\) dimensional affine space consisting of \(n\)-tuples of numbers in \(K\). Define a relation from \(R\) to \(X\) by stipulating that \(f \rho x\) if \(f(x) = 0\).

Given \(A \subset X\), \(\triangleleft A\) is the smallest algebraic variety (zero set of some polynomials) swallowing \(A\), while given \(S \subset R\), \(\Phi S = \sqrt{S}\) is the smallest radical ideal swallowing \(S\). Here, a radical ideal has the property that \(f \in I\) whenever some power
$f^n$ is. For example, if $S = (f)$ where $f$ factors into a product of irreducible polynomials as $f = \prod_{j=1}^{n} p_j^n$, then $\sqrt{S} = (g)$, where $g = \prod_{j=1}^{n} p_j$. The concepts are pairs of the form $(I, V)$ where $V = \mathcal{V} I$ is an algebraic variety (i.e., the set of zero set of some set of polynomials) and $I = \mathcal{V} V = \sqrt{V}$ is the ideal consisting of all polynomials which vanish on $V$.

The characterization of the closed sets in $R$ as being precisely the radical ideals deserves emphasis, since it is a nontrivial theorem; indeed, it is one of the fundamental theorems of algebraic geometry, the *Nullstellensatz* of Hilbert. (*Nullstelle* is the German word for the zero of a polynomial and *Satz* is the word for theorem.)

The *Zariski topology* on $X$ is the unique topology in which the closed sets are precisely the algebraic varieties in $X$. The radical ideals corresponding to irreducible varieties are particularly interesting: they are precisely the prime ideals of $R$; see Figure A.18. (Note that if $I, J$ are two radical ideals then $I \cap J = \sqrt{IJ}$.) In particular, each point of $X$ is an irreducible variety; the corresponding ideals are precisely the maximal ideals of $R$.

Reference: Chapter 4 of [21] contains a detailed discussion of the relationship between algebraic varieties and radical ideals.

$$
(R, \emptyset) \\
\downarrow \quad \downarrow \\
(\sqrt{I + J}, A \cap B) \\
\downarrow \quad \downarrow \\
(I, A) \\
\downarrow \quad \downarrow \\
(J, B) \\
\downarrow \quad \downarrow \\
(I \cap J, A \cup B) \\
\downarrow \quad \downarrow \\
(\sqrt{R}, X)
$$

Figure A.18: A schematic diagram of the concept lattice for algebraic varieties and radical ideals (Example 18), where $I = \mathcal{V} A, J = \mathcal{V} B$ for algebraic varieties $A, B$. 
Example 19 Let $X$ be a finite algebraic Galois extension of a field $K$. Let $G = \text{Gal}(X/K)$ be the group of all field automorphisms of $X$ which pointwise fix $K$. Define a relation from $G$ to $X$ by stipulating that $\gamma \rho x$ if $x$ is a fixed point of $\gamma$; that is, if $\gamma(x) = x$.

Given $A \subset X$, $\triangleright A$ is the smallest intermediate field containing $A$; dually, given $B \subset G$, $\triangleright B$ is the smallest subgroup of $G$ containing $B$. The concepts are pairs of form $(\text{Gal}(X/L), L)$. Here $\text{Gal}(X/L)$ is the group of all field automorphisms of $X$ which pointwise fix $L$; equivalently, $\gamma \in \text{Aut}(X)$ is in $\text{Gal}(X/L)$ iff the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\gamma} & X \\
\uparrow & & \uparrow \\
K & = & K
\end{array}
$$

(A.8)

commutes, where $\iota : L \to X$ is the inclusion of $L$ into $X$.

$$
(G, K) \\
\downarrow \\
((H_1 \cup H_2), L_1 \cap L_2) \\
\downarrow \\
(H_1, L_1) \quad \quad \quad \quad \quad \quad \quad (H_2, L_2) \\
\downarrow \\
(H_1 \cap H_2, L_1 L_2) \\
\downarrow \\
(1, X)
$$

Figure A.19: A schematic diagram of the concept lattice of intermediate fields and their Galois groups (Example 19), where $H_1 = \text{Gal}(X/L_1)$, $H_2 = \text{Gal}(X/L_2)$, $G = \text{Gal}(X/K)$, and $L_1 L_2$ is the intermediate field consisting of all finite sums of pointwise products of elements of $L_1$ and $L_2$. Here $(H)$ is the smallest subgroup swallowing $H \subset G$.

$G$ induces a well-defined action on concepts, because $\sigma$ fixes $x$ iff $\tau \sigma \tau^{-1}$ fixes $\tau x$, for every $\tau \in \text{Aut}(X)$. Moreover, $L$ is setwise fixed under $G$ iff $\text{Gal}(X/L)$ is a normal
subgroup, and if so

\[ \text{Gal}(L/K) \cong \text{Gal}(X/K)/\text{Gal}(X/L) \]

in which case the concept lattice for \( L \) as an extension of \( K \) is obtained by "pruning" out everything in the concept lattice for \( X \) which doesn't lie above \((K,G)\) and below \((L,\text{Gal}(L/K))\), and then moding out the subgroups by \( \text{Gal}(X/L) \).

The fact that every subgroup of \( G \) is closed (i.e., the automorphism group of some intermediate field) and every intermediate field between \( X, K \) is closed (i.e., the fixed field of some group of automorphisms of \( X \)) deserves emphasis, because, together with the remarkable identity \([G:\text{Gal}(X/L)] = \dim X/L\), it forms the nontrivial part of Galois's original duality theorem! The hypothesis that \( X/K \) is an algebraic Galois extension ensures that \( \triangleright \triangleleft L = L \) for every intermediate field, but unless \( X/K \) is also finite, it will not in general be true that \( \triangleleft \triangleleft H = H \).

Reference: [129] provides an excellent and readable introduction to classical Galois theory.

As the reader may have already realized, Example 19 can be generalized to obtain a concept lattice for an arbitrary group action.

**Example 20** Let \( G \) be a group acting on a set \( X \); that is, let \( \theta : G \to \text{Sym} X \) be a group homomorphism into the symmetric group on \( X \). Define a relation from \( G \) to \( X \) by stipulating that \( gpx \) if \( \theta(g)(x) = x \).

Here the closed sets of \( G \) are the (pointwise) stabilizers \( \triangleleft A \) of subsets \( A \subset X \), and the closed sets of \( X \) are the sets pointwise fixed by some stabilizer \( \triangleleft A \). The concepts are precisely the pairs \((H,A)\), where \( H = \triangleleft A \) is a stabilizer and \( A = \triangleright H \) is the corresponding fixed set. Once again, \( g \in G \) acts on the concepts by \((H,A) \mapsto (gHg^{-1},gA)\); here \( A \) is stable iff \( H \) is normal, and if so, the lattice for \( H \) acting on \( A \) is obtained by the obvious restriction. (See Appendix B for the Galois theory of \( G \)-sets.)
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Figure A.20: A schematic diagram of the concept lattice of stabilizer subgroups and fixsets (Example 20), where $A = \triangleright H$, $H = \triangleleft A$ and $B = \triangleright K$, $K = \triangleleft B$.

These considerations lead to a sort of "generalized information theory" [65], as follows. For a given action by $G$ on $X$, define a (pointwise!) motion of $A \subseteq X$ to be any mapping of the form $\theta(g)|A : A \rightarrow X$. In general, many different $g \in G$ will give the same motion of $A$. The complexion of $A$, written $G \cdot A$, is the set of all motions of $A$. Here, $G \cdot A$ is a new homogeneous $G$-set, isomorphic to $G/\triangleleft A$. We may think of the complexion as a "generalized orbit" of $A$; its "size" measures how much information we need to specify a particular motion of $A$. The meaning of the fixset $\triangleright \triangleleft A \supseteq A$ is very simple; it contains exactly those points $x$ whose motions are completely determined by the motions of $A$. (Note that $G \cdot \triangleright \triangleleft A$ is always isomorphic to $G \cdot A$, as must happen.)

Now suppose we know the motion of some $B \subseteq X$. The conditional complexion $\triangleleft B \cdot A$ is the set of all motions of $A$ of form $\theta(h)|A : A \rightarrow X$ where $h \in \triangleleft B$; its size measures the information we still lack about the motion of $A$ if we learn the motion of $B$. Also, $\triangleleft B \cdot A$ is isomorphic as a $\triangleleft B$-set to

$$\triangleleft B/\triangleleft(A \cup B) = \triangleleft B/\triangleleft(A \cap \triangleleft B)$$
For many familiar actions by Lie groups (see for instance [11][135]), the dimension of the complexes provide very reasonable measures of the "size" of the complexes; this gives a reasonable measure  

**derivation**. In the case of finite groups, the dimension may be replaced by the logarithm of the size of the complex. For actions by the symmetric groups $S_n$, we recover Shannon's notion of entropy [63][126] as a limiting case.

Our final example includes ideas from many of the previous examples. We need a considerable amount of structure, which will only be sketched here.

If $X$ be a compact Hausdorff space, let $C(X)$ denote the continuous functions on $X$. Note that $C(X)$ is a commutative (unital) Banach algebra with the sup norm topology (see section 10.3 of [118]). Let $P(X)$ denote the regular Borel probability measures on $X$. $P(X)$ is a compact Hausdorff space with respect to the vague topology in which $\mu_n$ converges to $\mu$ iff for all $f \in C(X)$ we have

$$\int f \, d\mu_n \to \int f \, d\mu$$

(See Prop. 7.19 of [36].) Moreover, $P(X)$ is a compact convex subset of $M(X)$ (the space of signed regular Borel measures) whose extreme points are precisely the delta measures;

$$\text{ext } P(X) = \{ \delta_x : x \in X \}$$

(See Prop. 2.5.7 of [105].) Moreover, $\text{ext } P(X) \approx X$ via a natural map $\eta_X$; this map is "natural in $X"$ in the sense that for any continuous map $\varphi$, the diagram

$$\begin{array}{c}
X \xrightarrow{\varphi} Y \\
\downarrow^{\eta_X} \quad \quad \quad \quad \quad \quad \downarrow^{\eta_Y} \\
\text{ext } P(X) \xrightarrow{\varphi_*} \text{ext } P(Y)
\end{array}$$

commutes. Indeed, for any closed subset $A \subset X$, we have $\text{ext } P(A) \approx A$. 
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Given $\mu \in P(X)$, define the support of $\mu$ to be $\text{supp} \mu = N^c$ where $N$ is the set of all $x$ which have an open neighborhood $U$ with $\mu(U) = 0$. Then $\mu$ is absolutely continuous with respect to $\nu$ iff $\text{supp} \mu \subseteq \text{supp} \nu$, and $\mu, \nu$ are mutually singular iff they have disjoint supports.

If $X \xrightarrow{\varphi} Y$ is a continuous map between compact Hausdorff spaces, then the pullback $C(X) \xleftarrow{\varphi^*} C(Y)$ is defined by $\varphi^* g = g \circ \varphi$. The pullback is a Banach algebra homomorphism (i.e. continuous and non-increasing). Next, the pushout $P(X) \xrightarrow{\varphi_*} P(Y)$ is defined by requiring that for $g \in C(Y)$, we have

$$\int g d(\varphi_* \mu) = \int g \circ \varphi d\mu$$

The pushout is a continuous affine map (see Thm. 6.7 of [137]) and thus takes convex sets to convex sets. Furthermore, $\varphi_*$ is one-one (onto) whenever $\varphi$ is one-one (onto), and $\varphi_* \mu$ is absolutely continuous with respect to $\varphi_* \nu$ whenever $\mu$ is absolutely continuous with respect to $\nu$, and then

$$\frac{d\nu}{d\mu} = \frac{d\varphi_* \nu}{d\varphi_* \mu} \circ \varphi$$

(where $d\nu/d\mu$ is the Radon-Nikodym derivative).

If $S \circ \mu$, then $(X, S, \mu)$ is called a process. If $(Y, T, \nu)$ is another process, and $\varphi : X \rightarrow Y$ is a continuous map such that the diagram

$$(X, \mu) \xrightarrow{S} (X, \mu)$$

$$\varphi \downarrow \quad \downarrow \varphi$$

$$(Y, \nu) \xrightarrow{T} (Y, \nu)$$

commutes (i.e. $\varphi \circ S = T \circ \varphi$ and $\varphi_* \mu = \nu$), then $\varphi$ is called a process homomorphism. Thus, $(X, S, \mu)$ is isomorphic to $(X, T, \nu)$ via $R \in \text{Aut}(X)$ iff

1. $RS = TR$,}


2. $R_\ast \mu = \nu.$

Given $X$ a compact metric space, let $\text{Aut}(X)$, the group of homeomorphisms on $X$, have the compact-open topology, or equivalently the topology of uniform convergence. Then $X$ is a homogeneous space of any closed subgroup $G \subset \text{Aut}(X)$ which acts transitively on $X$. The natural action by $\text{Aut}(X)$ on $X$ induces actions on $C(X)$ and $P(X)$ in the obvious way: $S \in \text{Aut}(X)$ takes $f \in C(X)$ to $S^* f$ and takes $\mu \in P(X)$ to $S_\ast \mu$.

**Example 21** Let $X$ be a compact metric space and $P(X)$ the compact convex set of regular Borel probability measures on $X$. Let $\text{Aut}(X)$ act on $P(X)$ as above. Define a relation from $\text{Aut}(X)$ to $P(X)$ by stipulating that $S \triangleright \mu$ iff $\mu$ is $S$-invariant; that is, $S_\ast \mu = \mu$.

The *Krylov-Bogoliubov Theorem* (see Cor. 6.9.1 of [137]) states that $\triangleright S$ is always a nonempty compact convex subset of $P(X)$; moreover, $\text{ext} \triangleright S$ consists exactly of the ergodic $S$-invariant regular Borel probability measures (see Thm. 6.10 of [137]). Indeed, $\mu$ is an ergodic $S$-invariant regular Borel probability measure iff the ergodic averages

$$\frac{1}{n} \sum_{k=0}^{n-1} (S_\ast)^k \nu$$

converge weakly to $\mu$ whenever $\nu$ is absolutely continuous with respect to $\nu$, i.e. whenever $\text{supp} \nu \subset \text{supp} \mu$. (See Thm. 6.12 of [137].)

It is easy to see that for any $S, T \in \text{Aut}(X)$, $\triangleright (ST) \approx \triangleright (TS)$. Indeed,

$$S_\ast \triangleright (TS) = \triangleright (STSS^{-1}) = \triangleright (TS)$$

and similarly for $\triangleright (ST)$. This in turn implies that $\triangleright S$ is setwise invariant under $T$ and vice versa whenever $S, T$ commute. If $S$ is not the identity map, $\triangleright S$ is nowhere
dense in \( P(X) \); in fact, it may be visualized as a sort of "flat slice" through the "simplex" \( P(X) \). This imagery is inspired by the remarkable *Ergodic Decomposition Theorem* (see p.153 of [137]), which says that for all \( \mu \in \triangleright S \), there is some \( \sigma \in P(\triangleright S) \) (note that this makes sense because \( \triangleright S \) is itself a compact Hausdorff space) such that \( \text{supp} \sigma = \triangleright S \) and such that for all \( f \in C(X) \),

\[
\int_X f d\mu = \int_{\text{ext } S} \hat{f} d\sigma
\]

where \( \hat{f}(\mu) = \int_X f d\mu \) is a continuous affine function on \( P(X) \). That is, \( \mu \) is a sort of "average" (involving an integral, not a sum) over the extreme points, which are something like the "vertices" of the "simplex" \( \triangleright S \).

Indeed, in the very special case when \( X \) is a finite space with the discrete topology, this imagery is literally true. (In general, \( \triangleright S \) is a *Choquet simplex*, an infinite dimensional object modeled on ordinary simplices [108][137].) For example, let \( X = \{x_1, x_2, x_3, x_4\} \). Then \( P(X) \) is the ordinary regular tetrahedron; the four vertices correspond to the four measures \( \delta_{x_1}, \delta_{x_2}, \delta_{x_3}, \delta_{x_4} \). Now for every \( S \in \text{Aut}(S) \), \( S_\ast \) is not only an affine map on \( P(X) \) but in fact a linear isometry of the tetrahedron. The three homeomorphisms (permutations, in this case)

\[
\begin{align*}
R &= (x_1, x_2)(x_3, x_4) \\
S &= (x_1, x_3)(x_2, x_4) \\
T &= (x_1, x_4)(x_2, x_3)
\end{align*}
\]

are mutually commuting; the compact convex sets \( \triangleright R, \triangleright S, \) and \( \triangleright T \) are the three medians of the tetrahedron, which intersect in the centroid, which of course represents the measure giving weight 1/4 to each point. The ergodic \( R \)-invariant measures are the two vertices

\[
\frac{\delta_{x_1} + \delta_{x_2}}{2}, \quad \frac{\delta_{x_3} + \delta_{x_4}}{2}
\]
For three cycles like $K = (x_1, x_2, x_3)$, $\triangleright K$ is the line segment from the unique fixed point $x_4$ through the centroid of the opposite face (and the ergodic $K$-invariant measures are the two vertices). For transpositions like $J = (x_1, x_2)$, $\triangleright J$ is the triangle from the edge containing the two fixed points through the midpoint of the opposite edge (and the ergodic $J$-invariant measures are the three vertices).
Appendix B

An Outline of the Theory of $G$-Sets

B.1 Introduction

In this paper, I present a rapid guided tour of some of the highlights of the theory of actions by groups on sets. The material presented here is either "well known" (in the sense that if you know just where to look, you can find it in the printed literature), or else part of the "folklore" (in the sense of something generally known but never published), but seems to be known in full to few people.

Group actions comprise a classical subject with eighteenth century roots, which nevertheless can be presented (as I have done here) in the efficient—but abstract—style of modern mathematics. Such actions first arose in the work of Lagrange and Galois on the problem of solving polynomials by radicals [32][133]. Later they were used by Felix Klein to state the thesis of his Erlangen Programme, according to which every geometry concerns the study of invariants under some transformation group [94]; and conversely, every group can be studied as a group of transformations of some "geometric" space. Klein was quite explicit about the first half of this claim but rather vague about the second half. In recent years, the development of the theory of buildings has finally realized the missing half of the Erlangen Programme [116][121]. Still more recently, the theory of automatic groups [33][120] has further invigorated the study of group actions.

In addition to the areas just mentioned, group actions are related to invariant theory [50][131] and algebraic geometry [21], the theory of (linear) group representations [77], combinatorial group theory [16], Lie theory [73] and differential equations [74], as well as enumeration theory [112]. I will indicate some of these con-
nections below. The category of $G$-sets also provides an important example of a topos [61]. For more information on the role of group actions in the development of modern algebra, see [134].

In [69] I proposed three “information-theoretical” problems which are fundamental to the study of a given category $\mathcal{C}$:

1. What information is needed to specify an isomorphism class in $\mathcal{C}$?

2. What information is needed to specify a morphism from $X$ to $Y$?

3. What information is needed to specify a subobject of $X$?

It is striking that these problems can be completely solved for the category of $K$-linear spaces and for the category of $G$-sets (defined below), but of course for most other categories which occur in mathematics they are practically or even theoretically intractable. The category of $G$-sets is also unusual in that the automorphism group of an arbitrary object can be described explicitly, just as for linear spaces. I think it is fair to say that, like the category of $K$-linear spaces, the category of $G$-sets strikes a nice balance between simplicity and complexity which (together with its ubiquity in modern mathematics) renders it an attractive candidate for inclusion in an undergraduate algebra course. (See [38][99].) Nevertheless, to my knowledge no short survey of the theory of $G$-sets has previously appeared.

This is an expository paper; hence, few proofs are offered but references to places where complete proofs may be found are given wherever possible. Familiarity with elementary group theory at the level of [99] or even [51] will be indispensable; some familiarity with the elements of category theory and universal algebra will be very helpful but not essential; see [61] for the relevant constructions and universal mapping properties (UMP's) in category theory. (Alternatively, see [44][45] for excellent
introductions to category theory at a fairly elementary level; more sophisticated introductions can be found in [76] [83].) See [35] for a brief but friendly introduction to universal algebra; a more sophisticated account is given in [76].

### B.2 Fundamental Concepts

In the study of vector spaces, it is natural to fix the field of scalars \( K \) and to study the class of vector spaces over \( K \). More generally, in the study of modules over rings, it is useful to fix the ring \( R \) and study the class of \( R \)-modules. Therefore, it is reasonable to expect that in the study of group actions, it will be useful to fix the group and to study the class of sets acted upon by \( G \).

**Definition 34** Let \( G \) be a group and let \( X \) be a set. Suppose \( \theta : G \to \text{Sym}(X) \) is a group homomorphism, where \( \text{Sym}(X) \) is the symmetric group of \( X \) (the group of all permutations of the elements of \( X \)). Then \( X \) is a \( G \)-set, and \( G \) is said to act on \( X \).

When the action \( \theta \) is understood, it is often convenient to abbreviate \( \theta(g)(x) \) as \( g.x \). Because \( \theta \) is a homomorphism, \( \theta(g_2g_1) = \theta_{g_2} \circ \theta_{g_1} \), or \( g_2.(g_1.x) = (g_2g_1).x \) for all \( x \in X \), and \( \theta(e) = \text{id}_X \), or \( e.x = x \) for all \( x \in X \). This is often used as an alternative definition of a group action [38].

Two particularly important examples of actions by \( G \) are as follows:

1. \( G \) acts on itself via left multiplication; i.e., \( \theta_1(g)(h) = gh \),

2. \( G \) also acts on itself via conjugation; i.e., \( \theta_2(g)(h) = ghg^{-1} \).

A presentation of a finite group \( G \) by means of generators and relations can be graphically described by means of a Cayley diagram [51] [23], a digraph in which the vertices correspond to elements of the group and an edge of type \( j \) leading from
$g_1$ to $g_2$ means that multiplying $g_1$ on the right by the $j$-th generator yields $g_2$. Given a presentation of $G$, we can graphically describe finite $G$-sets by means of Schreier diagrams [23]. These are digraphs in which the vertices correspond to the points of $X$ and an edge of type $j$ leading from $x_1$ to $x_2$ means that the $j$-th generator of $G$ takes $x_1$ to $x_2$. In drawing Schreier diagrams, it is helpful to observe the common conventions [23][51][132] that involutory (period two) generators are represented by one undirected edge instead of two parallel directed edges, and that all loops are deleted.

Definition 35 Let $X$ be a $G$-set and let $x \in X$. Then the orbit of $x$, written $x^G$, is the set of all points of $X$ which are accessible from $x$ via the action; that is, $x^G = \{gx : g \in G\}$.

The orbits of $X$ are easily read off from a Schreier diagram for $X$; they are the smallest subsets which are not connected by an edge to any larger subset. The collection of orbits in $X$ is the orbit space $X/G$. If $X$ has only one orbit, it is often called homogeneous. Homogeneous $G$-sets play a fundamental role in the theory of $G$-sets, as we shall see.

Definition 36 The stabilizer of $x$ is the subgroup consisting of all $g \in G$ which do not move $x$: $G_x = \{g : gx = x\}$.

Similarly, given $A \subset X$, the setwise stabilizer of $A$, written $G_A$, is the set of group elements taking $A$ into itself; that is, $g \in G_A$ iff $\theta(g)(A) \subset A$. Given $A \subset X$, the set of setwise translates of $A$ is written $A^G = \{g.A : g \in G\}$, where each $g.A = \theta(g)(A)$. (Note that $A^G$ is simply the orbit of $A$ and $G_A$ is the stabilizer of $A$, under the obvious action on $2^X$ induced from the action on $X$.)

Definition 37 A subset $A \subset X$ is stable under the action by $G$ if $G_A = G$. 
B.2. FUNDAMENTAL CONCEPTS

In a Schreier diagram for \( X \), the stable subsets are those which are not connected to any larger subset by any edges. It is easy to see that stable sets are unions of orbits, and that orbits are minimal nonempty stable sets. This has some important implications:

1. The complement \( X \setminus A \) of a stable subset \( A \) is another stable subset. More generally, if \( A \leq B \leq X \) where \( A, B \) are each stable, then \( B \setminus A \) is also stable.

2. The intersection \( A \cap B \) of two stable subsets \( A, B \leq X \) is another stable subset of \( X \).

3. The union \( A \cup B \) of two stable subsets \( A, B \leq X \) is another stable subset of \( X \).

These facts can be summed up by saying that the stable subsets of \( X \) form a complete Boolean lattice. (See [25] or [76] for the definition of lattice, completion, and Boolean lattice.)

Definition 38 Let \( X \) be a \( G \)-set and suppose \( \sim \) is an equivalence relation on \( X \). Then \( \sim \) is called a \( G \)-congruence if the action respects equivalence in the sense that \( x \sim y \) implies \( g.x \sim g.y \).

In this situation, the set of congruence classes \( \pi(x) \) forms a \( G \)-set, called the quotient of \( X \) by \( \sim \) and written \( X/\sim \), with the action \( g.\pi(x) = \pi(g.x) \). If \( \sigma, \tau \) are two \( G \)-congruences such that \( \sigma(x) \subset \tau(x) \) for every \( x \in X \), we say that \( \sigma \) refines \( \tau \), written \( \tau \leq \sigma \). Given any two \( G \)-congruences \( \sigma, \tau \), we can form two new \( G \)-congruences as follows. Define the join \( \sigma \vee \tau \) by setting \( (\sigma \vee \tau)(x) = \sigma(x) \cap \tau(x) \). That is, the congruence classes of the join are formed by taking the pairwise intersections of classes from \( \sigma \) with classes from \( \tau \), which implies that \( \sigma, \tau \leq \sigma \vee \tau \). Define the
meet \( \sigma \land \tau \) by taking the unions of overlapping atoms of \( \sigma, \tau \) to be the atoms of \( \sigma \land \tau \); thus, \( \sigma \land \tau \leq \sigma, \tau \). (See [30] for more detail). With respect to these operations, the set of \( G \)-congruences on \( X \) forms a complete lattice. However, there is no notion of complementation, so the \( G \)-congruences do not form a Boolean lattice.

A subset \( A \subseteq X \) is called a nucleus if for all \( g \in G \), either \( g.A = A \) or \( g.A \) is disjoint from \( A \). It is not hard to see that a subset \( A \) is a nucleus if it is the atom of a congruence relation, so this gives an alternative way of defining congruences [29].

The orbits of a \( G \)-set give one possible congruence relation on \( X \). If \( N \) is normal in \( G \), then the orbits \( x^N \) form another congruence relation on \( X \) (considered as a \( G \)-set). The costability class of \( x \in X \), written \([x]\), is the set of all \( y \in x^G \) such that \( G_y = G_x \). The costability classes always define a congruence relation on \( X \), which refines the congruence relation defined by the orbits.

**Definition 39** A simple \( G \)-set is one which has no nontrivial quotients.

Simple \( G \)-sets are also called irreducible or primitive) \( G \)-sets. Every simple \( G \)-set must be homogeneous, but the converse is false. In the next section, we shall see how every \( G \)-set can be “built up” from its orbits. and how each orbit can be constructed in turn from simple \( G \)-sets. Thus, simple \( G \)-sets are in a very real sense the “building blocks” of the theory of \( G \)-sets.

**B.3 The Category of \( G \)-Sets**

**Definition 40** Suppose \( \theta : G \rightarrow \text{Sym}(X) \) and \( \psi : G \rightarrow \text{Sym}(Y) \) define actions by \( G \) on \( X, Y \). If \( \varphi : X \rightarrow Y \) is a mapping which respects the actions in the sense that that for all \( g \in G \), \( \varphi \circ \theta(g) = \psi(g) \circ \varphi \), then \( \varphi \) is a \( G \)-homomorphism, hereafter abbreviated \( G \)-hom. \((G \text{-homs are often called } G \text{-equivariant maps.})\)
This condition is often expressed pictorially by stipulating that for all \( g \in G \), the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\theta(g)} & X \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
Y & \longrightarrow & Y
\end{array}
\]

commutes; that is, going across and then down has the same effect as going down and then across.

\( G \)-sets and \( G \)-homs (with composition being the ordinary composition of mappings) define the category of \( G \)-sets. Note well!: there is a distinct category of \( G \)-sets for every group \( G \), just as there is a distinct category of vector space over \( K \) for every field \( K \).

It is straightforward to show that a \( G \)-hom \( \varphi : X \rightarrow Y \) is

1. a monic morphism in the category of \( G \)-sets if and only if it is one-one,

2. an epic morphism in the category of \( G \)-sets if and only if it is onto,

3. an isomorphism in the category of \( G \)-sets if and only if it is bijective.

Furthermore, \( \eta : X \rightarrow X \) is an automorphism of \( X \) if and only if \( \eta \) is in the centralizer of \( \theta(G) \) (considered as a subgroup of \( \text{Sym} X \)). \( A \) is stable in \( X \) iff the inclusion map \( \iota : A \rightarrow X \) is a \( G \)-hom. We will call a monic \( G \)-hom a \( G \)-mono and an epic \( G \)-hom a \( G \)-epi for short.

**Definition 41** Two \( G \)-monos \( \alpha : A \rightarrow X \), \( \beta : B \rightarrow X \), each having codomain \( X \), are said to be equivalent, written \( \eta \sim \nu \), if there exists an isomorphism \( \xi : A \rightarrow B \) such that \( \beta = \alpha \circ \xi \). The equivalence classes are the subobjects of \( X \). Two \( G \)-epis \( \sigma : X \rightarrow S \), \( \tau : X \rightarrow T \), each having with domain \( X \), are said to be equivalent, written
\( \sigma \sim \tau \), if there is an isomorphism \( \xi : S \to T \) such that \( \tau = \xi \circ \sigma \). The equivalence classes are the quotient objects of \( X \).

The intuition behind these definitions is that \( G \)-monos "cut out a piece" of their codomain, while the codomain of a \( G \)-epi defines a a "simplified model" of its domain. It is not hard to check the stable subsets of \( X \) are precisely the subobjects of \( X \) in the category of \( G \)-sets. Thus, the notion of a stable subset provides a concrete characterization of the more abstract categorical notion of a subobject.

If \( \varphi : X \to Y \) is a \( G \)-hom., then \( \varphi(A) \) is stable in \( Y \) whenever \( A \) is stable in \( X \), and \( \varphi^{-1}(B) \) is stable in \( X \) whenever \( B \) is stable in \( Y \). The kernel \( \ker \varphi \) is the congruence relation on \( X \) defined by \( x_1 \sim x_2 \) if \( \varphi(x_1) = \varphi(x_2) \). That is, the congruence classes of \( \ker \varphi \) are precisely the preimages \( \varphi^{-1}(y) \).

From the categorical point of view, subobjects and quotient objects are dual notions (see [61] for duality in category theory). It is natural to ask whether we can find a concrete characterization of quotient objects, as we have done for subobjects. The following important theorem shows that the answer is "yes": the "quotient objects" of \( X \) in the sense of categories are precisely the "quotients" of \( X \) by \( G \)-congruences.

**Theorem B.1 (The Fundamental Theorem of \( G \)-sets)** Every \( G \)-epi \( \varphi : X \to Y \) induces a quotient \( X/\ker \varphi \), where \( G \) acts by setwise translation on the preimages \( \varphi^{-1}(y) \). Conversely, if \( \sim \) is a congruence relation on \( X \), then the map \( \pi : X \to X/\sim \) which takes each element to its congruence class, is a \( G \)-epi.

The proof is straightforward; see [99] for details. Suppose that \( \varphi : X \to Y \) and \( \psi : X \to Z \) are \( G \)-homs. Then \( \ker \varphi \) is refined by \( \ker \psi \) iff \( X/\ker \varphi \) is a quotient of \( X/\ker \psi \).

Any group \( G \) acts on any one point set \( \{x\} \) by the "trivial" action defined by \( \theta(g)(x) = x \) for all \( g \in G \). Because there is a unique \( G \)-epi from any other \( G \)-set
to \( \{x\} \), it satisfies the UMP (universal mapping property) for a final object in the category of \( G \)-sets (see [61] for this and other universal mapping properties). Any two such final objects are \( G \)-isomorphic, so we usually just write \( \ast \) for "the" one point \( G \)-set. Dually, the empty set satisfies the UMP for an initial object in the category of \( G \)-sets.

**Definition 42** Suppose \( \theta : G \to \text{Sym}(X) \) and \( \psi : G \to \text{Sym}(Y) \) define actions by \( G \) on \( X, Y \). Then the product of \( X, Y \) is the Cartesian product \( X \times Y \), equipped with the natural action \( g(x, y) = (g.x, g.y) \). The coproduct or sum of \( X, Y \) is the disjoint union \( X \sqcup Y = \{(z, \epsilon) : z \in X \text{ and } \epsilon = X \text{ or } z \in Y \text{ and } \epsilon = Y\} \) with the natural action

\[
g.(z, \epsilon) = \begin{cases} 
\theta(g)(z), & \epsilon = X \\
\psi(g)(z), & \epsilon = Y 
\end{cases}
\]

It is a routine exercise in "abstract nonsense" to show that these do satisfy the correct UMP's to qualify as products and coproducts, respectively, in the category of \( G \)-sets.

The coproduct is easiest to understand in terms of Schreier diagrams: simply collect the diagrams for \( X \) and for \( Y \) and declare the collection to constitute a single diagram. The concept of the coproduct is more important in the category of \( G \)-sets than that of the product, because every \( G \)-set is the coproduct of its orbits. It is straightforward to generalize the definitions of product and coproduct \( G \)-sets to arbitrarily many factors.

**Definition 43** Suppose \( \mu : X \to T \) and \( \nu : Y \to T \) are two \( G \)-homs with common codomain \( T \). Define a \( G \)-set \( \tilde{T} \) (sometimes called the fibered product) and \( G \)-homs

1. \( \tilde{\mu} \), called the pullback of \( \mu \) under \( \nu \),
2. \( \hat{\nu} \), called the pullback of \( \nu \) under \( \mu \),
as follows:

\[
\begin{align*}
\hat{T} & = \{(x, y) \in X \times Y : \mu(x) = \nu(x)\} \\
\hat{\mu}(x, y) & = y \\
\hat{\nu}(x, y) & = x
\end{align*}
\]

Here, \( \hat{T} \) inherits the product action from \( X \times Y \), and the diagram

\[
\begin{aligned}
\begin{array}{ccc}
\hat{T} & \longrightarrow & Y \\
\downarrow & \quad & \downarrow \nu \\
X & \longrightarrow & T
\end{array}
\end{aligned}
\]

commutes; indeed, \( \hat{T} \) is the "simplest" \( G \)-set for which such a diagram commutes.

A more formal version of this UMP defines a pullback square in category theory (see [61]), and the construction given above shows that pullback squares always exist in the category of \( G \)-sets.

Three special cases are noteworthy. First, the product of \( X, Y \) is given by the pullback square

\[
\begin{aligned}
\begin{array}{ccc}
X \times Y & \longrightarrow & Y \\
\downarrow & \quad & \downarrow \nu \\
X & \longrightarrow & *
\end{array}
\end{aligned}
\]

where \( \mu, \nu \) are the \( G \)-epis onto the one-point \( G \)-set \( * \).

Second, if \( \phi \) is a \( G \)-hom \( X \to Y \), then the pullback square

\[
\begin{aligned}
\begin{array}{ccc}
\hat{Y} & \longrightarrow & X \\
(\phi, 1) & \quad & \quad \quad \quad (\phi, 2) \\
(\phi, 2) & \quad & \quad \quad \quad (\phi, 2) \\
X & \longrightarrow & Y
\end{array}
\end{aligned}
\]

gives the congruence relation \( \text{ker} \phi \), considered as a subset of \( X \times X \).
Third, if $X \xrightarrow{\varphi} Y$ is a $G$-hom and $B$ is a stable subset of $Y$, then the pullback square

$$
\varphi^{-1}(B) \times B \xrightarrow{i} B \\
i \downarrow \hspace{2cm} \downarrow i
\begin{array}{c}
X \\
i \xrightarrow{\varphi} Y
\end{array}
$$

(where $i$ is the inclusion map, a $G$-mono) gives the preimage of $B$ in the sense of category theory; this is obviously essentially equivalent to the naive notion of preimage.

**Definition 44** Suppose $\alpha : S \to X$ and $\beta : S \to Y$ are two $G$-homs with common domain $S$. Define a $G$-set $\hat{S}$ and $G$-homs

1. $\hat{\alpha}$, called the pushout of $\alpha$ under $\beta$,

2. $\hat{\beta}$, called the pushout of $\beta$ under $\alpha$,

as follows:

$$
\hat{S} = (X \amalg Y)/\sim
$$

$$
\hat{\alpha}(x, y) = y
$$

$$
\hat{\beta}(x, y) = x
$$

where we define $\alpha(s) \sim \beta(s)$ for all $s \in S$ (this is a $G$-congruence relation on $X \amalg Y$).

Here, $\hat{S}$ has the quotient action induced from $X \amalg Y$, and the diagram

$$
\begin{array}{c}
S \xrightarrow{\beta} Y \\
\alpha \downarrow \hspace{2cm} \downarrow \hat{\alpha}
\begin{array}{c}
X \\
\xrightarrow{\hat{\beta}} \hat{S}
\end{array}
\end{array}
$$

commutes; indeed, $\hat{S}$ is the "most complex" $G$-set such that such a diagram commutes.

A more formal version of this UMP (see [61]) defines a pushout square in category
theory, and the construction above shows that pushout squares always exist in the
category of $G$-sets.

Three special cases are noteworthy: the coproduct, cokernel, and (direct) image
$\varphi(A)$ of a subobject $A$ all arise by trivializing the pullback squares giving the product,
kernel, and preimage respectively.

Definition 45 Suppose $X,Y$ are two $G$-sets. Then the exponential $Y^X$ is the set
of all mappings (not necessarily $G$-homs) from $X$ to $Y$. With the action

$$(g \xi)(x) = g \xi(g^{-1}x)$$

Note that a mapping $\varphi : X \to Y$ is a $G$-hom iff it is fixed under this action.
The exponential $Y^X$ satisfies the UMP for an exponential in the category of $G$-sets
(see [61]; note that the evaluation map $\text{ev}(\xi, x) = \xi(x)$ is a $G$-hom $Y^X \times X \to Y$).

Definition 46 Let $G$ act trivially on the binary digits $\mathbb{B} = \{0, 1\}$. Then $\mathbb{B}$ is called
the subobject classifier and the $G$-hom $\ast \xrightarrow{\top} \mathbb{B}$ which takes the single point of $\ast$ to
$1 \in \mathbb{B}$ is called truth. Any $G$-hom $\chi : X \to \mathbb{B}$ is called a character.

Here, $\mathbb{B}$ and the $G$-hom "truth" satisfy the UMP for a subobject classifier in the
category of $G$-sets (see [51]). Any category with the following properties

1. there exists an initial object, and pullback squares always exist,

2. there exists a final object, and pushout squares always exist,

3. exponentials always exist,

4. there is a classifying object
is called an **elementary topos**. Thus, the category of $G$-sets is a topos. Another example is the category of sets. Topoi are very special categories; roughly speaking, any topos can serve as the foundation for all of mathematics. (See [61] for a brief account of topoi and modeling first order logic in a topos: see [45] or [84] for a full treatment.)

**Definition 47** A projective system is a family of $G$-sets $\{X_j : j \in \mathbb{N}\}$ together with $G$-homs $X_j \xrightarrow{\varphi_j} X_k$ for all $j \leq k$ (note the direction of these arrows!) such that whenever $j \leq k \leq m$, we have $\varphi_j^m = \varphi_j^k \circ \varphi_k^m$. Given such a projective system, write $\prod_{j=1}^{\infty} X_j \xrightarrow{\varphi_k} X_k$ for the canonical $G$-epis. Then the projective limit of the projective system is the $G$-set

$$X = \left\{ x \in \prod_{j=1}^{\infty} X_j : \forall j \geq k, \varphi_j^k(x_k) = x_j \right\}$$

The projective limit satisfies the UMP of a limit (sometimes erroneously called the inverse limit) in category theory. Analogously, one may define a projective limit with respect to any poset.

*Warning:* $X$ might be the empty $G$-set.

A special case is particularly noteworthy. Let $\varphi : X \to X$ be any $G$-hom from $X$ to itself. We can take $\varphi^j = \varphi$ and $X_j = X$ for all $j, k \in \mathbb{N}$; this (trivially) gives a projective system. The projective limit $P$ is the set of prehistories of points in $X$ (or rather in the eventual image of $\varphi$, which is a stable subset of $X$). Specifically,

$$P = \left\{ x \in \prod_{j=0}^{\infty} : x_n = \varphi(x_{n+1}) \ \forall n \in \mathbb{N} \right\}$$

The pullback square

$$
\begin{array}{ccc}
P & \xrightarrow{\hat{\varphi}} & P \\
\pi \downarrow & & \downarrow \pi \\
X & \xrightarrow{\varphi} & X
\end{array}
$$
defines a $G$-automorphism $\hat{\varphi}$ on $P$, where
\[
(\hat{\varphi}x)_n = \begin{cases} 
  x_{n-1}, & n > 0 \\
  \varphi(x_0), & n = 0 
\end{cases}
\]

In the case $G = S^1$ (as a multiplicative subgroup of $\mathbb{C}$) and $\varphi(z) = z^2$, $P$ is called the solenoid. In general, the prehistory construction turns an arbitrary endomorphism into an automorphism, usually at the expense of increasing the size of the space where the map acts.

Definition 48 An injective system is a family of $G$-sets $\{X_j : j \in \mathbb{N}\}$ together with $G$-homs $X_j \xrightarrow{\eta_j} X_k$ for all $j \leq k$ such that whenever $j \leq k \leq m$, we have $\eta_{jm} = \eta_{jk} \circ \eta_{km}$. Given such an injective system, write $X \xrightarrow{i} \coprod_{j=1}^{\infty} X_j$ for the canonical $G$-monos. Then the injective limit of the injective system is the $G$-set
\[
X = \left( \coprod_{j=1}^{\infty} X_j \right) / \sim
\]
where for all $j \leq k$ we define $x_j \sim \eta_{jk}(x_k)$ (this defines a $G$-congruence on $\coprod_{j=1}^{\infty} X_j$).

The injective limit satisfies the UMP of a colimit (sometimes erroneously called a direct limit) in category theory. Analogously, one may define an injective limit with respect to any poset.

Warning!: $X$ might be the one-point $G$-set $\ast$.

Definition 49 Given a finite set $S$, the free $G$-set over $S$, $F$, consists of a coproduct of $|S|$ copies of $G$ acting on itself by $h \mapsto gh$.

Here, $F$ satisfies the UMP for a free object in the category of $G$-sets (see [61][44]). Note that every $G$-set is the quotient of some free $G$-set (take the free $G$-set over the orbit space of $X$).
B.4 Structure and Classification

The key to understanding the structure and classification of $G$-sets is the following simple result, discovered in embryonic form by Lagrange as early as 1770 [99].

**Theorem B.2 (Structure of Orbits)** Suppose that $x$ is a point in a $G$-set $X$. Then $x^G$ is $G$-isomorphic to the set of cosets $G/Ax$, where $G$ acts on $hAx$ by left multiplication.

Because every $G$-set is the coproduct of its orbits, this shows that in order to classify the allowed structure of $G$-sets it suffices to classify the orbits.

**Theorem B.3 (Classification of Orbits)** The orbits $x^G, y^G$ are $G$-isomorphic if and only if $Ax, Ay$ are conjugate subgroups.

Thus, the set of possible orbital structures is in bijection with the set of conjugacy classes of subgroups of $G$. For proofs of Theorems B.2 and B.3, see [99] or [89].

**Theorem B.4 (Classification of Orbital Quotients)** The orbit $x^G$ is a quotient of $y^G$ if and only if some conjugate of $Ay$ is a subgroup of $Ax$.

Theorems B.3, B.4 together say in effect that you can read off the subgroup lattice of $G$ the whole story about the possible structures of and quotient relations between homogeneous $G$-sets. The proof of Theorem B.4 is similar to that of Theorem B.3.

Let $\varphi : X \rightarrow Y$ be a $G$-hom. Then each restriction $\varphi |_{x^G} : x^G \rightarrow \varphi(x)^G$ is a $G$-epi. This implies that in order to define a $G$-hom out of a known $G$-set $X$, it suffices to name the target of just one point in each orbit. The following theorem gives the analogous “minimal description” for $X$ itself.
Theorem B.5 (Canonical Form for G-Sets) Suppose that the (subgroup) conjugacy classes of $G$ are indexed by $J$, and that a representative subgroup $H_j$ is chosen from each class. Then every $G$-set $X$ (having at most a finite number of orbits of each type) is $G$-isomorphic to a unique $G$-set of the form

$$\prod_{j \in J} \kappa(j)G/H_j$$

where $nZ$ denotes $\prod_1^n Z$ if $n > 0$ and is empty otherwise.

If $G$ is finite and the subgroups $H_1, H_2, \ldots H_r$ represent the conjugacy classes of $G$, then the number of non-isomorphic $G$-sets of size $n$ is given by the coefficient of $n$ in the power series expansion about zero of the rational function

$$\Psi_G(t) = \frac{1}{\prod_{j=1}^r 1 - t|G/H_j|}$$

(See [99].)

Theorem B.6 (Structure of Orbital Auto Group) Let $X$ be a $G$-set and let $x^G$ be one orbit of $X$. Then the group of all $G$-automorphisms of $x^G$, $\text{Aut}(x^G)$, acts on $x^G$ just like $N_G(\alpha x)$ acts by restriction on $x^G$. Put another way, the orbits of $\text{Aut}(x^G)$ on $x^G$ are the costability classes $[x]$, and $x^G$ is $N_G(\alpha x)$-isomorphic to

$$\prod_{1}^{m} N_G(\alpha x)/\alpha x$$

Here $m = [G : N_G(\alpha x)]$ is the number of conjugates of $\alpha x$ (which equals the number of the number of costability classes of $x$.) Therefore $\text{Aut}(x^G)$ is isomorphic as a group to $N_G(\alpha x)/\alpha x$, while $x^G$ is $\text{Aut}(x^G)$-isomorphic to the free $\text{Aut}(x^G)$-set

$$\prod_{1}^{m} \text{Aut}(x^G)$$
For a proof of Theorem B.6, see [99] or [89].

Suppose \( \theta : G \to \text{Sym} X \) and \( \psi : H \to Y \) define actions by \( G \) on \( X \) and by \( H \) on \( Y \), respectively. Then we can “wreath” the two actions by taking \( |Y| \) copies of \( X \), letting a copy of \( G \) act independently on each copy of \( X \) according to \( \theta \) (the “fine structure”), and letting \( H \) permute the copies according to \( \psi \) (the “coarse structure”). This produces a new group, the **wreath product** \( G \wr H \), and an action by \( G \wr H \) on the set \( \Pi_{y \in Y} X = X \times Y \). *(Warning! This “wreath product” group is a generalization of the usual wreath product defined group theory [99], which is the case where \( G \) and \( H \) act on themselves by translation.)*

The group \( G \wr H \) is defined as follows. Note that the group \( G^Y = \{ \mu : Y \to G \} \) is an abelian group under pointwise multiplication (using the definition of multiplication in \( G \)). \( G \wr H \) is the semidirect product of \( G^Y \) by \( H \); that is, we define multiplication in \( G \wr H = G^Y \times H \) as

\[
(\mu_2, h_2) (\mu_1, h_1) = (\mu_2 (\mu_1 \circ h^{-1}), h_2 h_1)
\]

The action of \( G \wr H \) on \( \Pi_{y \in Y} X = X \times Y \) is defined by

\[
(\mu, h).(x, y) = (\theta(\mu(hy))(x), \psi(h)(y))
\]

The subgroup \( G^Y \times e_G \) is normal in \( G \wr H \); this subgroup affects the “fine structure” of the wreathed actions. The subgroup \( e_{G^Y} \times H \) is *not* normal; it affects the “coarse structure” of the wreathed actions. Note that \( (G \wr H)/(G^Y \times e_H) \) is isomorphic as a group to \( H \).

The wreath product construction can be used to extend Theorem B.6 to a description of the automorphism group of a general \( G \)-set, which comes down to saying that an automorphism has a “coarse structure” which permutes isomorphic orbits, and a “fine structure” consisting of various orbital automorphisms acting on the different orbits.
A number of structural properties that may be possessed by a given orbit may be characterized in terms of the stabilizers of points in the orbit:

1. \( x^G \) is a free \( G \)-set if and only if \( \triangleleft x = 1 \),

2. \( x^G \) is a simple \( G \)-set if and only if \( \triangleleft x \) is a maximal subgroup,

3. \( x^G \) is a quotient group of \( G \) if and only if \( \triangleleft x \) is normal.

If we restrict the group hom \( \theta : G \to \text{Sym}(X) \) to \( H \leq G \), we obtain the action induced by \textbf{restriction to} \( H \). Note that \( H_x = H \cap G_x \). Usually, each orbit \( x^G \) will split up into several smaller orbits under \( H \).

**Lemma B.7 (Orbital Splitting Criterion)** Let \( H \leq G \) act by restriction. Then \( x^H \subset x^G \) with equality if and only if \( G = H \triangleleft x \).

The proof is very easy. If the orbits do split up under restriction to \( H \), it is possible to say exactly how this happens.

**Lemma B.8 (Orbital Splitting Formula)** Let \( H \) act by restriction on \( G/K \). Then \( G/K \) is \( H \)-isomorphic to the following coproduct taken over the double cosets \( HgK \):

\[
\coprod_{HgK} H/(H \cap gKg^{-1})
\]

See [99] for a proof. If either \( H \) or \( K \) should be normal, then Lemma B.8 says that the old orbit splits into \( [G : HK] \) identical pieces each \( H \)-isomorphic to \( H/H \cap K \). (Note that \( g.(x^H) = (g.x)^H \) whenever \( H \) is normal.) However, in general, the new orbits can be quite different from one another; see [138] for an example of this phenomenon. If \( x^G \) is finite, then \( |x^H| = |HG_x| \cdot |G_x| \).

It is interesting to note that virtually everything discussed in this section seems to have been known to Frobenius by 1887 [99].
B.5  Galois Theory

In this section we study a generalization of the “essentially trivial” aspects of classical Galois theory to arbitrary group actions.

The following notation [25] is so flexible and convenient that it is well worth introducing here, despite its initially strange appearance.

**Definition 50** Given \( A \subseteq X \), the (pointwise) stabilizer of \( A \) is

\[
\vartriangleleft A = \{ g \in G : gx = x \forall x \in A \}
\]

(In particular, \( \vartriangleleft x \) is nothing other than the stabilizer subgroup \( G_x \).) Given \( J \subseteq G \), the fixset of \( J \) is

\[
\triangleright J = \{ x \in X : gx = x \forall g \in J \}
\]

A simple mnemonic should assist the reader in using this notation: remember that \( G \) acts from the left on \( X \), whence \( \triangleright \) should take us from \( G \) to \( X \) and \( \vartriangleleft \) should take us from \( X \) back to \( G \).

**Lemma B.9** Let \( A, B \subseteq X \) a \( G \)-set. Then

\[
\vartriangleleft (A \cup B) = \vartriangleleft A \cap \vartriangleleft B.
\]

The pointwise stabilizer \( \vartriangleleft A \) is in this respect much better behaved than the setwise stabilizer

\[
G_A = \{ g \in G : gA = A \}.
\]

To see why, compare Lemma B.9 with the following:

**Lemma B.10** Let \( A, B \subseteq X \) a \( G \)-set. Then \( G_A \cap G_B \) is a subgroup of \( G_A \setminus B \), \( G_B \setminus A \), \( G_{A \cap B} \), and \( G_{A \cup B} \).
Corollary B.11 If $A, B$ are disjoint subsets, then $G_A \cap G_B$ is a subgroup of $G_A, G_B$.

It is easy to find examples to show that strict inclusion may hold in Lemma B.10 and Corollary B.11.

Proposition B.12 (Structure of Fixset)

$$
\triangleleft A = \bigcap_{g \in A} \triangleright g = \{x \in X : \triangleright x \supset \triangleleft A\} = \bigcup_{\{x\} \supset \triangleleft A} [x]
$$

This shows that each fixset consists of a selection (possibly empty) of costability classes from each orbit.

We will write the collection of all stabilizers of $X$ under the given action $\theta : G \to \text{Sym}(X)$ as $\text{Stab} \theta$, or $\text{Stab}(G, X)$ when the action is clear, and likewise write the collection of all fixsets of $X$ under $G$ by $\text{Fix} \theta$, or $\text{Fix}(G, X)$ when the action is clear. Note that $\text{Stab}(G, X) \leq 2^G$ and $\text{Fix}(G, X) \leq 2^X$. We can compose the maps $\triangleleft$ and $\triangleright$ to create two new maps $\triangleleft \triangleright : 2^X \to 2^X$ and $\triangleright \triangleleft : 2^G \to 2^G$. The diagram

$$
\begin{array}{ccc}
\text{Stab}(G, X) & \leftarrow \triangleleft & 2^X \\
\cong & \uparrow \cong & \downarrow \triangleright \triangleleft \\
2^G & \rightarrow & \text{Fix}(G, X)
\end{array}
$$

where each arrow denotes an onto map, should clarify the effect of the four maps $\triangleright, \triangleleft, \triangleright \triangleleft, \triangleleft \triangleright$. It is easy to see that $\triangleright \triangleleft A$ is the smallest fixset swallowing $A$; this set is called the galois closure of $A$. Similarly, $\triangleright \triangleleft H$ is the smallest stabilizer swallowing $H$. The maps $\triangleright : 2^G \to 2^X$ and $\triangleleft : 2^G \leftarrow 2^X$ define a galois connection. (See [25] for the definition of a galois connection; see [67] for the role of concept theory in this and many other notions of galois duality in mathematics.)
Lemma B.13 (Order Properties) Both $\triangleleft$ and $\triangleright$ reverse order in the sense that $A \subseteq B$ implies $\triangleleft A \triangleright \triangleleft B$. Moreover,

$\triangleright \triangleleft \triangleright = \triangleright$

$\triangleright \triangleleft \triangleleft = \triangleleft$

Definition 51 Let $X$ be a set. Then a closure operator on $X$ is a mapping $\langle \cdot \rangle : 2^X \rightarrow 2^X$ such that

- $A \subset \langle A \rangle$.
- $A \subseteq B$ implies $\langle A \rangle \subseteq \langle B \rangle$.
- $\langle \langle A \rangle \rangle = \langle A \rangle$.

Examples include the topological closure of subsets of a topological space, the convex hull of subsets in an affine space, and the mapping defined on the subsets of a group $G$ by declaring $\langle J \rangle$ to be the smallest subgroup containing $J \subseteq G$.

Definition 52 Let $\Omega$ be an operator domain in the sense of universal algebra [76], and suppose that $X$ is both an $\Omega$-algebra and a $G$-set under the action defined by $\theta : G \rightarrow \text{Sym} X$. Suppose further that for every $n$-ary operation $\omega \in \Omega$,

$$g.\omega(x_1, x_2, \ldots x_n) = \omega(g.x_1, g.x_2, \ldots g.x_n)$$

for all $g \in G$. Then $X$ is called a $G\Omega$-algebra. If $\mathcal{V}$ is a variety of $\Omega$-algebras, and if $X$ is both in $\mathcal{V}$ and also a $G\Omega$-algebra, then $X$ is said to be a $G\mathfrak{V}$ space. (The symbol $\mathfrak{V}$ is the fraktur letter $V$.)
Lemma B.14 (Closure Operators) The mappings $\triangleright\triangleleft$ and $\triangleright\triangleright$ are closure operators on $X, G$ respectively. If $\langle \cdot \rangle : 2^X \to 2^X$ is the closure operator defined by declaring $\langle A \rangle$ to be the smallest $\Omega$-subalgebra containing $A$, then

$$\triangleleft A = \triangleleft A,$$

$$\triangleright \triangleright A = \triangleright \triangleright A$$

Thus, $\triangleright \triangleright$ "absorbs" the weaker closure operators defined by any structure it may possess as a $G\mathfrak{W}$ space. For instance, if $X$ is a (ring) ($R$-module) (field extension) such that the action respects the (ring) ($R$-module) (field) structure, then $\triangleright \triangleright A$ is always a (subring) (submodule) (intermediate field).

Note that the lattices $2^G$ and $2^X$ are $G$-lattices under the setwise actions induced by the action on $G$ by conjugation, and the given action, respectively.

Lemma B.15 $\triangleleft$ and $\triangleright$ exchange the action on subsets, $A \mapsto gA$, with the conjugation action on subsets of $G$, $J \mapsto gJg^{-1}$. That is,

$$\triangleleft gA = g(\triangleleft A)g^{-1}$$

$$\triangleright (gJg^{-1}) = g\triangleright J$$

$$\triangleright \triangleright gA = g\triangleright \triangleright A$$

$$\triangleright \triangleleft (gJg^{-1}) = g(\triangleright \triangleleft J)g^{-1}$$

Lemma B.15 shows that $\text{Stab}(G, X)$ and $\text{Fix}(G, X)$ are $G$-sets, so the maps $\triangleright$, $\triangleleft$, $\triangleright \triangleright$ and $\triangleright \triangleleft$ are in fact $G$-epimorphisms.

We can turn $\text{Stab}(G, X)$ into a lattice by defining the meet of $\triangleleft A, \triangleleft B$ to be $\triangleleft A \cap \triangleleft B$ and defining their join to be

$$\triangleleft A \vee \triangleleft B = \triangleright \triangleright (\triangleleft A \cup \triangleleft A)$$

Similarly, we can turn $\text{Fix}(G, X)$ into a lattice by defining the meet of $\triangleright H, \triangleright K$ to be $\triangleright H \cap \triangleright K$ and defining their join to be

$$\triangleright H \vee \triangleright K = \triangleright \triangleright (\triangleright H \cup \triangleright K)$$
B.5. GALOIS THEORY

Lemma B.16 Stab\((G, X)\) and Fix\((G, X)\) are \(G\)-lattices. \(\triangleleft : 2^G \to \text{Stab}(G, X)\) and \(\triangleright : 2^X \to \text{Fix}(G, x)\) are \(G\)-epimorphisms, and \(\triangleright \triangleleft : 2^G \to \text{Stab}(G, X)\) and \(\triangleright \triangleleft : 2^X \to \text{Fix}(G, X)\) are \(G\)-lattice-epimorphisms.

All that one need check to prove Lemma B.16 is that

\[
\triangleright (H \cup K) = \triangleright (H \lor K) = \triangleright H \triangleright K, \quad \triangleright (H \cap K) = \triangleright H \lor \triangleright K
\]

\[
\triangleleft (A \cup B) = \triangleleft A \triangleleft B, \quad \triangleleft (A \cap B) = \triangleleft A \lor \triangleleft B
\]

This gives most of the following theorem:

Theorem B.17 Stab\((G, X)\) and Fix\((G, X)\) are complete \(G\)-lattices (i.e. the meet and join of arbitrarily many factors is well defined). Moreover, the lattice dual of Fix\((G, X)\) is \(G\)-lattice isomorphic to Stab\((G, X)\).

See [25] for the definitions of complete lattices and the dual of a lattice.

Let us denote (following [117][75]) the lattice of subgroups of \(G\) as \(L(G)\). Under the natural action by conjugation, \(L(G)\) is a \(G\)-lattice. Moreover, we can define a congruence relation on \(2^G\) (considered as a \(G\)-lattice) by stipulating that \(J \sim K\) iff \(J, K\) generate the same subgroup of \(G\). This shows that \(L(G)\) is a quotient of \(2^G\) (but not a sub-lattice, since the join operations \(\cup, \lor\) are distinct).

Similarly, we can define a congruence relation on \(L(G)\) by stipulating that \(H \sim K\) iff \(\triangleleft H = \triangleleft K\). This shows that Stab\((G, X)\) is a quotient of \(L(G)\) (but not a sublattice of \(L(G)\), or of \(2^G\), because all three join operations \(\cup, \lor, \lor\) are distinct.) Specifically, Stab\((G, X)\) is \(G\)-lattice isomorphic to \(L(G)/\ker \triangleright = L(G)/\triangleleft\).

If \(\triangleleft A\) acts by restriction of the original action, then Stab\((\triangleleft A, X)\) is simply the interval \([\triangleleft X, \triangleleft A]\) from Stab\((G, X)\). That is, take all stabilizers \(\triangleleft B\) such that \(\triangleleft X \subseteq \triangleleft B \subseteq \triangleleft A\).
If we know the lattices \( \text{Stab}(G, X) \) and \( \text{Stab}(G, Y) \), it is easy to find \( \text{Stab}(G, X \times Y) \) and \( \text{Stab}(Y \setminus \emptyset) \), as follows. First, observe that

\[
\wp_{x \times y} H = \wp_x H \times \wp_y H, \quad \wp_{x \uplus y} H = \wp_x H \mathbin{\uplus} \wp_y H
\]

and

\[
\forall_{x \times y} E = \forall_x p(E) \cap \forall_y q(E), \quad \forall_{x \uplus y} A \mathbin{\uplus} B = \forall_x A \cap \forall_y B
\]

Consequently,

\[
\forall_{x \times y} H = \forall_{x \uplus y} H = \forall_x H \cap \forall_y H
\]

This shows that \( \text{Stab}(G, X \times Y) = \text{Stab}(G, X \mathbin{\uplus} Y) \). Moreover, since \( X \) is the coproduct of its orbits, we see that \( \text{Stab}(G, X) \) depends only on the type of orbits \( G/H_j \) appearing in \( X \), not on the number of each type.

A complexion is a homogeneous \( G \)-set which represents the discrete motions of a given subset \( A \subset X \) under the action by \( G \) on \( X \). The theory of complexions [65] can be treated as an algebraic theory of geometric information [69]. In particular, there is a notion of Galois independence of subsets which plays the same role in the theory of complexions as does the notion of statistical independence in probability theory. Complexions are also closely related to the notions of entropy [63] and Hausdorff dimension [68].

### B.6 Partitions and Periods

Given a finite set \( X \), we denote the set of partitions of \( X \) by \( \Pi(X) \). We will write partitions as calligraphic letters. If \( A \) denotes the partition \( X = \bigcup_{j=1}^r A_j \), then the subsets \( A_j \) are called the atoms of \( A \). If \( B \) is the partition \( X = \bigcup_{k=1}^s B_k \), then the join of \( A \) and \( B \), written \( A \sqcup B \), is the partition \( X = \bigcup_{j=1}^r \bigcup_{k=1}^s A_j \cap B_k \), and the meet of \( A \) and \( B \), written \( A \sqcap B \), is the partition into subsets which are minimal unions over overlapping sets \( A_j \) and \( B_k \). (Warning! In the papers [117][75] the authors consider
\( \Pi(X) \) to be the dual of the lattice we have just defined.) It is not hard to see that \( \Pi(X) \) forms a lattice under these operations, with the associated partial order \( \mathcal{A} \leq \mathcal{B} \) iff \( \mathcal{B} \) refines \( \mathcal{A} \); that is, iff every \( B_k \) is included in some \( A_j \).

In particular, taking \( X = G \), because \( \triangleright_X \triangleright_Y H = \triangleright_X H \cap \triangleright_Y H \) for all \( H \in L(G) \), we have the identity

\[
\left( \operatorname{ker} \varphi \right)_X \triangleright \left( \operatorname{ker} \varphi \right)_Y = \operatorname{ker} \varphi_{XY}
\]

Consequently, for a given \( G \) we can obtain all possible stabilizer lattices of \( G \)-sets as follows: compute the lattices \( \text{Stab}(G, G/H_j) \) for all conjugacy classes of subgroups \( H_j \leq G \), then take all possible joins over these classes. The following observation can be helpful in carrying out this process.

**Lemma B.18** Suppose \( \sigma : G \to G \) is a group automorphism of \( G \), and suppose \( H \leq G \). Then \( \text{Stab}(G, G/H) = \text{Stab}(G, G/\sigma(H)) \).

The proof is virtually identical to a similar lemma in [75].

If \( X \) is a \( G \)-set, then we obtain an induced action on \( \Pi(X) \) by letting \( g \) take the partition \( X = \bigcup_{j=1}^r A_j \) to the partition \( X = \bigcup_{j=1}^r gA_j \). Thus, \( \Pi(X) \) is in fact a \( G \)-lattice. It is easy to see that the stabilizer of a partition is the intersection of the setwise stabilizers of its atoms. That is, if \( \mathcal{A} \) is the partition \( X = \bigcup_{j=1}^r A_j \), then

\[
\triangleleft \mathcal{A} = \bigcap_{j=1}^r \mathcal{G}_A
\]

Alternatively, we have

\[
\triangleleft A = \prod_{j=1}^r \left( \mathcal{G}_{A_j} \cap \triangleleft \left( \bigcup_{k \neq j} A_k \right) \right) = \prod_{j=1}^r \left( \mathcal{G}_{A_j} \cap \left( \bigcap_{k \neq j} \triangleleft A_k \right) \right)
\]

In other words, any element of \( G \) which independently permutes each \( A_j \) within itself fixes \( \mathcal{A} \), and vice versa. Put another way, we have

\[
\theta(\triangleleft A) = \theta(G) \cap \prod_{j=1}^r \text{Sym} A_j
\]
The fixset lattice $\text{Fix}(G, \Pi(X))$ has two remarkable features:

1. we can always replace collections of partitions with a single partition,

2. we can always replace the intersection of two collections of partitions with the meet of two partitions.

These phenomena are consequences of the following lemmas.

**Lemma B.19 (Refinement Lemma)** If $A \subseteq B$, then $\vartriangleleft A \supseteq \vartriangleleft B$.

**Proof:** Fix an atom $A_j$ of $A$ and suppose that $A_j = \cup_{k=1}^n B_k$. Then $\vartriangleleft A_j \supseteq \cap_{k=1}^n \vartriangleleft B_k$. Similarly for the other atoms of $A$. Now taking the intersection over the $\vartriangleleft A_j$ shows that $\vartriangleleft A \supseteq \vartriangleleft B$.

**Lemma B.20 (Join Lemma)** For all $A, B \in \Pi(X)$,

$$\vartriangleleft (A \vee B) = \vartriangleleft A \cap \vartriangleleft B = \vartriangleleft \{A, B\}$$

**Proof:** $G_{A_j} \cap G_{B_k} \subseteq G_{A_j \cap B_k}$ for all $j, k$, so on the one hand

$$\vartriangleleft (A \vee B) = \cap_j \cap_k G_{A_j} \cap G_{B_k} \supseteq (\cap_j G_{A_j}) \cap (\cap_k G_{B_k}) = \vartriangleleft A \cap \vartriangleleft B$$

On the other hand, $A \subseteq A \vee B$ implies that $\vartriangleleft A \supseteq \vartriangleleft (A \vee B)$ and similarly $\vartriangleleft B \supseteq \vartriangleleft (A \vee B)$, so $\vartriangleleft A \cap \vartriangleleft B \supseteq \vartriangleleft (A \vee B)$.

**Definition 53** Given $H \leq G$, the period of $H$, written $\mathcal{H}$, is the partition of $X$ into the orbits of $H$.

**Lemma B.21 (Maximal Partition Lemma)** Every fixset $\vartriangleleft H$ in $\text{Fix}(G, \Pi(X))$ contains a unique maximal element, namely the partition into the orbits of $H$. 


Proof: Let $\mathcal{H}$ be the period of $H$. Plainly $\mathcal{H} \in \triangleright H$.

CLAIM: $A \in \triangleleft H$ implies that $A \leq H$.

Reason: Suppose $A$ is the partition $X = \bigcup A_j$. Then every $A_j$ is $H$-invariant, and thus a union of $H$ orbits. Therefore, the period of $H$ refines $A$. 

Corollary B.22 For all subgroups $H, K \leq G$, we have

$$\triangleright (H \cap K) = \mathcal{H} \triangleleft \mathcal{K}, \quad \triangleleft (H \cup K) = \mathcal{H} \triangleright \mathcal{K}$$

That is, the periods form a lattice, called the lattice of periods of $X$, which is $G$-lattice isomorphic to $\text{Fix}(G, \Pi(X))$. The lattice of periods was used by Rota and Smith to state a generalization of Polya’s Enumeration Theorem [117], and has recently been studied in detail by Doran [75]. It is easy to see that $\triangleright H$ is the largest subgroup swallowing $H$ whose orbits agree with those of $H$. As Doran shows, the lattice of periods may be enumerated by computing $\triangleright$ for the homogeneous $G$-sets and taking all possible intersections to obtain all possible stabilizer lattices for $\Pi(X)$, and thus all possible period lattices for $G$. Since the lattice of periods is $G$-lattice isomorphic to $\text{Fix}(G, \Pi[X])$ and thus to $\text{Stab}(G, \Pi[X])$, it must depend only on the type of orbits appearing in $\Pi[X]$, not on the number of each type. Similarly, Doran proves that the period lattice depends only on the irreducible $G$-complex linear components of the $G$-complex linear space $C[X]$.

B.7 Pushouts and Pullbacks

Definition 54 Suppose $X$ is both a $G$-set an object in some category $\mathcal{C}$. (The symbol $\mathcal{C}$ is the fraktur letter $C$.) Then $X$ is said to be a $G\mathcal{C}$-space.

In particular, if $\mathcal{V}$ is a variety of $\Omega$-algebras, for a fixed operator domain $\Omega$, then we obtain the category of $G\mathcal{V}$-spaces.
Theorem B.23 (Induced Functors) Every covariant functor taking an object $X$ in $G\mathcal{C}$ to an object $X_*$ in a second category $\mathcal{C}'$ induces a functor taking $X$ to $X_*$ considered as an object in $G\mathcal{C}'$, where the original action $\theta : G \to \text{Sym}(X)$ induces the new action $\theta_* : X_* \to \text{Sym}(X_*)$ defined by $\theta_*(g) = \theta(g)_*$. Similarly, every contravariant functor taking $X$ to an object $X^*$ in $\mathcal{C}''$ induces a contravariant functor taking $X$ to $X^*$ considered as an object in $G\mathcal{C}''$, where the original action induces the new action $\theta^* : G \to \text{Sym}(X^*)$ defined by $\theta^*(g) = \theta(g^{-1})^*$.

The proof is left as an urgently recommended exercise (so that the reader can experience the pleasure of seeing how everything in the relevant definitions is used exactly once).

We now discuss four applications of this theorem.

Suppose we have some concrete category $\mathcal{C}$ and some fixed set $U$. Then we can define a covariant functor as follows. Take the object $X$ to $X^U = \{\sigma : U \to X\}$ and take the morphism $\varphi : X \to Y$ to the map $\varphi_* : X^U \to Y^U$ defined by $\varphi_*(\sigma) = \varphi \circ \sigma$. If $\mathcal{C}$ happens to be a variety of some $\Omega$-algebra, it may be possible to define certain operations on $X^U$ (such as pointwise multiplication) making $X^U$ into an object in some interesting category such as the category of rings. Another common variation is the case where $U$ is a fixed object in $\mathcal{C}$ and instead of $X^U$ we take the set of all morphisms $\sigma : U \to X$; this set is often a ring. Each of these variations is a different pushout construction.

Every pushout construction induces a new pushout construction in the presence of actions by $G$, as specified by Theorem B.23. Note that the action on $X^U$ (or whatever) is defined by $\theta_*(g)(\sigma) = \theta(g) \circ \sigma$. It is easy to see that with respect to this action, $\mathfrak{a}\sigma$ consists of the $g$ such that $\theta(g)$ permutes only points not in the image of $\sigma$, and that $\mathfrak{b}$$\mathfrak{a}\sigma$ consists of all $\sigma'$ having the same image as $\sigma$.

Similarly, we can define a contravariant functor as follows. Take the object $X$ to
$U^X = \{ \alpha : X \to U \}$ and take the morphism $\varphi : X \to Y$ to the map $\varphi^* : U^X \gets U^Y$ defined by $\varphi^*(\beta) = \beta \circ \varphi$. For instance, if $A$ is a subobject of $X$, i.e. the inclusion map $\iota : A \to X$ is a morphism of $\mathcal{C}$, then $\iota^* : U^A \gets U^X$ is the restriction map $\iota^*(\alpha) = \alpha|A$, which is onto. If $\pi : X \to Y$ is onto, $\pi^* : U^X \gets U^Y$ is one-one. If $U$ happens to be in some variety of $\Omega$-algebras, it is often possible to define operations pointwise on $U^X$ so that this becomes, say, a ring. If $U$ is an object in $\mathcal{C}$, we can replace $U^X$ with the set of morphisms $\alpha : X \to U$; as mentioned above this set is often a ring. Each of these variations is a different pullback construction.

Every pullback construction induces a new pullback construction in the presence of actions by $G$, as specified by Theorem B.23. Note that the action on $U^X$ (or whatever) is defined by $\theta^*(g)(\alpha) = \alpha \circ \theta(g^{-1})$.

The stabilizer lattices of pullback constructions tend to be closely related to the lattice of periods, for the following reason. Given $\alpha \in \mathcal{F}$, define the kernel of $\alpha$, written $\ker \alpha$, to be the partition $X = \biguplus_{u \in U} \alpha^{-1}(u)$. It is easy to see that $\triangleleft \alpha$ under the action $g.\alpha = \alpha \circ \theta(g^{-1})$ agrees with $\triangleleft \ker \alpha$ under the action on $\Pi(X)$. Moreover, the stabilizer of a set of functions agrees with the stabilizer of the join of their kernels. Unlike the case of $\Fix(G, \Pi(X))$, it need not happen that this join is itself in $\mathcal{F}$, but otherwise the analysis is similar; in particular, $\Fix(G, \mathcal{F})$ is a $G$-sublattice of $\Fix(G, \Pi(X))$.

It is easy to see that with respect to the pullback action, $\triangleleft \alpha$ consists of all $g \in G$ which permutes only within preimages of $\alpha$, and $\triangleright \triangleleft \alpha$ consists of all $\alpha'$ having the same kernel as $\alpha$.

Here is a concrete example of a pullback construction. We can define a contravariant functor taking finite sets to finite dimensional complex Hilbert spaces as follows. Given a set $X$, define $\mathbb{C}[X] = \{ \alpha : X \to \mathbb{C} \}$ with the Hermitian inner product

$$(\alpha_1, \alpha_2) = \sum_{x \in X} \overline{\alpha_1(x)} \alpha_2(x)$$
Given \( \varphi : X \to Y \), define \( \varphi^* : \mathbb{C}[X] \leftarrow \mathbb{C}[Y] \) by \( \varphi^*(\beta) = \beta \circ \varphi \); note that the inner product satisfies the identity \( (\varphi^*(\beta_1), \varphi^*(\beta_2)) = (\beta_1, \beta_2) \). Applying Theorem B.23, we obtain a contravariant functor, which I'll call the linear pullback functor, taking finite \( G \)-sets to finite dimensional complex Hilbert spaces which are also \( G \)-sets. Of course, \( \mathbb{C}[X] \) is nothing more than a representation of \( G \) in the usual sense; thus, the linear pullback functor gives a fundamental relation between the theory of finite \( G \)-sets and the theory of the representations of finite groups.

We actually get more structure than is given by Theorem B.23, since \( \mathbb{C}[X] \) is actually a linear algebra with the convolution product

\[
(\alpha_1 \ast \alpha_2)(x) = \sum_{g \in G} \alpha(x)\alpha_2(g^{-1}x)
\]

and \( \varphi^*(\alpha \ast \beta) = \varphi^*(\alpha) \ast \varphi^*(\beta) \), so in fact \( \varphi^* \) is a \( G \)-Hilbert algebra homomorphism.

If \( A \) is stable in \( X \), then the inclusion \( \iota : A \to X \) induces the restriction map \( \iota^* : \mathbb{C}[A] \leftarrow \mathbb{C}[X] \) defined by \( \iota^*(\alpha) = \alpha|A \). This is an epimorphism of linear spaces. If \( \pi : X \to Y \) is onto, then \( \pi^* : \mathbb{C}[X] \leftarrow \mathbb{C}[Y] \) is one-one and thus a monomorphism of linear spaces. The linear pullback functor behaves nicely with respect to the product and coproduct on \( G \)-sets:

\[
\mathbb{C}[X \amalg Y] = \mathbb{C}[X] \oplus \mathbb{C}[Y], \quad \mathbb{C}[X \times Y] = \mathbb{C}[X] \otimes \mathbb{C}[Y]
\]

Curiously, we can define a covariant functor, the linear pushout functor, also taking \( X \) to \( \mathbb{C}[X] \), by setting \((\varphi_* \alpha)(y) = \sum x \in \varphi^{-1}(y) \alpha(x)\). Then \( (\varphi^* \circ \varphi_*)(\alpha) \) is the function constant on each preimage \( \varphi^{-1}(x) \) with value the sum over the preimage of the values of \( \alpha \), whereas \( (\varphi_* \circ \varphi^*)(\beta) \) is the function which rescales \( \beta \) at each point \( y \in Y \) by the size of the fiber over \( y \).

Here is a concrete example of a pushout construction. We can define a covariant functor taking finite sets to (polynomial) rings, as follows. Given \( X \), let \( \mathcal{P}[X] \) be
B.7. PUSHOUTS AND PULLBACKS

the space of polynomial functions in the the components \( \alpha(g) \) of \( \alpha \in \mathbb{C}[X] \). (The symbol \( \mathfrak{P} \) is the fraktur letter \( P \).) Given \( \varphi : X \to Y \), we have \( \varphi^* : \mathbb{C}[X] \to \mathbb{C}[Y] \), so define \( \varphi_* : \mathfrak{P}[X] \to \mathfrak{P}[Y] \) by \( \varphi_*p = p \circ \varphi^* \), where \( p \in \mathfrak{P}[X] \). That is, if \( \beta \in \mathbb{C}[Y] \), then \( (\varphi_*p)(\beta) = p(\beta \circ \varphi) \). Let \( \mathfrak{P}[X] \) denote the homogeneous polynomials of degree \( d \) in \( \mathfrak{P}[X] \). Then \( \mathfrak{P}[X] = \oplus_{d=0}^{\infty} \mathfrak{P}[X] \); note \( \mathfrak{P}[X] \) is the set of constant polynomials. This shows that \( \mathfrak{P}[X] \) is a graded ring. By Theorem B.23, this gives a covariant functor, which I will call the algebraic pushout functor, taking finite \( G \)-sets to graded \( G \)-rings. Note that each subspace \( \mathfrak{P}(X) \) is stable in \( \mathfrak{P}X \) under the action by \( G \).

If \( A \) is stable in \( X \), then the inclusion \( \iota : A \to X \) induces the extension map \( \iota_* : \mathfrak{P}[A] \to \mathfrak{P}[X] \) defined by \( (\iota_*p)(\alpha) = p(\alpha \circ \iota) \) (notice this is constant on each atom of the kernel of the restriction map \( \iota^* \)). This is a monomorphism of rings. If \( \pi : X \to Y \) is onto, then \( \pi_* : \mathfrak{P}[X] \to \mathfrak{P}[Y] \) is onto and hence an epimorphism of rings.

The algebraic pushout functor gives a connection between finite \( G \)-sets and algebraic geometry [21]. Consider \( \triangleright_{\mathfrak{P}[X]} G \), the subring of polynomials fixed by the action. Suppose it has \( m \) generators (as a polynomial ring) \( p_1, p_2, \ldots, p_n \). There may be certain algebraic relations between them; let \( I[X] \) be the ideal generated by the polynomial functions \( \pi \) in the ring of formal polynomials in \( m \) indeterminates, \( \mathbb{C}[z_1, z_2, \ldots, z_m] \), such that \( \pi(p_1, p_2, \ldots, p_n) = 0 \). This is called the syzygy ideal of \( \triangleright_{\mathfrak{P}[X]} G \). Then the affine variety consisting of the common zeros of \( I[X] \) is isomorphic as an affine variety to the orbit space \( \mathbb{C}[X]/G \); this orbit space, considered as an affine variety, is called the Veronese variety of the action. Moreover, \( \mathbb{C}[X]/I \) is a quotient ring isomorphic as a ring to \( \triangleright_{\mathfrak{P}[X]} G \); this shows that \( \triangleright_{\mathfrak{P}[X]} G \) is the coordinate ring for the Veronese variety.
B.8 Structural Invariants

A structural invariant of a $G$-set is defined by a way of associating an object in a given category with any $G$-set (or perhaps only any finite $G$-set, or any $G$-set with finite orbits, etc.), such that isomorphic $G$-sets are always taken to isomorphic objects in the second category. The simplest types of invariants are numerical invariants, where we associate each $G$-set with a certain number, such that isomorphic $G$-sets are always associated with the same number. A complete set of invariants is a collection of invariants such that whenever $X, Y$ are taken to precisely the same set of values, then they are isomorphic. Complete sets of invariants are very important because they allow one to completely classify the possible $G$-sets.

The $G$-lattices $\text{Stab}(G, X)$ and $\text{Fix}(G, X)$ are algebraic invariants of the $G$-set $X$, because whenever $\varphi : X \to Y$ is a $G$-hom, we have

$$\varphi(\partial_x H) \subseteq \partial_y H$$

$$\partial_y \varphi(A) \supseteq \partial_x A$$

with equality whenever $\varphi$ is an isomorphism. Thus, if $X, Y$ are $G$-isomorphic then $\text{Stab}(G, X) = \text{Stab}(G, Y)$ and $\text{Fix}(G, X)$ is $G$-lattice isomorphic to $\text{Fix}(G, Y)$. However, $\text{Stab}(G, X)$ is certainly not a complete invariant, since as we saw it "forgets" the number of orbits of a given type (but "remembers" whether or not this number was zero).

Other examples of invariants of $G$-sets include:

1. The size of a finite $G$-set.

2. The canonical form of a finite $G$-set; that is, the number of orbits of a finite $G$-set isomorphic to $G/H_j$, where the various $H_j$ represent the conjugacy classes of subgroups of $G$. (This forms a complete set of invariants all by itself.)
3. The automorphism group of an arbitrary $G$-set.

4. The linearization $\mathbb{C}[X]$ of a finite $G$-set $X$.

5. The ring of invariant polynomials $\triangleright_{\mathbb{C}[X]} G$ on $\mathbb{C}[X]$.

6. The Veronese variety $\mathbb{P}[X]/G$.

7. The lattice of periods of an arbitrary $G$-set.

In this section we introduce some further invariants and explore the inter-relationships between them.

**Definition 55** Let $X$ be a finite $G$ set, $G$ finite. Then the character of $X$ is the function $\chi[X] : G \rightarrow \mathbb{N}$ defined by $\chi[X](g) = |\triangleleft g|$.

Note that $\chi[X]$ is precisely the character in the usual sense of representation theory for $\mathbb{C}[X]$; it is always constant on the conjugacy classes of $G$. Also, $\chi[X](e) = |X|$. The trivial character of $G$, written id, is precisely $\chi[G/G]$. It is defined by $\text{id}(g) = 1$ for all $g \in G$. Characters are well behaved on products and coproducts:

$$\chi[X \sqcup Y] = \chi[X] + \chi[Y], \quad \chi[X \times Y] = \chi[X] \chi[Y]$$

In representation theory the inner product of two characters is defined by

$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \overline{\chi_2(g)}$$

A classic result, universally called Burnside's Lemma but actually due to Cauchy, states that the number of orbits contained in $X$ (obviously a numerical invariant of $X$) is given by

$$(\chi[X], \text{id}) = \frac{1}{|G|} \sum_{g \in G} |\triangleleft g|$$
See any of [99][38][30] for a proof. This immediately implies that the number of orbits of \( X \times Y \) is given by the inner product

\[
(\chi[X], \chi[Y]) = \frac{1}{|G|} \sum_{g \in G} |\chi_X g| |\chi_Y g|
\]

We can define a function \( p_X \) taking \( G \) to integer coefficient formal monomials in the symbols \( s_1, s_2, \ldots, s_n \), where \( |X| = n \), as follows. Given \( g \in G \), define the cycle monomial of \( g \) as

\[
p[X](g) = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}
\]

where \( k_j(d) \) denotes the number of \( d \)-cycles contained in the permutation \( \theta(g) \). The cycle monomial of \( g \) completely determines the conjugacy class in \( \text{Sym} X \) of \( \theta(g) \), in the sense that \( \theta(g_1), \theta(g_2) \) are conjugate in \( \text{Sym} X \) iff \( p[X](g_1) = p[X](g_2) \).

Obviously \( \chi[X](g) = k_\theta(1) \) for all \( g \in G \), so \( p[X] \) completely determines \( \chi[X] \). On the other hand,

\[
\chi[X](g^m) = \sum_{d|m} d \ k_\theta(d)
\]

so by the Moebius inversion formula [76],

\[
k_\theta(m) = \frac{1}{m} \sum_{d|m} \mu(m/d) \ \chi[X](g^d)
\]

Thus, the invariant \( \chi[X] \) is equivalent to the labeled set of cycle monomials [87] [131].

The cycle index of \( X \) is the polynomial

\[
P_X = \frac{1}{|G|} \sum_{g \in G} p[X](g)
\]

Some of the most basic examples include:

1. \( C_n \) (the cyclic group with \( n \) elements) acting on \( \mathbb{Z}_n \) in the obvious way has cycle polynomial

\[
P_X(s_1, s_2, \ldots, s_n) = \frac{1}{n} \sum_{d|n} \phi(d) s_n^{n/d}
\]
where $\phi$ is Euler's phi function, which is defined by setting $\phi(p^k) = p^k - p^{k-1}$ if $p$ is prime and setting $\phi(jk) = \phi(j)\phi(k)$ whenever $j, k$ are relatively prime.

2. $D_n$ (the dihedral group with $2n$ elements) acting on $\mathbb{Z}_n$ has cycle polynomial

$$P_X(s_1, s_2, \ldots s_{2n}) = \frac{1}{2n} \left( \frac{n}{2} s_1^2 s_2^{(n/2) - 1} + \frac{n}{2} s_2^{n/2} + \sum_{d|n} \phi(d) s_d^{n/d} \right)$$

if $n$ is even or

$$P_X(s_1, s_2, \ldots s_n) = \frac{1}{2n} \left( n s_1 s_2^{(n-1)/2} + \sum_{d|n} \phi(d) s_d^{n/d} \right)$$

if $n$ is odd.

3. $S_n$ (the symmetric group on $n$ letters) acting on a set of $n$ elements has cycle polynomial

$$P_X(s_1, s_2, \ldots s_n)$$

$$= \frac{1}{n!} \sum_{\mathcal{A} \in \Pi(X)} 1^{-k_1(\mathcal{A})} 2^{-k_2(\mathcal{A})} \ldots n^{-k_n(\mathcal{A})} s_1^{k_1(\mathcal{A})} s_2^{k_2(\mathcal{A})} \ldots s_n^{k_n(\mathcal{A})}$$

where $k_m(\mathcal{A})$ is the number of blocks of size $m$ in the partition $\mathcal{A}$.

In applying the third formula, it is very helpful to realize that the number of partitions of $X$ having atoms of size $n_1, n_2, \ldots n_r$, where of course $n_1 + n_2 + \ldots n_r = n$, is given by the multinomial coefficient

$$\left( \begin{array}{c} n \\ n_1 \ n_2 \ \ldots \ n_{r-1} \ n_r \end{array} \right) = \frac{n!}{n_1!n_2!\ldots n_r!}$$

is the multinomial coefficient giving the number of partitions of $X$ with corresponding atom sizes; note that the $k_m(\mathcal{A})$ only depend on the sizes of the atoms of $\mathcal{A}$. 
If $G$ acts on $X$ and $H$ acts on $Y$, where $|X| = m$ and $|Y| = n$, say, then the cycle index for $G \wr H$ acting on $X \times Y$ is obtained by plugging

$$t_1 = P_X(s_1, s_2, \ldots s_m),$$

$$t_2 = P_X(s_2, s_4, \ldots s_{2m}),$$

$$\ldots$$

$$t_n = P_X(s_n, s_{2n}, \ldots s_{mn})$$

into the cycle index $P_Y(t_1, t_2, \ldots t_n)$ and expanding. This fact can be very helpful in computing the cycle index of certain geometry symmetry groups. In particular, the geometric symmetry group (including orientation reversing isometries) of an $n$-cube acting on its hyperfaces is the wreath product of $G = C_2$ acting on $X = \{ \pm 1 \}$ in the obvious way with $H = S_n$ acting by permuting $Y$ (the $n$ coordinate axes). For example, the geometric symmetry group of the four cube has cycle index

$$P_{X \times Y}(s_1, s_2, \ldots s_8) = \frac{1}{384} \left( s_1^8 + 4s_1^6s_2 + 18s_1^4s_2^2 + 28s_1^2s_2^3 + 25s_2^4 + 32s_1^2s_3^2 + 32s_2^2s_3^2 + 12s_1^4s_4 + 24s_1^2s_2s_4 + 36s_2^2s_4 + 60s_4^2 + 32s_1^2s_6 + 32s_2s_6 + 48s_8 \right)$$

This is obtained by plugging

$$t_1 = P_X(s_1, s_2) = \frac{1}{2} (s_1^2 + s_2)$$

$$\ldots$$

$$t_4 = P_X(s_4, s_8) = \frac{1}{2} (s_4^2 + s_8)$$

into

$$P_Y(t_1, t_2, t_3, t_4) = \frac{1}{24} \left( t_1^4 + 6t_4 + 8t_3t_1 + 6t_2t_1^2 + 3t_2^2 \right)$$

and expanding.
Let \( U \) be a set of colors. Then each coloring of \( X \) by colors in \( U \) defines a function \( \alpha : X \to U \) and vice versa. Now, \( G \) acts on the coloring space \( \{\alpha : X \to U\} \) by \( g \cdot \alpha = \alpha \circ \theta(g^{-1}) \). Given a coloring \( \alpha \), define its spectral monomial to be

\[
q[\alpha](u_1, u_2 \ldots u_r) = \prod_{j=1}^{r} u_j^{[\alpha^{-1}(u_j)]}
\]

where \( r = |U| \). The \( \text{Sym} \ X \) orbits on the space of colorings are obviously characterized by the spectral monomials in the sense that \( \alpha \) is in the same orbit as \( \beta \) iff \( q[\alpha] = q[\beta] \).

The Polya enumeration theorem [38] uses the cycle index to obtain information about how the orbits of the coloring space under the action by \( \text{Sym} \ X \) split into smaller orbits under \( G \). The theorem says that if we plug

\[
s_1 = \sum_{j=1}^{r} u_j, \ s_2 = \sum_{j=1}^{r} u_j^2 \ldots s_n = \sum_{j=1}^{r} u_j^n
\]

into the cycle index and expand, we obtain a polynomial in the \( u_j \) which is a sum of the spectral monomials \( q[\alpha] \), times certain integral coefficients, and the number of \( G \)-orbits contained in the \( \text{Sym} \ X \) orbit corresponding to \( q[\alpha] \) is precisely its coefficient in this sum. (Rota and Smith used the lattice of periods to obtain a generalization of this theorem [117].)

The cycle index is completely determined by the character of \( X \). On the other hand, it is easy to find examples of \( D_4 \)-sets showing that the cycle index of \( X \) does not determine its character.

Since the character is a complete invariant for \( G \)-linear spaces [77], it may come as a surprise that \( \chi[X] \) does not determine \( X \) up to \( G \)-isomorphism. (It is easy to find small counterexamples.) A classical theorem which may be found in Burnside's text [14] shows how the generalized character defined by \( \lambda(H) = |\cdot H| \) supplies the missing information. Indeed, the numbers \( \lambda(H_j) \) (one for each conjugacy class of subgroups) form a complete set of invariants for \( X \). To see this, define the marks
function \( \omega(H, K) = |\triangleright_{G/H} K| \). Like \( \kappa \) and \( \lambda \), this is a class function (that is, it is constant on each conjugacy class of subgroups); unlike them, it depends only on \( G \) itself and not on \( X \).

**Lemma B.24** Suppose that \( H_1, H_2, \ldots, H_r \) represent the conjugacy classes of subgroups of \( G \). Then

\[
\lambda(H_\ell) = \sum_{j=1}^r \kappa(H_j) \omega(H_j, H_\ell)
\]

**Proof:** We know that \( X \) is \( G \)-isomorphic to its canonical form \( \coprod_{j=1}^r \kappa(H_j)G/H_j \). Therefore,

\[\triangleright H_\ell = \prod_{j=1}^r \kappa(H_j)_{G/H_j} H_\ell \]

from which the assertion is immediate. \( \blacksquare \)

Note that \( \omega(H, H) = [N_G(H) : H] \), the size of the automorphism group of \( G \) acting on \( G/H \). The statement of Lemma B.24 can be written in matrix notation as \( \lambda = \kappa \Omega \), where \( \lambda \) and \( \kappa \) are written as row vectors. Moreover, the matrix \( \Omega \) is lower triangular, and the determinant \( \det \Omega = \prod_j [N_G(H_j) : H_j] \) is nonzero. Thus, we have \( \kappa = \lambda \Omega^{-1} \), which shows that \( X \) is determined up to \( G \)-isomorphism by the generalized character \( \lambda \).

**Definition 56** Let \( X \) be a \( G \)-set. The zeta function of \( g \in G \), written \( \zeta[g] \), is

\[
\zeta[g](t) = \exp \left( \sum_{d=0}^{\infty} \chi(g^d) \frac{t^d}{d} \right)
\]

The Bowen-Lanford formula (see [87],[52],[131]) says that

\[
\zeta[g](t) = \frac{1}{\det(I - t\Theta^*(g))} = \prod_{d=1}^n (1 - t^d)^{-\kappa_d(g)}
\]
where $n = \sum_{d=1}^{n} d \, k_{g}(d) = |X|$. (If the upper summation limit here looks suspicious, note that many of the $k_{g}(\cdot)$ will vanish for a given $g \in G$: for instance $k_{g}(n) = 1$ if $\theta(g)$ contains a single $n$-cycle and otherwise $k_{g}(n) = 0$.) Thus, the function $g \mapsto p[g]$ determines the function $g \mapsto \zeta[g]$. On the other hand, $g \mapsto \zeta[g]$ obviously determines $\chi[X]$, so the character, the labeled set of cycle monomials, and the labeled set of zeta functions give three equivalent sets of invariants for finite $G$-sets.

Recall that $\mathfrak{P}_{d}[X]$ is the space of homogeneous polynomials in the formal variables defined by the points of $X$, and of degree $d$, which are invariant under the given action by $G$ on $X$. The Molien function of $\theta$ is

$$
\Phi[\theta](t) = \sum_{d=0}^{\infty} \dim(\mathfrak{P}_{d}[X]^{G}) \, t^{d}
$$

Thus, the $d$-th coefficient of $\Phi[\theta]$ counts the dimension of the subspace consisting of the homogeneous polynomials of degree $d$ which are fixed by the induced action by $G$ on $\mathfrak{P}_{d}[X]$. Molien’s theorem [52][131] says that

$$
\Phi[\theta] = \frac{1}{|G|} \sum_{g \in G} \zeta[g]
$$

Thus, the Molien function is a rational function which is a $G$-set invariant of $X$, and which is completely determined by the character of $X$. 

Appendix C

Some Sturmian Software

Following is the complete listing of a Mathematica program called S23.m which was used to make all the figures involving $(2, 3)$ Sturmian systems in this thesis. Similar programs were used for other $(p, q)$ systems and are available from the author (it is convenient to have several separate systems because each case of $(p, q)$ involves different display techniques). The program calls a compiled C program called S23; the C code for this program is listed following the Mathematica code.

```plaintext
(* /ch/ABSLATEI/Sturm/33/S23.m

This program makes a list of oblique tiling cells in R^2 and allows user to perform various operations on these and on derived objects.

OPERATIONS WHICH CREATE A LIST OF OBLIQUE TILING CELLS

cells(y), where y in R^2 is usually a linear comb of the aj.
selects the cells hit by the cut plane y

OPERATIONS ON LISTS OF CELLS:
cutpyj[cells] projects to cut plane

OPERATIONS ON OUTPLANE CELLS:
excite[tiles] converts tiles to lists of two dim line segments
grayify[tiles] converts to light gray polygons
darken[tiles] converts to dark gray polygons

OPERATIONS ON COMPLANE CELLS:
nchange[tiles] converts tiles to lists of three dim line segments
polyhedronify[tiles] converts to list of three dim polyhedra

OPERATIONS ON GRAPHICS OBJECTS:
planeop[thing] makes a two dimensional picture of outplane things
spaceop[thing] makes a three dimensional picture of complane things
planeop[picture, aj] crops to a by a frame
unframe[picture] removes the frame

CONVENIENT HACKS:
   tiling[y], where y in R^2, draws tiling induced in cut plane by oblique tiling
   highlight[cells] shows tiles defined by list of cells as dark gray polygons
```
B.8. STRUCTURAL INVARIANTS

REMARK: We use the //F function to ensure we do not try to do exact rational arithmetic. *

(* DEFINE CONSTANTS *)

(* Will use (2*l+1)**6 multiscells in the oblique tiling *)

k = 16;

(* Steps will have "thickness" 2j *)

j = 0.0001;

(* The columns of A will span V *)

A =

\begin{pmatrix}
-1.1, & \text{GoldenRatio}, & 0.1, & \text{GoldenRatio} \\
0.1, & \text{GoldenRatio}, & -1.1, & \text{GoldenRatio}
\end{pmatrix};

(* SHADOW SOME POSSIBLE A MATRICES FOLLOW *)

Pearson W (includes non-Pearson tiling orbit closure)

\text{Transposed} \left[ \begin{pmatrix} 1, \cos(2 \pi / k), \cos(4 \pi / k), \cos(6 \pi / k)
\end{pmatrix},
\begin{pmatrix} 0, \sin(2 \pi / k), \sin(4 \pi / k), \sin(6 \pi / k)
\end{pmatrix} \right];

Some Periodic Approximations of the Pearson W

\text{Transposed} \left[ \begin{pmatrix} 3, 1, -3, -1, 1 \\
0, 3, 1, -1, -1
\end{pmatrix} \right];

\text{Transposed} \left[ \begin{pmatrix} 16, 5, -13, -13, 6 \\
0, 19, 12, -12, -18
\end{pmatrix} \right];

-----

One second order FAULT LINE

\text{Transposed} \left[ \begin{pmatrix} 1, 1.1, 1.1, \text{Sqrt}(22) \\
0, \text{Sqrt}(11), \text{Sqrt}(17), \text{Sqrt}(20), \text{Sqrt}(11)
\end{pmatrix} \right];

One first order FAULT LINE

\text{Transposed} \left[ \begin{pmatrix} 1, 1.1, \text{Sqrt}(22), \text{Sqrt}(13) \\
0, \text{Sqrt}(11), \text{Sqrt}(17), \text{Sqrt}(20), \text{Sqrt}(11)
\end{pmatrix} \right];

Two first order FAULT LINES

\text{Transposed} \left[ \begin{pmatrix} 1, 1.1, \text{Sqrt}(11), \text{Sqrt}(13) \\
0, \text{Sqrt}(11), \text{Sqrt}(17), \text{Sqrt}(20), \text{Sqrt}(11)
\end{pmatrix} \right];

Four first order FAULT LINES

\text{Transposed} \left[ \begin{pmatrix} 1, 1.1, \text{Sqrt}(22), \text{Sqrt}(22) \\
0, \text{Sqrt}(11), \text{Sqrt}(17), \text{Sqrt}(20)
\end{pmatrix} \right];
## Rule Examples

Another aperiodic tiling

\[
\text{Transpose} \left( \begin{array}{cccc}
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
0 & \sin(\pi/6) & \sin(\pi/6) & \sin(\pi/6) \\
0 & \sin(\pi/6) & \sin(\pi/6) & \sin(\pi/6) \\
\end{array} \right)
\]

Here's one with no slings except at origin in cartesain and some nearby tilings (??)

\[
\text{Transpose} \left( \begin{array}{cccc}
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
0 & 1 & 1 & 0 \\
\end{array} \right)
\]

Here's another with no slings except carreutilf etc.

\[
\text{Transpose} \left( \begin{array}{cccc}
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
\end{array} \right)
\]

Here's one with only a single vertical roll

\[
\text{Transpose} \left( \begin{array}{cccc}
2 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
0 & 3 & 2 & 1 \\
\end{array} \right)
\]

Here's one with a "line" of second order slings and a "grid" of first order slings in cartesain

\[
\text{Transpose} \left( \begin{array}{cccc}
1 & 1 & \sqrt{2} & 1 \\
0 & 2 & 1 & 0 \\
\end{array} \right)
\]

Another periodic tiling

\[
\text{Transpose} \left( \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 3 & 6 & 7 & 11 \\
\end{array} \right)
\]

### End of List of Possible A Matirces

(* Find standard basis for cutplane *)

\[a\] \[\text{Transpose}[\text{NormalSpace}[[\text{Transpose}[a]]]]\].

(* Find standard basis for enplane *)

\[b\] \[\text{Transpose}[\text{NormalSpace}[[\text{Transpose}[b]]]]\].

(* Use numerical form from now on *)

\[a\] \[\text{Eval}[a]\].

(* Compute ribbon ROW vectors for \(W\) and \(W_\text{type} \)*)

\[r_1 = \text{Simplify}[\text{Transpose}[a][1][1] - \text{Transpose}[a][1][1] - \text{Transpose}[a][1][1]]\].

\[r_2 = \text{Simplify}[\text{Transpose}[a][2][1] - \text{Transpose}[a][2][1] - \text{Transpose}[a][2][1]]\].

\[r_3 = \text{Simplify}[\text{Transpose}[a][3][1] - \text{Transpose}[a][3][1] - \text{Transpose}[a][3][1]]\].

\[r_4 = \text{Simplify}[\text{Transpose}[a][2][2] - \text{Transpose}[a][2][2] - \text{Transpose}[a][2][2]]\].

\[r_5 = \text{Simplify}[\text{Transpose}[a][1][1] - \text{Transpose}[a][1][1] - \text{Transpose}[a][1][1]]\].

\[r_6 = \text{Simplify}[\text{Transpose}[a][3][2] - \text{Transpose}[a][3][2] - \text{Transpose}[a][3][2]]\].

\[s_1 = \text{Transpose}[a][1][1]
\]

\[s_2 = \text{Transpose}[a][2][2]
\]

\[s_3 = \text{Transpose}[a][3][3]
\]

\[s_4 = \text{Transpose}[a][1][2]
\]

\[s_5 = \text{Transpose}[a][2][1]
\]

\[s_6 = \text{Transpose}[a][3][2]
\]

\[s_7 = \text{Transpose}[a][1][3]
\]

\[s_8 = \text{Transpose}[a][2][3]
\]

\[s_9 = \text{Transpose}[a][3][1]
\]

\[s_{10} = \text{Transpose}[a][1][3]
\]

\[s_{11} = \text{Transpose}[a][2][1]
\]

\[s_{12} = \text{Transpose}[a][3][2]
\]

\[s_{13} = \text{Transpose}[a][1][2]
\]

\[s_{14} = \text{Transpose}[a][2][3]
\]

\[s_{15} = \text{Transpose}[a][3][1]
\]

\[s_{16} = \text{Transpose}[a][3][1]
\]

\[s_{17} = \text{Transpose}[a][3][2]
\]

\[s_{18} = \text{Transpose}[a][2][1]
\]

\[s_{19} = \text{Transpose}[a][1][2]
\]

\[s_{20} = \text{Transpose}[a][1][3]
\]

\[s_{21} = \text{Transpose}[a][2][3]
\]

\[s_{22} = \text{Transpose}[a][3][1]
\]

\[s_{23} = \text{Transpose}[a][3][2]
\]

\[s_{24} = \text{Transpose}[a][1][2]
\]

\[s_{25} = \text{Transpose}[a][2][1]
\]

\[s_{26} = \text{Transpose}[a][1][3]
\]

\[s_{27} = \text{Transpose}[a][2][3]
\]

\[s_{28} = \text{Transpose}[a][3][1]
\]

\[s_{29} = \text{Transpose}[a][3][2]
\]

\[s_{30} = \text{Transpose}[a][1][2]
\]
B.8. STRUCTURAL INVARIANTS

(* Define four useful vector operations *)
norm[v_] := Sqrt[v . v];
normalize[v_] := v / norm[v];
coord[u_, v_] := u * Sqrt[v . v];
proj[u_, v_] := ( u * norm[v] * Chop[v . Chop[v]] )
perp[u_, v_] := Chop[u - proj[u, v]]; (* compute the vector norm *)
(* convert to unit vector *)
(* length of component of u parallel to v *)
(* component of u parallel to v *)
(* component of u perpendicular to v *)

(* Compute the projection P onto the cut plane and the projection Q onto the complementary plane *)
P := Chop[Inverse[Transpose[A].A].Transpose[A]] //N;

ze := {0, 0, 0, 0} //N;

(* Define the vectors ej, pj and qj *)
e1 := {1, 0, 0, 0} //N;
e2 := {0, 1, 0, 0} //N;
e3 := {0, 0, 1, 0} //N;
e4 := {0, 0, 0, 1} //N;
e5 := {0, 0, 0, 1} //N;

ej := Chop[Part[P, 1]] //N;
ej := Chop[Part[P, 2]] //N;
ej := Chop[Part[P, 3]] //N;
ej := Chop[Part[P, 4]] //N;
ej := Chop[Part[P, 5]] //N;

qj := Chop[Part[Q, 1]] //N;
qj := Chop[Part[Q, 2]] //N;
qj := Chop[Part[Q, 3]] //N;
qj := Chop[Part[Q, 4]] //N;
qj := Chop[Part[Q, 5]] //N;

(* Define matrix for projecting to cut plane, with answer in planar euclidean coordinates *)
s1 := Chop[normalize[e[j]]];
s2 := Chop[normalize[proj[e[j], w]]];
PP := Chop[{{coord[e[j], x1], coord[e[j], x2], coord[e[j], x3], coord[e[j], x4]},
            {coord[e[j], x5], coord[e[j], x6], coord[e[j], x7], coord[e[j], x8]},
            {coord[e[j], x9], coord[e[j], x10], coord[e[j], x11], coord[e[j], x12]},
            {coord[e[j], x13], coord[e[j], x14], coord[e[j], x15], coord[e[j], x16]},
            {coord[e[j], x17], coord[e[j], x18], coord[e[j], x19], coord[e[j], x20]},
            {coord[e[j], x21], coord[e[j], x22], coord[e[j], x23], coord[e[j], x24]},
            {coord[e[j], x25], coord[e[j], x26], coord[e[j], x27], coord[e[j], x28]},
            {coord[e[j], x29], coord[e[j], x30], coord[e[j], x31], coord[e[j], x32]},
            {coord[e[j], x33], coord[e[j], x34], coord[e[j], x35], coord[e[j], x36]},
            {coord[e[j], x37], coord[e[j], x38], coord[e[j], x39], coord[e[j], x40]},
            {coord[e[j], x41], coord[e[j], x42], coord[e[j], x43], coord[e[j], x44]},
            {coord[e[j], x45], coord[e[j], x46], coord[e[j], x47], coord[e[j], x48]}]}];

(* Define matrix for projecting to complementary plane, with answer in planar euclidean coordinates *)
v1 := Chop[normalize[v[j]]];
v2 := Chop[normalize[proj[v[j], w]]];
v3 := Chop[normalize[proj[v[j], w] - proj[v[j], w]]];
QQ := Chop[{{coord[v[j], x1], coord[v[j], x2], coord[v[j], x3], coord[v[j], x4]},
            {coord[v[j], x5], coord[v[j], x6], coord[v[j], x7], coord[v[j], x8]},
            {coord[v[j], x9], coord[v[j], x10], coord[v[j], x11], coord[v[j], x12]},
            {coord[v[j], x13], coord[v[j], x14], coord[v[j], x15], coord[v[j], x16]},
            {coord[v[j], x17], coord[v[j], x18], coord[v[j], x19], coord[v[j], x20]},
            {coord[v[j], x21], coord[v[j], x22], coord[v[j], x23], coord[v[j], x24]},
            {coord[v[j], x25], coord[v[j], x26], coord[v[j], x27], coord[v[j], x28]},
            {coord[v[j], x29], coord[v[j], x30], coord[v[j], x31], coord[v[j], x32]},
            {coord[v[j], x33], coord[v[j], x34], coord[v[j], x35], coord[v[j], x36]},
            {coord[v[j], x37], coord[v[j], x38], coord[v[j], x39], coord[v[j], x40]},
            {coord[v[j], x41], coord[v[j], x42], coord[v[j], x43], coord[v[j], x44]},
            {coord[v[j], x45], coord[v[j], x46], coord[v[j], x47], coord[v[j], x48]}]}];

(* Define projection functions *)
cutplane[vector_] := Chop[PP . vector];
complement[vector_] := Chop[QQ . vector];

(* Define the pj in Cartesian coordinates for the cutplane *)
pp1 = cutplane[x];
pp2 = cutplane[y];
pp3 = cutplane[z];
pp1 = complanify[x1];
pp2 = complanify[x2];

(* Define the qj in Cartesian coordinates for the complane *)
q1 = complanify[x1];
q2 = complanify[x2];
q3 = complanify[x3];
q4 = complanify[x4];
q5 = complanify[x5];

(* Define the l,p) in Cartesian coordinates for the complane *)
lp1 = complanify[Ch[p,L,p1]] //N;
lp2 = complanify[Ch[p,L,p2]] //N;
lp3 = complanify[Ch[p,L,p3]] //N;
lp4 = complanify[Ch[p,L,p4]] //N;
lp5 = complanify[Ch[p,L,p5]] //N;

(* OPERATIONS YIELDING LISTS OF CELLS *)

(* Define the multicell *)
multicell = ({(xs, 106), (xs, 406), (xs, 304), (xs, 203), (xs, 102),
(xe, 204), (xe, 103), (xe, 102), (xe, 406),
{ai+1, 103}, {ai-1, 103}, {ai+1, 104}, {ai-1, 104},
{ai+2, 103}, {ai-2, 103}, {ai+2, 104}, {ai-2, 104},
{ai+3, 103}, {ai-3, 103}, {ai+3, 104}, {ai-3, 104},
{ai+4, 104}, {ai-4, 104}, {ai+4, 105}, {ai-4, 105},
{a1+1, 102}, {a1-1, 102},
{a1+1, 103},
{a1+1, 104},
{a1+1, 105},
{a1+2, 103},
{a1+2, 104},
{a1+2, 105},
{a1+3, 103},
{a1+3, 104},
{a1+3, 105},
{a1+4, 105});

(* Define the step *)
step = ({(xs, 102), (xe, 103), (xs, 104), (xs, 105), (xs, 203),
(xe, 204), (xe, 205), (xe, 304), (xe, 305), (xe, 405),
{-a1, 203}, {-a1, 204}, {-a1, 205}, {-a1, 304}, {-a1, 305}, {-a1, 406},
{-a2, 103}, {-a2, 104}, {-a2, 105}, {-a2, 204}, {-a2, 205}, {-a2, 306},
{-a3, 103}, {-a3, 104}, {-a3, 105}, {-a3, 204}, {-a3, 205}, {-a3, 306},
{-a4, 103}, {-a4, 104}, {-a4, 105}, {-a4, 203}, {-a4, 204}, {-a4, 305},
{-a5, 103}, {-a5, 104}, {-a5, 105}, {-a5, 203}, {-a5, 204}, {-a5, 304},
{-a5-1, 304}, {-a5-1, 205}, {-a5-1, 206}, {-a5-1, 406},
{-a5-1, 406}, {-a5+1, 205}, {-a5+1, 406},
{-a5+1, 305}, {-a5+1, 406});
B.8. STRUCTURAL INVARIANTS

(-a1->a6, 203), (-a1->a6, 204), (-a1->a6, 304),
(-a2->a3, 103), (-a2->a3, 106), (-a2->a3, 106),
(-a2->a4, 103), (-a2->a4, 106), (-a2->a4, 304),
(-a3->a4, 103), (-a3->a4, 106), (-a3->a4, 106),
(-a3->a6, 103), (-a3->a6, 104), (-a3->a6, 204),
(-a4->a6, 103), (-a4->a6, 104), (-a4->a6, 203),
(-a1->a2->a3, 60G),
(-a1->a2->a4, 106),
(-a1->a3->a4, 304),
(-a1->a3->a6, 206),
(-a1->a4->a6, 303),
(-a2->a3->a4, 106),
(-a2->a3->a6, 104),
(-a2->a4->a6, 103),
(-a3->a4->a6, 103).

(* Make cubical array of cells *)
cubearray[x_] := Flatten[Table[move[ multicell[{x1, x2, x3, x4, x5}]],
          {x1, -4, 4}, {x2, -4, 4}, {x3, -4, 4}, {x4, -4, 4}, {x5, -4, 4}], 1];

(* Find the oblique tiling cells which intersect a given superline y = y*)
cutcell[y_] := RunThrough["222", y, 1, 1, 1]; (* the 222 and read in output *)

(* OPERATIONS ON LISTS OF CELLS *)

(* Select the tiles in a list which lie at a given "coordinate height" in the oblique tiling *)
terrace[cells_, x1, a1_] = Cases[cells, (a_, type_) /; Part[a, 1] == a1];
terrace[cells_, x2, a2_] = Cases[cells, (a_, type_) /; Part[a, 2] == a2];
terrace[cells_, x3, a3_] = Cases[cells, (a_, type_) /; Part[a, 3] == a3];
terrace[cells_, x4, a4_] = Cases[cells, (a_, type_) /; Part[a, 4] == a4];
terrace[cells_, x5, a5_] = Cases[cells, (a_, type_) /; Part[a, 5] == a5];

(* Select the tiles from alternate terraces to obtain "strata" *)
stratify[cells_, x1] := Flatten[Table[terrace[cells, x1, E], {x1, -4, 4}, 1];
stratify[cells_, x2] := Flatten[Table[terrace[cells, x2, E], {x2, -4, 4}, 1];
stratify[cells_, x3] := Flatten[Table[terrace[cells, x3, E], {x3, -4, 4}, 1];
stratify[cells_, x4] := Flatten[Table[terrace[cells, x4, E], {x4, -4, 4}, 1];
stratify[cells_, x5] := Flatten[Table[terrace[cells, x5, E], {x5, -4, 4}, 1];

(* Select intersection of a ribbon with given list of cells *)
ribbon[cells_, x1, a1_] := Cases[terrace[cells, x1, a1], (a_, type_) /;
          type == 102 || type == 103 || type == 105];
ribbon[cells_, x2, a2_] := Cases[terrace[cells, x2, a2], (a_, type_) /;
          type == 102 || type == 202 || type == 204 || type == 208];
ribbon[cells_, x3, a3_] := Cases[terrace[cells, x3, a3], (a_, type_) /;
          type == 103 || type == 203 || type == 304 || type == 308];
ribbon[cells_, x4, a4_] := Cases[terrace[cells, x4, a4], (a_, type_) /;
          type == 104 || type == 204 || type == 404 || type == 408];
ribbon[cells_, x5, a5_] := Cases[terrace[cells, x5, a5], (a_, type_) /;
          type == 105 || type == 205 || type == 305 || type == 405];
(* Have a list of cells by the INTEGER vector shift_in 8's *)

meta[cells, shift_in] := {meta[cells[1..4]]; type} := {meta[cells], type};

(* Convert oblique tiling cells to cutplane tiles *)

cutproj[cells[1..4]] := (x = cutplane[cells[1..4]] \ {type});

cutproj[cells[5..8]] := (x = cutplane[cells[5..8]] \ {type});

cutproj[cells[9..12]] := (x = cutplane[cells[9..12]] \ {type});

cutproj[cells[13..16]] := (x = cutplane[cells[13..16]] \ {type});

cutproj[cells[17..20]] := (x = cutplane[cells[17..20]] \ {type});

cutproj[cells[21..24]] := (x = cutplane[cells[21..24]] \ {type});

cutproj[cells[25..28]] := (x = cutplane[cells[25..28]] \ {type});

cutproj[cells[29..32]] := (x = cutplane[cells[29..32]] \ {type});

cutproj[cells[33..36]] := (x = cutplane[cells[33..36]] \ {type});

cutproj[cells[37..40]] := (x = cutplane[cells[37..40]] \ {type});

cutproj[cells[41..44]] := (x = cutplane[cells[41..44]] \ {type});

cutproj[cells[45..48]] := (x = cutplane[cells[45..48]] \ {type});

cutproj[cells[49..52]] := (x = cutplane[cells[49..52]] \ {type});

cutproj[cells[53..56]] := (x = cutplane[cells[53..56]] \ {type});

cutproj[cells[57..60]] := (x = cutplane[cells[57..60]] \ {type});

cutproj[cells[61..64]] := (x = cutplane[cells[61..64]] \ {type});

cutproj[cells] := Eap[cutproj[cells], cells];

(* Convert oblique tiling cells to cutplane tiles *)

cutproj[cells] := (x = cutplane[cells] \ {type});

(* OPERATIONS ON LISTS OF CUTPLANE TILES *)

point[tile] := (Point[1], Point[2], Point[3], Point[4]);

point[cells] := Eap[point[tiles], cells];

(* Convert a list of cutplane tiles to a list of edges in 8's *)

edgetiles[cells] := (Thickness[0.002], Line[1, cells[1..4], cells[5..8]]);

edgetiles := Eap[edgetiles, tiles];

(* Convert a list of cutplane tiles to a list of shaded polygons in 8's *)
grayedges[cells] := (Thickness[0.002], GrayLevel[0.85], Polygon[cells[1..4], cells[5..8]]);
B.8. STRUCTURAL INVARIANTS

grayify(tiles...) := Map(grayifytile, tiles);

darkenfill[x1, x2, y1, y2] := { thickness(0.001), graylevel(0.5),Polygon([x1, y1, x2, y2])};
darkenfill[] := Map(darkenfill, tiles);

(* Convert a list of cusp plane tiles to a list of red polygons in R^2 *)
redfill[x1, x2, y1, y2] := {RGBColor(1.0, 0.0, 0), Polygon([x1, y1, x2, y2])};
redfill[] := Map(redfill, tiles);

(* Convert a list of cusp plane tiles to a list of green polygons in R^2 *)
greenfill[x1, x2, y1, y2] := {RGBColor(0.0, 0.5, 0), Polygon([x1, y1, x2, y2])};
greenfill[] := Map(greenfill, tiles);

(* Convert a list of cusp plane tiles to a list of blue polygons in R^2 *)
bluefill[x1, x2, y1, y2] := {RGBColor(0.0, 0.0, 1), Polygon([x1, y1, x2, y2])};
bluefill[] := Map(bluefill, tiles);

(* Convert a list of cusp plane tiles to a list of brown polygons in R^2 *)
brownfill[x1, x2, y1, y2] := {RGBColor(0.5, 0.5, 0), Polygon([x1, y1, x2, y2])};
brownfill[] := Map(brownfill, tiles);

(* Convert a list of cusp plane tiles to a list of cyan polygons in R^2 *)
cyanfill[x1, x2, y1, y2] := {RGBColor(0.0, 0.5, 0.5), Polygon([x1, y1, x2, y2])};
cyanfill[] := Map(cyanfill, tiles);

(* Convert a list of cusp plane tiles to a list of yellow polygons in R^2 *)
yellowfill[x1, x2, y1, y2] := {RGBColor(0.0, 0.0, 0.5), Polygon([x1, y1, x2, y2])};
yellowfill[] := Map(yellowfill, tiles);

(* Convert a list of cusp plane tiles to a list of edges in R^2 of the derived tiles *)
polarityfill[x1, x2, y1, y2] := {thickness(0.001),Line([(1/2)(x1+y1, x1+y2, x2+y2, x2+y1)])};
polarityfill[] := Map(polarityfill, tiles);

(* OPERATIONS ON LISTS OF CUSP PLANE TILES *)

(* Convert a list of cusp plane tiles to a list of edges in R^3 *)
skeletonfill[x1, x2, y1, y2, z1, z2] :=
  { PointSize(0.00), Point(x1),
    PointSize(0.00), Point(y1),
    PointSize(0.00), Point(z1),
    PointSize(0.00), Point(x2),
    PointSize(0.00), Point(y2),
    PointSize(0.00), Point(z2),
    Thickness(0.001), Line([y1, z1]),
    Thickness(0.001), Line([y1, z2]),
    Thickness(0.001), Line([y2, z1]),
    Thickness(0.001), Line([y2, z2])}.

(* Convert a list of cusp plane tiles to a list of edges in R^3 *)
skeletonedgefill[x1, x2, y1, y2, z1, z2] :=
  { PointSize(0.00), Point(x1),
    PointSize(0.00), Point(y1),
    PointSize(0.00), Point(z1),
    PointSize(0.00), Point(x2),
    PointSize(0.00), Point(y2),
    PointSize(0.00), Point(z2),
    Thickness(0.001), Line([y1, z1]),
    Thickness(0.001), Line([y1, z2]),
    Thickness(0.001), Line([y2, z1]),
    Thickness(0.001), Line([y2, z2])}.
(* Convert a list of complete tiles to a list of thick edges in E^3 *)

healdensation[tiles_] := Map[(healdensation[tile]. tiles),
  (* Convert a list of complete tiles to a list of faces in E^3 *)
  polyhedrality[tiles_] :=
    (* OPERATIONS OF TWO DIMENSIONAL GRAPHICS OBJECTS *)

planepic[thing_] := Show[ Graphics[thing], PlotRange -> All,
  Frame -> True, FrameLabel -> {"x1", "x2"},
  AspectRatio -> Automatic];

labelplanepic[thing_] := Show[ Graphics[thing], PlotRange -> All,
  Frame -> True, FrameLabel -> {"x1", "x2", MatrixForm[\[ScriptA]], ""},
  AspectRatio -> Automatic];

(* Draw a background of every other Dirichlet cell in light gray."
  in the case of \[ScriptA] having RELATIVELY PRIME INTEGER ENTRIES *)

periodicpic[thing_] := ( vo = {0, 0}/N;
  v1 = FPPart[Transpose[vo], 1];
  v2 = FPPart[Transpose[vo], 2];
  unitcell = { GrayLevel[0.86], Polygon([ vo, v1, v1 + v2, v2, vo])};
  nroccell[GrayLevel[0.86], Polygon([ vo, v1, v1 + v2, v2, vo, v2, v1, vo + (v2 - v1)]), v1 + v2 - (v2 - v1 + (v2 - v1), v1 - (v2 - v1), vo)};
B.8. STRUCTURAL INVARIANTS

\[ v_2 = (v_w) v_1 + (m-a) v_3 \];

background = UnionTable[ novelcell[ unitcell[ (m-1).\text{+1} ], (m-1.5). (m-0.5) ] ];
planepic[ background, thing ] ];

(* Crop a picture using a square window *)
planecrop[picture_, a_, n_] := Show[picture, PlotRange \[Rule] ((-n,a).(-n,a)) ];

(* Remove the frame from a picture *)
unframe[picture_] := Show[picture, Frame \[Rule] False ];

(* OPERATIONS ON THREE DIMENSIONAL GRAPHIC OBJECTS *)

spacepic[thing_, viewpoint_] :=

spacepic[thing_, viewpoint_] :=

(* CONVENIENT HACKS *)

(* Draw the tiling defined by the cut plane y=W, where y is a five vector in complane *)
tiling[xy_] := planepic[ Edge[proj][cutcell[xy]] ];

(* Draw the derived tiling defined by the cut plane y=W, where y is a five vector in complane *)
polarxy[xy_] := planepic[ Polar[proj][cutcell[xy]] ];

(* Draw a darkened patch, defined as a list of oblique tiling cells, with edges and vertices *)
highlight[cells_] := (temp = cutproj[cells]; planepic[ Darken[temp], edgify[temp], pointify[temp] ] );

(* Draw tilings from a given list, shading the places where they DIFFER *)
compare[list_] := (nwx = Length[list];
cellarray = Table[ cutcell[Part[ list, i ], (n-1).nwx] ];
commachine = Apply[ Intersection, cellarray ];
allcells = Union[ Flatten[cellarray, 1] ];
discrepom = Complement[ allcells, commachine ];

(* Draw a tiling and shift of tiling; shade patches of discrepancy *)
shift should be an integral linear combination of the uj's *)
move[xy_, shift_] := compare[xy, y + shift ];

(* Draw the Christoffel tiling *)
sheetsheet := compare[ 5/2 ({qi, qj, qk, qL, qM, qN, qQ, qQ, qQ, qQ}, qQ) ];
printsheet := Display["temp.bmp", planecrop[sheetsheet, Calling[3 2/3]] ];

(* Draw the empire of a patch with given acceptance domain. *)
given as the convex hull of given extreme points (vertices in R^5) in complane *)
empire[domain_, a_, n_] := (nwx = Length[ domain ]; centroid = Apply[ Plus, domain]/nwx;
    shrink[xy_] := (1-a) x centroid + centroid );
Following is the C code for the routine S23. This routine is a much optimized and
streamlined program by Shawn Cokus, loosely based upon a routine written by the author.

//
// A Sturmian (2, 1) quasi-tiler by Shawn Cokus, July-September 1996.
// An elaboration in C of a Mathematica program by Chris Hillman.
//
// This program was automatically generated by version 1.0 of Shawn's meta-tiler
// at 12:27 PM PDT on Sunday, September 22, 1996.
//
// Compile it on (e.g., Neumann) with: 'gcc -lm -G3 -u 32203 323203.c'.
// For more warnings than can possibly be correct, add:
// -Wall -Wtraditional -Wshadow -Wpointer-without -Wcast-align -WWrite-strings -Wconversion
// -Wstrict-prototypes -Wredundant-else -Wnested-externs -Winline'.
// For more information in more files than you possibly want to read, add:
// -s -s -temp -Verbous-asm -da -dp'.
// (GCC may change my mind about GNU software. GCC is true cool, apologies to any Sx.)
//
// Shawn can be reached via e-mail at Smaltz-Cokus@Math.washington.edu. The web page
// http://www.math.washington.edu/~cokus/personal.html may contain additional information
// It approximately three billion years have elapsed since this comment was written.
//
// #define OUTPUT_PDF

#define hProgramName "S23"
#define H (H)
#define H (H)
#define H (H)

// --- INCLUDES ---------------------------------------------------------------
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

// --- TYPES & DEFINES --------------------------------------------------------

typedef double real;

#define AF211(a) ( \n if(1.0e-10 < a || a < 1.0e-10) || (a > 1.0e+10) || (a > 1.0e+10) ( \n fprintf(stderr, "Z: Numerical maxness (AF211) \n", hProgramName); 
 exit(EXIT FAILURE); \n ) \n )

// --- GLOBALS ---------------------------------------------------------------
static real a[3][3]; // given: specifies cut plane
static real PP[3][3]; // found: projector to cut plane
static real QQ[3][3]; // found: projector to complementary plane
static real XY; // given: specified translation of cut plane
static real Z; // given: Axiom of search; the points (-4..2)*Z are explored

static int [X]; // Index: current cell type (N x R of 0..(D-1), Incr. order)
static int [L]; // Index: complement of I in 0..(D-1) in increasing order

// EXTRIS FUNCTIONS

void computeMatrixInverse(int n, real *A, real *V)
{
    int (n > 0)
    #define 1 n
    if (n)
        #endif

    int row,i, j=0, k, n, pp; // the compiler is
    real W[1][2+n], x, xx, pp; // on its own...

    // minimal error checking

    if (!n) {
        printf(stderr, "Ex: ExEd matrix inversion", bProgramName, n, n):
            exit(EXIT_FAILURE);
    }

    // compose V by augmenting U with identity matrix
    // note: the pointer U (i.e., the stacked parameter) is moved

    for(i=0; i<n; i++) {
        for(j=0; j<n; j++)
            W[1][i] = 1.0;

        for(i=0; i<n; i++)
            W[1][i] = 0.0;

        W[1][i+1] = 1.0;
    }

    // perform elimination phase of Gaussian elimination with maximal column pivoting

    for(i=0; i<n; i++)
        row[i] = i;

    for(i=0; i<n-1; i++) {
        for(j=i; j<n; j++)
            printf("in[1][1]", j, i); j++;
        if (fabs(W[1][j]) > pp)
            printf("<1>", i, j); j++;
    }

}
#### B.8. STRUCTURAL INVARIANTS

```c
#define srew(p) : rre(p3) = rre(1); rre(1) = 0;
srew(rew); APZEA(s); s = s*0.0/s;

for(j = 0; j < n; j++) {
  s = srew(j);
  s = srew(rrew(j)) - s;
}

// perform backward substitution phase of Gaussian elimination

for(i = n - 1; i >= 0; i--)
  for(j = 0; j < n; j++)
    for(s = 0.0; s < (s = s - srew[j]));

F(srew[j]) = srew(WX(srew[j]));

F(n) = F(n)

void computeFandQ0FromA(int)
{
  real P[0][0], Q[0][0];

  // compute P from A

  for(i = 0; i < n; i++)
    for(j = 0; j < n; j++)
      for(s = 0.0; s < a[i][j]; s++)
        s = s + a[i][j] * a[i][j];

  A[i][j] = s;

  computeInverseA(s, A[0][0], A[0][0]);

  for(i = 0; i < n; i++)
    for(j = 0; j < n; j++)
      for(s = 0.0; s < a[i][j]; s++)
        s = s + a[i][j] * a[i][j];
```

```
data[i][j] = u;
}

for(i=0; i<n; i++)
for(j=0; j<n; j++) {
for(b=0, b<n; b++)
  s = data[i][b] * u[j][b];

p[i][j] = s;
}

// compute Q from P

for(i=0; i<n; i++)
for(j=0; j<n; j++)
  q[i][j] = p[i][j];

for(i=0; i<n; i++)
  q[i][i] = 1.0;

// compute PP from P

real u[i][j], s;

for(i=0; i<n; i++)
  for(b=0, b<n; b++)
    u[i][b] = p[i][b];

for(j=0; j<n; j++) {
  s = pp[j][j];

  for(b=0, b<n; b++)
    u[i][b] = s * u[i][b];
}

for(s=0.0, b=0; b<n; b++)
  s = s + u[i][b] * u[i][b];

s = sqrt(s);  APA(s);  s = 1.0 / s;

for(b=0, b<n; b++)
  u[i][b] = s;

for(j=0; j<n; j++) {
  for(s=0.0, b=0; b<n; b++)
    s = p[j][b] * u[i][b];
}

for(s=0.0, b=0; b<n; b++)
  s = s + u[i][b] * u[i][b];

s = sqrt(s);  APA(s);  s = 1.0 / s;

for(b=0, b<n; b++)
  u[i][b] = s;
B.8. STRUCTURAL INVARIANTS

```c
PP(I[J]) = s;
}
}
}

// compute Q0 from Q0

{
  real v[X][X], s;
  int i, j, k;

  for(i=0; i<C; i++) {
    for(k=0; k<C; k++)
      v(I[i][j]) = 0.0[v][X];

    for(j=0; j<C; j++) {
      s = Q0(I[j][j]);

      for(k=0; k<C; k++)
        v(I[i][j]) += s * Q0[j][k][k];
    }
  }

  for(s=0.0, k=0; k<C; k++)
    s += ||[I][j]|X|X|X|X|X|;

  s = sqrt(s); AP2[0] = s = 1.0 / s;

  for(k=0; k<C; k++)
    v[I][J] += s;

  for(j=0; j<C; j++) {
    for(k=0.0, k=0; k<C; k++)
      s += Q0[I][J] = X[I][J];

  Q0[I][J] = s;
}

// OUTPUT ROUTINES

static int setComplete = 0;
static int setSetFirst = 0;

void OB2(int test)
{
  if(test < 0) {
    fprintf(stderr, "%s: Error writing to std. output \n", bName);
    exit(EXIT_FAILURE);
  }
}
```
void outputCell(int x[3])
{
    //...
B.3. STRUCTURAL INVARIANTS

```c
if (l >= 0-1) GBC(printf("Eld. ", x[i]));
else GBC(printf("Eld. ", x[i]));

GBC(printf(" ", "");

for(i=0; i<z; i++)
GBC(printf("EO", i));

GBC(printf(" >"));
endif
}

void outputClose(void)
{

#ifdef OUTPUT_POV
GBC(printf("// End of tiling \n"));
#else
GBC(printf("\n\n"));
#endif
setComplete = 1;
}

void outputOpen(void)
{

#ifdef OUTPUT_POV
GBC(printf("Option 3.0\n"));
GBC(printf("#include "colors.inc\n"));
GBC(printf("\n"));
GBC(printf("camera \n"));
GBC(printf("\nlocation \-60=60=60\n"));
GBC(printf("\ndirection \1.6=1.6=20.0 / 0\n"));
GBC(printf("\n\n\n"));
GBC(printf("\n\n\n\n"));
GBC(printf("\n\n\n"));
GBC(printf("\n\n\n"));
GBC(printf("\n\n\n"));
GBC(printf("\n\n\n"));
GBC(printf("\n\n\n"));
GBC(printf("\n\n\n"));
GBC(printf("\n\n\n"));
GBC(printf("\n\n\n"));
#endif

defclare r = 1.6f\n", (0.04/0.0) = 0\n};
GBC(printf("\n"));
end;
GBC(printf("\n"));
endif
}

void outputEndSeek(void)
{

```
if(!exitComplete)
   fprintf(stderr, "E: WARNING—OUTPUT IS INCOMPLETE.", bProgramName);

void outputInitallNels(void)
{
   if(Exit(outputExitNels))
   {
      fprintf(stderr, "E: exit() functionality not available.". bProgramName);
      exit(EXIT_FAILURE);
   }
}

// -- TILES FUNCTIONS

void reLocateCells(real B[i][j][k]. real a[i]. real b[i])
{
   // identify those x in (x..x) for which B[i] is in the product of (a[i]. b[i])

   register int *x0, *x1, *x2, *x3, *x4;
   int i, j;
   
real x0=0, x1=0, x2=0, x3=0, x4=0;

   real B0000=B[0][0][0], B0001=B[0][0][1], B0002=B[0][0][2], B0003=B[0][0][3], B0004=B[0][0][4],
   real B0010=B[0][1][0], B0011=B[0][1][1], B0012=B[0][1][2], B0013=B[0][1][3], B0014=B[0][1][4],
   real B0020=B[0][2][0], B0021=B[0][2][1], B0022=B[0][2][2], B0023=B[0][2][3], B0024=B[0][2][4],
   real a00=a[0], a01=a[1], a02=a[2],
   real b00=b[0], b01=b[1], b02=b[2],
   real r0000, r0001, r0002;
   real r0100, r0101, r0102;
   real r0200, r0201, r0202;
   real r0300, r0301, r0302;
   real s0000, s0001, s0002;
   real s0100, s0101, s0102;
   real s0200, s0201, s0202;
   real s0300, s0301, s0302;

10000=E+B0000, 10001=E+B0010, 10002=E+B0020;
B.8. STRUCTURAL INVARIANTS

```c
int structure_invariant()
{
  int x = 0;
  int y = 0;

  while (true)
  {
    if (x < 0)
    {
      x = x + 1;
    }
    else
    {
      x = x - 1;
    }
  }
}
```
real B[0][0], QQ[0][0], x[0], b[0];

// calculate B from QQ and I

{ int i, j, k;
  real B[0][0];

  for(i=0; i<3; i++) {
    x = I[i][j];

    for(i=0; i<3; i++)
      B[i][j] = QQ[i][j];
  }

  computeMatrixInverse(N, &B[0][0], &QQ[0][0]);
}

// compute BB from B and QQ

{ int i, j, k;
  real a;

  for(i=0; i<3; i++)
    for(j=0; j<3; j++)
      for(k=0.0, f=0.0; f<3; f++)
        a = B[i][j]+QQ[0][j];

  BB[i][j] = a;
}

// compute a and b from J, B, and f

{ int l, i, k;
  real x;

  for(i=0; i<3; i++)
    for(k=0.0, f=0.0; f<3; f++)
      l = B[i][j]+f[j];

  x[i] = x - 1.0 - j;
  b[i] = x + j;
}

// do the work

realLocateCells(BB, a, b);
void locateAllCells(void)
{
    int I;

    // compute Initial I

    for(I=0; I<6; I++)
        I[I] = I;

    while(1) {
        // compute last from I

        for(I=0; I<6; I++)
            I[I] = I;

        for(I=0; I<6; I++)
            I[I] = 0;

        for(I=0; I<6; I++)
            if(I[I])
                goto;
    }

    // locate the cells of type I/I;

    fprintf(stderr, "Ie: Generating cells of type ", bProgramName);

    for(I=0; I<6; I++)
        fprintf(stderr, "%02d", I[I]);

    fprintf(stderr, " I/I: ");
    fflush(stderr);

    locateCells();

    // compute next I

    for(I=0; I<6; I++)
        I[I] = 0;

    if(I < 0)
        break;

    for(I=0; I<6; I++)
        I[I] = I[I-1];
}

// --- ELSE CONTROL

void doScan(const char *format)
{  
if (scanf(format) < 0) {
    fprintf(stderr, "%s: Error reading params. from stdin input:\n", ProgramName);
    exit(EXIT_FAILURE);
  }

void descan(void *format, void *target)  
{  
  if (scanf(format, target) < 0) {
    fprintf(stderr, "%s: Error reading params. from stdin input:\n", ProgramName);
    exit(EXIT_FAILURE);
  }
  
}

void main(void)  
{  
  outputInitialize();

  // Read parameters (in Mathematica format) from standard input as a 4-vector: the
  // first component is the A matrix, the second is the f vector (which will be
  // projected to the complementary plane before being used), the third is I, and
  // the fourth is E.

  // read initial opening brace
  descan(" { ");

  // read I and compute FF and QQ from it:
  
  int I, J;
  
  descan(" { ");

  for(j = 0; j < I; j++) {
    descan(" , ");
  }
  descan(" , ");

  for(i = 0; i < J; i++) {
    if(i > 0) descan(" IIF " " IIF ");
    else descan(" IIF ");

  }

  if(i > 0) descan(" } ");
  else descan(" } ");

  descan(" } ");

  computeFFandQQ(F,FQ);
}

// read f and project to the complementary plane
B.8. STRUCTURAL INVARIANTS

```c

real y[3], z;
int i, k;

dScanl(" ");

for(i=0; i<n; i++)
    if(i > 0) dScanl(" Ilf ", y[i]);
else dScanl(" Ilf ", y[0]);

dScanl(" ");

for(i=0; i<n; i++)
    for(k=0.0, k<=b; k++)
        z = Q0(1,0)+y[0];

y[i] = z;

// read j and k: the final closing brace isn't checked

dScanl(" Ilf ", j);
dScanl(" Id ", k);

// do minor error checking on parameters

if(j <= -0.8) {
    fprintf(stderr, "Is: J should be greater than -1/2.\n" ProgramName);
    exit(EXIT_FAILURE);
}

if(k < 1) {
    fprintf(stderr, "Is: K should be at least 1.\n" ProgramName);
    exit(EXIT_FAILURE);
}

// generate the tiling and exit:

fprintf(stderr, "Is: Preparing to generate tiles...\n" ProgramName);
ftile(stderr);
outputOpen();
locateAllCells();
outputClose();

fprintf(stderr, "Is: Generation complete/\n" ProgramName);
ftile(stderr);
exit(EXIT_SUCCESS);
```

Vita

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QUALITY OF CARE, ASYMMETRIC INFORMATION, AND PATIENT OUTCOMES IN U.S. FOR-PROFIT AND NOT-FOR-PROFIT RENAL DIALYSIS FACILITIES

by Renee A. Irvin

A dissertation submitted in partial fulfillment of the requirements for the degree of

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1998

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Program Authorized to Offer Degree Economics

Date 8/6/98
Doctoral Dissertation

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Abstract

QUALITY OF CARE, ASYMMETRIC INFORMATION, AND PATIENT OUTCOMES IN U.S. FOR-PROFIT AND NOT-FOR-PROFIT RENAL DIALYSIS FACILITIES

by Renee A. Irvin

Chairperson of the Supervisory Committee: Associate Professor Levis Kochin, Department of Economics

Economic theory suggests that investor-owned firms exhibit superior performance compared to their not-for-profit competitors due to efficiency gains realized from profit maximization incentives. Others argue that ownership type matters less than the incentives provided by the market in which the facilities operate. This dissertation examines the role of quality in comparative studies of sector performance in the health care industry. Quality variation is then analyzed with a cross-sectional study of almost 200,000 patients receiving renal dialysis treatments at over 2000 dialysis facilities nationwide. Multivariate regression analysis and propensity score analysis revealed higher patient mortality rates among patients treated at for-profit facilities, after adjusting for patient case-mix and market characteristics. Evidence of quality variation across ownership types suggests that researchers studying comparative efficiency of for-profit and not-for-profit health care firms must control for differing patient outcomes. Further testing was performed on a data sub-sample that included variables serving as proxies for patient knowledge. Dialysis facilities were found to treat high-knowledge patients differently than low-knowledge patients, but for-profit firms were not always found to exploit asymmetries of information more than their not-for-profit counterparts.
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DEDICATION

I wish to dedicate this dissertation to Jeff Cooper.
INTRODUCTION

The health care sector is one of several industries in the US where both for-profit and not-for-profit firms have substantial market share and power. For-profit ownership of US healthcare organizations has grown tremendously over the last two decades, and recent changes in healthcare payment systems have significantly heightened market competition levels while diminishing industry-wide profit margins. As competition mounts and financing for Medicare and Medicaid tightens, more attention has focused on comparative studies of healthcare performance by ownership type. While many studies have compared expenditures of for-profit and not-for-profit healthcare institutions, few studies have defined output correctly by controlling for quality of care. Thus, these comparative cost studies yield no meaningful results.

Consider two identical patients needing open-heart surgery. The patients are admitted for surgery on day one. Patient X dies on the operating table, while patient Y survives surgery, is hospitalized for one week, then discharged -- and goes on to live another 15 years. In this situation, patient X’s hospital may appear more “efficient” because it has taken only one day to treat the patient. Patient Y’s hospital is labeled “high cost” and even “inefficient” because the patient spent a week in the hospital for the identical illness. In effect, patient satisfaction or health production is ignored, hospital output is measured as patient-days or number of patients, and the resulting measures of efficiency are nonsensical.

Studies of health care quality have been the domain of the medical researcher, who researches the health outcome effects of one treatment versus another. Medical researchers have rarely included the ownership status of the health care facility in health outcome comparisons. Economists, on the other hand, have focused on comparative
financial performance among facilities of differing ownership types and have usually assumed away health outcome variation, or have used a proxy variable as a quality control variable. Common proxy variables for quality in health economics literature include such blunt instruments as hospital accreditation status and medical school affiliation, staffing hours, and market characteristics such as per capita personal income levels. Virtually no health economics research so far has attempted to combine the two domains to study comparative outcome quality by ownership status.

From the economics literature, we know that for-profit health care facilities use different levels of inputs in patient care than their not-for-profit counterparts utilize. For example, Eskoz and Peddecord (1985) and Pattison and Katz (1983) found that for-profit hospitals used more profitable ancillary services. The medical literature points to variations in patient health outcomes when different inputs are employed in the health care process. However, the implication that outcome quality differs by ownership type has not been specifically tested, and should not be assumed. This paper seeks to remedy this gap in the literature by testing outcome quality variation among health care providers of different ownership status.

Patients are said to choose not-for-profit health care providers because of an anticipated higher quality of care given at not-for-profit institutions. This “trust signal” function of the not-for-profit organizational form may aid in mitigating information asymmetries in health care. After two statistical approaches to an empirical study of mortality rates among US dialysis patients, a related study in this paper explores the role of asymmetric information in dialysis treatment inputs.
Chapter 1 provides a discussion of theoretical foundations of quality and ownership status, as well as a description of previous research on possible disparities in quality of health care by ownership type. An intertemporal model of for-profit and not-for-profit managerial utility maximization is introduced in Chapter 2. The US renal dialysis industry is described in Chapter 3 along with a review of pertinent medical and economics literature. Chapter 4 presents an empirical project, whereby almost 200,000 renal dialysis patients nationwide are found, in multivariate regression analysis, to have significantly higher mortality rates at for-profit dialysis facilities. Chapter 5 repeats the analysis for a smaller sample, but uses propensity score methodology to create comparable for-profit and not-for-profit patient sub-samples before analyzing the effect of for-profit versus not-for-profit treatment. A modification to existing propensity score methodology is proposed as part of the empirical analysis in Chapter 5. Chapter 6 introduces a new data set to empirically test differences in treatment for high-knowledge and low-knowledge patients, and constitutes one of the first such investigations of the possible effects of asymmetric information in the health care industry. Discussion of results and conclusions follow in Chapter 7.
CHAPTER 1: QUALITY AND OWNERSHIP STATUS: THEORETICAL FOUNDATION

SECTION 1: FORMATION AND RETENTION OF THE NOT-FOR-PROFIT FORM

PART A: PROVISION OF PUBLIC GOODS

Why donors choose the not-for-profit form over the for-profit form for provision of the public good is straightforward; the donor does not wish to enrich stockholders of the firm, but rather prefers to provide the public good. In the absence of complete monitoring to ensure that the good or service is provided (to a third party or to the community at large) the not-for-profit form with its non-distribution constraint and lack of residual claimants for profits has an efficiency advantage over the for-profit form of organization for donative goods and services (Fama and Jensen, 1983b).

Why donors choose the not-for-profit form over public provision of the public good is not well-researched, however. Considering that governments can obtain tax funding rather than having to rely at least partially on charitable contributions, the not-for-profit form appears to be at a long-term funding disadvantage compared to government. However, as Hansmann (1980) and Frank and Salkever (1994) elucidate, government provision of a public good is most feasible for goods preferred by the majority of the populace, not just a small subset. Furthermore, since not-for-profit firms can be structured more narrowly, they may be more responsive to their donors’ intentions, less constrained by bureaucratic restrictions, and more affected by market discipline than government-provided goods.
Weisbrod (1977) proposed that not-for-profit organizations arise when quantity demanded of a public good varies in a population. The voting process results in a level of governmental provision of a public good that is considered inadequate by part of the population, at the current price of the good. When a segment of the unsatisfied population chooses to voluntarily contribute to provide the good, instead of free-riding, the not-for-profit form of organization is chosen.

Since religion is not sponsored by the government in the U.S., churches comprise the largest sub-sector of the not-for-profit sector (Rose-Ackerman, 1996). Many other not-for-profit organizations, from hospitals to universities, were originally founded by religious organizations. Given the particularly heterogeneous religious preferences of the U.S. populace, it is also not surprising that religious organizations in the U.S. are organized as not-for-profits. Nonetheless, to differ with Weisbrod, it is not particularly accurate to characterize religion as a public good.

Not-for-profit hospitals do not necessarily specialize in provision of public goods today. Modern hospitals do provide some collective goods, such as treatment and prevention of communicable diseases. In addition, not-for-profit hospitals in particular provide a substantial amount of uncompensated care for uninsured and underinsured patients. Lewin et al. (1988) and Shortell et al. (1986) showed that not-for-profit hospitals provided more uncompensated care as a percentage of revenues or costs than for-profit hospitals in the 1980’s, particularly in competitive markets. Modern healthcare facilities, however, receive most of their compensation from producing private services, and are thus commonly classified as “commercial not-for-profits.” This is in contrast to the situation in the early 20th century, when most not-for-profit hospitals were formed and established dominance in the health care industry. Then, as Eli Ginzberg (1991) reports, large urban not-for-profit hospitals funded largely by philanthropists were formed to
provided medical care to the poor. Since charitable provision of aid to the needy is often considered a public good, Weisbrod's public good argument aids in explaining the historical foundation of not-for-profit health care in the U.S.

PART B: CONTRACT FAILURE

Hansmann (1980) argues that public goods are only a subset of goods where not-for-profit provision is appropriate. Whenever a purchaser is unable to evaluate the quality of the good (even after purchase), contract failure arises and free-market provision of the good will be inefficient. Public goods markets lead often to contract failure because the purchaser is not necessarily the recipient and is unable to verify if the public good was produced or distributed at all. The profit non-distribution constraint of not-for-profit firms gives some assurance that the funding provided for the good will indeed be used to produce and distribute the good. More importantly, public goods are a subset of goods that are largely consumed by a third party. Charity, which need not be classified as a public good, falls into this category. Aside from the public component of altruism in aiding the needy, charitable goods are difficult to monitor due to third-party consumption, and are subject to contract failure. Thus, for organizations whose output is charitable goods and services, virtually none that rely substantially on donor funding are organized as for-profit firms.

Another type of contract failure arises when the purchaser of the good or service is the consumer, but is still unable to evaluate the delivery or quality of the good. Many cite modern healthcare as an example where some forms of treatment are so specialized that patients cannot judge the quality of care (where the degree of patient sophistication varies by the disease and condition of the patient). Not-for-profit status is said to function as a trust signal for patients seeking some guarantee that the firm will not exploit its information advantage. Hansmann points out that physicians in hospitals serve as
"sophisticated purchasing agents" for patients, thus alleviating any contract failure problems caused by the profit status of the hospital (1980, page 867). Physicians are paid by the patient, are not employed by the hospital, and therefore serve as quality control agents for the layman patient. Furthermore, Hansmann (1996) argues, for-profit hospitals are restricted from taking advantage of information asymmetries because they have to maintain their professional reputation. However, when physicians become part owners of the healthcare facility, the purchasing agent role may be altered as partial compensation for physicians comes from facility profits. The empirical portion in Chapter 5 presents evidence that physicians, just as Hansmann theorized, are less affected by financial incentives created by ownership structure than are non-physician staff at renal dialysis facilities.

Two more situations affecting hospitals' incentives to exploit information asymmetries stem from the form of payment for medical care. When the hospital is part of an HMO or when the patient's insurer pays prospectively, rather than on a cost basis, the hospital has an incentive to reduce costs of patient care. Due to explosive growth of HMO medical care in the 1980's and 1990's, and the implementation of Medicare's DRG (diagnostic related group) reimbursement system in the mid 1980's, these two situations are no longer special cases but constitute the majority of hospital care payment mechanisms. Thus, whereas asymmetric information used to be more of a theoretical footnote for the discussions of market failure in the hospital industry, it is now of central concern to those studying physician and hospital staff incentives and performance.

Frank and Salkever (1994) note that the US nursing home industry is one sector of health care where patients are particularly vulnerable, compared to the hospital industry, and contract failure is thus more likely in a free market. The same vulnerability could be ascribed to psychiatric patients in inpatient facilities. One might expect that the not-for-
profit organizations in these industries would have a competitive advantage if consumers considered not-for-profit status as a signal of lower incentive to take advantage of the consumer’s vulnerability. Notably, however, not-for-profit nursing homes and inpatient psychiatric facilities have a much smaller market share than for-profit facilities, whereas not-for-profit hospitals have a higher market share than for-profit hospitals. Thus, factors other than the “trust signal” role of not-for-profits appear to determine competitive advantage by ownership status in the health care industry.

Contract failure may therefore take two forms that are prevalent in the healthcare industry. First, provision of charitable goods funded by donors is virtually always the role of a not-for-profit organization. Of course, these charitable goods are provided by non-free-riding donors who prefer more of the charitable good than is provided by the public sector. Inability of donors to monitor closely the distribution of charity prompts them to select or create a not-for-profit organization. Thus, charitable donations explain the foundation of not-for-profit healthcare, but may not provide sufficient explanation for the retention of the not-for-profit form, as donor funding has declined to about 1% of total operating costs and 5% of capital expenditures in 1990 (Ginzberg, 1991), from around 25% of not-for-profit urban hospital income in 1940. The main reasons for the precipitous decline in medical philanthropy were the transformation early in the 20th century of hospital care, from treatment of the poor to fee-for-service care of the general population, the growth of employer-paid medical insurance, and the passage of Medicare and Medicaid in 1965, for medical care of the elderly and impoverished.

Although the passage of Medicare and Medicaid eliminated a source of contract failure by reducing further the role of donor funding, Medicare and Medicaid funding provided not-for-profit hospitals with a substantially stronger financial footing, and hospital expenditure (both not-for-profit and for-profit) subsequently underwent astonishing
expansion for about two decades. This lends support to Mark Pauly's belief that healthcare organizational structure is less dependent on managerial objectives than on the prevailing reimbursement environment (1987).

The second form of contract failure prevalent in health care is the patient's lack of information regarding the quality of his treatment. Asymmetric information is not a solid argument for the formation of not-for-profit healthcare facilities, because most not-for-profit hospitals were founded when treatment was simpler. Contract failure due to asymmetric information may be a factor in the retention of the not-for-profit form, as medical knowledge becomes increasingly inaccessible to the layman. On the other hand, in the hospital setting, most physicians are not employees of the facility and are able to mitigate the asymmetric information problem. Whether or not physicians are part owners, the "trust signal" function purportedly performed by the not-for-profit health care facility may be utilized by some knowing patients, but is likely to be a very minor determination of retention of not-for-profit status, compared to other constraints and incentives facing the supply side of health care.

PART C: BENEFITS TO PHYSICIANS

In the first two decades of the twentieth century, physicians did not enjoy high incomes, and in particular, did not receive large direct financial rewards for being on staff at large modern (and almost invariably not-for-profit) hospitals. Ginzberg (1991, page 180) reports, "Physicians eagerly sought staff appointments at prestigious hospitals where many of them donated the equivalent of two days of work per week for the privilege of admitting private patients. Young physicians served for years in the hospitals' clinics to earn the right to be considered for a staff appointment." Physicians may have benefited from the prestige afforded by the staff appointment at not-for-profit hospitals, by attracting more patients than a physician at a smaller proprietary facility. It can also be
surmised that physicians of some religious groups were instrumental in founding religious not-for-profit hospitals when discriminatory practices limited the physicians' opportunities for staff appointments elsewhere. Remuneration for physician services has improved greatly since then, due in part to successful efforts by the American Medical Association to reduce the supply of graduating medical students (Ginzberg, 1991).

Pauly and Redisch (1973) developed a model of not-for-profit organization whereby, ignoring prestige and leisure effects, staff physicians cooperate to maximize the sum of the money income of all staff members. This seminal paper showed that not-for-profit hospital status can be advantageous for the physician, supporting the claim that physicians themselves can be advocates for retention of the not-for-profit form. Pauly and Redisch's paper also provides insight regarding variation of quality among for-profit and not-for-profit providers. Their model predicts that imperfect cooperation among medical staff members will cause overproduction of hospital services, i.e. quality. Whether or not physicians influence facility quality, it is apparent that to ensure the long-term survival of the health facility, physicians are often granted certain privileges by the facility administration. In an article detailing not-for-profit hospital privileges granted to medical staffs, Sullivan (1992) describes several quasi-legal schemes, such as subsidized office space and recruitment bonuses, undertaken to accommodate doctors' wishes in order to ensure a steady flow of paying patients to the hospital. Hansmann adds (1980, page 867) that doctors in not-for-profit hospitals may have more influence on hospital administration; "For example, a doctor may be able to induce a nonprofit hospital more easily than a for-profit hospital to buy an expensive piece of equipment that will help him increase considerably the size and profitability of his practice, even though the equipment is not cost-justified." Of course, the ability of the hospital to accommodate physicians' equipment desires and of the physician to increase his fees depends to a great extent on the reimbursement scheme in operation.
Thus, it can be concluded that although benefits to physicians may have played a minor role in the founding for not-for-profit healthcare organizations, a more convincing statement is that physicians have a potentially large stake in the retention of not-for-profit healthcare. As Hansmann (1980, page 868) writes, "It is rather as if a foundation, tax-exempt and supported in part by public contributions, were to build office space and then lease it at cost, or less, to Wall Street law firms. One would not expect to see the lawyers in a hurry to have the foundation converted into an ordinary profit-making landlord." Hansmann’s analogy is most relevant for the hospital sector, where doctors practicing at for-profit facilities are not as often part owners of the facility. In other sectors of the healthcare industry, such as renal dialysis and inpatient psychiatric care, physicians at for-profit facilities are more often part owners, and therefore are more likely to personally benefit from for-profit facility ownership relative to for-profit ownership.

PART D: MARKET RESTRICTIONS

Entry restrictions and other regulations have affected the strength of the not-for-profit sector over the past century. In health care, the American Medical Association, a not-for-profit organization, was given the authority shortly after the turn of the twentieth century to establish and oversee state regulations for medical schools (Ginzberg, 1991). Partly as a consequence of elimination of many small proprietary hospitals early in the twentieth century, not-for-profit hospitals continue to provide the vast majority of medical teaching programs to this day.

After Medicare was established and it became clear that healthcare costs were consuming ever larger shares of public expenditure, Certificate of Need (CON) programs were introduced by states to limit new investment in health care facilities and equipment (Friedman et al. 1990). Although the intent of CON legislation was to promote
economies of scale and thereby reduce local health care expenditures, a significant effect of CON laws was to protect established hospitals from entry of potential competitors. In many cases, CON laws protected not-for-profit facilities from entry by for-profit hospitals. Many, including Sloan (1988) concluded that CON programs failed to reduce the growth of health care expenditures. By the mid-1980’s, a majority of states had removed CON regulations and as of 1989, only 15 states had CON programs regulating both hospital-based and freestanding dialysis facilities (Rettig and Levinsky, 1991). In the dialysis industry, Rettig and Levinsky show evidence that when CON regulations were removed in the mid-1980’s in some states, significant increases in numbers of for-profit dialysis facilities occurred.

For-profit entry into a few states where rate reviews or budget approval programs are in place has also been slow. These “all-payer” rate review states force hospitals to share in the costs of uncompensated care, which would normally burden a small subset of hospitals. DeLew et al. (1992) report that states with rate-setting regulations successfully controlled hospital cost growth. The regulatory method of rate-setting creates an unfavorable regulatory environment for potential for-profit entrants, according to Friedman et al. (1990).

Market restrictions, whether aimed toward limiting costs of capital expansion, restricting physician licensure, or regulating budgetary processes, have appeared to provide a competitive advantage for existing not-for-profit health care firms. Although the aim of such state regulation is often cost control, the inadvertent consequence, similar to rent control, is that the current market players -- often not-for-profit facilities -- are provided some protection from market entrants.
PART E: TAX EXEMPTION

Not-for-profit organizations are exempt from corporate income taxes and, in most cases, local property taxes. Also, charitable contributions to not-for-profit organizations are deductible, and not-for-profit firms may utilize tax-exempt bond financing. However, Hansmann (1980) and Fama and Jensen (1983b) note that historically, tax benefits conferred to not-for-profit organizations have followed the founding of the not-for-profit sector, not precipitated it. Hansmann concludes in his 1980 article that taxation benefits aid the retention of not-for-profit form, but are not an important determination in the founding of not-for-profit organizations. Later, in an article using 1975 data, Hansmann measured the effect of state and local tax exemption on the market share of not-for-profit firms in the health care and education sectors. He found that the value of the state and local tax exemption was positively correlated with not-for-profit market share (Hansmann, 1987). Despite this finding it can be argued that much of the century’s for-profit expansion in health care occurred after 1975.

Friedman et al. (1990) estimate the value of tax exemptions in the US not-for-profit hospital industry is about 5 to 6 percent of total expenses. Even though some authors dismiss the tax exemption’s importance in ensuring the survival of the not-for-profit firm, some local governments are beginning to use tax exempt status (and the threat of revocation) as a tool to influence not-for-profit performance. Specifically, some local governments are demanding *quid pro quo* for hospital property tax exemption, in the form of uncompensated care quotas or demonstrated community benefits.¹ Thus, local governments are recognizing that tax exemption affects not-for-profit firm viability.

Hansmann (1996) supports the revocation of tax exemption for not-for-profit hospitals because he sees little difference between the output of for-profit and not-for-profit hospitals. Hansmann comments further that not-for-profit hospitals are not likely to quickly disappear if taxation is implemented because hospitals have considerable embedded capital and can survive even when their market rate of return is below that of a for-profit hospital. Rose-Ackerman (1996) and Frank and Salkever (1994) argue that tax benefits continue to be granted to not-for-profit healthcare providers because it is recognized that not-for-profit healthcare may consistently provide more goods and services that benefit the community, such as uncompensated care, medical education, research, and so on. The motives of government officials to establish or revoke tax benefits for not-for-profits and to design appropriate reimbursement mechanisms, are explored in the next section.

SECTION 2: US HEALTH CARE: OBJECTIVES BY ACTOR

Many discussions of not-for-profit theory ignore individual incentives in the larger consideration of organizational status. However, it is the individual donor, the hospital CEO, the patient, the physician, the government administrator, and the insurance company executive who faces incentives, maximizes objectives, and makes decisions leading to the production of health care. This section analyses individual decision making by economic actors, and discusses each actor’s consideration of quality in his production or consumption decisions. It is not instructive to conclude that not-for-profit firms produce higher quality because quality is in their objective function. In Pauly and Redisch’s (1973, page 99) words, “Appropriate choice of the variables to enter the utility function can make almost any observed behavior consistent with utility maximization.” Besides the obvious problem of modeling objectives by pulling oneself up by the
bootstraps, the definition of the decision maker is unspecified. To whom does this utility function belong?

PART A: THE DONOR OR FOUNDER

The strongest rationale for foundation of not-for-profit health care facilities historically has been the provision of charity. Charitable organizations are usually organized as not-for-profits in order to mitigate agency problems inherent in the acquisition of residual claims by for-profit managers when the recipient of a charitable donation is a third party and monitoring of charity production is incomplete (Fama and Jensen (1983b) and Hansmann (1980)). If the donor wants to purchase charity for others, the not-for-profit form is commonly believed to be the efficient conduit for such a transaction, in that managerial incentives to appropriate the donation as profits are reduced by the nondistribution constraint.

Weisbrod (1977) postulates that donors establish not-for-profit organizations when government provision of the good is inadequate. This provides a possible explanation why a donor would not simply donate to a public institution, whose managers are bound by the same nondistribution constraint as not-for-profit managers. Rose-Ackerman (1996) lists a few other reasons why a donor would select the not-for-profit form over a public agency; the not-for-profits might be easier to monitor than a government organization by a donor, not-for-profits might provide a greater diversity of goods than a government agency (providing public goods for which quantity demanded is very homogenous), and not-for-profits might provide the best operating environment for ideologues -- unconstrained by public agency bureaucracy and from for-profit shareholder demands. In addition, donors' selection of the not-for-profit form is aided by the tax deduction for charitable donations.
Grundfest's (1996) discussion of conversion from the not-for-profit form to the for-profit form suggests another powerful incentive for donors to select the not-for-profit form. Public agencies are more capable than not-for-profits of withdrawing from sectors where public provision is no longer in high demand (Hansmann 1996). Not-for-profits face formidable federal and state legal obstacles when converting from not-for-profit to for-profit status. Usually, the assets of the not-for-profit foundation must be repaid to the government or left within the not-for-profit sector upon conversion. "...(T)he decision to invest capital in the nonprofit form is effectively an irrevocable election of an organizational form, at least as to the current value of that capital." (Grundfest 1996, page 273). Donors wishing to preserve their charitable funding therefore select the not-for-profit firm -- the organizational form with the greatest potential for applying the capital over time to its original intended source.

When donors fund high-quality rather than charity health care, it is often a good that has a *public* component, such as research toward a cure for a disease or construction of an expensive treatment facility to aid those who would otherwise be unable to receive specialized care. Philanthropy, however, is now a small fraction of most not-for-profit health care organizations' revenues, so the motivation for founding not-for-profit institutions for charitable purposes is reduced. Even as early as 1964, philanthropy accounted for only 3.2% of private expenditures on health and medical services (Census Bureau, 1965). Donors played a pivotal role in the establishment of not-for-profit health care in the early part of the 20th century in the US. Other actors however, have been largely responsible for the continued strength of the not-for-profit sector.
PART B: THE BOARD OF DIRECTORS

One of the legal requirements for not-for-profit status is that an organization form a board of directors or trustees to oversee the management of the institution. Boards may often be composed of major donors to the organization. Fama and Jensen (1983a) point out that board members are usually outside agents, because a board composed of internal agents or outsiders chosen by internal agents is too vulnerable to expropriation of donations. Similarly, Callen and Falk (1993) hypothesized that outside board members perform a monitoring role whereas insider board members are more inclined to direct perquisites toward their own domain, causing the firm to have higher costs. Callen and Falk tested this hypothesis with data from 72 health charities, finding no evidence that the makeup of the board has an influence on the technical efficiency of the firm.

Molinari et al. (1995) also studied the influence of insider (medical staff) participation on boards of directors and found that hospitals with medical staff on boards performed better, i.e. had higher net operating margins. Molinari et al. conclude that their findings support a "managerialist" point-of-view -- insiders provide essential knowledge for efficient operation -- rather than the "agency" view of Fama and Jensen (1983 a and b). Molinari et al. did not discuss the possibility that the medical staff's desire to increase hospital profitability might align very closely with their desire to increase patient flow to their own practice, or to increase the prestige of their affiliation with the hospital. Bennett and DiLorenzo (1989) note that unlike boards of directors of private corporations, where board members are elected by shareholders, not-for-profit organization board members are appointed by those who are active in the organization.

Several researchers have noted that board members gain prestige as well as important networking opportunities by serving on boards. Middleton (1987, page 146) reports, "Their reasons for volunteering may also include heightening their own visibility in the
community, acquiring an opportunity to mobilize the resources of many organizations on behalf of policies and institutions they favor, and increasing their social and professional connections... Philanthropic work has become a career expectation for managers seeking to advance within corporations.” Thus, altruism may play little or even no part in a potential board member’s decision to serve on a not-for-profit board.

At the time of founding, a board may be entirely composed of donors, and considerable power, according to Perrow’s (1963) study of hospital boards, is wielded by trustees early in the life of the not-for-profit organization. Later, especially if donor financing is minimal, boards of directors perform a less supervisory role. Several studies, according to Middleton (1987), agree that not-for-profit boards in practice usually ratify policy suggested by the organization’s management, rather than propose policy changes themselves. Even though the board is the legal ultimate authority, in reality the management controls the flow of information to the board for decision making and the management carries out the recommended policy changes itself.

The give and take of long-term policy negotiations between the board and management varies, of course, by the make-up of the board and the nature of the organization. However, boards of directors all exercise important decision making authority when they hire new executive managers. In the health care industry where donor funding is minimal, the board may simply hire a CEO who appears to be the best candidate for furthering the interests of the board. Not involved in the day-to-day decisions of a not-for-profit organization, the board’s contribution to the organization is strongest in the screening process of potential managers, providing a link to external community decision-makers, and ratifying management policies regarding long-term expansion or contraction of facilities.
PART C: THE MANAGER

Regardless of how the not-for-profit manager passed the hurdle of being selected to his position, he now faces incentives uniquely different from a CEO or high-level manager of a for-profit corporation. A CEO of a for-profit firm is a shareholder, and his actions directly and almost immediately affect his wealth. The CEO of the not-for-profit firm, however, has no such direct reward for superior management, due to the non-distribution constraint. It is also instructive to note some of the incentives facing for-profit and not-for-profit managers that are similar. Both managers are fired if their performance is poor -- that is, if the long-term viability of the organization is threatened or if the board members' interests are not being served. Also, both managers may receive future salary enhancement if current job performance is good -- either from the same employer or from another employer. However, a not-for-profit manager's capitalized salary might include recognition for achievements that depart from profit-maximization.

Alchian and Demsetz discussed incentives to shirk facing not-for-profit managers in their 1972 paper. A related article by Alchian and Kessel (1962) discusses similar incentives facing publicly regulated monopolists, who operate within the same nondistribution constraint as not-for-profit managers. Although Alchian and Kessel's notions of desired on-the-job perquisites (i.e. "pretty secretaries") are dated, their central point remains powerful three decades later: Not-for-profit organization provides insufficient incentive for managers to work efficiently, and residual "profits" which would otherwise go to shareholders are consumed on-the-job by utility-maximizing managers as higher costs. Even if managers of not-for-profit hospitals value community health and other lofty ideals, it is also likely that they value also a more pleasant work environment (Young 1987). Since job perquisites are consumed rather than increasing wealth via stock earnings, the compensation system for a not-for-profit manager is inefficient, as cash income always beats in-kind transfers of equal value for utility maximization (Pauly 1987).
Hansmann (1996) contends that the much-maligned operating inefficiency of not-for-profits is usually exaggerated. He notes that for-profit managers commonly don’t share significantly in the firm’s residual earnings, and not-for-profit managers strive for cost-minimization in order to see their firm grow and prosper. Interestingly and perhaps to Hansmann’s credit, empirical studies of comparative hospital costs have not revealed lower costs at for-profit hospitals (Becker and Sloan 1985, Eskoz and Peddecord 1985, Grannemann, Brown and Pauly 1986, Pattison and Katz 1983, and Renn et al. 1985). This may very well be an artifact of the cost-based reimbursement system in place at the time of those studies, when profits were enhanced by maximizing services to patients and receiving reimbursement from third-party payers. Medicare reimbursements changed to a prospective payment in 1984, whereby patients’ diseases and severity levels are identified at admission, then reimbursed at a certain level (regardless of inputs utilized). This prospective reimbursement system, similar to prospective capitated payments for HMO patient care, greatly reduces incentives to “overtreat” in order to improve profits. Recent empirical studies with post-1984 data are bound to show a change in comparative cost-control, and if Alchian and Demsetz are correct, we expect for-profit hospitals to have lower costs due to clearer incentives facing managers. Efficiency is not defined over output with unknown differences in quality, however, so it is difficult to claim that not-for-profit managers will act “inefficiently” if they are raising costs by improving the quality of the product. Nevertheless if not-for-profit managers value a pleasant work environment in addition to producing a higher quality product, it can be predicted that not-for-profit managers act inefficiently in comparison to their for-profit counterparts.

When not-for-profit organizations are said to have different objectives than for-profit organizations, without specifically identifying the utility maximizing individual, the manager comes closest to the utility maximizer role. In the health care industry, his
decisions, within a loose framework of board of directors’ wishes, affect capital investments, medical staff satisfaction, and production of health by non-medical staff. Although several authors contend that managers of not-for-profit organizations are “more interested in providing high-quality service and less interested in financial rewards than are most individuals” (Hansmann 1980, page 876), one cannot conclude that not-for-profit status guarantees selection of managers valuing high quality. This is because the not-for-profit form may merely attract managers who value job perquisites more than profit-maximizing managers do. High quality is just one perquisite among many, including a more relaxed work environment, congenial coworkers, or even a sense of nonpecuniary noblesse oblige.

Although past studies measuring hospital industry costs have found no significant evidence of comparative efficiency, a few studies of managerial behavior have implied that the non-distribution constraint has measurable effects on managerial decision-making. In his study of hospital management, Clarkson (1972) found some empirical support for four property rights implications: 1) Rules governing not-for-profit management behavior are more extensive than owners’ rules for for-profit managers because for-profit manager behavior is partially controlled by linking their income to residual wealth. 2) Not-for-profit managerial effort is directed more toward acquiring on-the-job perquisites than for-profit managerial effort. 3) Not-for-profit managers base decisions on information that is easier to obtain than information used by for-profits in their decisions. 4) Not-for-profit managers vary their input mix more than for-profit managers in producing the same output. In a more recent study, Oswald and Gardiner (1994) purported to show expense preference among not-for-profit hospital managers indicating higher perquisite consumption than their for-profit counterparts. However, the paper’s main results could be interpreted as revealing a higher preoccupation with quality of care at not-for-profit facilities (for example, they report average full-time equivalent employees per occupied bed is higher for not-for-profits).
If not-for-profit managers are less interested in financial rewards, empirical studies should show that not-for-profit managers receive lower salaries than their for-profit counterparts. This hypothesis is supported by Preston (1989) in her study of for-profit and not-for-profit sector compensation. She found that average nonprofit wages were 20% lower for managers and 5% lower for clerical and sales workers, even after adjusting for implicit prices of fringe benefits (including job autonomy and flexibility), human capital, industry type and occupation type. Preston’s theoretical model of labor donations predicts that the negative wage differential results from not-for-profit workers accepting a lower wage in order to generate positive social externalities in their employment. This hypothesis was not proven by her findings, due to the inconclusive results of self-selectivity bias tests. That is, the not-for-profit worker may simply be of lower quality and may self-select into the less intense not-for-profit sector.

Young (1987) notes that compensation is likely to be comparable across ownership types in the rank and file workers, but that managerial salaries will differ. Preston’s results support this claim, and she writes (1989, page 449), “...results...support the prediction that workers less closely tied to social benefit provision in the organization are less likely to donate wages to nonprofit firms.” Why workers would be less tied to social benefit provision -- even as they carry out the socially beneficial policies chosen by the manager -- is left unexplained. Young adds an additional angle to consider in top management compensation differentials, reporting that nonprofit agencies pay premium salaries for top administrators in order to maintain their reputation for the highest quality in the profession.
Therefore, a preoccupation with quality in not-for-profit organizations may on one hand justify paying lower salaries to managers and workers (who seek out high-quality firms in which to work), and on the other hand may justify paying top dollar for the highest level administrators. Preston’s article appears to support the former notion, while Young’s article shows that not-for-profit hospital administrators’ salaries top their profession -- supporting the latter justification. Even rank and file hospital and nursing home staff, according to Oswald and Gardiner (1994) and Holtmann and Idson (1991) are paid more at not-for-profit facilities.

Alchian and Demsetz’ (1972) view that not-for-profit managers face insufficient incentive to operate efficiently is supported in part by Clarkson (1972) and Oswald and Gardiner (1994). However, others contend that not-for-profit’s production of a higher quality product offsets some, if not all of the predicted inefficiency (Pauly 1987). The defining factor appears to be in the screening process and self-selection of managers into their profession. The not-for-profit sector attracts either quality-obsessed ideologues (Rose-Ackerman 1996, Hansmann 1980), perquisite-consuming shirkers (Bennett and DiLorenzo 1989), or managers valuing some unknown combination of high quality and a pleasant workplace. Testing of comparative quality produced by for-profit and not-for-profit firms would aid in determining the nature of differing managerial objectives.

PART D: THE PHYSICIAN

If objectives of donors, board members and managers are difficult to elucidate, physician objectives can appear virtually unapproachable. Conflicting incentives, varying with remuneration form, face the physician, and widely differing physician objectives are modeled in the literature. Pauly and Redisch (1973), on one end of the spectrum, model physicians’ efforts to maximize income and find that higher quality at not-for-profit hospitals may be the result of non-cooperative behavior among medical staff members
who dominate hospital administration. Young (1987, page 173) lists several other authors agreeing with this premise, stating “nonprofits tend to be domains in which professions and professional thinking dominate, and agendas are shaped by a quest for professional excellence and prestige... (not-for-profit) hospitals compete for patients on the basis of quality reputation as perceived by physicians who seek affiliation.” On the other end of the spectrum, Arrow (1963 and 1986) and Hansmann (1980) grant a socially ethical role for doctors whereby they act in the patients' behalf. “Clearly, there is a whole world of rewards and penalties in social rather than monetary form. Professional responsibility is clearly enforced in good measure by systems of ethics, internalized during the education process and enforced in some measure by formal punishments and more broadly by reputations” (Arrow, 1986, page 1194).

Mooney and Ryan (1993) lament the lack of clear separation of patients' and physicians' utility functions in their paper discussing agency theory in health care. Usually it is assumed that the principal and agent utility functions are independent, leading agency theorists to devise complicated incentive-compatible fee structures in markets involving asymmetric information. However, in health care, physicians are said to incorporate their patient's utility into their own utility. If such incorporation is perfect, complicated fee structures are unnecessary. If incorporation is incomplete, physician actions can be predicted to vary according to the remuneration structure. Indeed, many studies note changes in treatment utilization rates when fee structures are changed. Examples are Hughes and Yule (1992), Melnick and Zwanziger (1988), and Custer et al. (1990).

The conditions under which a physician can be expected to provide high quality patient care vary widely. 1) The physician may be more altruistically motivated. 2) The physician may be better trained than his peers. 3) The physician may note that a patient
is particularly knowledgeable (Mooney and Ryan, 1993). 4) The physician’s compensation may be closely linked to service provided (for example, fee-for-service). 5) High-quality treatment may result in lower malpractice insurance premiums and fewer lawsuits. 6) A reputation for high-quality care may attract more patients to his practice. 7) A reputation for high-quality care may grant him a certain amount of prestige and allow him to have staff privileges at high-quality hospitals. 8) High-quality care prolongs the life of his patients, especially those with chronic diseases, and prolongs the receipt of fees for treatment.

Among the conditions outlined above, only the first presupposes an ethical constraint on treatment behavior. A myriad of external influences can also positively influence quality of care. Since evidence appears to reject the hypothesis that physicians are motivated solely by ethical considerations in choosing treatments for their patients, agency theory can be of use in determining physician reactions to different remuneration schemes and constraints on behavior resulting from the ownership status of the facility in which they practice. Hospital studies with cost-based reimbursement era data by Eskoz and Peddecord (1985) and Pattison and Katz (1983) showed that for-profit hospitals utilized ancillary services more than not-for-profit hospitals, prompting concerns about overutilization. Similarly, McCue et al. (1993) expressed concern that patients at for-profit psychiatric hospitals had longer stays and were possibly ‘overtreated.’ Physicians have the authority to admit patients, order ancillary services and discharge patients, so some of the blame for possible over- or underutilization must rest on the physician, rather than on the facility’s administration. Thus, ownership status of the facility, and in particular any differences in physician compensation stemming from ownership status, may affect physicians’ choices of health care inputs for the patients.
PART E: THE GOVERNMENT

It is presumed the objectives of the government or specifically of the government official are to promote social welfare and get reelected. The government pursues these objectives by producing public goods, intervening in markets where goods have production or consumption externalities, and correcting other market failures in the presence of asymmetric information and uncertainty.

Market inefficiencies in health care are abundant, prompting a multi-faceted response from government agencies. Federal and local governments are involved in subsidizing several health goods that have positive externalities, such as medical education and research, immunization, treatment of infectious diseases, and treatment of the indigent and uninsured for reduction of misery in the general population. The market failure cited most often in health care is asymmetric information. Several regulatory approaches established in the early 20th century address information asymmetries; physician licensure, health care facility accreditation, and malpractice liability law. Governments, however, have rarely sought to bridge information gaps by disseminating information on provider quality to health care consumers. Lastly, the Federal government is involved in mitigating uncertainty and moral hazard problems in health care by providing Medicaid and universal Medicare insurance coverage for the poor or elderly, who would otherwise be selected out of the medical insurance market due to their high probability of costly care.

The market inefficiencies described above imply two possible motives for government intervention to subsidize not-for-profit organizations; the public good motive and the asymmetric information motive. Intervention may at times take the form of entry barriers to for-profit facilities or explicit subsidies for not-for-profit capital construction. The
most significant intervention influencing not-for-profit viability is exemption from federal and local taxes.

I. Public Good Motive
Very early in the twentieth century, not-for-profit hospitals were exempted from paying taxes to provide some compensation for treatment of large numbers of indigent patients. Not-for-profit hospitals were important providers of charity care to the urban poor. At that time, patients with infectious diseases formed the bulk of the patient population, so charity care was not only important for altruistic purposes, but also for reducing disease in the wealthier population as well.

The inception of Medicaid in 1965 reduced the charitable motivation for not-for-profit and public charitable care of uninsured patients. In addition, by the latter part of the twentieth century hospital care was no longer concentrated on treatment of conditions resulting from infectious diseases, so much of the public component of health care was reduced. A few public good justifications for subsidy of not-for-profit and public providers remain today: Not-for-profit and public health care organizations perform the bulk of the nation’s medical education and research. For-profit health care participation in medical education and research per facility is far below that of the other ownership types. Additionally, most studies of uncompensated care report that not-for-profit facilities are providing more unreimbursed care to uninsured patients than for-profit facilities.

Even though not-for-profit organizations are dominant providers of medical education and research, many individual not-for-profit facilities provide no medical education for health professionals or research, so a blanket assumption that not-for-profit tax exemption is justified on the basis of provision of public goods is on shaky ground. So too, is the assumption that not-for-profit health care facilities provide more “community benefit”
than their for-profit counterparts. Community benefit is achieved by providing services that not only satisfy patient needs, but also provide a public good by serving the local community as well. An example would be a hospital offering subsidized flu shots for senior citizens at the local mall. One could argue that this service is a form of marketing and thus provides a net benefit to the hospital, not to the community. Perhaps a far stronger contribution to community benefit is from providing not services with a public good component, but private goods efficiently -- higher quality care or lower prices. Thus, not-for-profit advocates would do well to research pricing and quality when calculating comparative social benefits provided by different ownership types.

II. Asymmetric Information Motive

Some advocates of not-for-profit health care attribute not-for-profit tax exemption to a deliberate regulatory strategy to increase quality of care. However, even if not-for-profit providers' objectives lead them to produce a higher quality product, this approach to regulatory quality is as indirect as physician licensure and hospital accreditation. Certainly such indirect methods are cheaper than monitoring quality of care patient by patient. But ultimately, the asymmetric information problem is not entirely solved by quality monitoring in the form of inspections and accreditation procedures because what is monitored are variables most easily measured; physical characteristics of the building, staff education and other health care inputs.

Although promotion of not-for-profit care is another indirect attempt to correct for asymmetric information by improving quality of care, it may be a method nevertheless more likely to enhance quality variables that are hard to measure (patient satisfaction and health outcomes) normally neglected by government programs to monitor and enforce quality. In other words, the managerial behaviors that not-for-profit organization is purported to encourage – devotion to service and high quality -- may convince governments to support not-for-profit health care with tax subsidies in order to improve
quality of care and eliminate problems resulting from asymmetric information. Whether intentionally or not, governments' exemption of not-for-profit health care organizations from taxes and related subsidies perpetuates the strength of the not-for-profit sector in US health care. The argument that not-for-profit facilities supply more public goods to the community is true for the industry as a whole (in particular, for medical education and research, and proportionally more uncompensated care), but not necessarily for individual facilities. It is this lack of individual contribution to the public by some not-for-profit facilities that has prompted some local tax authorities to challenge not-for-profit tax exemptions.

Others may argue that government subsidy of not-for-profit health care arises from the government's efforts to control quality problems arising from asymmetries of information. The supposition that governments subsidize not-for-profit health care in order to maintain quality depends on the critical and unproven hypothesis that not-for-profit managers produce a higher quality product and do not merely consume on-the-job perquisites. Furthermore, the government is rarely endorsed by voters to increase an industry's product quality, except in cases involving externalities.

Finally, if provision of medical education, research, and uncompensated care -- as well as asymmetries of information -- are of such a concern to governments, why isn't greater emphasis placed in direct funding of public health care facilities? Medicare and Medicaid funding reduced the government's role as a direct provider, allowing elderly and indigent patients to choose not-for-profit and for-profit facilities for their care. The number of public health care facilities declined notably throughout the 1970s and 1980s, and was largely replaced by growing numbers of for-profit facilities, especially chain hospitals.
With Medicare and Medicaid programs in place, financing of medical care by federal and state governments changed to a voucher system rather than direct provision, and the government essentially subcontracted care. This movement away from direct provision is discussed by Salamon (1987), who notes that support for the not-for-profit sector allows the government to reconcile the public's "hostility to the governmental apparatus" and the desire of the government to increase social welfare. Thus, promotion of not-for-profit health care may be a way for the government to maximize quality and quantity of health care in the community without enlarging the government payroll or sacrificing consumer choice.

If government objectives have appeared to focus on any one aspect of health care since 1965, the likely candidate would not be quality, but costs. State Certificate of Need programs popular in the 1970's and early 1980's were designed to slow expansion in an industry seen as over-capitalized. All-payer rate review regulations in some states were somewhat more successful at restricting health care costs during those decades as well. By the 1980's, though, costs of medical care had increased enough to convince Medicare to replace the inflationary cost-based reimbursement system with the Diagnostic Related Group (DRG) prospective-price reimbursement system. This change produced almost immediate results in the industry -- not only were patients coded to be "much sicker" upon admittance to hospitals to ensure maximum reimbursement from Medicare, but also utilization decreased dramatically. For example, patient lengths of stay decreased significantly (Melnick and Zwanziger, 1988). This important change in financing, affecting hospital costs and utilization, may eventually have an effect on the quality of patient care and patient health outcomes.
PART F: PATIENTS

Patients in the health care industry often differ from consumers of other products because of the presence of third-party payers. Most often, the price the patient pays for health care at one facility or another is identical (zero or a negligible percentage of the total bill as a co-payment). Decisions by the patient regarding prices paid for medical care and quality expected are made in advance of actual purchases. That is, an individual insured at his place of employment chooses an insurer or health maintenance organization based on price and perceived quality of care, or accepts government financing (if over age 65 or indigent). At a later date, the individual falls ill and chooses a facility or doctor for treatment, but does not face further price decisions because the insurer pays according to the pre-arranged agreement.

At the point of consumption, the patient has relatively more choice regarding quality, as several physicians or facilities are usually funded by insurance or Medicare/Medicaid. One would expect *a priori* that patients automatically seek the highest quality provider, since price is no longer a relevant issue. However, patients are often incapacitated or not knowledgeable enough to judge the quality of the provider's product before or even after consumption of the product. Patients differ considerably in their ability to judge quality, so at least a portion of the population is unaware of quality of care. For these patients, the choice of provider may be determined by amenities such as location and physical attributes of the facility. Another factor contributing to patient heterogeneity is varying personal discount rates; some patients have a low personal discount rate and a strong desire to remain in good health over a long period, while other patients with higher discount rates are willing to trade survival probability for immediate pain relief or less aggressive treatment of their illness. Patients with high marginal rates of discount may choose facilities similar to those chosen by the patients unaware of quality -- facilities with convenient locations and attractive appearances.
Not-for-profit status may serve as a signal of higher quality in service industries where information on quality is not readily available to the consumer. This trust signal from a not-for-profit organization provides a proxy for patients to evaluate prior to treatment. Hansmann (1980) explains that it is not necessary for all consumers to be aware of an organization's ownership status in order for a not-for-profit to use its trust signal status successfully. There will also be a number of patients who are able to judge quality through research, personal connections, and so forth. For these particularly knowledgeable patients, a provider's ownership status is unimportant next to actual evidence of clinical quality. Thus, the proportion of patients unaware of quality issues but aware of ownership status -- and choosing their provider based on that status -- is likely to be small.

A patient's utility is affected by his quality of health, length of life, and amenities consumed in the course of treatment. The patient's choice of provider does not depend merely on the quality of clinical care. The empirical content of this dissertation measures mortality as the clinical quality variable, and admittedly makes no attempt to measure total patient utility by including amenity variables such as provider location in the empirical analysis. However, for at least a subset of the patient population, health and longevity are dominant variables in utility, and some patients are consequently very aware of the quality of their care. If not-for-profit facilities offer higher quality care because they attract managers or physicians less influenced by cash incentives, one would expect knowledgeable patients to deliberately choose not-for-profit facilities when a choice is possible.
PART G: OBJECTIVES BY ACTOR: CONCLUSION

Examining objectives of individuals in the provision of health care reveals many wordy discussions, but few refutable implications. Certainly, it is clear that not-for-profit health care played a critical role when public and charitable goods were a majority of its output prior to the establishment of Medicare and Medicaid and widespread employer-based health care insurance. Then, the motives for donors, boards of directors, and government officials to promote not-for-profit health care were straightforward. Presently, increasing numbers of uninsured individuals are seeking care at not-for-profit hospitals as public hospitals decline in number, but the volume of charity care at most hospitals is not comparable to the large number of charity patients treated at hospitals a century ago. As a justification for continued subsidy of not-for-profit health care, some now assert that not-for-profit organizations provide comparatively more benefits to the community and provide a higher quality of care to their patients, due to their differing objectives from strict profit maximization.

Government motives for promoting not-for-profit care now sometimes rest on the assertion that the presence of not-for-profit care in a community elevates quality of patient care. Patient motives for choosing not-for-profit providers over for-profit providers rely on the assumption that not-for-profit facilities provide better care, or their not-for-profit status provides at the very least a signal of higher quality care. Physician motives for choosing to be on staff at not-for-profit facilities rather than for-profit facilities depend on compensation systems in place. Differences in physician compensation structures exist in for-profit and not-for-profit organizations, and managers have some influence, especially in for-profit facilities, to alter physician compensation structures in order to encourage quality-enhancing or cost-controlling physician behavior.
However, the assertion that not-for-profit health care facilities provide higher quality care because they are motivated to do so has not been rigorously tested. As discussed in Part C. above, Armen Alchian and colleagues provided perhaps the strongest implication for testing in his examination of not-for-profit managerial objectives. Alchian hypothesized that the not-for-profit organizational form encourages managers to shirk and consume cost-increasing on-the-job perquisites, because managerial effort is not as closely linked to cash income as in for-profit firms. Expressed in a more positive light, Preston adds that the not-for-profit organization is likely to attract managers who are less motivated by cash incentives and more motivated by the opportunity to provide social benefits. Note that provision of social benefit is an on-the-job perquisite, so Preston’s research focused on a subset of Alchian’s hypothesis. Providing a higher-quality product than that which would maximize profits is another possible on-the-job perquisite for not-for-profit managers, and thus constitutes another subset of Alchian’s hypothesis.

A test of Alchian’s hypothesis is straightforward: One must compare costs of producing output by the two ownership types, while holding the quality of the output constant across the types. Many studies in health economics have purported to compare “efficiency” of for-profit and not-for-profit health care facilities, but have neglected to measure quality of output; instead assuming that no quality differences existed or that input measures of quality were sufficient controls. Rosko et al. (1995) provide the rare exception. In their study of nursing home efficiency, they utilized data envelopment analysis methodology to compare costs while holding three outcome quality controls constant; pressure ulcer rate, catheter use rate, and use of restraints. They found that although not-for-profit nursing homes produced higher quality outcomes overall, for-profit nursing homes were more efficient, holding quality constant. However, in the nursing home industry, many patients self-pay, and previous studies describe the not-for-profit nursing home sector as occupants of a high-price, high-quality market niche. Thus, results of the Rosko et al. study may be interpreted as mere confirmation that not-for-
profits deliver a different product and cannot be compared in efficiency with for-profit nursing homes. Other health care sectors such as hospitals and dialysis centers, for which price is fixed by Medicare or other third-party payers, are better candidates for cross-ownership comparisons of efficiency.

The hypothesis that not-for-profit managers consume more on-the-job perquisites does not posit any preference on the part of the manager for increasing perquisites that accrue just to the manager himself, for producing more output beneficial to the health care facility's community, or for producing output of a higher quality. This paper assumes that Alchian's hypothesis is correct in that managers of for-profit firms will pursue profit-maximizing opportunities more aggressively than their not-for-profit counterparts will. If not-for-profit facilities are indeed producing output of a higher quality when reimbursement is determined by a third party, it can be concluded that not-for-profit managers are not spending all the financial cushion created by preferential tax treatment on personal perquisites, and at least some of the financial cushion accrues to the patient of the facility. This result offers assurance that individual decision making by patients, government officials, and physicians in health care is indeed rational. Patients seeking higher quality base their choice of provider on ownership status, governments subsidize not-for-profits in order to elevate quality or mitigate asymmetric information problems, and physicians choose to affiliate with not-for-profits to provide higher quality care even though compensation may be lower.

The theoretical model in the next chapter shows that over time, the existence of the non-distribution constraint alone induces not-for-profit managers to offer higher quality care, even though for-profit and not-for-profit managers may be equally altruistic, valuing quality of care identically.
SECTION 3: EMPIRICAL EVIDENCE OF QUALITY DIFFERENCES BY OWNERSHIP TYPE

Since the contention exists that not-for-profit firms produce a higher quality product than their for-profit counterparts, it is a worthwhile exercise to examine the medical literature to determine if indeed a disparity in quality exists across ownership types.

PART A: LONG-TERM CARE

Studies of the nursing home sector suggest that for-profit nursing homes use different inputs in the production of long-term care (Davis, 1991, Greene and Monahan, 1981, and Zinn, Aaronson and Rosko, 1993). In one of the few studies focusing on outcome measures of quality, Aaronson, Zinn and Rosko (1994) indicated that quality of care differs by ownership structure in the nursing home industry. However, in the nursing home industry, a significant portion of patients self-pay, and previous research describes the not-for-profit nursing home sector as occupants of a high-price, high-quality market niche (Cohen and Dubay, 1990). Thus, results of the Aaronson, Zinn and Rosko study may be interpreted as mere confirmation that not-for-profits deliver a different product for a different price. Other health care sectors such as hospitals and dialysis centers, for which prices are often fixed by Medicare or other third-party payers, are better candidates for cross-ownership comparisons of quality and efficiency.

PART B: GENERAL ACUTE CARE HOSPITALS

After reviewing studies of hospital performance by ownership, several researchers have come to the conclusion that quality does not differ significantly by ownership type (Gray, 1986, Frank and Salkever, 1994). Comparisons of outcome quality among hospitals are difficult due to the large number of products hospitals produce. Outcomes produced by
one department for one type of illness may be very different from the outcomes produced for another department. The best quality comparison would take clinical outcomes by illness type, adjusted for severity of the disease and patient case-mix characteristics, and then compare survival rates across hospitals. The enormity of this task has discouraged researchers from focusing on outcomes. Instead, researchers have often measured inputs as indicators of quality. Outcomes and patient satisfaction research have recently undergone a remarkable transformation, due to significant improvements in computing capacity. We expect that future comparative studies of hospital quality by ownership will focus much more on actual clinical outcomes.

A second reason for the lack of significant results in comparative quality studies across ownership types is the change in reimbursement systems from cost-based to prospective-price financing. Prior to the mid-1980's, hospitals received payment by third-party payers largely on the basis of fee-for-service. Thus, hospitals were encouraged to provide premium care in the form of extensive diagnostic treatment, long hospital stays, and so on. Any firm that pursued profits aggressively found it in their best financial interest to provide the highest quality care at the highest cost. In their study of services provided by for-profit and not-for-profit hospitals, Pattison and Katz (1983) and Eskoz and Peddecord (1985) found that for-profit hospitals provided more profitably reimbursed ancillary services to patients than not-for-profit hospitals. Certainly, the distinctive incentives created by cost-based reimbursement encouraged high cost care which may have resulted in high quality care.

Now, however, prospective price reimbursement systems dominate hospital care financing. Not surprisingly, immediately after Medicare initiated its prospective price Diagnostic Related Group financing in the mid-1980's, average lengths of stay for patients declined (Melnick and Zwanziger, 1988). High cost care is still with us, but
hospitals are now competing for managed care contracts on the basis of price, and incentives now exist to reduce costs. It is in this new reimbursement environment that we may see distinct quality differences across hospitals arise. Care should be taken, when viewing comparative quality studies, to note whether the financing of the care was fee-for-service or prospective price.

PART C: QUALITY DIFFERENCES: CONCLUSION

For the above reasons, it cannot be concluded from previous research that a significant quality difference exists by ownership type in the hospital industry. Some research points to a quality gap in the nursing home industry, but this may be attributable to the existence of self-pay patients and the availability of a quality decision by patients based on price. The renal dialysis industry, introduced in Chapter 3, offers an ideal alternative laboratory for testing comparative quality across ownership types, due to the homogeneity of patients and treatment protocol and due to its financing, which is almost solely prospective-price.
CHAPTER 2: HEALTH CARE QUALITY AND UTILITY MAXIMIZING MANAGERS

SECTION 1: AN INTERTEMPORAL MODEL OF UTILITY MAXIMIZATION BY HEALTH CARE FACILITY MANAGER/OWNERS

As a formalization of managerial objectives discussed in Chapter 1, this chapter examines the effect of the non-distribution constraint on not-for-profit managerial choice of quality level over time. It is found that when at least some of the patients choose their provider on the basis of quality, the existence of the non-distribution constraint causes not-for-profit managers to select a higher level of patient quality of care than for-profit manager/owners choose. This result holds even though for-profit and not-for-profit providers are assumed to obtain identical amounts of utility from quality of care provided to their patients. This conclusion indicates that at least some of the slack created by preferential tax treatment for not-for-profit firms is used to produce a higher quality product, and therefore not all of the perquisite-consuming behavior predicted by Alchian and Demsetz (1972) is enjoyed solely by the manager.

The following model is a departure from many other models of not-for-profit/for-profit objective functions. It is not assumed that not-for-profit managers seek to break even, since many instances have been observed where not-for-profits have consistently maintained profits and have used their profits in subsequent capital expansion. Nor is it assumed that not-for-profit firm managers derive utility from quality (or number of patients, etc.) more than their for-profit counterparts. Here, identical objective functions are modeled, with only one very fundamental difference between the two ownership
forms: The not-for-profit managers take home less cash income from profits than for-profit managers.²

The utility-maximizing manager seeks to maximize a combination of cash income and quality. The cash income is modeled as a percentage of profits generated by the firm. For-profit managers receive more cash income tied to the firm’s profits than not-for-profit managers. Not-for-profit managers do not explicitly receive cash income as a percentage of profits, but income linked to profitable performance does accrue to not-for-profit managers over time in the form of higher salaries.

Managers maximize \[ \int_0^\infty e^{-n} \{ \alpha [P_x - C(u)] + f(u) \} dt, \] such that \[ x' = s(u) + i(u) + E - ax. \]

where \( \alpha \) is percentage of profits that managers receive as cash:

\[ 0 \leq \alpha_{\text{NFP}} < \alpha_{\text{FP}} \leq 1. \]

\( P \) is the reimbursement per treatment (or patient) that facilities receive.

\( x \) is the firm’s number of patients or treatments.

\( u \) is quality of care.

\( C(u) \) are costs, dependent on quality of care.

\( f(u) \) are benefits accruing to the manager (prestige, etc.), dependent on quality.

\( s(u) \) is survival, dependent upon quality of care.

² It is assumed that managers and owners of for-profit firms are indistinguishable in behavior in this analysis.
i(u) is inflow of patients, dependent upon quality of care.

E is exogenous inflow of patients, regardless of quality.

The manager, caring about the quality of care given and her cash income, seeks to maximize utility over an infinite horizon, discounted by r, the interest rate. The control variable is quality and the number of patients is the state variable. Increased quality of care results in a higher survival rate of current patients, and higher inflows of knowledgeable patients. The distinction between knowledgeable and unknowledgeable patients is relevant to the analysis of asymmetric information and treatment by ownership type in Chapter 6. Inflow of patients is also dependent on the number of patients already being treated.

It is assumed:

- costs are increasing and convex in u; \( C'(u) > 0, \ C''(u) > 0 \)
- survival, inflow, and prestige benefits are increasing and concave in u;
  \( s'(u) > 0, i'(u) > 0, f'(u) > 0, \) and \( s''(u) < 0, i''(u) < 0, f''(u) < 0. \)

The Hamiltonian is \( e^\eta [\alpha(Px - C(u)] + f(u) + \lambda(t)[s(u) + i(u) + E - ax]. \) The current-value Hamiltonian is

\[
H^* = \alpha[Px - C(u)] + f(u) + m[s(u) + i(u) + E - ax], \text{ where } m = e^\eta \lambda(t).
\]

Optimality conditions are:

\[
\frac{\partial H^*}{\partial u} = 0: \quad - \alpha C'(u) + f'(u) + m[s'(u) + i'(u)] = 0 \quad (1)
\]

\[
- \frac{\partial H^*}{\partial x} = m' - rm: \quad - \alpha P + am = m' - rm \quad (2)
\]
\[
\frac{\partial H^*}{\partial m} = x' : \quad x' = s(u) + i(u) + E - ax
\]

From 1), \( C'(u) = f'(u) + m[s'(u) + i'(u)] \)

\( \alpha \)

The manager chooses quality level \( u \) to equate marginal costs with marginal benefits from increased future patient survival and inflow, plus marginal personal benefits from quality, all weighted by the percentage \( \alpha \) of profits accruing to the manager as cash income. Similarly,

\[
\alpha = \frac{f'(u) + m[s'(u) + i'(u)]}{C'(u)}
\]

The for-profit manager with the larger percentage of cash income from profits (\( \alpha \)) must utilize a smaller quality of care (\( u \)), using the assumptions given above. The important result here is that despite identical manager tastes for quality, the cash income constraint results in the not-for-profit manager providing a higher quality product, with correspondingly higher survival rates of not-for-profit patients.

SECTION 2: AN EXAMPLE SPECIFYING FUNCTIONAL FORM

Letting \( C(u) = cu^2 \), \( f(u) = fu \), \( s(u) = su \), and \( i(u) = iu \), the optimality conditions are:

- \( 2\alpha cu + f + m(s + i) = 0 \)
- \( -\alpha P + ma = m' - rm \)
- \( x' = su + iu + E - ax \).

In the steady state when \( x' \) and \( m' = 0 \), the optimality conditions reveal:
\[ m_s = \frac{\alpha P}{r+a} \]

\[ x_s = \frac{1}{a} \left[ \frac{E + (s+i)f + (s+i)^2 P}{2\alpha c} + \frac{(s+i)^2 P}{2c(r+a)} \right] \]

\[ u_s = \frac{f + (s+i)P}{2\alpha c} + \frac{(s+i)P}{2c(r+a)} \]

Thus, in the steady state, for-profit firms have fewer patients and provide a lower quality of care. Note that in the absence of any altruism on the part of the managers (quality does not enter independently into the utility function), when \( f = 0 \), managers simply maximize profits and there is no difference between for-profit and not-for-profit managerial behavior, even though the residual earnings constraint differs for not-for-profit managers.

Further analysis with the functional forms specified above produces the Euler equations,

\[ x(t) = A_1 e^{\beta_1 t} + A_2 e^{\beta_2 t} + \frac{1}{a} \left[ \frac{E + (s+i)f + (s+i)^2 P}{2\alpha c} + \frac{(s+i)^2 P}{2c(r+a)} \right] \]

\[ m(t) = \left( \frac{(\beta_1 + a)2\alpha c}{(s+i)^2} \right) A_1 e^{\beta_1 t} + \left( \frac{(\beta_2 + a)2\alpha c}{(s+i)^2} \right) A_2 e^{\beta_2 t} + \frac{\alpha P}{r+a} \]

The roots of the equations, \( \beta_1 \) and \( \beta_2 \), are \( r + a \) and \(-a\). Since they are of opposite sign, the model is stable and a steady state can be reached. Assuming that \( x(0) = x \), the path for quality over time is:

\[ u(t) = \frac{3f}{4\alpha c} + E - ax + \frac{P[1/2 + (s+i)]}{2c(r+a)} \quad 4) \]
Thus, over time, the for-profit manager chooses a lower level of quality, and survival rates $s(u)$ are lower for patients in for-profit facilities. If quality does not enter the utility function for the managers, they both choose an identical level of quality, regardless of the percentage of income derived from profit generated. Considering that managers who value cash income more than other perquisites will gravitate toward for-profit firms, it is plausible that a utility function of a for-profit manager will contain just the cash income component, while the not-for-profit manager's utility function will contain both cash income and an independent quality variable. In that case, the first term of $4)$ is absent for the for-profit manager, and the quality produced over time is lower for the for-profit firm. However, this assumption of differing utility functions for for-profit and not-for-profit managers is not necessary for the quality variation result to hold. The opposite assumption -- that not-for-profit managers value cash income more than for-profit managers -- is not plausible given the not-for-profit non-distribution constraint. Managers will self-select into organizations that provide more of their favored form of compensation.
CHAPTER 3: THE US RENAL DIALYSIS INDUSTRY

SECTION 1: DESCRIPTION OF THE DIALYSIS INDUSTRY

To test the effect of the health care provider’s ownership status on quality of care, as measured by mortality rates, we utilized Medicare data on US kidney dialysis patients in the End-Stage Renal Disease Program. The US kidney dialysis industry is ideal for studying the effects of ownership status on clinical quality of care for several important reasons:

1) Patients in the US End-Stage Renal Disease (ESRD) program all have the same symptoms and the same need for dialysis or transplant to remain alive. Thus, complex data adjustment to account for case-mix and facility differences is reduced.

2) There are over 2000 dialysis facilities treating over 200,000 patients annually in the US. An adequate sample size of both facility ownership types exists to determine significant differences in clinical quality produced.

3) Mortality, an unambiguous measure of health outcome, is high in dialysis patients; close to 20% annually.

4) Medicare payment for dialysis is a fixed fee per treatment session, regardless of treatment duration and costs. This creates financial incentives similar to incentive created by Medicare’s prospective-price Diagnostic Related Groups payments for hospital care. Results can therefore yield insights for the majority of US health care sectors.
5) Since virtually all of the patients with ESRD are eligible to receive Medicare compensation for treatment 90 days after starting treatment, the data set encompasses virtually all patients undergoing dialysis in the U.S.

SECTION 2: HISTORY OF THE U.S. END-STAGE RENAL DISEASE PROGRAM

Dialysis treatment for patients with chronic renal failure was pioneered in 1960 by Dr. Belding Scribner and colleagues at the University of Washington (Rettig, 1976). Dialysis treatments were beyond the financial reach of most renal failure patients then, but were undeniably successful in keeping alive patients who would otherwise not survive. The first dialysis patient, a machinist at Boeing in Seattle, continued to work and lived for eleven years while on dialysis. The Veterans Administration started a dialysis program in its network of hospitals in 1963, and in the late 1960's, the Public Health Service operated several dialysis clinics as a research project (Levinsky, 1993).

By 1972, Congress voted to extend Medicare benefits to some 5,000 patients on dialysis, regardless of age. Thus, renal failure patients, classified as disabled, are the only recipients of Medicare funding who can be under age 65. Health care costs comprised a significantly lower portion of the US gross national product, so the decision to fund a proven life-saving treatment was attractive (Levinsky, 1993). Interesting changes occurred in the average characteristics of patients undergoing dialysis and transplant. In the 1960's, treatment was funded by the patients themselves or by donated funds earmarked for the most worthy of patients. It became clear by the 1980's that the decision to extend Medicare benefits to all end-stage renal disease patients improved the
access to treatment for women, minorities, children, and patients over age 55 (Daniels, 1991).

In 1973, reimbursement for dialysis treatment was set at a “reasonable-charge basis” of $138 per treatment for freestanding dialysis units and somewhat higher cost-based reimbursement for dialysis units in hospitals. In the 1980’s the per-treatment reimbursement rates were reduced and adjusted to a composite rate. By 1989, accounting for inflation, the average composite reimbursement rate had declined to $54 (in 1974 dollars) (Rettig and Levinsky, 1991). Nevertheless, the industry continued to be profitable, and Garella (1997) notes that while the average Medicare composite rate was $126, the audited average cost per treatment in 1991 was $107.21, producing an astonishing 18% profit per treatment.

The first for-profit freestanding dialysis facility, named National Medical Care, Inc. (NMC) opened in New York in 1970. Initially the dialysis industry was composed primarily of not-for-profit dialysis facilities in hospitals, but the fastest growth after 1972 was seen in freestanding for-profit facilities. NMC grew rapidly to become the industry’s largest provider of dialysis services, and by 1985 operated in 31 states (Daniels, 1991). In 1980, not-for-profit hospital-based providers still treated the most dialysis patients (58.1%) and for-profit freestanding facilities treated 32.4% of the nation’s dialysis patients. Phenomenal growth of freestanding for-profit units in the 1980’s occurred, and by 1988, freestanding for-profit facilities were treating 51.4% of the nation’s dialysis patients, and hospital-based not-for-profit facilities treating only 37% (Rettig and Levinsky, 1991, chapter 6).
Rettig and Levinsky provide further revealing information about the transformation of the dialysis and transplant industry in the 1980's. Besides the growth of for-profit and freestanding dialysis centers, from 1980 to 1988 the total U.S. dialysis patient population increased from 52,364 to 105,958 patients. Not-for-profit facilities also increased during the 1980's, but at a much slower pace. Large multi-unit chains such as NMC grew rapidly during this time as well. Note that the growth of this industry occurred despite the reductions in the composite per-treatment reimbursement rate.

The 1990's have witnessed a continuation of the growth of investor-owned chains, and hospital-based not-for-profit and small physician-owned units are the most common facilities purchased by for-profit chains. At the beginning of 1994, freestanding for-profit facilities treated 55.5% of all U.S. dialysis patients (Garella, 1997). The patient-mix has continued to change, with ever more elderly patients receiving dialysis treatments. Today, over 160,000 patients currently receive Medicare funding for dialysis treatments. Gross mortality rates have declined in the past decade, but when patient age, primary diagnosis, and other case-mix characteristics are controlled for, it is found that overall mortality rates have declined during this period (Rettig, 1996).

Medicare spent approximately $8.9 billion on the end-stage renal disease (ESRD) program in 1995 (Garella, 1997). Although total expenditures for ESRD treatment continue to rise, the increase is due to increases in numbers of ESRD patients treated, rather than increasing expenditures per patient, as is typical of many other health expenditures. Levinsky (1993) noted that in 1990, expenditures per capita for ESRD patients was ten times the average medical expenditure for the U.S. population. Friedman (1996) and others fear that the considerable size of the ESRD program may make it a target for budget cutting in the coming years, as the Medicare fund faces possible future bankruptcy.
Treatment usually consists of three weekly hemodialysis sessions of about three hours duration at an outpatient facility. During these sessions, patients are connected to dialyzing machines that filter waste products from the bloodstream. Nurses and technicians monitor the treatment process and also draw patients' blood and take blood pressure. This time-consuming procedure has its costs for both the patients and the facility, which incurs significant capital and labor expenses. Garella (1997) reports that each dialysis treatment provided requires approximately 2.2 hours of direct patient care. Since facilities are paid a fixed amount per treatment session by Medicare, regardless of the duration or expenses incurred, facilities have an incentive to economize on inputs, especially duration of treatment. With a given amount of dialysis machines and staff, treating patients for a shorter duration can allow a greater volume of patients to be treated weekly, for virtually the same cost. As will be introduced in the next section, duration of treatment may play an important role in outcome quality. Other noted methods of cost-control in dialysis facilities are replacing highly trained nursing staff with less well-trained technicians, reusing dialyzers, and delaying replacement of old equipment (Rettig, 1996).

Medicare compensation for dialysis has two main components. Medicare pays a composite rate fee of approximately $125-$130 depending on the geographic region to dialysis centers for each dialysis treatment. Aside from geographic variation, this composite rate is fixed, even though treatment sessions and complications vary from treatment to treatment and from patient to patient. Medicare actually pays 80% of the composite rate, with the remaining 20% coming from additional insurers, Medicaid, or very rarely, the patient himself. The composite rate includes payment for treatments, specified weekly and monthly laboratory tests, and services by dieticians and social workers (Garella, 1997). Other services for medically necessary tests and procedures are reimbursed separately. Recent research by the U.S. General Accounting Office (1997)
indicates that some facilities are over-prescribing these separately-billed laboratory tests, but the research did not explore the possibility that the widely varying lab test rates were related to ownership of the facility.

Physicians are also paid a capitated fee per month per dialysis patient of approximately $240, again depending on the geographic region. Physicians affiliated with not-for-profit dialysis centers receive only the capitated fee as compensation. Physicians affiliated with for-profit dialysis centers receive the capitated fee, but may also receive a portion of the dialysis facility's proceeds from the composite rate. Thus, physicians affiliated with for-profit dialysis centers can in some instances receive compensation that is partially linked to the dialysis facility's efficiency, unlike not-for-profit physicians. Note that if this occurs, the distinction between manager objectives and physician objectives are blurred, if not at times nonexistent, because for-profit physicians can own their dialysis centers or serve as part-owners in the organization.

In the dialysis industry, patients rarely select a facility for treatment on their own. Initially, their primary care physician refers them to nephrologists, who then refer patients to the dialysis facility where they are on staff. Once treatment commences, patients seldom switch providers. Lundin (1996) states, "(H)emodialysis patients are the most intimidated and intimidatable of patients, fearing that transfer to another dialysis center may only make their situation worse." On the other hand, conversations with several nephrologists suggested that patients do switch facilities now and then, not only for relocation purposes, but because those patients are dissatisfied with their treatment.

Costs of care vary by treatment modality. Garella (1997) reported the 1991 average annual expense per patient on dialysis as $35,000. Patients receiving a transplant cost
$86,000 in the first year, followed by $7,000 annual costs per patient with a well-functioning transplant and $43,000 annual costs for patients whose previous transplant failed during the year. Since costs over time differ for dialysis and transplant patients, facilities may have incentives to limit or encourage transplants, depending on reimbursement structures.

SECTION 3: EVIDENCE OF QUALITY VARIATION IN THE DIALYSIS LITERATURE

Held et al. (1991) report that dialysis treatments of shorter duration contribute to higher mortality of ESRD patients. Other studies showing a linkage between short duration and higher mortality are Charra et al. (1992); Charra, Calemard and Laurent (1996) and Kjellstrand (1985). Furthermore, Held et al. (1991) indicate that for-profit dialysis centers are more likely to treat dialysis patients with shorter sessions.

The USRDS (US Renal Data System) measured facility average hospital admission and mortality rates, adjusted for patient sex, age, race, and primary disease causing ESRD in the 1995 Annual Data Report. They found that for 1991-1993, not-for-profit dialysis centers had lower hospital admission rates and lower mortality rates than for-profit dialysis centers. However, their results were not adjusted for other factors such as differences in local market competition level, market share, patient income and education level, or geographic region. Hospital admission rates were not measured in this study as an outcome measure because the decision to hospitalize a dialysis patient may be motivated by a concern for quality care for a somewhat sick patient, or may be necessary care for an extremely sick patient. Thus, the hospitalization decision can be measured as
an input or a health outcome. Another important distinction between this study and the USRDS measurement of facility-level mortality rates is that here, we consider the possibility of a third element of quality; the opportunity to cease dialysis treatments by receiving a functioning transplant.

Griffiths et al. (1994) studied the US End-Stage Renal Disease program using dialysis facility cost reports in 1990 and determined that for-profit facilities produced more treatments per month than not-for-profit facilities after weighting the data for quantities of capital and labor used in production, and adjusting for facility and patient case-mix characteristics. However, Griffiths et al. did not compare or control for patient outcomes across ownership type, nor did they control for inputs such as duration of treatment. Held et al.'s (1991) results linking shorter treatment sessions with higher mortality imply that Griffiths et al.'s conclusion that for-profit dialysis centers are more efficient may be insupportable due to a differing level of outcome quality between the two ownership types.

Held and Pauly (1983) measured an important quality feature for the dialysis patient; the facility's number of dialysis machines per patient. That is, patients are more able to schedule convenient dialysis sessions (an amenity) at facilities with greater capacity and smaller peak loads. Held and Pauly found that in markets with greater levels of competition, facilities offer more amenities (higher quality). Furthermore, in competitive areas, for-profit dialysis centers offer more amenities than not-for-profit centers. Interestingly, Held and Pauly's amenity results by ownership contrast with USRDS clinical quality results by ownership. One explanation may be that in areas of high competition where patients are able to switch dialysis centers more easily, for-profit dialysis centers may improve quality features that are most apparent to the patient. That is, dialysis facility scheduling, location, music system at the facility, and even duration of
treatments may be most important or most visible to the patient, not the facility's comparative mortality and hospitalization rates. Since Held and Pauly did not control for duration of treatment in their study, it is not guaranteed that the facilities with higher machine per patient ratios had more convenient dialysis scheduling available for patients. In addition, convenience of scheduling is only one facet of quality a dialysis patient notices.

In a study of market competitiveness on treatment strategies of facilities, Farley (1993) measured patient outcomes (hospitalization rates and mortality rates) among dialysis-only patients in 1990, for a total sample of 72,445 patients. A sub-sample of 58,437 patients who survived was used in an analysis of patient hospitalization rates. Her results suggested that not-for-profit ownership was linked with lower hospitalization rates. However, this could reflect three phenomena: 1) Not-for-profit firms take better care of their patients, requiring fewer hospitalizations; 2) Patients at not-for-profit firms are healthier, experiencing fewer complications requiring hospitalization; and 3) Not-for-profit firms resist discharging revenue-generating patients to hospitals. Notably, the third reason listed portends very different patient outcomes than the first two reasons. This potential endogeneity of the hospitalization variable makes interpretation of the results as patient outcomes risky.

Farley’s multivariate regression analysis of mortality rates included treatment input variables and interactions between competition levels and ownership type. Her analysis was performed on a subset of dialysis-only patients, and did not include outcomes for patients with functioning transplants. Farley’s patient case-mix variables were similar with the exception of age, for which she used four discrete groups, rather than a continuous variable. Farley collapsed three measures of competition into one composite variable. This composite variable may have masked different effects on mortality rates of
Herfindahl, market share, and Certificate of Need regulations. In Chapter 4 here, Certificate of Need regulations were found to have different effects on mortality than other measures of competition. Further discussion of Farley's results is provided at the end of Chapter 4.

The research in Chapter 4 analyzes mortality outcomes for a sample 2.5 times as large as Farley's 1990 sample, and includes analysis of transplant status as an independent variable influencing mortality, and transplant status as a possible third (preferred) outcome. Chapter 5 provides a radically different investigation of the estimated causal effect of for-profit ownership on patient mortality. Chapter 6 investigates a new topic – asymmetric information – and provides some evidence that patients' sophistication level may be related to the treatment they receive at dialysis facilities.
CHAPTER 4: OWNERSHIP AND MORTALITY IN THE DIALYSIS INDUSTRY:
MULTIVARIATE REGRESSION ANALYSIS

SECTION 1: DATA SOURCES AND ATTRIBUTES

The study described in this chapter is an examination of possible variation in mortality rates by ownership status. Besides controlling for patient case-mix, this study controls for market competition level (using a Hirschman-Herfindahl index), market share of the firm, local income and education levels, entry regulations and geographic region. After controlling for these factors, remaining differences in the quality of care are more likely to originate from differences in objectives of the two ownership types. As facility size is likely to be an endogenous variable, this study does not control for number of patients at each facility, nor does it omit small facilities. Transplant status is examined in two ways – as an independent variable affecting the dependent mortality variable, and as a third dependent variable.

Before the empirical project is described, it is necessary to point out that differing quality of clinical care does not imply greater patient satisfaction, and thus we fall short of specifying the quality of the complete product. For the average patient, renal dialysis consists of three-hour dialysis sessions three times per week. For some patients, the long treatment sessions may seem too great a price to pay for prolonging their lives. Their satisfaction could theoretically be increased by reducing the duration and thus the clinical quality of their treatments.
The sample for the multinomial logistic regression analysis in this chapter consists of 197,710 patients being treated at for-profit and not-for-profit dialysis units nationwide in 1993. This includes all patients who had been on dialysis for three or more months and receiving Medicare benefits as of December 31, 1993. Patients who start dialysis or transplant treatment after age 65 are included in the sample from the beginning of their treatment. Since patients who are under age 65 may be covered by other insurance until Medicare coverage starts after three months of treatment, these patients may be missing from the USRDS database. Therefore, all 4453 patients under age 65 who had less than 90 days of treatment are omitted from the sample. This introduces a possible sample bias toward overrepresentation of patients aged 65 and older, and will limit analysis of very short-term younger patients who died less than three months after starting treatment.

The sample includes all patients receiving dialysis treatments or a transplant in 1993 at dialysis-only facilities, dialysis and transplant and transplant-only facilities (either hospital-based or freestanding). Omitted from the study are patients whose main treatment facility was government-owned, patients whose main treatment modality for dialysis was not center-based hemodialysis or transplant, and patients for whom a treatment facility was unidentifiable. Data programming and regression analysis was performed on mainframe computer with version 6.12 of SAS. Some patients were treated at two or more facilities in 1993. For these patients, their main facility was designated as the facility where they spent the most days during 1993. A related empirical project described in Chapter 6 uses a subset of this data consisting of black and white dialysis patients with hypertension or diabetes as their primary diagnosis, being treated at freestanding facilities only.

The data are from several sources. Patient and facility-level data were from the Patient, Residence, Treatment History, and Facility Standard Analysis Files from the USRDS.
The USRDS is funded by the National Institute of Diabetics and Digestive and Kidney Diseases (NIDDK) of the National Institutes of Health. The Health Care Financing Administration of the US Department of Health and Human Services supplies the NIDDK with the original data. Data by zipcode on education, household income, and urban population are from the 1990 US Census. Data on state Certificate of Need regulations affecting renal dialysis provision were obtained from the National Directory of Health Planning Policy and Regulatory Agencies.

Summary statistics for the data are provided in Tables 1-3. Noteworthy is the very high number of African American ESRD patients. According to the USRDS (1995), among black Americans one person in 399 U.S. residents is an ESRD patient. Among white Americans, one person in 1,672 U.S. residents is an ESRD patient. Another factor explaining the high rates of African Americans in the sample is the choice of treatment modality by black patients and their physicians. Black ESRD patients are less likely to be treated with home hemodialysis than whites (Kendix, 1997). Home hemodialysis patients, as well as continuous ambulatory peritoneal dialysis and continuous cycling peritoneal dialysis patients, comprised 14.4% of US ESRD patients in 1993, but are not included in this study because their treatment is largely performed at home without dialysis center staff assistance. Removal of home dialysis patients from the study may introduce a bias against not-for-profit facilities, since not-for-profit facilities are more likely to place patients on home dialysis than for-profit facilities. Furthermore, home dialysis patients are healthier as a population group than center-based dialysis patients (Kendix, 1997). Thus, removing home dialysis patients from the sample may cause for-profit firms in the sample to have healthier center-based dialysis patients than not-for-profit firms.
**Table 1: Patient Characteristics: n = 197,710**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>57.76</td>
<td>16.78</td>
<td>60</td>
<td>28</td>
<td>81</td>
</tr>
<tr>
<td>Vintage (days on dialysis)</td>
<td>1519.12</td>
<td>1487.08</td>
<td>1004</td>
<td>116</td>
<td>4780</td>
</tr>
<tr>
<td>Household income (by patient’s zipcode)</td>
<td>29,119</td>
<td>11,278</td>
<td>27,644</td>
<td>14,016</td>
<td>50,471</td>
</tr>
<tr>
<td>Avg. years of schooling (by patient’s zipcode)</td>
<td>12.54</td>
<td>.99</td>
<td>12.52</td>
<td>11.07</td>
<td>14.36</td>
</tr>
</tbody>
</table>

**Gender:**
- Female (dummy variable = 1): n = 92,377 (46.7%)
- Male: n = 105,333 (53.3%)

**Race:**
- Native American: n = 2,465 (1.2%)
- Black: n = 63,873 (32.3%)
- White: n = 123,741 (62.6%) (omitted reference category)
- Asian: n = 5,209 (2.6%)
- Other race: n = 2,422 (1.2%)

**Primary Diagnosis Causing End-Stage Renal Disease:**
- Diabetic: n = 57,767 (29.2%)
- Hypertension: n = 52,498 (26.6%)
- Glomerulonephritis: n = 31,057 (15.7%)
- Cystic Kidney: n = 8,238 (4.2%)
- Other Primary Diagnoses: n = 48,150 (24.4%) (omitted reference category)

**Transplant Status:**
- Transplants > or = 1 (dummy = 1): n = 38,517 (19.5%)
- Transplants = 0: n = 159,193 (80.5%)
### Table 2: Market Characteristics

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5&lt;sup&gt;th&lt;/sup&gt; %</th>
<th>95&lt;sup&gt;th&lt;/sup&gt; %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Herfindahl Index of Facility (by county&lt;sup&gt;3&lt;/sup&gt;)</td>
<td>.2561</td>
<td>.1770</td>
<td>.2206</td>
<td>.0431</td>
<td>.6100</td>
</tr>
<tr>
<td>Market Share of Facility (in patient’s county)</td>
<td>.2559</td>
<td>.2576</td>
<td>.1489</td>
<td>.0085</td>
<td>.8061</td>
</tr>
<tr>
<td>Percent Urban (in patient’s zipcode)</td>
<td>69.01%</td>
<td>42.75%</td>
<td>99.10%</td>
<td>0%</td>
<td>100%</td>
</tr>
</tbody>
</table>

### Table 3: Facility Characteristics: n = 2,258

**South:**
- Facility in South (dummy = 1): n = 654
- Facility not in South: n = 1604

**Certificate of Need (CON) entry regulation status:**
- Facility in state with CON (dummy = 1): n = 868
- No CON: n = 1390

**Free-standing:**
- Facility free-standing (dummy = 1): n = 1632
- Facility in hospital: n = 626

**Ownership:**
- For-profit facility (dummy = 1): n = 1439
- Not-for-profit facility: n = 819

---

<sup>3</sup> Market information and facility characteristics are attached to each patient observation in the empirical analysis.

<sup>4</sup> The patient origin Hirschman-Herfindahl index (HHI) calculation follows Gruber (1994), p. 209. The HHI is calculated for each county, then a weighted average of the index is created for each county served by a facility. The HHI was also calculated using the first three digits of the zipcode. Regressions using the zipcode-based HHI yielded results similar to those below.
Patient case-mix characteristics that have been noted as important predictors of mortality are age, race, and primary diagnosis. From previous research, it is expected that older patients, patients on dialysis for a long period, white patients (especially when compared to black patients) and patients with a primary diagnosis of diabetes will have a higher mortality rate. The age variable had a distinctly convex shape with respect to mortality, so in the logistic regression analysis discussed below, an additional age term, deviation from the mean age, was added to capture the convexity. Transplant status was also an important indicator of a patient’s health, because primarily younger and healthier patients are selected to receive a transplant instead of continuing dialysis. Transplant status enters as an independent variable in the first logistic regression analysis. Later, for the successive cumulative logistic regression analysis, transplant status will be considered as a dependent variable for one of three possible outcomes; death, living on dialysis, and living with a transplant.

Several economic variables were included in the study, but had no a priori hypotheses regarding their potential effect on patient mortality. Household income and average years of schooling may be related to patient sophistication level and ability to monitor quality of care. Held et al. (1991) note that higher-income patients had a shorter average duration of dialysis treatment sessions, so higher income may be linked to higher mortality. The Hirschman-Herfindahl index measures the competition level of the facility’s market. Stronger competition may induce either greater quality, as facilities compete for patients on the basis of quality, or facilities may cut quality to reduce costs in order to remain viable in a competitive market. Certificate of Need entry regulations affect the level and nature of competition in a market. The market share variable measures the monopoly power a firm may have in a market. Percent urban was included to capture possible urban/rural quality differences, while the south variable may signal regional differences in population besides race, or regional differences in physician
practice. A variable indicating freestanding status was also included, due to possible differences in hospital and freestanding patient populations.

Since the populations of patients at for-profit and not-for-profit facilities may differ with respect to variables that affect mortality rates, the data were divided by ownership type to examine the population characteristics separately. Tables 1-3 were repeated for patients at for-profit facilities in Tables 4-6, and for patients at not-for-profit facilities in Tables 7-9 below.

**Table 4. Patient Characteristics: n = 108,520**

**For-profit Facilities Only**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>61.04</td>
<td>15.53</td>
<td>64</td>
<td>32</td>
<td>82</td>
</tr>
<tr>
<td>Vintage (days on dialysis)</td>
<td>1249</td>
<td>1295</td>
<td>816</td>
<td>102</td>
<td>4052</td>
</tr>
<tr>
<td>Household income (by patient’s zipcode)</td>
<td>28,004</td>
<td>10,737</td>
<td>26,681</td>
<td>13,630</td>
<td>48,235</td>
</tr>
<tr>
<td>Avg. years of schooling (by patient’s zipcode)</td>
<td>12.44</td>
<td>1.00</td>
<td>12.41</td>
<td>10.92</td>
<td>14.27</td>
</tr>
</tbody>
</table>

**Gender:**
- Female (dummy variable = 1): n = 52,796 (48.7%)
- Male: n = 55,724 (51.3%)

**Race:**
- Native American: n = 1,328 (1.2%)
- Black: n = 41,088 (37.9%)
- White: n = 62,126 (57.2%) (omitted reference category)
- Asian: n = 2,560 (2.4%)
- Other: n = 1,418 (1.3%)

**Primary Disease Causing End-Stage Renal Disease:**
- Diabetic: n = 34,868 (32.1%)
- Hypertension: n = 33,982 (31.3%)
- Glomerulonephritis: n = 12,999 (12.0%)
- Cystic Kidney: n = 3,385 (3.1%)
- Other Primary Diagnoses: n = 23,286 (21.5%) (omitted reference category)

*Table 4 continues on the next page.*
Table 4, continued.

Transplant Status:
Transplants > or = 1 (dummy variable = 1): n = 1,928 (1.8%)
Transplants = 0: n = 106,592 (98.2%)

Table 5. Market Characteristics

For-profit Facilities Only

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5&lt;sup&gt;th&lt;/sup&gt; %</th>
<th>95&lt;sup&gt;th&lt;/sup&gt; %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Herfindahl index of facility (by county)</td>
<td>.2443</td>
<td>.1851</td>
<td>.2089</td>
<td>.0330</td>
<td>.6157</td>
</tr>
<tr>
<td>Market share of facility</td>
<td>.2593</td>
<td>.2641</td>
<td>.1429</td>
<td>.0081</td>
<td>.8046</td>
</tr>
<tr>
<td>Percent urban (in patient’s zipcode)</td>
<td>69.7%</td>
<td>42.3%</td>
<td>99.3%</td>
<td>0%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Table 6. Facility Characteristics: n = 1439

For-Profit Facilities Only

South:
Facility in South (dummy variable = 1): n = 508 (35.3%)
Facility not in South: n = 931 (64.7%)

Certificate of Need (CON) entry regulation status:
Facility in state with CON (dummy variable = 1): n = 446 (31.0%)
No CON: n = 993 (69.0%)

Free-standing:
Facility free-standing (dummy variable = 1): n = 1410 (98.0%)
Facility in hospital: n = 29 (2.0%)
Table 7. Patient Characteristics: n = 89,190

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>53.79</td>
<td>17.38</td>
<td>54</td>
<td>25</td>
<td>80</td>
</tr>
<tr>
<td>Vintage (days on dialysis)</td>
<td>1848</td>
<td>1633</td>
<td>1340</td>
<td>142</td>
<td>5298</td>
</tr>
<tr>
<td>Household income (by patient’s zipcode)</td>
<td>30,477</td>
<td>11,761</td>
<td>29,100</td>
<td>14,763</td>
<td>52,305</td>
</tr>
<tr>
<td>Avg. years of schooling (by patient’s zipcode)</td>
<td>12.67</td>
<td>0.97</td>
<td>12.54</td>
<td>11.24</td>
<td>14.44</td>
</tr>
<tr>
<td><strong>Gender:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Female (dummy variable = 1): n = 39,581 (44.4%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male: n = 49,609 (55.6%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Race:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Native American: n = 1,137 (1.3%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Black:</td>
<td>n = 22,785 (25.5%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>White:</td>
<td>n = 61,615 (69.1%) (omitted reference category)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Asian:</td>
<td>n = 2,649 (3.0%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other:</td>
<td>n = 1,004 (1.1%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Primary Disease Causing End-Stage Renal Disease:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diabetic:</td>
<td>n = 22,899 (25.7%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hypertension:</td>
<td>n = 18,516 (20.8%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Glomerulonephritis:</td>
<td>n = 18,058 (20.2%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cystic Kidney:</td>
<td>n = 4,853 (5.4%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other Primary Diagnoses: n = 24,864 (27.9%) (omitted reference category)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Transplant Status:**
| Transplants > or = 1 (dummy variable = 1): n = 36,589 (41.0%) | |
| Transplants = 0: n = 52,601 (59.0%) | |
Table 8. Market Characteristics

Not-for-Profit Facilities Only

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Herfindahl index of facility (by county)</td>
<td>.2705</td>
<td>.1655</td>
<td>.2278</td>
<td>.0677</td>
<td>.6099</td>
</tr>
<tr>
<td>Market share of facility</td>
<td>.2518</td>
<td>.2493</td>
<td>.1535</td>
<td>.0090</td>
<td>.8061</td>
</tr>
<tr>
<td>Percent urban (in patient’s zipcode)</td>
<td>68.2%</td>
<td>43.3%</td>
<td>98.8%</td>
<td>0%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Table 9. Facility Characteristics: n = 819

Not-for-Profit Facilities Only

South:
Facility in South (dummy variable = 1): n = 146 (17.8%)
Facility not in South: n = 673 (82.2%)

Certificate of Need (CON) entry regulation status:
Facility in state with CON (dummy variable = 1): n = 422 (51.5%)
No CON: n = 397 (48.5%)

Free-standing:
Facility free-standing (dummy variable = 1): n = 222 (27.1%)
Facility in hospital: n = 597 (72.9%)

Tables 4-9 reveal important population differences between the samples of for-profit and not-for-profit patients. Because the majority of transplant facilities are not-for-profit, and transplant patients are substantially healthier than the remaining population of patients with end-stage renal disease, the data above show markedly younger patients, fewer diabetic patients at not-for-profit facilities, and many more transplant patients. On the other hand, for-profit facilities have more female patients, patients on dialysis for a shorter length of time, and more African-American patients. If the multivariate
regression model is correctly specified, the effects of population bias are eliminated by including independent variables such as age, transplant status, diabetic status, etc. If that is true, the estimate of the effect of for-profit ownership should be correct in its direction and magnitude of effect on the probability of mortality. The confounding influence of transplant status may not be adequately controlled for, however, so after the initial multivariate analysis of ownership and mortality rates, the data are once more examined, using the subset of patients who are treated by dialysis alone. Furthermore, Chapter 5 presents an analysis of data whereby a data sample is created that has no population differences in observable independent variables.

SECTION 2: BINOMIAL LOGIT REGRESSION RESULTS

First, a binomial logistic regression was run, with results shown in Table 10. The continuous and dummy variables above were regressed against a binary variable for death. Parameter estimates were well behaved, with signs following results of previous research. For example, age was the strongest variable, with a strongly positive effect on mortality. The longer the patients had been on dialysis, the higher the mortality risk. Patients with a primary diagnosis of diabetes showed greater mortality rates, while patients with other primary diagnoses showed more favorable mortality results. White patients, the omitted reference group for race variables, had higher mortality results than all other ethnic groups shown. Especially favorable mortality results were shown for African American patients and patients who had received a transplant. Supporting the results of the theoretical model in Chapter 2, the parameter estimate for the effect of a dummy variable indicating for-profit ownership was positive, indicating a higher mortality rate for patients in for-profit facilities. The results linking for-profit ownership with higher mortality rates were robust for a sample excluding patients who started
dialysis or transplant treatment during 1993 and for a sample including patients under age 65 who had less than 90 days of treatment.

Economic variables for which no hypothesis existed regarding sign of the effect on mortality were competition (Hirschman-Herfindahl index, market share) and Certificate of Need regulatory status. Interestingly, the results of these economic variables could be considered contradictory. The Hirschman-Herfindahl index, where 1.0 designates no competition and smaller values designate more intense competition, had no effect on mortality. The market share variable, where 1.0 designates full market share and smaller values designate a smaller market share, had a slightly negative effect on mortality. The explanatory power of the Herfindahl and market share variables may be weak because multiple facilities in markets are considered competitors, when in fact they may be owned by the same corporation. Chain ownership status was not available in this data. The presence of Certificate of Need regulations in the facility’s state, limiting entry of new dialysis facilities, had a weakly positive effect on mortality. Urban/rural status appeared not to matter, nor did the average years of schooling in the patient’s zipcode, yet higher patient income by zipcode seemed to predict lower mortality. Patients treated at facilities in the South and facilities in hospitals rather than freestanding units fared worse.
Table 10: Binomial Logit with Death=1, Living=0 as Dependent Variable

Number of observations: 197,710

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Intercept Only</th>
<th>Covariates</th>
<th>$\chi^2$ for Covariates</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2 ln L</td>
<td>249,668</td>
<td>211,148</td>
<td>38,520 with 22 DF (p=0.0001)</td>
</tr>
<tr>
<td>Score</td>
<td></td>
<td></td>
<td>33,896 with 22 DF (p=0.0001)</td>
</tr>
</tbody>
</table>

$R^2 = \frac{38,520/249,668}{2} = .1543$ (Lemeshow’s pseudo-$R^2$)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter Estimate</th>
<th>Standard Error</th>
<th>Wald $\chi^2$</th>
<th>Prob. &gt; $\chi^2$</th>
<th>Standardized Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-3.2924</td>
<td>0.0931</td>
<td>1251.40</td>
<td>0.0001</td>
<td>.</td>
</tr>
<tr>
<td>Age</td>
<td>0.0441</td>
<td>0.0004</td>
<td>12161.30</td>
<td>0.0001</td>
<td>0.4084</td>
</tr>
<tr>
<td>Age Deviation$^2$</td>
<td>0.0003</td>
<td>0.000002</td>
<td>165.62</td>
<td>0.0001</td>
<td>0.050</td>
</tr>
<tr>
<td>Transplant</td>
<td>-1.6868</td>
<td>0.0260</td>
<td>4220.50</td>
<td>0.0001</td>
<td>-0.3683</td>
</tr>
<tr>
<td>Days on dialysis</td>
<td>0.0001</td>
<td>4.21E-6</td>
<td>368.53</td>
<td>0.0001</td>
<td>0.0663</td>
</tr>
<tr>
<td>South</td>
<td>0.0455</td>
<td>0.0139</td>
<td>10.72</td>
<td>0.0011</td>
<td>0.0106</td>
</tr>
<tr>
<td>Freestanding</td>
<td>-0.1456</td>
<td>0.0198</td>
<td>54.08</td>
<td>0.0001</td>
<td>-0.0386</td>
</tr>
<tr>
<td>Herfindahl index</td>
<td>-0.0043</td>
<td>0.0536</td>
<td>0.01</td>
<td>0.9368</td>
<td>-0.0004</td>
</tr>
<tr>
<td>Market share</td>
<td>-0.0579</td>
<td>0.0337</td>
<td>2.95</td>
<td>0.0860</td>
<td>-0.0082</td>
</tr>
<tr>
<td>For-profit</td>
<td>0.1362</td>
<td>0.0170</td>
<td>64.10</td>
<td>0.0001</td>
<td>0.0374</td>
</tr>
<tr>
<td>Female</td>
<td>-0.0998</td>
<td>0.0107</td>
<td>86.43</td>
<td>0.0001</td>
<td>-0.0274</td>
</tr>
<tr>
<td>Native American</td>
<td>-0.4154</td>
<td>0.0486</td>
<td>72.98</td>
<td>0.0001</td>
<td>-0.0254</td>
</tr>
<tr>
<td>Black</td>
<td>-0.3922</td>
<td>0.0128</td>
<td>931.75</td>
<td>0.0001</td>
<td>-0.1011</td>
</tr>
<tr>
<td>Asian</td>
<td>-0.5212</td>
<td>0.0359</td>
<td>211.24</td>
<td>0.0001</td>
<td>-0.0460</td>
</tr>
<tr>
<td>Other non-white</td>
<td>-0.1538</td>
<td>0.0516</td>
<td>8.88</td>
<td>0.0029</td>
<td>-0.0089</td>
</tr>
<tr>
<td>Diabetic</td>
<td>0.4100</td>
<td>0.0151</td>
<td>741.49</td>
<td>0.0001</td>
<td>0.1028</td>
</tr>
<tr>
<td>Hypertension</td>
<td>-0.0080</td>
<td>0.0154</td>
<td>0.28</td>
<td>0.5998</td>
<td>-0.0020</td>
</tr>
<tr>
<td>Glomerulonephritis</td>
<td>-0.3332</td>
<td>0.0194</td>
<td>295.30</td>
<td>0.0001</td>
<td>-0.0668</td>
</tr>
<tr>
<td>Cystic Kidney</td>
<td>-0.6748</td>
<td>0.0335</td>
<td>406.87</td>
<td>0.0001</td>
<td>-0.0743</td>
</tr>
<tr>
<td>HH income</td>
<td>-1.96E-6</td>
<td>7.60E-7</td>
<td>6.66</td>
<td>0.0099</td>
<td>-0.0122</td>
</tr>
<tr>
<td>Avg. years schooling</td>
<td>0.0096</td>
<td>0.0083</td>
<td>1.32</td>
<td>0.2504</td>
<td>0.0052</td>
</tr>
<tr>
<td>Certificate of Need</td>
<td>0.0468</td>
<td>0.0112</td>
<td>17.57</td>
<td>0.0001</td>
<td>0.0128</td>
</tr>
<tr>
<td>Percent Urban</td>
<td>-0.0002</td>
<td>0.0002</td>
<td>1.21</td>
<td>0.2718</td>
<td>-0.0039</td>
</tr>
</tbody>
</table>
Logistic regression results are not noted for their particular ease in interpretation because the dependent variable in the regression is the natural log of the odds ratio, where the odds ratio = \text{Probability(death)}/[1-\text{Probability(death)}]. Since the dependent variable, death, is either 0 or 1, when multiplying $\beta$ by 100\%, the parameter estimates are interpreted as the percentage change in the odds ratio for a one-unit percentage change in the independent variable. At the very least, the parameter estimates are useful because they indicate the direction of effect from a change in an independent variable on the dependent variable. Standardized coefficients are preferred by some in interpreting logistic regression results, but are rarely seen in econometric studies due to the dearth of binary variables in Economics. Standardized coefficients represent the effect on the probability of death for a one standard deviation change in the independent variable. One may rank the standardized coefficients by absolute value of size to obtain a snapshot of the various independent variables' effect on the dependent variable (Table 11). Patient case-mix variables profoundly affect mortality, while economic variables appear less important. The economic variable with the strongest effect on mortality is ownership status of the facility.
Table 11: Independent Variables Ranked by Standardized Coefficient

1. Age
2. Transplant status
3. Diabetic
4. Black
5. Cystic Kidneys
6. Glomerulonephritis
7. Days on dialysis (vintage)
8. Age deviation squared
9. Asian
10. Free-standing facility
11. For-profit ownership
12. Female
13. Native American
14. Certificate of Need status
15. Household income
16. South
17. Other non-white race
18. Market share (significant at p < .10)
19. Average years of schooling (insignificant)
20. Percent urban (insignificant)
21. Hypertension (insignificant, compared to reference category of "other diagnoses")
22. Hirschman-Herfindahl index of competition (insignificant)

PART A: SELECTED POPULATION PROBABILITY OF DEATH

The parameter estimates can be examined, for selected groups of patients, to determine their effect on death. The logistic equation estimated is

$$\ln \left[ \frac{P}{(1-P)} \right] = \beta_0 + \sum \beta_i X_i,$$

where \( i = 1 - 22 \), and \( P \) is the probability of death during the year and \( X_i \) are independent variables.
As a sample reference case we can enter mean values for all continuous variables and select dummy values of interest to calculate the natural log of the odds. Then, solving for the probability of death in each case, we can determine the effect of a selected dummy variable (ownership) on the probability of death:

For white males, non-South, no transplant, diabetic, treated at a freestanding for-profit facility with no state Certificate of Need entry regulations, with average age, days on dialysis, household income, schooling, percent urban, and average facility Hirschman-Herfindahl index and market share,

\[
\ln \left( \frac{P}{1-P} \right) = -0.0746 \quad \text{solving for the probability of death, } P = .4814 \quad 1)
\]

For identical patients at not-for-profit facilities,

\[
\ln \left( \frac{P}{1-P} \right) = -0.2108 \quad \text{the probability of death } = .4475 \quad 2)
\]

Subtracting 2) from 1) we obtain the increase in the probability of death during the year from being treated at a for-profit facility rather than a not-for-profit facility. Here, for the reference group above (white males), that increase is .0339 or 3.39%. Performing a similar operation with the same reference group values but for Black females with hypertension, the increase in the probability of death associated with being treated at a for-profit facility is 2.61%.

**PART B: RESULTS FOR DATA SUB-SAMPLES**

In the above binomial logistic regression, transplant status of the patient was utilized as an independent variable. The parameter estimate for transplant status was negative, suggesting that patients selected for transplants are healthier than average dialysis patients. Having a functioning transplant is less stressful to the body than long-term
dialysis. Patients with cadaveric transplants have adjusted death rates during year 1 of only 9.2%, compared to death rates of 26.4% for patients on dialysis, where the data have been adjusted for age, race, sex, and primary disease causing end-stage renal disease (USRDS, 1995). Since the populations of transplant and dialysis patients are likely to be substantially different, an additional logit regression was run with dialysis-only patients in the sample. For this sample of 159,193 dialysis-only patients shown in Table 12, results showing poorer patient outcomes at for-profit facilities were consistent, with the for-profit ownership variable ranking ninth among the independent variables’ effect on the probability of death in 1993 (Table 13). Note that the independent variable parameter estimates and their significance level as reported in Table 12 are virtually identical to the parameter estimates reported in Table 10. Because results showed little change from the dialysis and transplant sample to the dialysis-only sample, this shows some support for the ability of Table 10’s regression model to correct for bias in the for-profit and not-for-profit patient populations.
Table 12: Binomial Logit, Dialysis-Only Patients

Number of observations: 159,193

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Intercept Only</th>
<th>Covariates</th>
<th>( \chi^2 ) for Covariates</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2 ln L</td>
<td>212,893</td>
<td>194,158</td>
<td>18,735 with 22 DF (( p=0.0001 ))</td>
</tr>
<tr>
<td>Score</td>
<td></td>
<td></td>
<td>17,638 with 22 DF (( p=0.0001 ))</td>
</tr>
</tbody>
</table>

\[ R^2 = \frac{18,735}{212,893} = 0.0880 \] (Lemeshow's pseudo-\( R^2 \))

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter Estimate</th>
<th>Standard Error</th>
<th>Wald ( \chi^2 )</th>
<th>Prob. &gt; ( \chi^2 )</th>
<th>Standardized Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-3.2554</td>
<td>0.0961</td>
<td>1147.69</td>
<td>0.0001</td>
<td></td>
</tr>
<tr>
<td>Age</td>
<td>0.0433</td>
<td>0.0004</td>
<td>11.003.64</td>
<td>0.0001</td>
<td>0.3729</td>
</tr>
<tr>
<td>Age Deviation(^2)</td>
<td>0.0003</td>
<td>0.00002</td>
<td>183.79</td>
<td>0.0001</td>
<td>0.0479</td>
</tr>
<tr>
<td>Days on dialysis</td>
<td>0.0001</td>
<td>4.44E-6</td>
<td>322.65</td>
<td>0.0001</td>
<td>0.0580</td>
</tr>
<tr>
<td>South</td>
<td>0.0517</td>
<td>0.0142</td>
<td>13.26</td>
<td>0.0003</td>
<td>0.0124</td>
</tr>
<tr>
<td>Freestanding</td>
<td>-0.1506</td>
<td>0.0200</td>
<td>56.83</td>
<td>0.0001</td>
<td>-0.0340</td>
</tr>
<tr>
<td>Herfindahl index</td>
<td>-0.0124</td>
<td>0.0553</td>
<td>0.05</td>
<td>0.8221</td>
<td>-0.0013</td>
</tr>
<tr>
<td>Market share</td>
<td>-0.0651</td>
<td>0.0347</td>
<td>3.53</td>
<td>0.0604</td>
<td>-0.0097</td>
</tr>
<tr>
<td>For-profit</td>
<td>( 0.1346 )</td>
<td>( 0.0172 )</td>
<td>( 61.15 )</td>
<td>( 0.0001 )</td>
<td>( 0.0349 )</td>
</tr>
<tr>
<td>Female</td>
<td>-0.0904</td>
<td>0.0111</td>
<td>66.80</td>
<td>0.0001</td>
<td>-0.0249</td>
</tr>
<tr>
<td>Native American</td>
<td>-0.4261</td>
<td>0.0498</td>
<td>73.19</td>
<td>0.0001</td>
<td>-0.0263</td>
</tr>
<tr>
<td>Black</td>
<td>-0.4126</td>
<td>0.0131</td>
<td>992.37</td>
<td>0.0001</td>
<td>-0.1094</td>
</tr>
<tr>
<td>Asian</td>
<td>-0.5530</td>
<td>0.0368</td>
<td>223.26</td>
<td>0.0001</td>
<td>-0.0479</td>
</tr>
<tr>
<td>Other non-white</td>
<td>-0.1628</td>
<td>0.0527</td>
<td>9.55</td>
<td>0.0020</td>
<td>-0.096</td>
</tr>
<tr>
<td>Diabetic</td>
<td>( 0.3856 )</td>
<td>( 0.0156 )</td>
<td>( 612.76 )</td>
<td>( 0.0001 )</td>
<td>( 0.0990 )</td>
</tr>
<tr>
<td>Hypertension</td>
<td>-0.0054</td>
<td>0.0158</td>
<td>0.12</td>
<td>0.7310</td>
<td>-0.0014</td>
</tr>
<tr>
<td>Glomerulonephritis</td>
<td>-0.3416</td>
<td>0.0205</td>
<td>277.92</td>
<td>0.0001</td>
<td>-0.0626</td>
</tr>
<tr>
<td>Cystic Kidney</td>
<td>-0.7125</td>
<td>0.0359</td>
<td>393.73</td>
<td>0.0001</td>
<td>-0.0696</td>
</tr>
<tr>
<td>HH income</td>
<td>( -1.62E-6 )</td>
<td>( 7.89E-7 )</td>
<td>4.22</td>
<td>0.0400</td>
<td>-0.0099</td>
</tr>
<tr>
<td>Avg. years schooling</td>
<td>0.0104</td>
<td>0.0086</td>
<td>1.48</td>
<td>0.2243</td>
<td>0.0056</td>
</tr>
<tr>
<td>Certificate of Need</td>
<td>0.0541</td>
<td>0.0116</td>
<td>21.94</td>
<td>0.0001</td>
<td>0.0147</td>
</tr>
<tr>
<td>Percent Urban</td>
<td>-0.0001</td>
<td>0.0002</td>
<td>0.47</td>
<td>0.4929</td>
<td>-0.0025</td>
</tr>
</tbody>
</table>
Using the same reference groups as noted above to provide a snapshot of the effects of for-profit treatment on mortality, white males in the dialysis-only group with diabetes were 3.34% more likely to die during the year when treated at for-profit facilities (for a probability of death of .4941 at for-profit facilities), while Black females with hypertension were 2.68% more likely to die when treated at for-profit facilities (with a probability of death at for-profit facilities of .2895).

Table 13: Independent Variables Ranked by Standardized Coefficient, Dialysis-Only Patients

1. Age
2. Black
3. Diabetic
4. Cystic Kidney
5. Glomerulonephritis
6. Days on dialysis (vintage)
7. Asian
8. Age deviation squared
9. **For-profit ownership**
10. Freestanding facility
11. Native American
12. Female
13. Certificate of Need status
14. South
15. Household income (significant at p < .05)
16. Market share (significant at p < .10)
17. Other non-white race
18. Average years of schooling (insignificant)
19. Percent urban (insignificant)
20. Hypertension (insignificant, compared to reference category of “other diagnoses”)
21. Hirschman-Herfindahl index of competition (insignificant)

Since the data set size is substantial, it was possible to perform the same logistic regression on sub-samples of the dialysis-only data by population group. Table 14 shows the sub-samples that were used to run the logistic regressions.
Table 14 expresses strong results, suggesting that no matter how the dialysis-only data was subdivided and analyzed with separate multivariate logistic regressions, all groups of patients appeared to fare worse at for-profit facilities. Results of the sub-sample regressions were consistent regarding the effects of for-profit ownership on patient mortality. In all sub-samples except one, the parameter estimate on the for-profit ownership dummy variable was positive and statistically significant at the p < .01 level. The one sub-sample that showed no statistically significant effect of ownership on mortality was the subset of 33,799 patients who were treated at hospital-based facilities. The lack of significance for the hospital subset may be caused by the very small number of dialysis patients treated at for-profit hospitals. For-profit ownership effects on mortality were particularly strong (surpassing some patient characteristics in importance) for the following groups; the sub-samples of patients who had been on dialysis for more
than 1600 days (about 4 years), non-diabetic patients, white patients, and patients in states with Certificate of Need entry regulations.

PART C: TRANSPLANT STATUS AS A DEPENDENT VARIABLE: SUCCESSIVE CUMULATIVE LOGISTIC REGRESSION

Instead of treating transplant status as an independent variable, one could also consider an ESRD patient with a transplant, off dialysis, to be experiencing a different outcome than those patients still on dialysis. Therefore, three ordered outcomes could be used for dependent variables: death, alive on dialysis, and alive with a transplant. An ordered multinomial logistic regression was run in SAS, and the score test soundly rejected the possibility that outcomes are necessarily ordered as above. To determine the source of unspecified data irregularities causing the rejection of the ordered logistic regression, two successive logistic regressions were run, with the following parameter estimates shown in Table 15.
Table 15: Successive Cumulative Logistic Regression Analysis

<table>
<thead>
<tr>
<th>Variable</th>
<th>Logit I Alive = 0, Died = 1</th>
<th>Wald $\chi^2$</th>
<th>Logit II Transplant = 0, No Transplant = 1</th>
<th>Wald $\chi^2$</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-3.8687 **</td>
<td>1774</td>
<td>-2.4037 **</td>
<td>248</td>
<td></td>
</tr>
<tr>
<td>Age</td>
<td>0.0515 **</td>
<td>17.257</td>
<td>0.0952 **</td>
<td>10.172</td>
<td></td>
</tr>
<tr>
<td>Age Deviation $^2$</td>
<td>0.0003 **</td>
<td>130</td>
<td>0.0015 **</td>
<td>2.194</td>
<td></td>
</tr>
<tr>
<td>Days on Dialysis</td>
<td>3.725E-6 **</td>
<td>.83</td>
<td>-0.00036 **</td>
<td>4.437 *</td>
<td></td>
</tr>
<tr>
<td>South</td>
<td>0.0257 *</td>
<td>3.5</td>
<td>0.3875 **</td>
<td>206</td>
<td></td>
</tr>
<tr>
<td>Herfindahl index</td>
<td>-0.2119 **</td>
<td>16</td>
<td>-1.8209 **</td>
<td>463</td>
<td></td>
</tr>
<tr>
<td>Market share</td>
<td>0.3049 **</td>
<td>84</td>
<td>3.5920 **</td>
<td>4.168</td>
<td></td>
</tr>
<tr>
<td>For-profit</td>
<td>0.3501 **</td>
<td>915</td>
<td>3.6088 **</td>
<td>16,409</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>-0.0803 **</td>
<td>57</td>
<td>0.1399 **</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>Native American</td>
<td>-0.3471 **</td>
<td>52</td>
<td>0.4226 **</td>
<td>31</td>
<td></td>
</tr>
<tr>
<td>Black</td>
<td>-0.2921 **</td>
<td>523</td>
<td>1.1440 **</td>
<td>2498</td>
<td></td>
</tr>
<tr>
<td>Asian</td>
<td>-0.4673 **</td>
<td>172</td>
<td>0.4108 **</td>
<td>71</td>
<td></td>
</tr>
<tr>
<td>Other race</td>
<td>-0.0453 **</td>
<td>0.8</td>
<td>1.0339 **</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>Diabetic</td>
<td>0.4127 **</td>
<td>774</td>
<td>0.3193 **</td>
<td>164</td>
<td></td>
</tr>
<tr>
<td>Hypertension</td>
<td>0.0168</td>
<td>1.2</td>
<td>0.0732 **</td>
<td>7.1</td>
<td></td>
</tr>
<tr>
<td>Glomeruloneph.</td>
<td>-0.3628 **</td>
<td>364</td>
<td>-0.2663 **</td>
<td>127</td>
<td></td>
</tr>
<tr>
<td>Cystic Kidney</td>
<td>-0.7400 **</td>
<td>514</td>
<td>-0.6765 **</td>
<td>341</td>
<td></td>
</tr>
<tr>
<td>HH income</td>
<td>-9.43E-7</td>
<td>1.6</td>
<td>8.296E-6 **</td>
<td>53</td>
<td></td>
</tr>
<tr>
<td>Avg. yrs. school</td>
<td>-0.0172 *</td>
<td>4.4</td>
<td>-0.2459 **</td>
<td>346</td>
<td></td>
</tr>
<tr>
<td>Cert. of Need</td>
<td>0.0737 **</td>
<td>45</td>
<td>0.3660 **</td>
<td>419</td>
<td></td>
</tr>
<tr>
<td>Percent Urban</td>
<td>0.0004 *</td>
<td>6.1</td>
<td>0.0036 **</td>
<td>234</td>
<td></td>
</tr>
</tbody>
</table>

* Statistically significant at $p < .05$.
** Statistically significant at $p<.01$

• These parameter estimates switched signs from Logit I to Logit II.
Logit I is the same as the regression reported in Table 10 above, but without the dummy variable for transplant status among the independent variables. Lemeshow’s pseudo-$R^2$ for Logit I is .1330, which is less than the former model’s $R^2$, due to the omission of the important transplant variable on the right-hand side. Logit II tests for explanatory power of the remaining independent variables on the possibility of transplant, and has a high pseudo-$R^2$ value of .5298. Note that Logit II is ordered such that a positive parameter estimate means that the patient is less likely to receive a transplant. This ensures consistency across the two logit regressions, in that a zero value for the dependent variable is favorable to the patient, and a value of one for the dependent variable is an unfavorable outcome for the patient.

The successive cumulative logistic regression results offer interesting insights due to differing rates of transplantation among different population groups: Minorities are more likely to survive (positive parameter estimates in Logit I) but are less likely to have transplants. 13.7% of African American ESRD patients have transplants, whereas 33% of white Americans with ESRD have transplants (USRDS, 1995). The parameter estimate for years of schooling becomes more negative with a transplant vs. non-transplant regression, indicating that the more schooling a patient has, the more likely she is to receive a transplanted kidney. The parameter estimate for household income switches to positive, indicating that there does not appear to be a positive income bias for obtaining transplants.

Market competition may be important, as here we see (contrary to results for the binomial logistic regression above) the higher the market share of the firm, the less likely it is that the patient receives a transplant. The strength of the market share parameter estimate in Logit II might be explained by the fact that transplant patients undergo transplant in
facilities located primarily in large cities. Paradoxically, the higher the Herfindahl index of the firm, the higher the transplant rate. Freestanding status of the firm was not included as an explanatory variable in Logit II because no transplants take place at freestanding facilities. The effect of for-profit ownership is steady across the two logistic regressions — that is, patients are less likely to receive transplants at for-profit institutions and are less likely to survive at for-profit institutions. Both regressions suggest that patient outcomes are worse at for-profit facilities. However, the transplant results should be interpreted cautiously, as patients who are ideal transplant recipients are probably referred to the nearest transplant facility, which is usually a not-for-profit facility.

SECTION 3: LOGISTIC REGRESSION ANALYSIS SUMMARY

Multivariate logistic regression supported the hypothesis generated by the theoretical model of managerial utility maximization presented in Chapter 2. In this chapter, we find a positive influence on patient mortality for patients treated at for-profit facilities, compared to the reference group of patients treated at not-for-profit facilities. Binomial logistic regression of twenty-two independent patient, facility, and market characteristic variables on a binary variable indicating death in 1993 revealed statistically significant results of higher for-profit patient mortality. Sample probability statistics for two selected population groups were calculated, indicating that the estimated causal effect (increased probability) associated with for-profit treatment was around three percent. Analysis of various subsets of the population data suggested a consistent theme of higher mortality among patients in for-profit facilities.
Results indicating higher mortality among patients of for-profit facilities were consistent with similar studies by Farley (1993) and the USRDS (1995). Farley included ownership in a logistic regression of treatment inputs, market characteristics, and patient characteristics on the probability of patient mortality. Her model had substantial specification differences than the model discussed in Table 10 here, but her results showed not-for-profit ownership to be a negative influence on patient mortality, statistically significant at the p <= .001 level. Given the information in Farley’s results, it was not possible to calculate sample probabilities of death for selected patient groups from her logistic regression results because average values for certain variables were not reported.

The USRDS presented standardized mortality ratios (SMRs) only for freestanding facilities with more than 20 dialysis patients. The SMR is obtained by dividing the observed number of deaths for the patient group by the expected number of deaths for that patient group, given age, sex, race, and primary disease characteristics. Although the USRDS did not include adjustments for market characteristics, their facility-level analysis indicated evidence of higher outcome quality at not-for-profit facilities. The SMR for freestanding not-for-profit facilities was .94 and the SMR for freestanding for-profit facilities was 1.01, where an SMR of 1.00 indicates that the same number of patients die during the study period as is statistically predicted by the case-mix characteristics of the patient group. This research suggests a difference in the probability of death due to for-profit treatment of approximately 7%, but note that their facility-level data were not weighted for facility size. In addition, hospital-based patients and transplant patients were not included in the USRDS analysis. The logistic regression results reported in this chapter are therefore consistent with previous literature with respect to the direction of effect that for-profit treatment has on patient outcomes. Here, though, it was possible to present sample probabilities for certain patient groups in order to characterize the magnitude of the effect of for-profit treatment.
There remains a question, however, of whether the logistic models were correctly specified to adjust for baseline differences in the population of for-profit and not-for-profit patients. The next chapter introduces a relatively new technique, propensity score methodology, for correcting differences in the data and avoiding the problem of misspecification of the model, as well as presenting a clearly-defined measure of the magnitude of the effect of for-profit treatment on patient mortality.
CHAPTER 5: OWNERSHIP AND MORTALITY IN THE DIALYSIS INDUSTRY:
PROPENSITY SCORE ANALYSIS

SECTION 1: INTRODUCTION

A common criticism of many empirical research projects, especially in health and social
sciences, is that the data are based on observations of subjects non-randomly assigned to
treatment and control groups. Baseline differences in important control variables may
exist between the treatment and control groups. Multivariate regression analysis with
inclusion of the relevant control variables will correct for the existing differences in the
treatment and control groups if the model is correctly specified. However, the search for
a correctly specified model is often difficult and unrewarding. The multivariate
regression analysis in Chapter 4 estimated the causal effect of for-profit ownership for the
entire population of patients being treated for end-stage renal disease in the US.
Nevertheless, there is no guarantee that the logistic regression models were correctly
specified in order to control for possible differences in the populations of for-profit and
not-for-profit patients. Furthermore, the logistic analysis produced parameter estimates
showing the direction of effect of for-profit treatment on mortality of patients, but these
parameter estimates could not be used to specify the magnitude of that effect, unless
specific population groups were targeted for analysis.

One tactic to avoid the lack of randomization would be the use of identical twins in
research, because twins have almost identical opportunities and abilities, and they
approximate a pair of identically matched observations. Assignment of each identical
twin to the treatment and control groups will result in differing effects of the treatment.
The causal effect of the treatment can then be calculated by subtracting the dependent
variable value (reading score, income, 100-yard dash time, etc.) of the control group twin from the dependent variable value of the treatment group twin. Propensity score methodology provides a way of creating such matched groups without the twins. Observations are evaluated and arranged according to their similarities between treatment and control groups until no statistically significant difference exists between treatment and control group observations for any of the independent variables. Then, the estimated causal effect of the treatment is easily calculated. In the study presented in this chapter, propensity score methodology is employed as an alternative to the multivariate study in the previous chapter. Results from a sample of 15,422 patients showing no discernable differences among for-profit and not-for-profit patients indicate that the estimated causal effect of treatment at a for-profit facility is a 5.86% higher annual mortality rate than the mortality rate experienced by patients at not-for-profit facilities.

The main advantage of propensity score methodology is that it is possible to create treatment and control data subsets that have no discernable differences in the independent variables that affect outcomes. Consequently, there are no lingering concerns about incorrect specification of the model. If the treatment and control groups are proven to have virtually identical characteristics, then the main task is simplified to measuring the difference in outcomes between the treatment and control groups. Another important advantage of propensity score methodology is its ease of interpretation. The difference in outcomes between the treatment and control groups is the estimated causal effect of the treatment. Of course, these propensity score methodology results are dependent on the inclusion of all variables affecting outcome that may differ among treatment and control group observations, but the same dependence exists in multivariate regression models.
SECTION 2: PROPENSITY SCORE METHODOLOGY: THEORETICAL BACKGROUND

When working with a sample, the objective, as Rubin (1974) describes, is to determine the typical causal effect of treatment at a for-profit facility versus the typical causal effect of a patient being treated at a not-for-profit facility, on a dichotomous dependent variable, Y, indicating death during the year of study. However, in attempting to measure $Y_{\text{for-profit treatment}} - Y_{\text{not-for-profit treatment}}$, we are unable to observe both simultaneously. Therefore, we observe $y_{j}(\text{FP}) - y_{j}(\text{NFP})$ in M trials, where the mean causal effect is

$$(1/M)\Sigma[y_{j}(\text{FP}) - y_{j}(\text{NFP})], \text{ where } j = 1\ldots M$$

As long as patients are randomly assigned to for-profit and not-for-profit treatment, the difference in Y, death, is an unbiased estimate of the average causal effect of for-profit ownership on mortality.

Randomization is impossible in this case, however, so our estimate of ownership type's effect on mortality may be biased. Patients are more or less free to choose their treatment facility, and some patients may be aware of the facility's ownership status. Nevertheless, the lack of randomization would be irrelevant if the two groups of patients at for-profit and not-for-profit facilities were identical for every variable affecting mortality. Rubin and others have suggested that using propensity score methodology might be a viable way to create identical patient sub-groups. Propensity score methodology may even be preferred to analysis using randomized data because perfectly matched groups of patients in for-profit and not-for-profit assignments would guarantee that the estimated causal effect of treatment would be completely unbiased, whereas a random assignment of patients to the treatment group may introduce population bias due to random error. Of course, this claim is predicated on the assumption that the observations matched on
observed variables are not influenced by existing bias in unobserved variables. In this chapter, we limit the sample to a population of patients with case-mix similarities. We then find comparable groups of patients using the propensity score methodology and use those matched groups to estimate the average causal effect of ownership on mortality.

SECTION 3: PROPENSITY SCORE METHODOLOGY IN PRACTICE

PART A. CHOOSE SAMPLE AND CHOOSE RELEVANT INDEPENDENT VARIABLES

The first step was to reduce the sample size to hemodialysis patients being treated at freestanding facilities, black and white patients, and patients with either hypertension or diabetes (Table 16). This reduction in the size of the sample was necessary to simplify the process by which sub-samples are identified as "comparable" across ownership types, and to enhance ability to meaningfully characterize average causal effects of ownership for the majority of patients nationwide. Further, it was necessary to eliminate strictly facility-level characteristics from the analysis, since propensity score methodology requires a simple "control versus treatment" assignment with no secondary variables affecting assignment, which, in this case, would mean characteristics of the facility itself that are unrelated to the patient. Therefore, patients being treated at hospital-based facilities were eliminated from the sample so that all facilities remaining were freestanding.
Table 16. Propensity Score Project Sample Characteristics

Total sample size = 77,110.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>62.95</td>
<td>14.07</td>
<td>66</td>
<td>36</td>
<td>82</td>
</tr>
<tr>
<td>Vintage (days on dialysis)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1045</td>
<td></td>
<td>997</td>
<td>746</td>
<td>97</td>
<td>3082</td>
</tr>
<tr>
<td>Household Income (in patient’s zipcode)</td>
<td></td>
<td>27,005</td>
<td>10,475</td>
<td>25,786</td>
<td>13,162</td>
</tr>
<tr>
<td>Average Years of Schooling (in patient’s zipcode)</td>
<td></td>
<td>12.38</td>
<td>0.98</td>
<td>12.35</td>
<td>10.92</td>
</tr>
<tr>
<td>Gender</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Female: n = 38,471 (49.9%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male: n = 38,639 (50.1%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Race</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Black: n = 34,967 (45.3%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>White: n = 42,143 (54.7%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Primary Diagnosis</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diabetes: n = 38,536 (50.0%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hypertension: n = 38,574 (50.0%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Area</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Patients in South: n = 26,164. Facilities in South:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>584. Patients not in the South: n = 50,946. Facilities not in the South: 1032</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Certificate of Need (CON) Status</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Patient’s facility in state with CON entry regulations: n = 28,924.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of facilities in states with CON entry regulations: n = 550.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Patient’s facility in state without CON entry regulations: n = 48,186.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of facilities in states without CON entry regulations: n = 1066.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ownership</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Patients in for-profit facilities: n = 65,825.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of for-profit facilities: n = 1395.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Patients in not-for-profit facilities: n = 11,285.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of not-for-profit facilities: n = 221.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 17: Binomial Logit with Death = 1, Living = 0 as Dependent Variable

(Includes black and white hemodialysis patients with the primary diagnosis of diabetes or hypertension, being treated at free-standing facilities during 1993.)

Number of observations:  
Died: = 32,435  
Living: = 44,675  
Total: = 77,110

<table>
<thead>
<tr>
<th>Criterion &amp;</th>
<th>Intercept Only</th>
<th>Covariates</th>
<th>( \chi^2 ) for Covariates</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2 ln L Score</td>
<td>104,946</td>
<td>96,762</td>
<td>8183 with 14 DF (p=0.0001)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>7772 with 14 DF (p=0.0001)</td>
</tr>
</tbody>
</table>

Logistic Procedure: Analysis of Maximum Likelihood Estimates

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter Estimate</th>
<th>Standard Error</th>
<th>Wald ( \chi^2 )</th>
<th>Prob. &gt; ( \chi^2 )</th>
<th>Standardized Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-4.0502</td>
<td>0.1376</td>
<td>866</td>
<td>0.0001</td>
<td>.</td>
</tr>
<tr>
<td>Age</td>
<td>0.0461</td>
<td>0.0006</td>
<td>4923</td>
<td>0.0001</td>
<td>0.3573</td>
</tr>
<tr>
<td>Age Deviation(^2)</td>
<td>0.0004</td>
<td>0.00004</td>
<td>122</td>
<td>0.0001</td>
<td>0.0573</td>
</tr>
<tr>
<td>Female</td>
<td>-0.0569</td>
<td>0.0157</td>
<td>13</td>
<td>0.0003</td>
<td>-0.0157</td>
</tr>
<tr>
<td>Vintage (days on dialysis)</td>
<td>0.0002</td>
<td>8.04E-6</td>
<td>432</td>
<td>0.0001</td>
<td>0.0920</td>
</tr>
<tr>
<td>Diabetic</td>
<td>0.3924</td>
<td>0.0164</td>
<td>572</td>
<td>0.0001</td>
<td>0.1082</td>
</tr>
<tr>
<td>Black</td>
<td>-0.5803</td>
<td>0.0176</td>
<td>1084</td>
<td>0.0001</td>
<td>-0.1593</td>
</tr>
<tr>
<td>South</td>
<td>0.0647</td>
<td>0.0177</td>
<td>13</td>
<td>0.0003</td>
<td>0.0169</td>
</tr>
<tr>
<td>Cert. of Need</td>
<td>0.0649</td>
<td>0.0165</td>
<td>16</td>
<td>0.0001</td>
<td>0.0173</td>
</tr>
<tr>
<td>HH Income /10,000</td>
<td>-0.0293</td>
<td>0.0117</td>
<td>6</td>
<td>0.0124</td>
<td>-0.0169</td>
</tr>
<tr>
<td>Market Share</td>
<td>-0.0746</td>
<td>0.0421</td>
<td>3</td>
<td>0.0767</td>
<td>-0.0136</td>
</tr>
<tr>
<td>Avg. Schooling</td>
<td>0.0462</td>
<td>0.0122</td>
<td>14</td>
<td>0.0001</td>
<td>0.0250</td>
</tr>
<tr>
<td>Percent Urban</td>
<td>-0.0003</td>
<td>0.0002</td>
<td>2</td>
<td>0.2014</td>
<td>-0.0067</td>
</tr>
<tr>
<td>Herfindahl</td>
<td>0.0706</td>
<td>0.0588</td>
<td>2</td>
<td>0.2141</td>
<td>0.0102</td>
</tr>
<tr>
<td>For-Profit</td>
<td>0.1316</td>
<td>0.0223</td>
<td>34</td>
<td>0.0001</td>
<td>0.0256</td>
</tr>
</tbody>
</table>

Table 17 reports the results of the first logistic regression of all the remaining variables (after removing the less common race and primary diagnosis patients) on the dependent variable, death during 1993. Performing this logistic regression was necessary to determine which independent variables have a statistically significant effect on death and should therefore be included in the process of identifying "identical" sub-samples by
ownership type. Results of chi-square tests indicate that a Herfindahl index of market competition and the percent urban area in the patient's zipcode are not statistically significant for this sample of 77,110 patients, and thus will not be used in the propensity score analysis. Including irrelevant variables such as these could bias the results in determining the average causal effect of ownership on mortality. Market share of the facility in the patients' zipcode was not significant at the five percent level, so it was also dropped from the analysis. The statistical contribution of the market share variable did not justify the computational requirements of inclusion.

Similar to the results reported in Tables 10 and 12, Table 17 reveals that for-profit ownership is positively associated with higher mortality for patients in this sample. For-profit ownership is ranked sixth among the fourteen covariates listed above, using the absolute value of the standardized parameter estimates, which represent the effect on the probability of death for a one standard deviation change in the independent variable. The independent variables with stronger effects on the probability of death are age, black, diabetic, number of days on dialysis, and deviation from the average age—all of which are important patient case mix characteristics.

After removing the Herfindahl index, percent urban and market share variables, the additional independent variables have effects on the probability of death that are similar to the effects seen in Tables 10 and 12, which described the logistic regression for virtually all patients in the US. Patients in the south have higher mortality rates. Patients in states with Certificate of Need entry regulations appear to have higher mortality rates. Patients in zipcodes with higher income levels have lower mortality rates, yet patients in zipcodes with more years of schooling have a higher probability of death.
PART B. Compute Propensity Scores and Separate Sample into Identical For-Profit and Not-for-Profit Strata

Following the techniques outlined in Rosenbaum and Rubin (1983) and Dehejia and Wahba (1997), to calculate the propensity of each observation to be a patient treated at a for-profit facility, the independent variables other than ownership were regressed using binomial logistic regression on the ownership dummy variable. The estimates of the dependent variable, propensity score, were retained for each observation. The propensity score thus represents the estimate of the natural log of the odds, where the odds refer to the probability of being a for-profit facility over the probability of being a not-for-profit facility. Then, the observations were ranked by propensity score and divided into strata by propensity score groupings. Separating the data into strata by propensity score enables one to create new smaller data sets whose population of treatment and control observations (for-profit and not-for-profit patients) are comparable. The closer the propensity score for two patients, the more likely it is that their independent variables are identical in distribution.

Before the data are separated into strata, it is necessary to remove any for-profit observations that are substantially different from not-for-profit observations, which is analogous to removing a population of control group observations that are dissimilar to treatment group observations and are therefore irrelevant to the analysis. Here, removing observations substantially different from the rest of the population is accomplished by eliminating for-profit observations that have propensity scores above the highest not-for-profit propensity score observations, and eliminating not-for-profit observations with propensity scores below for-profit observations' propensity scores, where the propensity score refers to the probability of the patient being a for-profit patient. Curiously, in this data set, the observation with the highest propensity score is a not-for-profit patient, and the observation with the lowest propensity score is a for-profit patient. Therefore, all the observations could be considered part of the relevant population. However, the
population becomes more homogeneous by eliminating the upper propensity range and
the lower propensity range, where for-profit and not-for-profit patients cluster,
respectively.

To determine if the for-profit and not-for-profit patients in a strata had identical values
for independent variables, a difference in means t-test was performed between the for-
profit and not-for-profit subsets of each strata. Strata j had “identical” for-profit and not-
for-profit patient observations if the absolute value of the $t_\alpha$-statistic for strata j was less
than 1.645, where $\alpha = .05$ and the null hypothesis is $H_0: (\mu_1 - \mu_2) \neq 0$:

$$t = \frac{(y_1 - y_2)}{[s(1/n_1 + 1/n_2)]^{1/2}},$$

where $y_1$ and $y_2$ are the independent variable means for the for-profit and not-for-profit
portions, respectively, of the stratum, $n_1$ and $n_2$ are the number of for-profit and not-for-
profit observations in the stratum, and $s$ is the independent variable’s standard deviation
in the stratum (Mendenhall, 1987, page 414). Difference in means t-tests were computed
for each independent variable in the stratum. If all t-tests failed to reject the null
hypothesis, the for-profit and not-for-profit observations were statistically indistinct and
comparable to a completely randomized data set.

Rosenbaum and Rubin (1984) and Cochran (1968) assert that separating a data set into
five strata of equal sizes results in a 90% reduction in bias between treatment and control
group populations. Several conditions must hold for this to be true, such as normality of
distribution of the independent variables. For this particular project, separating the
sample into five strata by propensity score was not enough to eliminate statistically
significant differences in independent variable estimates by ownership. Difference in
means t-tests rejected the null hypothesis for several of the independent variables in all
five strata, suggesting the need for further subdivision of the data set by propensity score.
The following chart, Table 18, illustrates the process whereby the t-statistics are calculated for each independent variable in each stratum. Table 18 shows these calculations for a stratum that comprises the upper tenth of the data set, when ranked by propensity to be a for-profit facility. The propensity scores in this stratum range from .917 to .979. The standard deviation for each variable is calculated for the entire stratum, while the variable means are calculated for each ownership subset of the stratum. T-tests fail to reject the null hypothesis for the variables age, number of days on dialysis, female, and south, meaning that for-profit patients and not-for-profit patients have essentially identical values for these variables. T-tests reject the null hypothesis, however, for the variables diabetic, black, Certificate of Need (entry) regulations, household income, and average years of schooling. When t-tests fail to reject the null hypothesis, for-profit and not-for-profit patients in this stratum have characteristics that differ substantially from one another.

<table>
<thead>
<tr>
<th>Stratum Variable</th>
<th>Std. Dev.</th>
<th>F-P Subset Mean (n = 7055)</th>
<th>N-f-P Subset Mean (n = 657)</th>
<th>t-statistic</th>
<th>Reject H⁰?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>13.29</td>
<td>66.56</td>
<td>66.28</td>
<td>0.51</td>
<td></td>
</tr>
<tr>
<td>Days on Dialysis</td>
<td>608.22</td>
<td>691.31</td>
<td>700.72</td>
<td>-0.38</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>0.50</td>
<td>0.50</td>
<td>0.48</td>
<td>1.17</td>
<td>reject</td>
</tr>
<tr>
<td>Diabetic</td>
<td>0.49</td>
<td>0.40</td>
<td>0.43</td>
<td>-1.27</td>
<td></td>
</tr>
<tr>
<td>Black</td>
<td>0.47</td>
<td>0.34</td>
<td>0.29</td>
<td>2.56</td>
<td>reject</td>
</tr>
<tr>
<td>South</td>
<td>0.32</td>
<td>0.12</td>
<td>0.10</td>
<td>0.83</td>
<td></td>
</tr>
<tr>
<td>Cert. of Need</td>
<td>0.04</td>
<td>0.0014</td>
<td>0.0061</td>
<td>-2.69</td>
<td>reject</td>
</tr>
<tr>
<td>Household Income/10,000</td>
<td>1.47</td>
<td>3.69</td>
<td>3.98</td>
<td>-4.97</td>
<td>reject</td>
</tr>
<tr>
<td>Avg. Years Schooling</td>
<td>1.38</td>
<td>12.29</td>
<td>12.43</td>
<td>-2.65</td>
<td>reject</td>
</tr>
</tbody>
</table>
If t-tests are rejected for any of the variables, it cannot be claimed that the data set's for-profit patients are the same as the not-for-profit patients, so the sample, or stratum, should be broken into smaller samples by propensity score. However, it can be argued that a stratum is acceptable if variables are biased in the correct direction. For example, the stratum above has more black patients at for-profit units than not-for-profit units. In Table 17, it is clear that black patients have a lower mortality risk than non-diabetic patients. Therefore, it biases the results in favor of for-profit facilities if we accept a stratum where for-profit units have significantly more black patients, as above. If final results indicate that for-profit facilities have a higher death rate, knowledge that for-profit units have more black patients would only serve to strengthen that conclusion. Unfortunately, accepting a stratum where the t-tests are rejected with a bias in favor of for-profits implies that the final estimate of the average causal effect of for-profit ownership on patient mortality becomes not a consistent estimate but a lower limit of the estimated causal effect.

Similar to black race, Certificate of Need regulations and average years of schooling variables also favor for-profit firms in the sample stratum described above. However, household income is a problematic variable. In Table 17, household income (by patient’s zipcode) is negatively correlated with patient mortality. Here, however, we see in Table 18 that not-for-profit patients have higher incomes in the featured stratum. Thus, due to the household income variable, results may be biased in favor of not-for-profit facilities. Even though the positive bias exhibited by the three other variables that rejected the t-test may balance out the negative bias from the household income variable, the results are currently indeterminate. Since all of the other strata from the division of the data set into ten subsets have similar t-test problems to those presented in Table 18, the next step is to subdivide the data into yet more strata by propensity score.
Division of the data above into twenty strata instead of ten resulted in slightly fewer t-test rejections and somewhat smaller t-statistics on the rejected variables. In nineteen out of twenty strata, one or more covariates had t-test rejections. Dehejia and Wahba (1997) suggest further stratification, or restructuring of the logistic regression that creates the propensity scores. Further stratification may not yield results until the sample size is down to two digits. If one views the household income variable means for the for-profit and not-for-profit subsets of the stratum showed above in Table 18, it is not clear that further stratification would ever solve the problem of substantially different household income levels in that stratum. At any rate, division of the data set into twenty or more strata requires substantial computing effort, with little promise that certain variables would ever equalize across ownership types. It is also theoretically unsound to divide strata that have substantially different for-profit and not-for-profit subsets into tiny strata just to get good t-test results on the basis of the very small number of observations.

The other option suggested by Dehejia and Wahba is to respecify the logistic regression in obtaining the propensity scores. Rosenbaum and Rubin (1984) describe a process of cycling between reformulating the original propensity score model and checking the composition of the sub-samples. The goal is to specify a logistic equation (using, for example, higher order terms and interactive terms for some variables) which produces propensity scores that aid in equalizing the for-profit and not-for-profit variable means in each stratum. There are no theoretical restrictions on specification of the propensity score regression, since it is simply a tool for ordering the observations in a useful manner. On the other hand, there are few specific guidelines for the ideal specification of the propensity score regression. Thus, this option can be even more labor-intensive than tiny stratification, for it involves respecifying the logistic regression in several ways, and testing the viability of each specification by observing the number of t-tests rejected for various numbers of strata.
The data set used here has a large number of covariates and a fair number of population differences between for-profit and not-for-profit patients, so the problems encountered in finding strata without t-test rejections may be more numerous than with other studies using propensity score methodology. The propensity score ranking of the data described in Table 16 has numerous patterns of variable means. Some variables have an identifiable linear or quadratic trend in their means when the observations are ranked by propensity score. Other variables means show no such discernable pattern. A data set with fewer independent variables would exhibit fewer nonlinear fluctuations in variable means in a propensity score ranking, and creating identical samples of for-profit and not-for-profit observations would require fewer strata. Nevertheless, all of the independent variables used in this analysis have a statistically significant effect on patient mortality, and must be included in the analysis.

However, the data set used here has the advantage of being very large. With a large data set, there is more opportunity to use subsets of the data without much loss of explanatory power. An alternative to propensity score methodology is to break the data down into categories and examine different population groups separately. Thus, it may be instructive to examine only white, male, diabetic patients who do not live in the south. There are 8908 such patients in the U.S. end-stage renal disease program. Unfortunately, with such a division of the population into cells, this ignores the role continuous variables play in the analysis. Household income, length of time on dialysis, and especially age are very important determinants of mortality with dialysis patients. If age, for example, is divided into groups for separation of the population into cells, what sort of division of age groups is needed? It is not certain whether a young-middle-old division is meaningful, or age group divisions by ten or even five years is better. Since there are only 8908 white, male, diabetic, northern patients, division by ten age groups reduces the data to subsets of only 891 patients on average, and still ignores the roles of entry regulations, household income, average years of schooling, and so on. Thus, the propensity score methodology
looks more promising in comparison than a simple division of the data into population cell groups, despite the computational difficulties.

The method described above for obtaining identical ownership subsets of strata is summarized as follows:

1. Perform a logistic regression on the ownership variable to obtain propensity scores for each observation.
2. Rank the observations by propensity score.
3. Remove from the data the observations whose propensity scores do not overlap propensity scores from patients in the other ownership group (the very highest and very lowest propensity score observations).
4. Divide the remaining data into strata.
5. Perform t-tests on the means of the for-profit and not-for-profit subsets of each strata for each covariate (age, diagnosis, race, etc.).
6. If no t-tests are rejected for any covariates in a stratum, that stratum has essentially identical for-profit and not-for-profit patient populations.
7. If one or more t-tests are rejected inside a stratum:
   a. Divide the stratum into sub-strata and perform t-tests again. Repeat until no t-tests fail for any variable in any of the sub-strata.
   Or, b. Respecify the variables in the original logistic regression and repeat the stratification process.

The problem with this process is that the original propensity score logistic regression may not be adequate for obtaining for-profit and not-for-profit means of covariates in the strata unless extremely small strata are created. Plus, the removal of the very highest and lowest propensity score observations described in number three above changes the make-
up of the data set. The sub-stratification process described in number seven does not take advantage of the new, less extreme characteristics of the smaller data set. Two main improvements in the stratification process can be achieved from utilizing more than one logistic regression to order the data.

PART C. TWO SUGGESTED IMPROVEMENTS OF THE PROPENSITY SCORE STRATIFICATION PROCESS

I. Reduction of the Data Set by Iterative Logistic Regression

For analysis of very large data sets, it could be a computationally expedient option to simply discard more data than the data described in number three. For example, a researcher might save time by dividing the data into five strata and discarding all the data except for the middle (third) stratum. However, the middle stratum is based on the original propensity score ranking of the entire data set, and observations in that middle stratum could still differ substantially, due to the inclusion of extreme covariate values in the original propensity score ordering.

Therefore, if the data set is large, allowing reduction of the data set without too much reduction of explanatory power, it would be better to use an iterative logistic process to arrive at a more successful ranking of observations by propensity score. For example, assume that for-profit facilities have older patients. A propensity score logistic regression would give older patients a higher propensity score (propensity to be a patient in a for-profit facility). The strata with the highest and lowest propensity scores could be discarded, leaving the middle strata remaining with fewer very old and very young patients. Further analysis of the remaining middle strata may not easily yield identical sub-strata because the logistic regression used to order the data was based on the complete data set, which included the extreme observations. It is a simple and intuitively
attractive option, then, to perform the logistic regression again, using the new smaller data subset that does not contain the extremely old or extremely young observations. A new ranking of patients by propensity score is obtained from the second logistic regression, and the stratification process will be more successful.

Performing more than one logistic regression in an iterative process is allowable because the intent is merely to find the optimal ordering of observations for groupings into operationally identical for-profit and not-for-profit subsets. An iterative process was very important for analysis of this particular data set, which contained nine covariates. Propensity score rankings differed substantially as different levels of logistic regression were used, each time eliminating the highest and lowest propensity score observations. The following is a summary of the iterative stratification process:

1. Perform a logistic regression on the ownership variable to obtain a propensity score for each observation.
2. Rank the observations by propensity score.
3. Remove from the data the observations whose propensity scores do not overlap propensity scores from patients in the other ownership group (the very highest and very lowest propensity score observations).
4. Divide the remaining data into strata.
5. Remove the highest and lowest strata from the analysis (here, the upper and lower 10% were removed).
6. For the remaining data perform a second logistic regression on the ownership variable to obtain a new propensity score for each observation.
7. Rank the observations by the new propensity score.
8. Divide the data into strata.
9. Perform t-tests on the means of the for-profit and not-for-profit subsets of each strata for each covariate (age, diagnosis, race, etc.).
10. If no t-tests are rejected for a stratum, that stratum has essentially identical for-profit and not-for-profit patient populations.

11. If one or more t-tests are rejected inside a stratum:
   a. Divide the stratum into sub-strata and perform t-tests again. Repeat until no t-tests fail for any variable in any of the sub-strata.
   Or,   b. Repeat steps 5 through 11a again as needed.

This iterative process reduces the data set size each time a new logistic regression is performed, so the researcher should carefully weigh the advantages of having a large data set against the necessity of dividing the data into extremely small strata to analyze. On the other hand the iteration process provides the researcher with an alternative to the prospect of analyzing hundreds of strata, and provides a methodology for reducing a data set to identical treatment/control subsets without ad hoc elimination of observations that have unusual variable values.

II. Recursive Iteration without Reducing Data Set Size

Since successive logistic regression is possible, and in many cases preferred when propensity scores depend heavily on the observations remaining in a data subset, it is useful to re-order the observations within a troublesome stratum by performing the propensity score logistic regression on the observations in that stratum alone. If t-tests fail within a stratum, and further stratification of that stratum appears to yield persistent t-test rejections, another option is to perform the logistic regression again to produce new propensity scores for that stratum alone. The data will then be re-ordered, taking into account the new propensity scores calculated for the stratum alone. Stratification of this stratum alone will yield a more successful ordering of observations in the form of fewer t-test rejections because the ordering did not reflect as many influences from dissimilar observations. This tactic is particularly useful if it is inadvisable to reduce the entire data
set further. A within-stratum reordering from a new logistic regression will produce fewer t-test rejections in sub-strata without reducing the size of the overall data set.

PART D. ALTERNATIVES TO SUB-STRATIFICATION BY PROPENSITY SCORE

Several other approaches to propensity score methodology exist. Dehejia and Wahba (1997) describe matching and weighting by propensity score as two such alternatives to sub-stratification of the data. Matching by propensity score entails computing the propensity score, then finding treatment and control observations that have identical or very close propensity scores. To balance the data when propensity scores differ slightly, Connors et al. (1996) ensure that pairs of observations with positive propensity score differences were matched with pairs of observations with corresponding negative propensity score differences. Matching pairs of observations requires that the propensity score regression be correctly specified. The analysis presented above, in contrast, uses the propensity score calculation as a means of ordering and does not assume or require that it is derived from a correctly specified logistic equation. The second alternative to sub-stratification is weighting each observation by the propensity score. If the propensity score regression was correctly specified, weighting the observation by propensity score, then performing the estimation of the causal effect should presumably correct for population bias. Again, however, the same mis-specification argument against multivariate regression analysis holds for matching and weighting observations by propensity score. There is no guarantee that the regression equation to produce the propensity score is correctly specified.

Several others have investigated the feasibility of combining propensity score methodology with stratification based on category response (for example, black or white, diabetes or hypertension). Cook and Goldman (1988) stratified their data in an
asymmetric fashion on the basis of category response, finding that the number of strata created were smaller than simple sub-stratification would require, and the strata created were more meaningful in interpretation. Stone, et al. (1995) used first a classification tree method of defining seven major groups of observations, then performed propensity score adjustment. The category divisions reduced the overall number of strata required significantly, but even after propensity score adjustment, Stone et al. found that some data bias remained, and they suggested further regression to control for the remaining bias. Both studies used categorical response (or classification tree) stratification in an advantageous way to reduce the number of strata required. Inasmuch as strata are defined whose data are comparable in treatment and control groups, categorical response methods are useful and intuitively appealing. However, introducing categorical response methods introduces \textit{ad hoc} divisions of strata, and the study loses the objectivity associated with letting the propensity score ordering arrange the strata. If many covariates are involved, categorical response methods may become even more unwieldy than sub-stratification methods. Furthermore, stratification by categorical response requires creating artificial cut-off points for categorical division of continuous variables. Finally, if bias still remain in the data, as Stone et al. experienced, the use of multivariate regression to control for the remaining bias introduces the possibility once more of model mis-specification.

\section*{SECTION 4: RESULTS USING PROPENSITY SCORE METHODOLOGY}

With the data described in this study, four separate logistic regressions on successively smaller subsets of the data were performed (Section 3.C.I.). Each time the extreme propensity score strata were removed and a new, smaller data set was used, the covariates’ chi-square values (influence on the propensity score) reported in the logistic
regression results declined. After four such iterations, strata were delineated for the remaining data set of 15,422 observations. Division of the data into five strata yielded "identical" for-profit and not-for-profit independent variable values for two of the five strata, but the other three strata had t-test rejections, so the remaining three strata were each divided into two more strata. Of these new smaller strata, each containing one tenth of the data (1542 observations), three were successful, while the other three strata had t-test rejections. The strata with t-test rejections were re-ordered using a new logistic regression for the propensity scores (Section 3.C.II.), then further subdivided. The flowchart on the following page shows the iteration and subdivision process for obtaining strata with statistically similar for-profit and not-for-profit observations.
From the initial data set of 77,110 observations, four logistic regressions were run in succession, each time eliminating the top ten percent and bottom ten percent of the observations ranked by propensity score. The remaining data set has 15,422 observations.

<table>
<thead>
<tr>
<th># Observations</th>
<th>Strata Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>3085</td>
<td>Stratum 1</td>
</tr>
<tr>
<td>1542</td>
<td>Fifth level logit</td>
</tr>
<tr>
<td></td>
<td>Stratum 2</td>
</tr>
<tr>
<td>771</td>
<td>Sixth level logit</td>
</tr>
<tr>
<td></td>
<td>Stratum 3</td>
</tr>
<tr>
<td>771</td>
<td></td>
</tr>
<tr>
<td>1542</td>
<td>Stratum 4</td>
</tr>
<tr>
<td>1542</td>
<td>Fifth level logit</td>
</tr>
<tr>
<td></td>
<td>Stratum 5</td>
</tr>
<tr>
<td>1542</td>
<td>Sixth level logit</td>
</tr>
<tr>
<td></td>
<td>Stratum 6</td>
</tr>
<tr>
<td>1542</td>
<td></td>
</tr>
<tr>
<td>1542</td>
<td>Stratum 7</td>
</tr>
<tr>
<td>289</td>
<td>Fifth level logit</td>
</tr>
<tr>
<td>289</td>
<td>Sixth level logit</td>
</tr>
<tr>
<td></td>
<td>7th level logit</td>
</tr>
<tr>
<td></td>
<td>Stratum 8</td>
</tr>
<tr>
<td>289</td>
<td></td>
</tr>
<tr>
<td>289</td>
<td>Stratum 9</td>
</tr>
<tr>
<td>289</td>
<td>7th level logit</td>
</tr>
<tr>
<td>289</td>
<td>Stratum 10</td>
</tr>
<tr>
<td>386</td>
<td></td>
</tr>
<tr>
<td>386</td>
<td>Stratum 11</td>
</tr>
<tr>
<td>3085</td>
<td></td>
</tr>
<tr>
<td>3085</td>
<td>Stratum 12</td>
</tr>
<tr>
<td>Total Observations = 15,422</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 1: Stratification Flow Chart*
One of the best features of propensity score methodology is the ease of interpretation of the results. The estimate produced for the expected causal effect of for-profit treatment as opposed to not-for-profit treatment (or treatment versus control), is a consistent, unbiased estimate as long as all the t-tests for each covariate fail to reject the null hypothesis of a zero difference in means between for-profit and not-for-profit patients in each stratum. Once the stratification process is finished, calculation of the causal effect is straightforward. For each ownership subset of a stratum, the mean of the death variable is obtained. The difference in means is calculated for each stratum, and multiplied by the number of observations in that stratum. Then, these weighted differences in means are summed and divided by the total sample size. The estimated causal effect, \( E \), is calculated as follows:

\[
E = \frac{1}{N} \Sigma \frac{n_j}{n} [d(\text{FP}) - d(\text{NFP})],
\]

where \( j = 1, \ldots, 13 \) strata, \( n \) are the individual strata sizes, \( N = 15,422 \), and \( d(\text{FP}) \) and \( d(\text{NFP}) \) are the stratum means for the probability of death (number of deaths divided by number of patients) at for-profit and not-for-profit facilities.

The estimated causal effect on mortality for patients treated at for-profit facilities, corresponding to the data set used, is 5.86%. That is, operationally identical patients were 5.86% more likely to die in 1993 if they were treated at a for-profit facility rather than a not-for-profit facility. This estimate is quite large – of the thirteen strata in the analysis, only one stratum had a higher death rate at not-for-profit facilities.

Since the sample used in the propensity score analysis was only 15,422 patients, the sample means of all the variables should be noted, so that general differences in the data
set sample and the entire US data set can be observed. The iterative process that was used to reduce the data set to one that produced homogenous population characteristics among for-profit and not-for-profit patients had the effect of changing the general make-up of the data set.

Table 19. Propensity Score Project Sample Characteristics, After Reduction by Iteration

Total sample size = 15,422.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>63.84</td>
<td>14.39</td>
<td>66</td>
<td>36</td>
<td>84</td>
</tr>
<tr>
<td>Vintage (days on dialysis)</td>
<td>1009</td>
<td>1024</td>
<td>699</td>
<td>82</td>
<td>3060</td>
</tr>
<tr>
<td>Household Income (in patient’s zipcode)</td>
<td>26,225</td>
<td>10,240</td>
<td>24,705</td>
<td>13,162</td>
<td>46,658</td>
</tr>
<tr>
<td>Average Years of Schooling (in patient’s zipcode)</td>
<td>12.40</td>
<td>0.94</td>
<td>12.36</td>
<td>11.10</td>
<td>14.13</td>
</tr>
</tbody>
</table>

Gender: Female: n = 7,378 (47.8%) Male: n = 8,044 (52.2%)

Race: Black: n = 7,299 (47.3%) White: n = 8,123 (52.7%)

Primary Diagnosis: Diabetes: n = 6,321 (41.0%) Hypertension: n = 9,101 (59.0%)

Area:
Patients not in South: n = 4,071. Facilities not in South: n = 691.

Certificate of Need (CON) Status:
Patient’s facility in state with CON entry regulations: n = 1807.
Number of facilities in state with CON entry regulations: n = 248.

Patient’s facility in state without entry regulations: n = 13,615.
Number of facilities in state without CON entry regulations: n = 833.
*Table 19 continued on next page.*
Table 19, continued.
Ownership:
Patients at for-profit facilities: n = 13,481.
Number of for-profit facilities: n = 944.

Patients at not-for-profit facilities: n = 1,941.
Number of not-for-profit facilities: n = 137.

The iterative logistic regression process of eliminating upper and lower strata with extreme propensity scores produced this new, smaller sample resembling the population data of 77,110 observations described in Table 16 for household income, average years of schooling, and days on dialysis. However, comparing Tables 16 and 19, it is clear that the smaller sample has proportionally more patients in the south and patients in states without Certificate of Need entry regulations. Furthermore, the new data set has fewer diabetic patients and slightly more older, white and male patients than the larger population data. Therefore, a drawback to the iterative method of defining a sub-sample with more identical characteristics between for-profit and not-for-profit patients is that the process may create a new sample that does not closely resemble the population.

The advantage of iterative elimination of data with extreme characteristics is shown below in Table 20, where difference in means t-tests are reported for the for-profit and not-for-profit subsets of the large and small data sets. For the larger population of 77,110 patients, t-tests on differences in means for the for-profit and not-for-profit subsets are strongly rejected for seven of the nine independent variables. The smaller data set, however, shows only two t-test rejections. It would be tempting to accept the smaller data set without stratification, noting that the t-test rejection on the south variable indicates a positive bias for for-profit units ("south" is positively associated with higher mortality, as reported in Table 17), and the age variable counters that with a bias in favor of not-for-profit facilities. However, age has the strongest effect on mortality of all the
independent variables, so an age discrepancy in the data is a serious flaw, and should not be dismissed because the t-statistic was rather small. Hence, stratification of the data was necessary to achieve an unbiased and consistent estimate of the causal effect of for-profit treatment on mortality.

Table 20. Difference in Means Tests for the Population and Reduced Sample

<table>
<thead>
<tr>
<th>Variable</th>
<th>Population: n = 77,110</th>
<th>Reduced Sample: n = 15,422</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Difference in Means t-test</td>
<td>Difference in Means t-test</td>
</tr>
<tr>
<td>Age</td>
<td>4.28 reject: Bias favors NFP</td>
<td>2.20 reject: Bias favors NFP</td>
</tr>
<tr>
<td>Days on Dialysis</td>
<td>- 9.53 reject: Bias favors FP</td>
<td>0.07</td>
</tr>
<tr>
<td>Female</td>
<td>- 0.24</td>
<td>0.23</td>
</tr>
<tr>
<td>Diabetic</td>
<td>- 1.64 (slight bias favoring FP)</td>
<td>- 0.27</td>
</tr>
<tr>
<td>Black</td>
<td>- 8.10 reject: Bias favors NFP</td>
<td>- 1.18</td>
</tr>
<tr>
<td>South</td>
<td>- 4.11 reject: Bias favors FP</td>
<td>- 2.86 reject: Bias favors FP</td>
</tr>
<tr>
<td>Cert. of Need</td>
<td>-44.34 reject: Bias favors FP</td>
<td>0.26</td>
</tr>
<tr>
<td>Household Income</td>
<td>12.01 reject: Bias favors FP</td>
<td>0.08</td>
</tr>
<tr>
<td>Avg. Years Schooling</td>
<td>- 3.65 reject: Bias favors FP</td>
<td>0.18</td>
</tr>
</tbody>
</table>

It is also instructive to estimate the difference in means for the samples above for the mortality variable in order to gauge how general population differences noted above in Tables 16 and 19 affect average mortality rates. Table 21 below shows that the reduced sample of 15,422 has a somewhat larger proportional number of deaths, despite the fewer numbers of diabetic patients in the sample. Most of the differences in the smaller sample as compared to the population are correlated with higher mortality, such as southern status, older and more white patients. For-profit facilities appear to have higher death rates in the reduced sample, but interestingly, not-for-profit facilities have lower death rates in the reduced sample.
Table 21. Average Number of Deaths for the Population and Reduced Sample

<table>
<thead>
<tr>
<th>Sample/Ownership</th>
<th>Population: n = 77,110</th>
<th>Reduced Sample: n = 15,422</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>For-profit</td>
<td>Not-for-profit</td>
</tr>
<tr>
<td>Number of Deaths</td>
<td>42.5%</td>
<td>39.4%</td>
</tr>
<tr>
<td>Difference in Means</td>
<td>3.17%</td>
<td>6.20%</td>
</tr>
</tbody>
</table>

The reason that the difference in means calculated for the reduced sample is 6.2%, while the estimated causal effect calculated after the stratification process above is 5.86% is straightforward. The difference in means of death rates for the two ownership subsets is higher in the entire data set before stratification because the data have not been adjusted for the potentially damaging bias due to a slightly older population in the for-profit facilities. Simply reporting the difference in means as presented in Table 21 would overstate the effect on mortality from treatment at a for-profit unit. The stratification process succeeds in eliminating the age bias in each stratum, so the estimated causal effect of for-profit treatment after stratification is smaller and unbiased.

SECTION 5: PROPENSITY SCORE ANALYSIS SUMMARY

The propensity score methodology offers a way to duplicate and perhaps improve upon a completely randomized study by estimating the average causal effect of treatment from trials of closely matched populations. When the initial population is not well-matched in patients characteristics and many significant covariates must be controlled for in the
study, such as with this nationwide data of renal dialysis patients, the stratification process suggested by others for achieving identically matched samples of treatment and control group patients is laborious and may not be successful until over 100 subsets of the original data set are separately evaluated.

Two improvements were introduced to aid in creating data subsets with identically matched characteristics. The first proposal is to reduce the data set for analysis in size with an iterative logistic process, repeating a logistic regression for ordering the data by propensity score, and each time removing the upper and lower-ranked strata. As the logistic regression re-ordering and removal of extreme data values progresses, the new sample’s treatment and control subsets will have variable means that are progressively closer.

After the final sample size is reached, the second proposed improvement can be employed in the stratification process. Dividing the data into five subsets may reveal some subsets that have no statistically significant differences in means. These strata should be retained as-is for calculation of the average causal effect of treatment later. The other strata with rejected t-tests should be subdivided further. When further subdivision fails to produce identical treatment and control subsets, the second proposed improvement in the stratification process is to first perform a logistic regression to reorder the observations in the strata that failed, then subdivide the data. Performing an additional logistic regression to reorder the observations in a stratum before subdividing greatly increases the likelihood that the new strata will have closely matched population characteristics. These two proposed improvements in the propensity score methodology save computing effort and avoid the intuitively unattractive prospect of basing the calculation of the causal effect of treatment on numerous tiny subsets of the original data.
set. The variance of the estimated causal effect is inversely related to the sizes of the individual strata from which it was calculated.

Using these two proposed changes to the propensity score stratification process, the final data set size was chosen to be 15,422 observations and the number of strata analyzed was thirteen, with the smallest stratum containing 289 observations. Using strictly sub-subdivision of the data to find adequate strata would have necessitated creating over thirty strata for analysis, so the technique introduced in this chapter significantly reduced the number of strata from which the estimated causal effect was calculated.

The estimated causal effect of treatment at a for-profit facility was calculated to be a 5.86\% increase in patient mortality. Although the sample of 15,422 patients had statistically identical independent variable values for for-profit and not-for-profit patients, the overall characteristics of the sample's patients were different than the general population of patients. The sample had more white patients, more patients in the south, fewer diabetic patients, and slightly older patients than the population from which it came. Thus, results are unbiased and consistent for the sample analyzed, but may not easily be generalized to the entire population of patients receiving dialysis for end-stage renal disease in the US. Nevertheless, analysis of the data revealed a persistent pattern of higher mortality rates in for-profit facilities. For this sample of 15,422 patients alone, if all patients had been treated at for-profit facilities, approximately 100 additional patients would have died during the year. If all patients in the sample had been in not-for-profit facilities, roughly 750 more patients would have survived the year.
CHAPTER 6: ASYMMETRIC INFORMATION, OWNERSHIP, AND QUALITY OF CARE IN THE US RENAL DIALYSIS INDUSTRY

SECTION 1: INTRODUCTION

Several theorists have asserted that not-for-profit firms exist to mitigate agency problems in donor-financed charitable organizations (Fama and Jensen, 1983[b] and Hansmann, 1980). This was a strong rationale for the founding of charity hospitals in the U.S. during the late 19th to early 20th centuries. However, since most not-for-profit health care organizations now earn most of their revenue from commercial sources, theorists turn to other reasons for the continued strength of the not-for-profit sector. The 1960’s through the mid-1980’s saw expansion of both not-for-profit and for-profit health care sectors due to the generous cost-based reimbursement financing of health care from Medicare and employer-provided health insurance. Favorable tax conditions, too, have aided the survival of not-for-profit health care organizations in increasingly competitive times. Occasionally the asymmetric information rationale has been used to explain the existence of not-for-profit firms in commercial health care markets. That is, where consumers have incomplete information about the quality of the product, consumers may seek a not-for-profit firm under the assumption that, due to the non-distribution constraint, not-for-profit managers have less incentive to exploit information advantages by reducing quality to increase profits.

This chapter describes an empirical project using data from the U.S. renal dialysis industry. End-stage renal disease (ESRD) patients are ranked by level of sophistication, then patient treatment data are adjusted for case mix differences and examined to see if
treatment patterns differ by patient sophistication level. Evidence indicates that unsophisticated patients receive lower quality care from their dialysis facility in the form of more shortened and skipped dialysis treatments. Also, patients at for-profit facilities have more skipped dialysis treatments than patients at not-for-profit facilities. Physicians appear to prescribe shorter hemodialysis treatments for more sophisticated patients, case-mix adjusted, but the effects of facility ownership on physician hemodialysis prescriptions is indeterminate. Thus, facility ownership may affect patient care at the institutional level, but has no significant effect on care provided by the physician. Results imply that asymmetric information in the renal dialysis industry accounts for a differing quality of treatment provided to patients of differing knowledge levels, but results lend mixed support for the theoretical proposition that for-profit firms exploit information asymmetries more often than not-for-profit firms.

SECTION 2: THE ASYMMETRIC INFORMATION RATIONALE FOR NOT-FOR-PROFIT HEALTH CARE

Some economists have claimed that patients select not-for-profit health care providers in an effort to alleviate information asymmetries. That is, not-for-profit organizational status serves as a trust signal for patients unable to evaluate quality of care (Frank and Salkever, 1994 and Weisbrod, 1989). The patient assumes that not-for-profit providers have less incentive than for-profit providers to reduce unmonitored quality in order to increase profits. Although this line of reasoning may seem attractive for static analysis, it ignores intertemporal effects on the firm’s reputation and ability to attract future customers. Since both for-profit and not-for-profit firms currently co-exist in health care, it appears that asymmetric information problems may be solved by time effects, and therefore, quality should be uniform among for-profit and not-for-profit providers.
Permut (1981) tested the trust signal theory with surveys of customers of not-for-profit organizations and found that very few actually knew the ownership status of the firm, or believed that a not-for-profit organization would provide a higher quality product. We are presented, then, with a possible continuum of patient ability to judge and respond to quality: First, a subset of patients know the quality of care they are receiving and go to the highest quality provider they can, given equivalent prices paid by third-party payers. Second, other patients are unable to judge quality, but do know the ownership status of the firm and select not-for-profit firms on the basis of the trust signal. Third, the rest of the patients can not judge quality and do not know the ownership status of the firm. The second subset is probably very small indeed. Also, the mechanics of hospital selection are probably quite removed from the for-profit versus not-for-profit decision, since patients usually go wherever their physician is on staff.

If the trust signal argument holds, we would expect to see more knowledgeable patients who are aware of ownership status selecting not-for-profit facilities. Assuming equal prices paid by third party payers, sophisticated patients would seek out the highest quality provider and the lower quality firms would leave the market, regardless of ownership status. If not-for-profit firms produced a higher-quality product, for-profit firms would eventually be forced out of the market.

In fact, for-profit health care firms are thriving in the US. One explanation could be that patients are heterogeneous enough in their knowledge levels to allow low-quality firms to exist in the market, serving low-knowledge patients. A second reason for a presence of both low and high-quality providers in the health care market may be that health care firms are competing on the basis of quality that is visible to the patient rather than clinical quality. Thus, parking, appointment scheduling, clinic décor, and waiting room times
may be more immediately important to the patient than clinical treatment. Firms providing clinical care of a lower quality may be providing amenities that attract and keep customers. This may be considered a subset of the first reason – that is, patients who place more emphasis on amenities rather than clinical care might be considered by physicians to be less knowledgeable than patients intent on receiving the best clinical care.

A third reason for the co-existence of low and high-quality firms is that some health care sectors such as long term care operate with substantial self-paying patients. In this case, patients may be selecting into separate niches – the higher-priced not-for-profit niche and the inexpensive but lower quality for-profit niche. The comparative absence of fixed payments by third party payers in the nursing home sector makes it impossible to fault providers for providing a level of care that differs from their competitors. Similarly, at the level of insurance and/or managed care plan selection, consumers simply get what they pay for, and a differing level of quality provided may be merely attributable to the different prices paid for premiums. For this reason, the analysis in this paper focuses on one sector where costs of care are paid for almost exclusively by Medicare and virtually no self-pay patients exist. Patients’ choice of facility is removed from the price decision in this sector, so firms compete on the basis of quality.

Finally, persistent quality variance by ownership may be attributed to differing preferences of the medical staff. Where physician compensation can be tied to residual earnings, physicians more inclined to favor income may gravitate toward facilities offering higher claims on residual earnings. Physicians with a strong preference for higher quality of care may be on staff at institutions producing high quality clinical care, even though they receive less compensation. Preston (1989) studied the salaries of not-for-profit and for-profit professionals and suggested that employees of not-for-profit
institutions trade higher salaries for jobs that produce more social benefits. However, Preston was unable to rule out the possibility that not-for-profit professionals are lower-quality employees self-selecting into lower-paying professions.

Using the mathematical model of managerial utility maximization described in Chapter 2, the notion of high versus low-knowledge patients appears in the form of two separate populations of patients: patients who enter regardless of the quality of the firm (exogenous variable E) and patients who enter higher quality facilities more often (variable i(u)). When results are derived from the model, the existence of no-knowledge patients, as symbolized by E, does not appear as a variable predicting differing behavior between for-profit and not-for-profit firms. However, since the model predicts a higher quality level over time by not-for-profit firms (u(t)), these firms are thus predicted to have a greater population of knowledgeable patients (i(u)).

Since the results in Chapters 4 and 5 suggest lower quality at for-profit firms in the form of higher patient mortality, and since a third party (Medicare) is paying for patients’ dialysis treatments, one would expect that patients would avoid for-profit facilities, ceteris paribus, because the patients are shopping on the basis of quality, not price. This would eventually lead to the decline of for-profit facilities. In fact, the number of for-profit facilities, especially in states with no Certificate of Need entry regulations, is increasing and the number of not-for-profit facilities is decreasing. The presence of asymmetric information could account for the strength of the for-profit sector despite the outcome quality disparity. If patients have no knowledge about clinical quality, they may be basing their choice of provider on other, more visible aspects of quality such as location and appearance of the facility.
The empirical project described in this chapter has two tiers: First, the project investigates the question of whether facilities are taking advantage of asymmetric information by providing lower quality care to low-knowledge patients. Although this question is not central to the issue of comparative quality by ownership type, it is of very high interest in the health economics field, and needs exploration. Second, the project seeks to determine if for-profit facilities are more aggressive about taking advantage of information asymmetries than their not-for-profit counterparts. If this is true, the asymmetric information (or more generally, the contract failure) rationale for the not-for-profit form of organization is valid as a justification for subsidy of not-for-profit health care.

SECTION 3: PREVIOUS STUDIES OF ASYMMETRIC INFORMATION AND QUALITY OF CARE

Weisbrod and Schlesinger (1986) controlled for price differences and reviewed regulatory violations and consumer complaint data for nursing homes in Wisconsin. They found that for-profit institutions had more consumer complaints but fewer regulatory violations than not-for-profit institutions. They conclude that not-for-profits operate with more organizational slack, leading to more regulatory violations of quality that are easily monitored. Because for-profits were found to have more consumer complaints, they conclude that for-profits are lowering quality attributes that are more costly to monitor by regulatory agencies, but are noticed by the consumers. In their study, for-profits appear to be promising one level of quality to consumers, but delivering another and are thus taking advantage of information asymmetries.
Another study (Holtmann and Ulmann, 1993) used marital status, number of living children, extent of disability, living arrangement, and mental health as independent variables describing transactions costs to monitoring uncertain quality of care. Holtmann and Ulmann predicted that high-cost (that is, low knowledge) patients would choose high-cost, not-for-profit nursing home providers in an effort to avoid future opportunistic behavior by for-profit firms. They found that patients with low-knowledge characteristics, such as those who are not married, are more likely to be in not-for-profit nursing homes. On the other hand, they found that patients with more mental health problems were found in for-profit institutions. Holtmann and Ulmann conclude that low-knowledge patients are selecting into not-for-profit firms because they recognize that they are unable to monitor quality deficiencies, while more knowledgeable patients are selecting the relatively cheaper for-profit nursing homes. Their conclusion rests, however, on the somewhat shaky assumption that patients with lower knowledge levels still are able to recognize their disability and choose their provider strategically.

Scant evidence exists in economics literature to suggest quality differences among for-profit and not-for-profit health care providers, except in the renal dialysis industry, where mortality rates appear to be higher among patients in for-profit facilities. Although a number of empirical studies have explored the relation between ownership status and quality of care as measured by treatment inputs, data are scarce on patient knowledge levels, so virtually no studies have attempted to analyze health care providers’ treatment of patients with different knowledge levels by ownership. Rather than analyzing patient choice of provider, the study described in this chapter instead focuses on provider treatments as they differ by patient knowledge levels.
SECTION 4: DATA ANALYSIS AND METHODS

PART A. DATA SOURCES AND DESCRIPTIONS

The sample consists of approximately 4900 patients being treated at for-profit and not-for-profit dialysis units nationwide in 1990-92. This includes all patients who had been on dialysis for three or more months and receiving Medicare benefits during 1990-92. Omitted from the study are patients whose main treatment facility was government-owned, and patients whose main treatment modality for dialysis was not center-based hemodialysis. Data programming and regression analysis was performed on a PC with version 6.12 of SAS for Windows. Least squares and logistic regressions were run in SAS and negative binomial regressions were run in version 6.0 of LIMDEP (MS-DOS).

The data are from several sources. Patient and facility-level data were from the Patient, Residence, Treatment History, and Facility Standard Analysis Files (SAFs) from the US Renal Data System (USRDS), and were from the same data as described in Table 1 of Chapter 4. Sophistication variables and treatment variables were from the USRDS’ Case Mix Adequacy SAF. The Case Mix Adequacy SAF, unlike the other files containing patient data nationwide, contained data from a small sample of 7000 patients. Data collection by the USRDS began on April 1, 1992. The purpose of collecting the Case Mix Adequacy data was to examine the relationship between dialysis dose, delivered dialysis therapy, reuse of dialyzers, and patient mortality. The Case Mix Adequacy sample was randomly selected from an incident sample of patients starting dialysis during 1990 and a prevalent sample of hemodialysis patients with onset of ESRD prior to 1990. The facilities were chosen by the USRDS to be representative of the nation. Patients were then chosen from the facilities by random. ESRD regional network offices collected these data in conjunction with their Medical Case Review data abstraction. Included in the data are variables that can be assumed to vary with a patient’s knowledge.
or sophistication level. These variables include patient education level, job status, living situation, and other key predictors of knowledge, as explained later in Table 24. The data also included patient treatment variables, from which we selected number of hemodialysis hours prescribed per week, number of treatments skipped and number of treatments shortened as indicators of physician treatment quality and facility treatment quality, respectively.

The size of the final data sets used in regressions below was determined by the following adjustments: The case mix adequacy file consisted of approximately 7000 patients. 179 patients were lost to follow-up during the study. 347 patients were missing identification numbers, so no case-mix information could be matched to their file from the nationwide databases described above. Patients who were being treated at government-owned facilities, patients who were not receiving center-based hemodialysis, and patients who had missing facility identification numbers (and thus had no information on facility ownership status) were eliminated from the sample. The approximately 4900 remaining patients still had various data missing. As described below for each of the three main data sets created, some observations were excluded for lack of an important missing variable, while others were retained by using sample means.

Summary statistics for the data are provided in Tables 22 and 23. Age, days on dialysis, gender, race, and primary diagnosis were included as they are important patient case-mix characteristics. The variable South was included to catch regional differences in physician and facility staff treatment styles. The Certificate of Need (CON) regulations variable noted the presence of entry regulations in the facility's state. It was included to capture important competitive effects between existing facilities and potential entrants into the market. The effects of competition may show up as an increase in patient treatment quality, since patients do not pay for dialysis treatments themselves and have
the incentive to shop for a facility on the basis of quality. On the other hand, competition may induce firms to reduce unmonitored treatment quality to low-knowledge patients. The freestanding variable was included because patient case-mix may differ substantially between freestanding and hospital-based facilities.

Table 22. Patient Characteristics: n = 4939

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
<th>Missing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>61.45</td>
<td>16.09</td>
<td>64</td>
<td>32</td>
<td>84</td>
<td>0</td>
</tr>
<tr>
<td>Days on dialysis</td>
<td>1802</td>
<td>1217</td>
<td>1481</td>
<td>413</td>
<td>4399</td>
<td>0</td>
</tr>
<tr>
<td>Knowledge Score</td>
<td>6.67</td>
<td>1.20</td>
<td>6.67</td>
<td>4.33</td>
<td>8.67</td>
<td>0</td>
</tr>
<tr>
<td>Skipped Treatments</td>
<td>.54</td>
<td>1.45</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3245</td>
</tr>
<tr>
<td>Shortened Treatments</td>
<td>.90</td>
<td>2.05</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>3134</td>
</tr>
<tr>
<td>Dialysis Hours/week</td>
<td>8.90</td>
<td>2.05</td>
<td>9</td>
<td>6</td>
<td>12</td>
<td>117</td>
</tr>
</tbody>
</table>

Gender: Female (dummy variable = 1): n = 2406 (48.7%)   Male: n = 2533 (51.3%)

Race:
Native American: n = 77 (1.6%)
Black: n = 1757 (35.6%)
White: n = 2950 (59.7%) (omitted reference category)
Asian: n = 102 (2.1%)
Other race: n = 43 (.9%)

Primary Diagnosis Causing End Stage Renal Disease:
Diabetic: n = 1542 (31.2%)
Hypertension: n = 1386 (28.1%)
Glomerulonephritis: n = 716 (14.5%)
Cystic Kidney: n = 177 (3.6%)
Other Primary Diagnoses: n = 1118 (22.6%) (omitted reference category)
Table 23. Facility Characteristics: n = 513

Facility in South (dummy = 1): n = 152   Facility not in South: n = 361

Facility in state with Certificate of Need regulations (dummy = 1): n = 187
Facility in state with no Certificate of Need regulations: n = 326

Facility free-standing (dummy = 1): n = 388   Facility in hospital: n = 125

For-profit facility (dummy = 1): n = 336   Not-for-profit facility: n = 177

PART B. DEPENDENT VARIABLES

The three dependent variables of analysis in this study are introduced in Table 22 as number of treatments shortened and number of treatments skipped during a specified one-month period and the number of dialysis hours prescribed weekly by the patient's physician. The number of treatments shortened skipped offer a perspective on quality of treatment by the dialysis facility. Regardless of physician prescription, facility nurses, technicians, and other staff can affect the level of adherence to the prescribed number of dialysis hours. Staff at a high-quality facility may work closely with patients to ensure that few treatments are skipped or shortened. The skipped variable refers to the number of treatments skipped during December 1-23, 1993, and the shortened variable refers to the number of treatments during December 1993 that were shortened by more than ten minutes, not including skipped treatments. Hemodialysis is a long process, and patients less concerned with their health may seek to skip or shorten the procedure frequently. With effort, facility staff can reduce this behavior to a minimum. Facilities are not compensated for treatments that are skipped, but are compensated for treatments that are shortened.

The main problem with the Skipped and Shortened variables is missing answers to the survey. Over half of the observations for those variables are blank. Data sets with the
subset that contains completed surveys were constructed for each of the independent
to complete sets of observations. Differences in sample case-mix and
facility variables (such as age, sex, ownership, etc.) were slight, indicating that the
smaller sub-samples had slightly more minorities and fewer white patients, a tighter
distribution on the patient knowledge variable, and fewer for-profit firms. Tables 22 and
23 above are repeated for the sub-samples in Appendices A and B. Since proportionally
fewer for-profit firms are in the sub-samples, it means that for-profit firms were more
likely to leave the Skipped and Shortened survey questions blank. Including the blank
answers as zeros introduces a bias in favor of for-profit firms. Therefore, when the
regressions were run, a complete set of observations was also tested, whereby all blank
answers to Skipped and Shortened were considered zero. Results were consistent for
either data set specification.

The dialysis prescription variable can be considered an indicator of quality care stemming
from the physician's decisions. Held et al. (1991) report that dialysis treatments of
shorter duration contribute to higher mortality of ESRD patients. Other studies showing
a linkage between short duration and higher mortality are Charra et al. (1992), Charra,
Calemard and Laurent (1996) and Kjellstrand (1985). The duration of treatment is also
an important element in facility costs. When patients can be dialyzed for shorter periods,
more patients can be scheduled per week and average fixed costs per patient can be
reduced. Reimbursement per dialysis treatment is fixed prospectively, regardless of the
length of the dialysis session.

Since physician behavior may be influenced by compensation structures, it is necessary
to examine financing of dialysis. All of the patients in the study receive Medicare
funding to pay for dialysis. Medicare compensation for dialysis has two main components. Medicare pays a composite rate fee to dialysis centers for each dialysis treatment. This composite rate does not vary, even though treatment sessions and complications vary from treatment to treatment and from patient to patient. Physicians are also paid a capitated fee per month per dialysis patient. Physicians affiliated with not-for-profit dialysis centers receive only the capitated fee as compensation. Physicians affiliated with for-profit dialysis centers receive the capitated fee, but may also receive a portion of the dialysis facility’s proceeds from the composite rate if they serve as a medical director. Thus, the compensation physicians receive at for-profit dialysis facilities may be partially linked to the dialysis facility’s efficiency, unlike not-for-profit physicians. It is hypothesized that for-profit physicians will prescribe shorter dialysis treatment hours per week than their not-for-profit counterparts due to stronger pressure by administration to reduce costs and due to possible income effects from facility profitability if the physician serves as the medical director.

PART C. CONSTRUCTING THE KNOWLEDGE SCORE

Because the data included an unusual number of demographic variables, it was possible to create a composite score roughly indicating the patient’s level of sophistication and probable awareness of quality. The following were the knowledge indicators used and the values assigned to survey responses, as well as the methodology used to adjust for missing responses when necessary. Note that variables indicating higher levels of knowledge are assigned higher index scores.
Table 24: Knowledge Score Components

<table>
<thead>
<tr>
<th>Variable</th>
<th>Survey Response</th>
<th>Index Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Living Situation</td>
<td>missing or other</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Live alone</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Live with others</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Nursing home, Institution</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Homeless</td>
<td>0</td>
</tr>
<tr>
<td>Education Level</td>
<td>missing or other</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Less than 12 years</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>High school graduate</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Some college</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>College graduate</td>
<td>3</td>
</tr>
<tr>
<td>Number of House-Members</td>
<td>missing</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>&gt; or = 4</td>
<td>3</td>
</tr>
<tr>
<td>Walk Independently</td>
<td>missing</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Yes</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>0</td>
</tr>
<tr>
<td>Eat Independently</td>
<td>missing</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Yes</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>0</td>
</tr>
<tr>
<td>Transfer Independently</td>
<td>missing</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Yes</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>0</td>
</tr>
<tr>
<td>Nutritional Status</td>
<td>missing or other</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Obese/overweight</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Under-nourished/cachetic</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Well-nourished</td>
<td>1</td>
</tr>
</tbody>
</table>

*Table 24 continued on next page.*
Table 24, continued.

<table>
<thead>
<tr>
<th>Occupation</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>missing or other</td>
<td>1</td>
</tr>
<tr>
<td>Clerical</td>
<td>2</td>
</tr>
<tr>
<td>Professional</td>
<td>3</td>
</tr>
<tr>
<td>Tradesperson</td>
<td>2</td>
</tr>
<tr>
<td>Manual labor</td>
<td>1</td>
</tr>
<tr>
<td>Housewife</td>
<td>1</td>
</tr>
<tr>
<td>Student</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Smoking Status</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>missing or other</td>
<td>3</td>
</tr>
<tr>
<td>Active smoker</td>
<td>0</td>
</tr>
<tr>
<td>Former smoker</td>
<td>2</td>
</tr>
<tr>
<td>Smoker, time unknown</td>
<td>1</td>
</tr>
<tr>
<td>Non-smoker</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Employment Status</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>At Start of Study</td>
<td></td>
</tr>
<tr>
<td>missing or other</td>
<td></td>
</tr>
<tr>
<td>use values from another variable indicating highest employment level ever reached</td>
<td></td>
</tr>
<tr>
<td>still missing or other</td>
<td>1</td>
</tr>
<tr>
<td>full-time work or study</td>
<td>3</td>
</tr>
<tr>
<td>part-time work or study</td>
<td>2</td>
</tr>
<tr>
<td>homemaker</td>
<td>2</td>
</tr>
<tr>
<td>retired</td>
<td>1</td>
</tr>
<tr>
<td>unemployed</td>
<td>0</td>
</tr>
<tr>
<td>disabled</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Self-hemodialysis</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>1</td>
</tr>
<tr>
<td>No</td>
<td>0</td>
</tr>
</tbody>
</table>

Most variables had missing data responses of two to three percent, and were corrected for by inputting the mode value for the missing responses. For example, 2.1% of the Eat Independently survey questions were blank. By far, most patients can eat independently so the missing responses were assumed to be “yes.” Less straightforward was the treatment of missing values for the Employment Status variable. A very similar variable, highest employment status ever, was used as a backup to provide responses for missing values. If values were still missing, it was assumed that the patient was retired, or had an...
employment status comparable to being retired. "Retired" is also the mode for that variable. The self-hemodialysis variable, unlike the others above it, was obtained from data in the nationwide sample of all ESRD patients receiving treatment at facilities in the US. A patient performing self-hemodialysis inserts her own needle to access the graft. Nationwide, the number of patients who do this is less than two percent of those receiving treatment for ESRD.

Most of the variables in the knowledge score could be rejected as true indicators of patient sophistication level, but in general point to a greater or lesser likelihood that the patient has knowledge about her quality of care. Smoking status and nutritional status are probably the least indicative of knowledge. If a person smokes or is very overweight, this could signal not a lack of knowledge, but a lack of desire to adhere to a healthy lifestyle. In effect, smoking and obesity could signal a lack of interest in getting well, and thus a lack of interest in gaining information about the quality of care the patient receives.

The seemingly different index scores given to "housewife" and "homemaker" responses on the occupation and employment status variables were chosen to reflect the lifetime occupation choice as opposed to the current status of the patient. That is, a person whose occupation over her lifetime was that of a housewife might be less knowledgeable than a student, tradesperson, clerical worker, or professional. Conversely, a homemaker was judged to have more ability to judge health care quality than a retired person (a retired homemaker, for example) or an unemployed or disabled person. Clearly, the assignment of index scores required ad hoc generalizations about knowledge status with several of the knowledge score variable components.
Index scores for the eleven variables above were then divided by their maximum value so that each index score ranged from zero to one. The calibrated scores were then summed for an overall score, now referred to as the knowledge score. No attempt was made to weight the individual variables for relative importance, as the weighting process would likely be as ad hoc as the process above in assigning values to survey responses. The addition of health-related variables such as smoking and nutritional status is thought to capture attitudinal differences among patients. Similarly, variables signaling independence such as Eat Independently are considered strong indicators of frailty and awareness of care being received. Education, occupation, and employment status were fairly straightforward indicators of patient sophistication level. The patient’s living situation and number of household members were included on the advice of practicing health care workers, who felt strongly that the presence of partners positively affects the quality of care patients receive.

PART D. REGRESSION METHODS

The three dependent variables under study required use of two main regression techniques. The dependent variables stemming from facility care, Skipped and Shortened, were count data truncated at zero. Most of the variable values were zero, but their means were significantly above zero due to higher nonzero observations. These two variables could be analyzed with either Poisson regression or negative binomial regression. OLS regression would not be appropriate in these cases because OLS can produce nonsensical negative predicted values for the dependent variable when used with count data truncated at zero (Roncek, 1998). Poisson regression requires the assumption that the variance of the dependent variable is equal to its mean. Considering the severity of the Poisson requirement, negative binomial regression offers the best choice for analysis of the Skipped and Shortened treatment data (Greene, 1993). The third dependent variable
under study, the number of hemodialysis hours prescribed weekly by the patient's physician (shown in Table 25) could be analyzed by ordinary least squares regression, but since the data resembled counts rather than continuous data, polytomous logit was also employed in analyzing the data.

Table 25: Hemodialysis Hours per Week Prescribed

<table>
<thead>
<tr>
<th>Value</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>86</td>
</tr>
<tr>
<td>6</td>
<td>910</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>65</td>
</tr>
<tr>
<td>9</td>
<td>2780</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>940</td>
</tr>
<tr>
<td>15</td>
<td>14</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
</tr>
</tbody>
</table>

Total Frequency: 4822
(117 observations with missing values were eliminated from the sample.)

SECTION 5: ASYMMETRIC INFORMATION ANALYSIS RESULTS

PART A. SKIPPED TREATMENTS
The results below in Table 26 indicate that case-mix differences in age and length of time on dialysis do have a relationship with number of treatments skipped during the study period. Knowledgeable patients, females, older patients, and patients who had been on
dialysis for a long time were all found to have fewer skipped treatments, as indicated by negative values for coefficients. The existence of Certificate of Need (CON) regulations and for-profit ownership was a statistically significant determinant of greater skipped treatments.

### Table 26: Skipped Treatments: Negative Binomial Regression

Main sample with no missing observations included as zero values.

N = 1694, Log-likelihood = -1445.795

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coeff.</th>
<th>Standard Error</th>
<th>T-ratio</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>1.0656</td>
<td>0.5959</td>
<td>1.799</td>
<td>.0737</td>
</tr>
<tr>
<td>Age</td>
<td>-0.0117</td>
<td>0.0048</td>
<td>-2.436</td>
<td>.0149</td>
</tr>
<tr>
<td>Age Deviation²</td>
<td>0.0001</td>
<td>0.0003</td>
<td>0.349</td>
<td>.7274</td>
</tr>
<tr>
<td>Days on Dialysis</td>
<td>-0.0002</td>
<td>0.530E-04</td>
<td>-3.865</td>
<td>.0001</td>
</tr>
<tr>
<td>Female</td>
<td>-0.4005</td>
<td>0.1368</td>
<td>-2.928</td>
<td>.0003</td>
</tr>
<tr>
<td>Native American</td>
<td>-1.1038</td>
<td>0.7980</td>
<td>-1.383</td>
<td>.1666</td>
</tr>
<tr>
<td>Asian</td>
<td>0.6701</td>
<td>0.4870</td>
<td>1.376</td>
<td>.1688</td>
</tr>
<tr>
<td>Black</td>
<td>0.1265</td>
<td>0.1553</td>
<td>0.814</td>
<td>.4154</td>
</tr>
<tr>
<td>Other race</td>
<td>0.3437</td>
<td>0.6833</td>
<td>0.503</td>
<td>.6149</td>
</tr>
<tr>
<td>Diabetic</td>
<td>-0.1952</td>
<td>0.1775</td>
<td>-1.100</td>
<td>.2715</td>
</tr>
<tr>
<td>Hypertension</td>
<td>-0.0117</td>
<td>0.1815</td>
<td>-0.064</td>
<td>.9487</td>
</tr>
<tr>
<td>Glomeruloneph.</td>
<td>-0.1741</td>
<td>0.2158</td>
<td>-0.807</td>
<td>.4199</td>
</tr>
<tr>
<td>Cystic Kidney</td>
<td>0.2564</td>
<td>0.3589</td>
<td>0.714</td>
<td>.4750</td>
</tr>
</tbody>
</table>

**For-Profit Ownership**

| CON Entry Regs.   | 0.6787   | 0.1439         | 4.718   | .0001   | ***   |
| Facility in South | -0.3194  | 0.1837         | -1.739  | .0821   | *     |
| Free-Standing     | 0.1648   | 0.2238         | 0.737   | .4613   |       |

**Knowledge Score**

| -0.2087           | 0.0548   | -3.809          | .0001   | ***    |

The parameter $\alpha$ measures overdispersion and is used in computation of negative binomial log-likelihoods (Greene, 1992). This dispersion parameter corrects for the variance of the dependent variable not equaling its mean.

$^5$ The parameter $\alpha$ measures overdispersion and is used in computation of negative binomial log-likelihoods (Greene, 1992). This dispersion parameter corrects for the variance of the dependent variable not equaling its mean.
The main results from Table 26 are that for-profit facilities are more likely to have patients with skipped treatments (significant at the p < .01 level) and that high-knowledge patients are less likely to have skipped treatments (significant at the p < .01 level). While it is worthwhile to know that for-profit ownership is linked to greater numbers of skipped treatments, it is also important to determine if ownership determines different treatment of patients by knowledge level. For-profit facilities are hypothesized to take greater advantage than not-for-profit firms of lower-knowledge patients by providing lower quality care to them. If this were true, we would see a significantly more negative coefficient on the knowledge score variable for for-profit facilities, as compared to not-for-profit facilities. Summarized results in Table 27 are provided for the separate ownership types. The data were separated by ownership type, then a regression of remaining independent variables was run on the dependent variable, number of skipped treatments.

<table>
<thead>
<tr>
<th>Ownership Type</th>
<th>Sample Size</th>
<th>Coefficient for Knowledge Score</th>
<th>t-statistic for Knowledge Score</th>
<th>p-value for Knowledge Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>For-Profit</td>
<td>N = 957</td>
<td>-0.2129</td>
<td>-3.379</td>
<td>.0007***</td>
</tr>
<tr>
<td>Not-for-Profit</td>
<td>N = 737</td>
<td>-0.1534</td>
<td>-1.140</td>
<td>.2543</td>
</tr>
</tbody>
</table>

*** = Statistically significant at p < .01

Results indicate support for the hypothesis that for-profit firms are taking advantage of information asymmetry by allowing more skipped treatments for low-knowledge patients. The negative sign of the coefficients indicates that both ownership types have fewer
skipped treatments for high-knowledge patients. This result is statistically significant for for-profit firms, but is not statistically significant for not-for-profit firms.

The second set of regressions results, shown below in Table 28, were produced by using the larger data set described in Tables 22 and 23 above, where missing values for Skipped were counted as zero (zero treatments skipped during the study period). Since for-profit firms were more likely to have missing answers, this larger data set may favor for-profits, if their blank answers did not refer to true zero values. Thus, this larger data set is more conservative. Results show that for-profit facilities have statistically significantly fewer skipped treatments for high-knowledge patients, while not-for-profit firms again show fewer skipped treatments, but the result is not statistically significant. These results imply that for-profit firms are taking advantage of information asymmetries and allowing more skipped treatments among low-knowledge patients. Of course, this phenomenon can easily stem from the patient’s own determination to persevere with treatment, but effective facility staff can play a large role in exhorting patients to be conscientious about their treatment program. The following passage from Garella (1997) illustrates well the relationship between patient desires and facility staff’s role as health care provider:

“By and large, patients insist on receiving the least possible amount of dialysis, especially as it relates to time spent on treatment. The pressure from patients to receive shorter treatments is great, especially when they observe that their neighbour is staying ‘on the machine’ for only 2 h. while they are prescribed a longer treatment, and especially when they become aware that a nearby facility (or another nephrologist) tends to prescribe shorter treatment times. To convince patients of the desirability of longer dialysis takes time and effort. In the face of reductions in supporting staff, this important component of quality of care, namely patient education and emotional support, is being progressively shortchanged. An atmosphere of ‘let them do what they want’ is established, compliance decreases, and smaller dialysis doses are delivered.”
Table 28: Skipped Treatments Regression Results for Knowledge Score Variable By Ownership Type
Missing observations included as zero values.

<table>
<thead>
<tr>
<th>Ownership Type</th>
<th>Sample Size</th>
<th>Coefficient for Knowledge Score</th>
<th>t-statistic for Knowledge Score</th>
<th>p-value for Knowledge Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>For-Profit</td>
<td>N = 2976</td>
<td>-0.2546</td>
<td>-3.334</td>
<td>.0009***</td>
</tr>
<tr>
<td>Not-for-Profit</td>
<td>N = 1963</td>
<td>-0.2192</td>
<td>-1.525</td>
<td>.1273</td>
</tr>
</tbody>
</table>

*** = Statistically significant at p < .01

PART B. SHORTENED TREATMENTS

The data sub-sample with non-missing values for the number of treatments shortened during the study period was, like the skipped treatment data, count data truncated at zero. Negative binomial regression was the appropriate analytical tool here as well, so Table 29 below shows the results of the negative binomial regression of the independent variables on the number of treatments shortened by ten minutes or more during the study period.
Table 29: Shortened Treatments: Negative Binomial Regression
Main sample with no missing observations included as zero values.

N = 1805, Log-likelihood = -2166.864

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Standard Error</th>
<th>T-ratio</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>1.2610</td>
<td>0.4516</td>
<td>2.792</td>
<td>.0052</td>
</tr>
<tr>
<td>Age</td>
<td>-0.0107</td>
<td>0.0037</td>
<td>-2.913</td>
<td>.0358</td>
</tr>
<tr>
<td>Age Deviation$^2$</td>
<td>-0.260E-03</td>
<td>0.184E-03</td>
<td>-1.416</td>
<td>.1567</td>
</tr>
<tr>
<td>Days on Dialysis</td>
<td>-0.106E-03</td>
<td>0.459E-04</td>
<td>-2.313</td>
<td>.0207</td>
</tr>
<tr>
<td>Female</td>
<td>-0.5473</td>
<td>0.0979</td>
<td>-5.591</td>
<td>.0001</td>
</tr>
<tr>
<td>Native American</td>
<td>-0.7551</td>
<td>0.4986</td>
<td>-1.514</td>
<td>.1299</td>
</tr>
<tr>
<td>Asian</td>
<td>0.0296</td>
<td>0.3067</td>
<td>0.096</td>
<td>.9232</td>
</tr>
<tr>
<td>Black</td>
<td>0.0636</td>
<td>0.1041</td>
<td>0.611</td>
<td>.5415</td>
</tr>
<tr>
<td>Other race</td>
<td>0.8512</td>
<td>0.3603</td>
<td>2.363</td>
<td>.0181</td>
</tr>
<tr>
<td>Diabetic</td>
<td>-0.1283</td>
<td>0.1386</td>
<td>-0.926</td>
<td>.3546</td>
</tr>
<tr>
<td>Hypertension</td>
<td>-0.0339</td>
<td>0.1346</td>
<td>-0.252</td>
<td>.8013</td>
</tr>
<tr>
<td>Glomerulonephritis</td>
<td>-0.2724</td>
<td>0.1563</td>
<td>-1.648</td>
<td>.0994</td>
</tr>
<tr>
<td>Cystic Kidney</td>
<td>0.2159</td>
<td>0.2546</td>
<td>0.848</td>
<td>.3964</td>
</tr>
<tr>
<td>For-Profit Ownership</td>
<td><strong>0.9714</strong></td>
<td><strong>0.1517</strong></td>
<td><strong>6.405</strong></td>
<td><strong>.0001</strong></td>
</tr>
<tr>
<td>CON Entry Regs.</td>
<td>0.3272</td>
<td>0.1015</td>
<td>3.222</td>
<td>.0001</td>
</tr>
<tr>
<td>Facility in South</td>
<td>-0.2276</td>
<td>0.1185</td>
<td>-1.922</td>
<td>.0547</td>
</tr>
<tr>
<td>Free-Standing</td>
<td>-0.8319</td>
<td>0.1713</td>
<td>-4.857</td>
<td>.0001</td>
</tr>
</tbody>
</table>

Knowledge Score

| $\alpha$            | 2.7254      | 0.1832         | 14.876  | .0001   | *** |

*** = Statistically significant at p < .01
**  = Statistically significant at p < .05
*   = Statistically significant at p < .10

Just like the results from the analysis of skipped treatments, here the main results from the analysis of shortened treatments are that for-profit facilities are more likely to have patients with shortened treatments (significant at the p < .01 level) and high-knowledge patients are less likely to have shortened treatments (significant at the p < .01 level). Results from the regression of independent variables on the number of treatments
shortened by ten minutes or more during the study period have more explanatory power than the Table 26 regression involving skipped treatments because the data set with shortened treatments is somewhat larger and there are more non-zero values for the dependent variable. For-profit ownership is a strong positive influence on the number of shortened treatments, as is the presence of Certificate of Need entry restrictions. As before with skipped treatments, older patients and female patients have fewer shortened treatments, signaling greater compliance among females and the elderly.

To determine if for-profit firms are taking advantage of information asymmetries for high-knowledge patients by allowing more shortened treatments, the sample was divided into for-profit and not-for-profit subsets and the knowledge score and significance was recorded. Summarized results in Table 30 are provided for the separate ownership types. The data were separated by ownership type, then a regression of remaining independent variables was run on the dependent variable, number of shortened treatments.

**Table 30: Shortened Treatments Regression Results for Knowledge Score Variable By Ownership Type**
Main sample with no missing observations included as zero values.

<table>
<thead>
<tr>
<th>Ownership Type</th>
<th>Sample Size</th>
<th>Coefficient for Knowledge Score</th>
<th>t-statistic for Knowledge Score</th>
<th>p-value for Knowledge Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>For-Profit</td>
<td>N = 1050</td>
<td>-0.0441</td>
<td>-0.758</td>
<td>.4487</td>
</tr>
<tr>
<td>Not-for-Profit</td>
<td>N = 755</td>
<td>-0.1527</td>
<td>-1.719</td>
<td>.0856*</td>
</tr>
</tbody>
</table>

* = Statistically significant at p < .10

In regression results separated by ownership type we do not find that for-profit firms are taking advantage of information asymmetry to allow more shortened treatments. If anything, not-for-profit providers here appear to be allowing more shortened treatments
for low-knowledge patients, as the negative coefficient on the knowledge score shows some statistical significance.

<table>
<thead>
<tr>
<th>Ownership Type</th>
<th>Sample Size</th>
<th>Coefficient for Knowledge Score</th>
<th>t-statistic for Knowledge Score</th>
<th>p-value for Knowledge Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>For-Profit</td>
<td>N = 2976</td>
<td>-0.0456</td>
<td>-0.681</td>
<td>.4961</td>
</tr>
<tr>
<td>Not-for-Profit</td>
<td>N = 1963</td>
<td>-0.2458</td>
<td>-2.468</td>
<td>.0136**</td>
</tr>
</tbody>
</table>

** = Statistically significant at p < .05

For the more conservative data set, where missing observations of the Shortened variable are treated as zero values, not-for-profit firms appear even more likely to allow shortened treatments for low-knowledge patients, since the coefficient for the knowledge score is strongly negative. Although for-profit firms may allow more shortened treatments overall (Table 29), this does not appear to be related to the patients' level of sophistication (Tables 30 and 31).

One may argue that not-for-profit firms, if they are of higher quality, may simply have a larger population of knowledgeable patients who are more conscientious about getting full treatments. To test this, the for-profit and not-for-profit sample mean values for the knowledge score were examined. Slightly more knowledgeable patients are being treated at not-for-profit facilities, as shown in Table 32.
Table 32: Knowledge Score Mean Values by Ownership and Data Set

<table>
<thead>
<tr>
<th>Ownership</th>
<th>Mean Value for Knowledge</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Main sample with no missing observations included as zero values:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For-Profit</td>
<td>6.634</td>
<td>1.165</td>
</tr>
<tr>
<td>Not-for-Profit</td>
<td>6.677</td>
<td>1.176</td>
</tr>
<tr>
<td><strong>Missing observations included as zero values:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For-Profit</td>
<td>6.636</td>
<td>1.202</td>
</tr>
<tr>
<td>Not-for-Profit</td>
<td>6.726</td>
<td>1.205</td>
</tr>
</tbody>
</table>

For the analysis of number of shortened treatments, therefore, we cannot conclude that for-profit firms are taking advantage of information asymmetries, but do have some evidence that quality of care is higher at not-for-profit facilities.

**PART C. HEMODIALYSIS HOURS PRESCRIBED BY PHYSICIAN**

The third dependent variable under study is the physician’s prescription for hemodialysis. If the physician’s income comes at least partially from the facility’s earnings in the case of medical directors at for-profit facilities, or if pressure from administration to reduce costs is stronger at for-profit facilities, there may be a stronger incentive for the physician with patients at for-profit facilities to prescribe a shorter duration of treatment so that more patients can be scheduled per dialysis station and average costs per patient can be reduced. Data, as reported in Table 25, are continuous, a product of number of hours prescribed per week and the number of dialysis hours per week. However, the data resemble count data due to the high frequency of certain values. Therefore, in addition to ordinary least squares regression, several forms of logistic regression analysis were also tested for appropriateness. Negative binomial regression was not used with this variable.
because the data are not truncated at zero. Results of the initial least squares regression are shown in Table 33.

Table 33: Physician Hemodialysis Hours per Week Prescribed by Physician, Ordinary Least Squares Regression

N = 4822, F value = 31.142, Prob. > F = 0.0001.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter Estimate</th>
<th>Standard Error</th>
<th>T for H₀: Parameter = 0</th>
<th>Prob. &gt;</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>10.6511</td>
<td>0.2530</td>
<td>42.106</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Age</td>
<td>-0.0207</td>
<td>0.0021</td>
<td>-9.956</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Age Deviation²</td>
<td>-0.0004</td>
<td>0.0001</td>
<td>-3.779</td>
<td>0.0002</td>
<td>***</td>
</tr>
<tr>
<td>Days on Dialysis</td>
<td>1.88E-04</td>
<td>2.50E-05</td>
<td>7.525</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Female</td>
<td>-0.5537</td>
<td>0.0573</td>
<td>-9.672</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Native American</td>
<td>-0.4398</td>
<td>0.2313</td>
<td>-1.901</td>
<td>0.0574</td>
<td>*</td>
</tr>
<tr>
<td>Asian</td>
<td>-0.8498</td>
<td>0.1980</td>
<td>-4.293</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Black</td>
<td>0.2498</td>
<td>0.0638</td>
<td>3.917</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Other Race</td>
<td>0.2627</td>
<td>0.3052</td>
<td>0.861</td>
<td>0.3898</td>
<td></td>
</tr>
<tr>
<td>Diabetic</td>
<td>0.3662</td>
<td>0.0822</td>
<td>4.457</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Hypertension</td>
<td>0.1594</td>
<td>0.0833</td>
<td>1.913</td>
<td>0.0558</td>
<td>*</td>
</tr>
<tr>
<td>Glomerulonephritis</td>
<td>0.2050</td>
<td>0.0954</td>
<td>2.148</td>
<td>0.0318</td>
<td>**</td>
</tr>
<tr>
<td>Cystic Kidney</td>
<td>-0.0147</td>
<td>0.1614</td>
<td>-0.092</td>
<td>0.9271</td>
<td></td>
</tr>
<tr>
<td>For-Profit Ownership</td>
<td>0.0632</td>
<td>0.0835</td>
<td>0.757</td>
<td>0.4492</td>
<td></td>
</tr>
<tr>
<td>CON Entry Regulations</td>
<td>0.4774</td>
<td>0.0593</td>
<td>8.053</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Facility in South</td>
<td>-0.2391</td>
<td>0.0716</td>
<td>-3.339</td>
<td>0.0008</td>
<td>***</td>
</tr>
<tr>
<td>Free-Standing Facility</td>
<td>-0.4498</td>
<td>0.0943</td>
<td>-4.770</td>
<td>0.0001</td>
<td>***</td>
</tr>
<tr>
<td>Knowledge Score</td>
<td>-0.0798</td>
<td>0.0243</td>
<td>-3.279</td>
<td>0.0010</td>
<td>***</td>
</tr>
</tbody>
</table>

*** = Statistically significant at p < .01  
** = Statistically significant at p < .05  
* = Statistically significant at p < .10

¹ 177 observations with missing values for the dependent variable were deleted from this sample. A similar regression was run for a sample with the missing observations equal to 9, the mode. Results from that regression were virtually identical, with the same significance levels of variables.
The clear message from the regression above is that case-mix variables are the deciding factor in the physician's selection of dialysis prescription. Curiously, patients with high knowledge scores have lower hemodialysis prescriptions. The knowledge score elements most related to physical condition (walk, eat, transfer independently as well as weight status) may predominate here, as physicians prescribe the longest hemodialysis hours for the patients who are the most frail. Physicians in CON-regulated states appear to prescribe more dialysis, but freestanding facilities and facilities in the South prescribe shorter treatments. Here, the addition of a geographic variable indicates possible regional differences in physician practice.

The knowledge score may be correlated strongly with the patient's physical condition, which makes interpretation of these results regarding hemodialysis prescription somewhat unreliable. Unlike the Skipped and Shortened variables, which most likely have little relation to patient condition, the physician's hemodialysis prescription is probably longer for patients in poor condition, corresponding to the results presented above.

It was hypothesized that the financial incentives created by for-profit ownership would result in shorter hemodialysis prescriptions among for-profit physicians. However, we fail to accept this hypothesis, since the coefficient on the for-profit ownership parameter is not statistically significant. Therefore, there is no evidence that facility ownership and its differing financial incentives created for the patient's physician have an effect on physician treatment. The least squares regression was repeated above for for-profit and not-for-profit subsets of the data. For-profit firms were not found to prescribe different treatment duration for high-knowledge patients. Not-for-profit firms, however, were found to prescribe significantly shorter dialysis times for high-knowledge patients. These
results refute the hypothesis that physicians may be reducing costly dialysis times for low-knowledge patients.

Since the physician prescription variable resembles count data, logistic regression analysis provides an alternative to least squares. Ordered logit was first tried with no success. The score test for the proportional odds assumptions (that the dialysis prescription data are ordered as given) failed for the data as listed in Table 25, and for combined subsets of the data as well. The next step was to try successive cumulative logit, whereby the first logistic regression, Table 34, is between "less than nine" and "more or equal to nine" hours of dialysis. The second logistic regression, Table 35, is between "less than or equal to nine" and "more than nine" hours of dialysis. Breaking the logistic regression into two separate regressions in this fashion allows us to spot changes in parameter estimates as the values of the dependent variable increase.
Table 34: Successive Cumulative Logit for Prescribed Hours of Treatment
Logit I: Less than 9 Hours Versus 9 or More Hours Prescribed

N = 4822, Lemeshow’s $R^2 = .0555$, -2 Log L = 5144.428 (intercept only), 4859.108 (intercept and covariates) and $\chi^2 = 285.320$ with 17 df ($p = 0.0001$).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter Estimate</th>
<th>Standard Error</th>
<th>$\chi^2$</th>
<th>Prob. &gt; $\chi^2$</th>
<th>Standardized Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>2.7932</td>
<td>0.3218</td>
<td>75.32</td>
<td>0.0001***</td>
<td></td>
</tr>
<tr>
<td>Age</td>
<td>-0.0199</td>
<td>0.0026</td>
<td>61.10</td>
<td>0.0001***</td>
<td>-0.1771</td>
</tr>
<tr>
<td>Age Deviation$^2$</td>
<td>-0.0004</td>
<td>0.0001</td>
<td>10.30</td>
<td>0.0013***</td>
<td>-0.0702</td>
</tr>
<tr>
<td>Days on Dialysis</td>
<td>1.33E-04</td>
<td>3.4E-05</td>
<td>14.99</td>
<td>0.0001***</td>
<td>0.0887</td>
</tr>
<tr>
<td>Female</td>
<td>-0.5095</td>
<td>0.0728</td>
<td>48.93</td>
<td>0.0001***</td>
<td>-0.1404</td>
</tr>
<tr>
<td>Native American</td>
<td>-0.4999</td>
<td>0.2677</td>
<td>3.49</td>
<td>0.0619*</td>
<td>-0.3411</td>
</tr>
<tr>
<td>Asian</td>
<td>-0.9657</td>
<td>0.2102</td>
<td>21.10</td>
<td>0.0001***</td>
<td>-0.0766</td>
</tr>
<tr>
<td>Black</td>
<td>0.2836</td>
<td>0.0835</td>
<td>11.53</td>
<td>0.0007***</td>
<td>0.0748</td>
</tr>
<tr>
<td>Other</td>
<td>-0.3293</td>
<td>0.3646</td>
<td>0.82</td>
<td>0.3663</td>
<td>-0.0169</td>
</tr>
<tr>
<td>Diabetic</td>
<td>0.4088</td>
<td>0.1033</td>
<td>15.66</td>
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</tr>
<tr>
<td>Hypertension</td>
<td>0.1824</td>
<td>0.1032</td>
<td>3.12</td>
<td>0.0772*</td>
<td>0.0453</td>
</tr>
<tr>
<td>Glomerulonephritis</td>
<td>0.2437</td>
<td>0.1221</td>
<td>3.98</td>
<td>0.0460**</td>
<td>0.0470</td>
</tr>
<tr>
<td>Cystic Kidney</td>
<td>-0.1146</td>
<td>0.1899</td>
<td>0.36</td>
<td>0.5462</td>
<td>-0.0118</td>
</tr>
<tr>
<td>For-Profit Ownership</td>
<td>-0.0279</td>
<td>0.1068</td>
<td>0.07</td>
<td>0.7937</td>
<td>-0.0075</td>
</tr>
<tr>
<td>CON Entry Regs.</td>
<td>0.5566</td>
<td>0.0789</td>
<td>49.71</td>
<td>0.0001***</td>
<td>0.1480</td>
</tr>
<tr>
<td>South</td>
<td>-0.2113</td>
<td>0.0892</td>
<td>5.61</td>
<td>0.0178**</td>
<td>-0.0492</td>
</tr>
<tr>
<td>Free-Standing</td>
<td>-0.0112</td>
<td>0.1213</td>
<td>0.01</td>
<td>0.9262</td>
<td>-0.0027</td>
</tr>
<tr>
<td>Knowledge Score</td>
<td>-0.0810</td>
<td>0.0310</td>
<td>6.85</td>
<td>0.0089***</td>
<td>-0.0539</td>
</tr>
</tbody>
</table>

*** = Statistically significant at $p < .01$
** = Statistically significant at $p < .05$
* = Statistically significant at $p < .10$
Table 35: Successive Cumulative Logit for Prescribed Hours of Treatment
Logit II: Less Than or Equal to 9 Hours Versus 10 or More Hours Prescribed

N = 4822, Lemeshow’s R² = .0754, -2 Log L = 4802.5 (intercept only), 4400.356 (intercept and covariates) and χ² = 362.144 with 17 df (p = 0.0001).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter Estimate</th>
<th>Standard Error</th>
<th>Wald χ²</th>
<th>Prob. &gt; χ²</th>
<th>Standardized Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.4484</td>
<td>0.3425</td>
<td>1.71</td>
<td>0.1904</td>
<td>.</td>
</tr>
<tr>
<td>Age</td>
<td>-0.0229</td>
<td>0.0030</td>
<td>56.52</td>
<td>0.0001***</td>
<td>-0.2034</td>
</tr>
<tr>
<td>Age Deviation¹</td>
<td>-0.0004</td>
<td>0.0001</td>
<td>7.39</td>
<td>0.0066***</td>
<td>-0.0659</td>
</tr>
<tr>
<td>Days on Dialysis</td>
<td>2.26E-04</td>
<td>3.0E-05</td>
<td>55.63</td>
<td>0.0001***</td>
<td>0.1505</td>
</tr>
<tr>
<td>Female</td>
<td>-0.6285</td>
<td>0.0780</td>
<td>64.86</td>
<td>0.0001***</td>
<td>-0.1732</td>
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<tr>
<td>Native American</td>
<td>-0.4065</td>
<td>0.3546</td>
<td>1.31</td>
<td>0.2515</td>
<td>-0.0277</td>
</tr>
<tr>
<td>Asian</td>
<td>-0.2856</td>
<td>0.3089</td>
<td>0.85</td>
<td>0.3552</td>
<td>-0.0226</td>
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<tr>
<td>Black</td>
<td>0.2647</td>
<td>0.0835</td>
<td>10.06</td>
<td>0.0015***</td>
<td>0.0699</td>
</tr>
<tr>
<td>Other</td>
<td>0.4733</td>
<td>0.3648</td>
<td>1.68</td>
<td>0.1945</td>
<td>0.0242</td>
</tr>
<tr>
<td>Diabetic</td>
<td>0.3288</td>
<td>0.1113</td>
<td>8.72</td>
<td>0.0031***</td>
<td>0.0840</td>
</tr>
<tr>
<td>Hypertension</td>
<td>0.1767</td>
<td>0.1133</td>
<td>2.43</td>
<td>0.1188</td>
<td>0.0439</td>
</tr>
<tr>
<td>Glomerulonephritis</td>
<td>0.1373</td>
<td>0.1250</td>
<td>1.21</td>
<td>0.2721</td>
<td>0.0265</td>
</tr>
<tr>
<td>Cystic Kidney</td>
<td>0.1798</td>
<td>0.2192</td>
<td>0.67</td>
<td>0.4119</td>
<td>0.0185</td>
</tr>
<tr>
<td>For-Profit Ownership</td>
<td>0.2510</td>
<td>0.1201</td>
<td>4.37</td>
<td>0.0366**</td>
<td>0.0676</td>
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<tr>
<td>CON Entry Regs.</td>
<td>0.3191</td>
<td>0.0775</td>
<td>16.95</td>
<td>0.0001***</td>
<td>0.0848</td>
</tr>
<tr>
<td>South</td>
<td>-0.2791</td>
<td>0.1025</td>
<td>7.41</td>
<td>0.0065***</td>
<td>-0.0651</td>
</tr>
<tr>
<td>Free-Standing</td>
<td>-0.8586</td>
<td>0.1283</td>
<td>44.76</td>
<td>0.0001***</td>
<td>-0.2072</td>
</tr>
<tr>
<td>Knowledge Score</td>
<td>-0.0633</td>
<td>0.0326</td>
<td>3.78</td>
<td>0.0519*</td>
<td>-0.0421</td>
</tr>
</tbody>
</table>

*** = Statistically significant at p < .01
**  = Statistically significant at p < .05
*   = Statistically significant at p < .10

The differences between the least squares regression results in Table 33 and the logistic regression results here in Tables 34 and 35 are small. Coefficients on most parameters are the same sign for either form of regression. Logistic regression coefficients are not interpretable as marginal effects of the variables on the dependent variable, but do indicate the sign and statistical significance of the effect on the dependent variable. If
standardized estimates are used, one can rank the absolute value of the variables’ standardized effects to obtain a relative ranking of the variables’ effect on the dependent variable for a one standard deviation change in the independent variable. Ownership has one of the smallest effects on hemodialysis prescription of any of the independent variables.

An interesting result obtained from the successive cumulative logistic regression analysis is the revealed pattern in the effect of for-profit ownership on hemodialysis prescription. Ownership was one of the few variables to show a change in sign from Logit I to Logit II. Results of Logit I indicate (with no statistical significance) that for-profit firms may prescribe fewer hemodialysis hours, as indicated by the negative parameter estimate. However, Logit II shows a statistically significant positive parameter estimate for for-profit ownership. Therefore, for this data sample, for-profit firms are more likely to prescribe longer hemodialysis hours at higher than average levels (greater than nine hours per week). But this effect is canceled out in the entire data sample due to a somewhat higher likelihood of prescribing fewer than nine hours for for-profit facility patients. It appears, therefore, that the distribution of hours prescribed for for-profit firms is larger than physicians associated with not-for-profit firms.

SECTION 6: DISCUSSION AND CONCLUSION

Little explicit evidence of patient knowledge levels exists in most medical databases. The data set used here, however, had an unusual amount of demographic data that could be combined to fashion a knowledge index score for each patient. This knowledge level was found to be a statistically significant determinant of certain aspects of care. The
results that high-knowledge patients have fewer shortened and skipped treatments may depend entirely on the high-knowledge patients' own determination to get the best possible care. However, in an idealized world of perfect health care, treatment should depend entirely on patient health needs, not on socioeconomic status or presence of a relative serving as a patient advocate. Thus, at least some of the differences in treatment by knowledge level are due to a failure of institutional staff to encourage low-knowledge patients to be more conscientious, or conversely due to relatively over-attentive treatment of high-knowledge patients.

Results were mixed, but did show evidence of quality of care differences at for-profit and not-for-profit facilities, as well as quality of care differences in relation to patient knowledge levels. The first variable tested, number of skipped dialysis treatments during the study period, showed that for-profit facilities allow more skipped treatments than not-for-profit firms. Results also showed that more knowledgeable patients have fewer skipped treatments. When tested separately by ownership type, not-for-profit firms did not appear to treat knowledgeable patients better than unknowledgeable patients. For-profit firms, however, showed statistically significantly fewer skipped treatments for high-knowledge patients, indicating that for-profit firms may be taking advantage of information asymmetries and providing more conscientious care to knowledgeable patients.

The second variable tested, number of shortened dialysis treatments during the study period, showed again that for-profit facilities allow more shortened treatments than not-for-profit facilities. In addition, knowledgeable patients experienced fewer shortened treatments. However, these effects were not in evidence when testing ownership types separately. For-profit firms may allow more shortened treatments than not-for-profit firms, but for-profit firms do not appear to allow more shortened treatments specifically
for low-knowledge patients. Of the two variables testing treatment of patients by facility staff, Skipped and Shortened, Skipped is more important. That is, Shortened refers to the number of dialysis treatments shortened by 10 minutes or more (a dialysis session lasts usually three hours), whereby Skipped refers to an entire dialysis session skipped on a scheduled day. Held et al. (1996) report that one skipped treatment in a month containing 13 total treatments results in a 14% higher annual mortality risk. So although for-profit facilities do not appear to be taking advantage of information asymmetries in terms of shortened treatments, it is more important to patients’ health outcome that the data suggest that for-profit facilities are taking advantage of information asymmetry in terms of skipped treatments.

The numbers of skipped and shortened treatments indicate the ability of the dialysis facility staff to provide quality care and to carry out the prescription ordered by the patient’s off-site doctor. The third variable tested, number of hemodialysis hours prescribed weekly by the patient’s physician, was included to determine if financial incentives due to ownership differences would affect the duration of the hemodialysis prescription. Results indicate that physicians’ dialysis hours prescribed were not affected by ownership type, but that physicians tended to order longer dialysis for low-knowledge patients. Physicians with patients at not-for-profit facilities prescribed significantly fewer hours for high-knowledge patients. Therefore, no evidence exists that physicians affiliated with either ownership type were taking advantage of low knowledge levels among patients to prescribe shorter dialysis treatments.

In summary, though facility treatment quality does appear to differ by ownership type, evidence is mixed that for-profit firms in the renal dialysis industry are specifically taking advantage of information asymmetries in order to reduce costs. Physicians do not appear to be affected by ownership type when prescribing dialysis hours of treatment and also do
not appear to be prescribing more dialysis for more knowledgeable patients. Therefore, further studies on this topic should take care to examine not just prescribed treatment differences by the physician, but deviation from prescribed treatment by institutional staff.
CHAPTER 7: QUALITY OF CARE AND OWNERSHIP: FINDINGS AND CONCLUSIONS

SECTION 1: DISCUSSION OF FINDINGS

This dissertation was commenced to correct a perceived lack of both rigorous theoretical models of not-for-profit organizations and proper control of outcome quality in studies of comparative cost efficiency by ownership type. In the process, an improved methodology for an econometric technique was devised, and a study of asymmetric information as a possible cause of health care quality disparity was completed. Contributions to the literature from this dissertation are noted below, in order of their appearance in the dissertation.

PART A. THEORETICAL MODEL

Little effort has gone into the daunting task of modeling not-for-profit objective mathematically. Some theorists have tried a model of maximizing output while breaking even, while others have attempted to model not-for-profit firms as maximizing profit with a break-even constraint. Many have simply assumed that not-for-profit managers value quality or numbers of patients served, as opposed to for-profit managers who merely maximize profits. In Chapter 2, an alternative model is provided, whereby patient survival depends on quality produced over time. The model is attractive in its incorporation of intertemporal effects and also in its conservative assumption that both not-for-profit and for-profit managers are equally altruistic, and both have quality of care in their utility functions. The key feature separating not-for-profit from for-profit
managers is the non-distribution constraint: For-profit managers are able to receive a higher percentage of firm profits in the form of income. Of course, not-for-profit firm managers cannot legally have their salary tied directly to residual income, but in practice they do benefit more when the organization is more profitable, so the assumption that they take home some of the profits (albeit indirectly) is sound. The mathematical model is conservative, yet the steady state conditions predict that, over time, for-profit firms will provide care of a lower quality than their not-for-profit counterparts will provide.

PART B: MULTIVARIATE REGRESSION ANALYSIS

The renal dialysis industry provided an excellent opportunity to test whether outcome quality differs by ownership type. The data sets were very large – over 200,000 patients receive care for end-stage renal disease (ESRD) nationwide – and the treatment protocol is straightforward for ESRD. Mortality, the main “quality” variable under scrutiny, is indisputable as a measure of outcome quality, and was frequent enough among ESRD patients that results were easily interpreted. The method of reimbursement for ESRD patients was prospective-price, and thus immediately comparable to prospective-price reimbursement systems used in hospital and other medical care.

Logistic regression analysis in Chapter 4, using virtually all patients undergoing center-based dialysis or transplant in 1993 nationwide, suggested that patients whose main treatment facility was a for-profit firm had higher mortality rates than patients whose main treatment facility was a not-for-profit firm, case-mix and market characteristics adjusted. The increase in mortality appeared to be around three percent, depending on the population group studied. A review of medical literature and discussions with nephrologists indicated that this three percent figure was not at all trivial. A three percent increase in the probability of death translates to roughly five months of life lost for a
teenager, whereas a similar increase in the probability of death for an 85-year-old patient reduces life expectancy by approximately two weeks (USRDS, 1995).

The controls for market and firm-level characteristics were found to be an important part of the model’s specification, as characteristics such as geographic location and entry regulations were statistically significantly related to higher mortality rates. Certificate of Need regulations’ positive influence on mortality was somewhat of a surprise. These regulations restricting the entry of new (primarily for-profit) firms could be inducing current market participants in those areas to treat only the sickest, neediest patients, whereas firms in saturated markets may be starting patients on dialysis who are not yet so debilitated. Further testing of the effect of Certificate of Need regulations on patient mortality would be an interesting future project, but the issue was peripheral to this particular study except for its appearance as a control variable.

PART C: PROPENSITY SCORE ANALYSIS

Several lingering issues remained after performing multivariate regression analysis. First, the data were not randomly generated, and there was no particular guarantee that the data were identical in characteristics of for-profit and not-for-profit patients. Second, the logistic regression models may not have been correctly specified to account for any possible differences in for-profit and not-for-profit patient populations. Third, some independent variables were clearly facility-level covariates, such as freestanding or hospital based status, whereas other independent variables were patient case-mix characteristics such as age, length of time on dialysis, and so on. Neither patient-level regression analysis nor facility-level analysis of patient mortality rates appeared accurate in modeling the effects of these multi-level variables.
Propensity score methodology provided a very different approach to measuring differences in mortality across ownership types. By stratifying the data set on the basis of propensity score (propensity to be a for-profit facility), it was possible to create subsets of data whereby for-profit and not-for-profit patients had comparable characteristics. That is, for-profit and not-for-profit patients in each data subset had means of variables that were not significantly different from each other. These data subsets could be used to calculate the overall mean probability of death for each ownership type. In addition to correcting for biases due to differences in populations of for-profit and not-for-profit patients, propensity score methodology solved the multi-level independent variable problem by reducing the analysis to a subset of patients where facility characteristics were identical (freestanding facilities only). Results for the subset of white or black dialysis patients with diabetes or hypertension indicated that patients being treated at for-profit facilities had a 5.86% greater probability of dying during the year than patients treated at not-for-profit facilities. This figure exceeds that of the approximate 3% difference in mortality rates for dialysis and transplant sample population groups analyzed in Chapter 4, yet falls below the 7% difference in facility average mortality rates suggested by the USRDS study. Since the propensity score methodology model accounted for population bias in not only patient case-mix variables, but also relevant market variables, the 5.86% figure is most likely a better estimate of the causal effect of for-profit treatment for dialysis patients at freestanding facilities.

**PART D: IMPROVEMENTS TO THE PROPENSITY SCORE STRATIFICATION PROCESS**

Despite the attractiveness of propensity score methodology's ease of interpretation and calculation of causal effects from identical and unbiased subsets of data, the methodology was difficult to implement with this renal dialysis data due to the many number of independent variables to control for and persistent differences in patient populations across ownership types. The data had to be simplified by eliminating observations of non-black and non-white race and uncommon primary diagnoses. Still, though, the
stratification methodology used by previous researchers – weighting by propensity score, matching on the propensity score, or sub-sub-stratification – was not adequate to reduce the data to comparable for-profit and not-for-profit subsets unless the data were divided into extremely small subsets. The prospect of basing the estimated causal effect on a myriad of tiny subsets of data, passing the "identical" test on the strength of low numbers of observations alone, was unappealing.

Two alternative methods of stratification were proposed. The first method entails reducing the data to a homogenous subset by recursive formulation of propensity scores and discarding the highest and lowest strata when the observations are ranked by propensity score. This method provided a quick way of creating a data set for analysis whereby the for-profit and not-for-profit subsets are very similar. Remaining differences in for-profit and not-for-profit populations could be easier eliminated by the common stratification, weighting, or matching methods.

The second improvement to the propensity score stratification process may be potentially more attractive to researchers because it does not involve reduction of the overall data set. This second method entails subdividing the data into strata (for example, five initial strata). The strata that have comparable subsets of for-profit and not-for-profit populations are retained for final calculation of the estimated causal effect, while strata rejecting difference in means tests for any of the variables are subdivided again into smaller subsets. They key difference proposed is that the logistic regression creating the propensity score is repeated for the strata that failed, creating new "propensity scores" and thus new rankings of observations. It was found that this employment of recursive propensity score regression quickly produced homogenous populations of for-profit and not-for-profit subsets of data and consequently reduced the number of final strata produced significantly. Using this recursive method of logistic regression propensity
score ordering, only thirteen strata were needed to estimate the causal effect of for-profit treatment on mortality, while the traditional sub-stratification method would have necessitated creating over thirty strata.

PART E. ASYMMETRIC INFORMATION AND OWNERSHIP

Since the empirical analysis in Chapters 4 and 5 suggested that a disparity in clinical quality exists by ownership type, the question arose as to how this quality gap could exist over time. Chapter 6 investigated the possibility that differences in quality are emanating from the practice of treating low-knowledge patients with low cost, lower quality care than the care provided to high-knowledge patients. It was hypothesized that for-profit facilities would be more likely to exploit information advantages than not-for-profit facilities. Analysis took the form of specifying a knowledge index, composed of responses to variables indicating patient ability to judge quality of care, and analyzing the number of skipped or shortened dialysis treatments during a specified time period. Physician prescriptions of dialysis treatment duration were also examined. Due to the difficulty of obtaining data on patient knowledge levels, this study represents one of the first attempts in the literature to measure differing treatment in response to patient knowledge levels.

Results of the regressions of patient case-mix and market characteristics, ownership, and knowledge scores on the three dependent variables under scrutiny showed the following: First, for-profit providers were found to allow more skipped treatments for their patients than not-for-profit providers. Regardless of ownership type, higher knowledge patients had fewer skipped dialysis treatments. When the results were divided by ownership type, for-profit firms were found to allow significantly fewer skipped treatments for high-knowledge patients, while not-for-profit firms showed no such effect with respect to
knowledge level. Therefore, for the variable indicating numbers of treatments skipped, the results could be interpreted to say that for-profit firms are taking advantage of information asymmetry to provide sub-standard care to low-knowledge patients.

Second, when the number of treatments shortened by ten minutes (besides skipped treatments) were examined, it was found that for-profit firms allowed more shortened treatments for their patients than not-for-profit firms allowed. Also, regardless of ownership type, high-knowledge patients had fewer shortened treatments. When the data were separated by ownership type, though, for-profit firms did not show a tendency to allow more shortened treatments for low-knowledge patients. Not-for-profit firms, however, did appear to allow more shortened treatments among low-knowledge patients. In this case, it appeared that not-for-profit firms, rather than for-profit firms, were taking advantage of information asymmetry to provide sub-standard care to low-knowledge patients. Of the two variables under scrutiny, treatments skipped or shortened, the number of skipped treatments has a much greater significance to the patient’s health. Treatments last on average a little less than three hours, so a treatment skipped means a substantial reduction in weekly dialysis, whereas a treatment shortened by ten minutes is a relatively minor reduction.

The third variable studied in the asymmetric information analysis was physician hemodialysis prescription. No evidence emerged that physicians prescribe shorter treatment duration for patients in for-profit facilities as opposed to not-for-profit facilities. In addition, physicians appeared not to be influenced by patient sophistication levels, and in fact were found to prescribe longer treatments for low-knowledge patients, which may be caused by patient health conditions’ relation to the knowledge score. Results of the asymmetric information analysis imply that differences in clinical quality may be emanating from facility staff rather than physician care. Physicians may be
acting as patient advocates to insure that contract failure in health care does not occur, but their advocacy role may not be adequate to overcome differences in facility administration.

SECTION 3: CONCLUSIONS

The theoretical model in Chapter 2 underscored the importance of the non-distribution constraint in observed behavior between not-for-profit and for-profit health care firms. Empirical evidence supported the theoretical hypothesis that quality of care will be higher at not-for-profit facilities, and also provided insight into treatment differences for high- and low-knowledge patients. Further empirical work in Chapter 5 using an innovative method of eliminating population bias also pointed to an outcome quality gap between for-profit and not-for-profit renal dialysis providers. The quality disparity found in this research suggests that outcome quality is a key control variable for future studies of cost efficiency in the health economics literature. In addition, previous theoretical assumptions of a not-for-profit quality advantage in health care, though in the past largely unsubstantiated, may in fact be justified.
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APPENDIX A: "SKIPPED" SMALL SAMPLE DATA

Table A1. "Skipped" Data Patient Characteristics: n = 1694
Observations with missing values for Skipped not included in the sample.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>60.93</td>
<td>16.03</td>
<td>64</td>
<td>32</td>
<td>83</td>
</tr>
<tr>
<td>Days on dialysis</td>
<td>1838</td>
<td>1220</td>
<td>1497</td>
<td>417</td>
<td>4369</td>
</tr>
<tr>
<td>Knowledge Score</td>
<td>6.61</td>
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<td>6.67</td>
<td>4.67</td>
<td>8.33</td>
</tr>
<tr>
<td>Skipped Treatments</td>
<td>.54</td>
<td>1.45</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

**Gender:** Female (dummy variable = 1): n = 823 (48.6%)  
Male: n = 871 (51.4%)

**Race:**
- Native American: n = 36 (2.1%)
- Black: n = 659 (38.9%)
- White: n = 933 (55.1%) (omitted reference category)
- Asian: n = 46 (2.7%)
- Other race: n = 16 (.9%)

**Primary Diagnosis Causing End Stage Renal Disease:**
- Diabetic: n = 555 (32.8%)
- Hypertension: n = 460 (27.2%)
- Glomerulonephritis: n = 242 (14.3%)
- Cystic Kidney: n = 52 (.31%)
- Other Primary Diagnoses: n = 385 (22.7%) (omitted reference category)
Table A2. “Skipped” Data Facility Characteristics: \( n = 298 \)

Facility in South (dummy = 1): \( n = 97 \)   Facility not in South: \( n = 201 \)

Facility in state with Certificate of Need regulations (dummy = 1): \( n = 106 \)
Facility in state with no Certificate of Need regulations: \( n = 192 \)

Facility freestanding (dummy = 1): \( n = 245 \)   Facility in hospital: \( n = 53 \)

For-profit facility (dummy = 1): \( n = 217 \)   Not-for-profit facility: \( n = 81 \)

Table A3: Treatments Skipped

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<td>12</td>
<td>1</td>
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<tr>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>

Total Frequency: 1694
(Observations with missing variables not included.)
APPENDIX B: “SHORTENED” SMALL SAMPLE DATA

Table B1. “Shortened” Data Patient Characteristics: n = 1805
Observations with missing values for Shortened not included in the sample.

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<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Median</th>
<th>5th %</th>
<th>95th %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>60.65</td>
<td>16.07</td>
<td>64</td>
<td>31</td>
<td>83</td>
</tr>
<tr>
<td>Days on dialysis</td>
<td>1886</td>
<td>1240</td>
<td>1540</td>
<td>453</td>
<td>4374</td>
</tr>
<tr>
<td>Knowledge Score</td>
<td>6.65</td>
<td>1.17</td>
<td>6.67</td>
<td>4.67</td>
<td>8.33</td>
</tr>
<tr>
<td>Shortened Treatments</td>
<td>.90</td>
<td>2.05</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

Gender: Female (dummy variable = 1): n = 878 (48.6%)  Male: n = 927(51.4%)

Race:
Native American: n = 36 (2.0%)
Black: n = 687 (38.1%)
White: n = 1023 (56.7%) (omitted reference category)
Asian: n = 41 (2.3%)
Other race: n = 14 (.8%)

Primary Diagnosis Causing End Stage Renal Disease:
Diabetic: n = 564 (31.2%)
Hypertension: n = 486 (26.9%)
Glomerulonephritis: n = 277 (15.3%)
Cystic Kidney: n = 58 (3.2%)
Other Primary Diagnoses: n = 420 (23.3%) (omitted reference category)
Table B2. “Shortened” Data Facility Characteristics:  n = 329

Facility in South (dummy = 1):  n = 100  Facility not in South:  n = 229

Facility in state with Certificate of Need regulations (dummy = 1):  n = 113  Facility in state with no Certificate of Need regulations:  n = 216

Facility free-standing (dummy = 1):  n = 261  Facility in hospital:  n = 68

For-profit facility (dummy = 1):  n = 231  Not-for-profit facility:  n = 98

Table B3: Treatments Shortened

<table>
<thead>
<tr>
<th>Value</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1169</td>
</tr>
<tr>
<td>1</td>
<td>337</td>
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<td>2</td>
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<td>3</td>
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<td>15</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
</tr>
</tbody>
</table>

Total Frequency: 1805
(Observations with missing variables not included.)
VITA

Renee A. Irvin

University of Washington

1998

EDUCATION

Date of completion: August 6, 1998
Fields of specialization: Microeconomics, Health Economics, Public Finance, Natural Resource and Environmental Economics, Not-for-Profit Organizational Theory
Thesis: Quality of Care, Asymmetric Information, and Patient Outcomes in U.S. For-Profit and Not-for-Profit Renal Dialysis Facilities
Thesis Advisor: Dr. Levis Kochin


Additional advanced coursework: Chinese language, creative writing and mathematics
Exchange programs: Studied one year at Universitaet Stuttgart (Stuttgart, Germany) and one semester at Beijing Normal College of Foreign Languages (Beijing, China).

Academic Honors:

Ensley Graduate Fellowship for Economic Policy: 1992-93
Economics Graduate Teaching Assistantship: Contract awarded on merit basis, 1990-1996.
Undergraduate Honors: Phi Beta Kappa, Dean’s List, and five separate merit scholarships

Languages:

Fluent German, some proficiency in Japanese, Chinese and Spanish.

Software Experience:

SAS (PC and mainframe), SPSS (mainframe), LIMDEP (MS-DOS), Eviews, MS-Word and EXCEL
TEACHING

Primary Teaching Areas: Classes Taught:

Economics
- Introductory Microeconomics (UW, 1990 – 1996)
- Introductory Macroeconomics (UW, 1990 – 1996)
- Intermediate Microeconomics (UW, Fall 1993)
- Intermediate Macroeconomics (UW, Summer 1992)

Public Economics
- The Public Economy (UNO, Fall 1997 and Spring 1998)

Health Economics & Finance
- Health Care Finance (UNO, Fall 1997 and Fall 1998)

Economics of Not-for-Profit Organizations
- Not-for-Profit Firms in a Market Economy
  (will teach Spring 1999 at UNO)

New Courses Developed:

- Health Care Finance (masters-level course)
- Not-for-Profit Firms in a Market Economy (masters and doctoral-level course)

PUBLICATIONS

Papers Submitted for Publication:


Unpublished Reports and Manuscripts:

- Quality Differences in For-Profit and Not-for-Profit Renal Dialysis Facilities: An Application of Propensity Score Methodology, June 1998.
- Cost and Efficiency Performance of For-Profit and Not-for-Profit Health Care Providers, December 1995.
- The Provision of Community Benefit in an Era of Health Care Competition with Dr. Douglas R. Conrad and Dr. Carolyn Madden, October 1995.
- Regulating Industrial Dischargers of Multiple Pollutants, June 1995.
- One Fish, Two Fish... Fiscal Analysis Using Comparison States Methodology, April 1995.
FUNDED RESEARCH

Not-for-Profit Health Care Research:  

The Impact of Competition on the Provision of Community Benefit: 
Critiqued literature pertaining to health care organizations' provision of community benefits, comparative cost efficiency and quality of care. Co-wrote *The Provision of Community Benefit in an Era of Health Care Competition*, for use in aiding national and state-level health care policy discussions, especially with regard to tax exemption of not-for-profit health care providers. Wrote *Cost and Efficiency Performance of For-Profit and Not-for-Profit Health Care Providers*. This research was funded by the Catholic Health Association, June 1995 to September 1995 and the University of Washington Department of Health Services, September 1995 to December 1995.

Institute for Public Policy & Management Fiscal Policy Center:  

PROFESSIONAL ASSOCIATION ACTIVITIES

American Economic Association (AEA)

AEA Committee on the Status of Women in the Economics Profession (CSWEP)  
Presented paper at MEA conference as part of CSWEP-sponsored section on the Economics of Information, March 1998.

American Society for Artificial Internal Organs (ASAIO)  
Invited and fully funded to present paper, *Quality of Care Differences by Ownership Form: Implications for Cost Efficiency Analysis*, at annual conference in New York City, April 1998.

American Society for Public Administration (ASPA)  
ASPA Section on Health and Human Services Administration
Association for Public Policy Analysis and Management (APPAM)  
Will present Quality Differences in For-Profit and Nonprofit Renal Dialysis Facilities: An Application of Propensity Score Methodology at the October 1998 APPAM national conference in New York City.

Northwest Regional Economics Association (NREA)  
Presented paper, One Fish, Two Fish....Fiscal Analysis Using Comparison States Methodology at annual conference in Missoula, MT, April 1995.

Midwest Economics Association (MEA)  

Western Economics Association (WEA)  

PUBLIC AND UNIVERSITY SERVICE

Recent Seminars Presented:

Does Quality of Care Differ by Ownership in Health Care? Seminar presented to University of Nebraska Medical Center faculty at the Nebraska Center for Rural Health Research, December 1997.

Does Quality of Care Differ by Ownership in Health Care? Seminar presented to University of Nebraska at Omaha College of Business Administration faculty, November 1997.

Served as water resources, waste management, and energy and information systems panelist at Omaha’s Integrated Approaches to Infrastructure Issues conference, October 1997.

The Economics of Not-for-Profits Seminar presented to University of Washington Department of Economics faculty and students, November 1996.

Volunteer Experience:

Served as member of the City of Seattle’s Citizen’s Water Quality Advisory Committee, and chaired its Industrial and Household Hazardous Waste Sub-Committee, 1990-1993.

Served as citizen representative for the City of Seattle’s Industrial Waste Advisory Board, a group organized to hear appeals from businesses penalized for industrial waste regulations violations, 1991-1993.
PROFESSIONAL EXPERIENCE


**Pre-Graduate School Employment:**
*Language Specialist,* Japan Ministry of Education, Hyogo Prefecture, Japan, 1987-88
*Foreign Study Assistant,* Oregon State System of Higher Education, Corvallis, OR, 1985-87
*Financial Aid Needs Analyst,* University of Oregon, Eugene, OR, 1984-85