Generalizations of the Exterior and Symmetric Power Functors on Categories of Modules using Coinvariants of Tensor Power Functors

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Abstract

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This work, broadly speaking, is a study of coinvariants of abstract group actions on functors. We discuss background on coinvariants of group actions on objects in categories in general, as we benefit from taking the perspective of a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ as an object of the category of functors from $\mathcal{C}$ to $\mathcal{D}$. That said, the focus of this work concerns the nth tensor power functor T^n on the category of modules over a commutative unital ring k. We show that the k-algebra of endomorphisms of T^n is isomorphic to the group algebra $k\mathrm{S}_n$. This allows us to identify specific groups of automorphisms of T^n and investigate the resulting functors of coinvariants. For example, the nth symmetric power functor Sym^n and the nth exterior power functor $\wedge^n$ are coinvariants of T^n with respect to actions of the symmetric group S_n. Hence, the study of coinvariants of T^n with respect to group actions is a way of generalizing these familiar functors.

We give examples of groups G_1 and G_2 of automorphisms of T^n over $\mathbb{Z}$ such that $|G_1|$ is countably infinite whilst $|G_2|$ is finite, and they give rise to canonically isomorphic coinvariants of the functor T^n.

We also explore the topic of sequences of groups G_n with actions on components of a graded algebra S. We provide sufficient conditions for when the resulting direct sum of modules of coinvariants is a quotient algebra of S. For example, this condition implies that

for any R-module M, the actions of the sequence of alternating groups A_n on the components of the tensor algebra $\mathrm{T}(M)$ induces an algebra C of coinvariants. While C is distinct from the algebras $\mathrm{Sym}(M)$ and $\wedge(M)$, we observe that for finitely generated R-modules M and sufficiently large n, there are isomorphisms $\mathrm{C}_n \cong \mathrm{Sym}^n(M)$.

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Chapter 1

INTRODUCTION

For any object c of a category $\mathcal{C}$, we consider group actions as group homomorphisms $\alpha\colon G \to \operatorname{Aut}_{\mathcal{C}}(c)$. In this framework, a morphism $f\colon c \to d$ of $\mathcal{C}$ is G-equivariant with respect to actions α and β of a group G on c and d if for all $g \in G$ the following diagram commutes,

$$\begin{array}{ccc} c & \xrightarrow{f} & d \\ \downarrow{\scriptstyle \alpha(g)} & & \downarrow{\scriptstyle \beta(g)} \\ c & \xrightarrow{f} & d \end{array}$$

We are particularly interested in G-equivariant morphisms such that the action on the codomain d is trivial. For a given object c and group action α of G on c, we define the G-coinvariants of c to be the universal G-equivariant morphism with trivial action on its codomain, Definition 3.9.

As an example, let k be a commutative unital ring and consider a k-module M. Fix a positive integer n. There is an action of the symmetric group S_n on the nth tensor power $M^{\otimes n}$, such that a permutation σ permutes the factors of a simple tensor,

$$\sigma \cdot x_1 \otimes \cdots \otimes x_n = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \tag{1.1}$$

As we discuss in Section 3.8, the k-module of coinvariants of $M^{\otimes n}$ with respect to this action of S_n is isomorphic to the nth symmetric power of M,

$$\left(M^{\otimes n}\right)_{\mathrm{S}_n} \simeq \operatorname{Sym}^n(M)$$

The kernel of the canonical homomorphism $M^{\otimes n} \to \operatorname{Sym}^n(M)$ has the following generating set,

$$\left\{\, x_1 \otimes \cdots \otimes x_n - x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \;:\; \sigma \in \mathrm{S}_n \,\right\}$$

which is recognized to be the kernel of the coinvariants map in the more familiar setting of G-modules, such as that described in [Wei94].

In fact, if $\mathcal{C}$ is an abelian category with all small direct sums, then our definition of coinvariants by universal property agrees with the direct generalization of the familiar construction of coinvariants via the kernel.

Proposition 1.1 (Proposition 3.18)**.** *Let $\mathcal{C}$ be an abelian category that has all small direct sums. Let G be an abstract group action on an object c of $\mathcal{C}$. Then, the coinvariants of c with respect to G arises as the following cokernel,*

$$c_G = \operatorname{coker}\left(\bigoplus_{\sigma\in G} \operatorname{im}(\sigma - id_c) \to c\right)$$

Provided that 2 is a unit in k, the exterior powers arise as coinvariants with respect to a different action of S_n on $M^{\otimes n}$, namely

$$\sigma \cdot x_1 \otimes \cdots \otimes x_n = \operatorname{sgn}(\sigma)\, x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}$$

so that, with this action, we have $\left(M^{\otimes n}\right)_{S_n} \simeq \wedge^n(M)$.

The fact that alternating groups are subgroups of the symmetric groups means that there are induced actions of A_n on $M^{\otimes n}$. Both of the actions of S_n on $M^{\otimes n}$ that we have contemplated so far restrict to the same action of A_n on $M^{\otimes n}$. The modules of coinvarians $\left(M^{\otimes n}\right)_{A_n}$ are distinct from the corresponding nth exterior power and the nth symmetric power of M in gereral. However, for finitely generated k-modules M, with minimal number of generators $\mu(M)$, and all $n > \mu(M)$, we show

$$\left(M^{\otimes n}\right)_{A_n} \simeq \operatorname{Sym}^n(M)$$

in Proposition 5.10.

We have so far considered coinvariants with respect to three separate group actions on individual k-modules M. An organizing mechanism for the above discussion is to work with the nth tensor power functor T^n on the category of k-modules. The group actions of S_n and

A_n on nth tensor powers of modules arise from related group actions on the functor T^n by natural automorphisms. The group $\mathrm{Aut}(\mathrm{T}^n)$ is difficult to study in general. However, the following theorem gives a classification of all natural endomorphisms.

Theorem 1.2 (Theorem 3.8)**.** *The action (1.1) of S_n on $\mathrm{T}^n(M)$ extends to a k-algebra isomorphism ρ of the group algebra kS_n to the k-algebra* $\mathrm{End}(\mathrm{T}^n)$ *of natural endomorphisms of the functor* T^n*.*

Thus, $\mathrm{Aut}(\mathrm{T}^n)$ is isomorphic to the group of units, $U(k\mathrm{S}_n)$.

The proof of Theorem 3.8 is developed out of the thought that every element of every object in the image of the category Mod_k under the functor T^n can be described using a map $k^n \to M$ in the domain category and an element $\eta \in \mathrm{T}^n(k^n)$. Using naturality squares, we reduce to understanding the image of the element $\eta \in \mathrm{T}^n(k^n)$ under a natural endomorphism of T^n.

Having described the ring of endomorphisms of T^n we set ourselves up to explore examples of coinvariants of T^n. Working with the group of units of the integral group ring $\mathbb{Z}\mathrm{S}_n$, which has been studied in [PMS02], we obtain an infinite order automorphism σ and a finite order automorphism τ of the same functor T^n, such that the coinvariants with respect to the subgroups they generate are naturally isomorphic.

Theorem 1.3 (Theorem 4.7)**.** *The pair of subgroups H_1 and H_2 of the group algebra $\mathbb{Z}S_5$ given by*

$$H_1 = \langle (1) - (13524) - (14253) \rangle \quad \textit{and} \quad H_2 = \langle -(12345) \rangle$$

satisfy the following: $|H_1| = \infty$, $|H_2| = 10$, and for the tensor power functor $\mathrm{T}^5\colon Mod_{\mathbb{Z}} \to Mod_{\mathbb{Z}}$, there is a natural isomorphism of coinvariants,

$$\mathrm{T}^5_{H_1} \cong \mathrm{T}^5_{H_2}$$

Given a module M over a commutative unital ring k, the symmetric and (when 2 is invertible in k) exterior algebras arise as quotients of the tensor algebra. Componentwise, they are the quotient maps $\mathrm{T}^n \to \mathrm{Sym}^n$ and $\mathrm{T}^n \to \wedge^n$ that we have been considering above. We investigate the problem of whether a functor $\mathcal{F}_G$ that arises from a graded functor $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ by a sequence of group actions of $G = (G_n)$ is an algebra. We give a sufficient condition, lr-compatibility of the group actions, that ensures the componentwise coinvariants comprise an algebra.

Definition 1.4 (Definition 5.3)**.** A sequence of groups G_i, $i \in \mathbb{N}$ is **lr-compatible** with respect to a functor $\mathcal{F}\colon \mathcal{C} \to \mathrm{GrAlg}_k$ if for all $i, j \in \mathbb{N}$ and all $g \in G_i$ there exist two elements g' and g'' of G_{i+j} such that the following diagroms commute,

$$\begin{array}{ccc} \mathcal{F}_i \otimes \mathcal{F}_j & \xrightarrow{\mu} & \mathcal{F}_{i+j} \\ \downarrow{\scriptstyle g\otimes \mathrm{id}} & & \downarrow{\scriptstyle g'} \\ \mathcal{F}_i \otimes \mathcal{F}_j & \xrightarrow{\mu} & \mathcal{F}_{i+j} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_j \otimes \mathcal{F}_i & \xrightarrow{\mu} & \mathcal{F}_{i+j} \\ \downarrow{\scriptstyle \mathrm{id}\otimes g} & & \downarrow{\scriptstyle g''} \\ \mathcal{F}_j \otimes \mathcal{F}_i & \xrightarrow{\mu} & \mathcal{F}_{i+j} \end{array}$$

The purpose of lr-compatibility is to set up an ideal from the componentwise kernels of the coinvariants maps. We shown in Proposition 5.5 that lr-compatibility is indeed sufficient for the coinvariants of a functor to the category of graded algebras to be an algebra. In light of Theorem 1.2, lr-compatibility has a concrete description for the functor T, as described in Lemma 5.8, and we show how to form the coinvariants of the tensor algebra functor T with respect to the sequence of alternating groups A_n.

Chapter 2
PRELIMINARIES

2.1 Categories and Functors

We use facts about category theory and follow conventions from standard references such as [ML98]. A (locally small) category $\mathcal{C}$ consists of a class of objects and for each pair of objects c and d, a set of morphisms, $\mathrm{Mor}(c, d)$ along with a rule of composition,

$$\circ\colon \mathrm{Mor}(d, e) \times \mathrm{Mor}(c, d) \to \mathrm{Mor}(c, e)$$

that is associative, and for every object $c \in \mathcal{C}$, there is a corresponding identity morphism $\mathrm{id}_c \in \mathrm{Mor}(c, c)$.

A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a mapping on objects and morphisms such that, for $f \in \mathrm{Mor}_{\mathcal{C}}(c_1, c_2)$, the image $\mathcal{F}(f)$ belongs to $\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(c_1), \mathcal{F}(c_2))$. The conditions that $\mathcal{F}$ preserves composition and identities must also be satisfied.

Example 2.1. Let k be an associative unital ring. Then, Mod_k is a category, consisting of the k-modules and k-module homomorphisms.

Example 2.2. Given categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathcal{D}^{\mathcal{C}}$ of all functors from $\mathcal{C}$ to $\mathcal{D}$. For a pair of functors $\mathcal{F}, \mathcal{G}\colon \mathcal{C} \to \mathcal{D}$, morphisms from $\mathcal{F}$ to $\mathcal{G}$ in $\mathcal{D}^{\mathcal{C}}$ are natural transformations η, which are collections of morphisms $\eta_c\colon \mathcal{F}(c) \to \mathcal{G}(c)$ in $\mathcal{D}$, indexed by objects c of $\mathcal{C}$, such that for any morphism $f\colon c \to d$ in $\mathcal{C}$ the following diagram commutes,

$$\begin{array}{ccc} \mathcal{F}(c) & \xrightarrow{\eta_c} & \mathcal{G}(c) \\ \downarrow{\scriptstyle \mathcal{F}(f)} & & \downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(d) & \xrightarrow{\eta_d} & \mathcal{G}(d) \end{array}$$

Lemma 2.3 (Yoneda Lemma)**.** *For a category $\mathcal{C}$, there is a natural isomorphism,*

$$\mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(c,\cdot),\mathcal{F}) \simeq \mathcal{F}$$

natural with respect to objects c of $\mathcal{C}$ and functors $\mathcal{F}\colon \mathcal{C} \to Set$.

Corollary 2.4 (Yoneda Embedding)**.** *The functor $y\colon \mathcal{C}^{op} \to Set^{\mathcal{C}}$ that sends an object c of $\mathcal{C}^{op}$ to the covariant hom-functor $\mathrm{Mor}(c,\cdot)$ is fully faithful.*

Remark 2.5. Fully faithful functors reflect isomorphisms.

Definition 2.6. A morphism $f\colon c \to d$ in a category $\mathcal{C}$ is a

- **monomorphism** if the natural transformation $y(f)\colon \mathrm{Mor}(\cdot,c) \to \mathrm{Mor}(\cdot,d)$ consists of injective functions.

- **epimorphism** if the natural transformation $y(f)\colon \mathrm{Mor}(d,\cdot) \to \mathrm{Mor}(c,\cdot)$ consists of injective functions.

Inherited structure on functor categories

In many cases the category of functors from $\mathcal{C}$ to $\mathcal{D}$ inherits properties from the target category $\mathcal{D}$. As an illustration, we show how the property of having k-module structures on sets of morphisms is inherited.

Proposition 2.7. *For any category $\mathcal{C}$ and functors $\mathcal{F},\mathcal{G}\colon \mathcal{C} \to Mod_k$ the set of natural transformations $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ has the structure of a k-module that is induced by k-module structures on hom-sets in Mod_k. Moreover, composition in the functor category $Mod_k^{\mathcal{C}}$ is k-linear.*

Proof. Let η,τ be natural transformations from $\mathcal{F}$ to $\mathcal{G}$, and let $\lambda,\mu \in k$. We claim that the collection,

$$\lambda\eta_c + \mu\tau_c \in \mathrm{Hom}_{\mathrm{Mod}_k}(\mathcal{F}(c),\mathcal{G}(c)) \qquad (c \in \mathcal{C})$$

is a natural transformation.

Given a morphism $f\colon c\to d$ in $\mathcal{C}$, we have,

$$\begin{aligned}\mathcal{G}(f)\circ(\lambda\eta_c+\mu\tau_c) &= \lambda\mathcal{G}(f)\circ\eta_c+\mu\mathcal{G}(f)\circ\tau_c\\ &= \lambda\eta_d\mathcal{F}(f)+\mu\tau_d\mathcal{F}(f)=(\lambda\eta_d+\mu\tau_d)\circ\mathcal{F}(f)\end{aligned}$$

and this shows that $\lambda\eta+\mu\tau$ is a natural transformation.

That $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ is a k-module follows from the fact that each $\mathrm{Hom}_{\mathrm{Mod}_k}(\mathcal{F}(c),\mathcal{G}(c))$ is a k-module.

Also, let $\mathcal{E}$ and $\mathcal{H}$ be functors from $\mathcal{C}$ to Mod_k along with natural transformations

$$\varepsilon\colon\mathcal{E}\to\mathcal{F}\quad\text{and}\quad\sigma\colon\mathcal{G}\to\mathcal{H}$$

For each object c of $\mathcal{C}$, the component of the natural transformation $\sigma\circ(\lambda\eta+\mu\tau)\circ\varepsilon$ satisfies,

$$\begin{aligned}(\sigma\circ(\lambda\eta+\mu\tau)\circ\varepsilon)_c &= \sigma_c\circ(\lambda\eta_c+\mu\tau_c)\circ\varepsilon_c\\ &= \lambda(\sigma_c\circ\eta_c\circ\varepsilon_c)+\mu(\sigma_c\circ\tau_c\circ\varepsilon_c)\\ &= \big(\lambda(\sigma\circ\eta\circ\varepsilon)+\mu(\sigma\circ\tau\circ\varepsilon)\big)_c,\end{aligned}$$

because composition is k-linear in Mod_k. Hence, composition is k-linear in $\mathrm{Mod}_k^{\mathcal{C}}$. □

Corollary 2.8. *For any category $\mathcal{C}$ and any functor $\mathcal{F}\colon\mathcal{C}\to Mod_k$ the endomorphisms of $\mathcal{F}$ have a k-algebra structure induced by the k-algebra structures on the components $\mathrm{End}_{Mod_k}(\mathcal{F}(c))$.*

If limits and colimits with fixed index category exist for all diagrams of that shape in the target category $\mathcal{D}$, then for any domain category $\mathcal{C}$, the functor category $\Phi=\mathcal{C}^{\mathcal{D}}$, has all limits and colimits of that shape.

Let $\mathcal{I}$ be a small category. Functors $\delta\colon\mathcal{I}\to\mathcal{D}$ represent diagrams in the category $\mathcal{D}$.

Proposition 2.9 ([KS06, p. 38])**.** *Let $\mathcal{C}$ be any category and suppose $\mathcal{D}$ is a category that has all limits (resp. colimits) of shape $\mathcal{I}$. Then, $\mathcal{D}^{\mathcal{C}}$ has all limits (resp. colimits) of shape $\mathcal{I}$.*

Proposition 2.10 ([Wei94, p. 25])**.** *If $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is an arbitrary category, then the functor category $\mathcal{C}^{\mathcal{A}}$ is an abelian category.*

2.2 Tensor products

Proposition 2.11. *For a k-module M, the nth tensor power of M is generated by* ***simple tensors*** $x_1 \otimes \cdots \otimes x_n$.

Proof. This follows from the standard construction of the tensor product of a pair of k-modules, see [Mat86, p. 266]. □

Example 2.12. The nth tensor power of the free module k^m is a free module of rank m^n and it has a free basis $\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}$ for tuples $(i_1, \ldots, i_n)$ of indices in $[m]$.

2.3 Tensor products of functors

We show how to construct tensor products in an object-by-object manner. This is fimiliar due to the fact that monoidal structure is inherited by categories of functors. For any category $\mathcal{C}$ and a monoidal category $\mathcal{D}$, the functor category $\mathcal{D}^{\mathcal{C}}$ is monoidal with induced monoidal structure, by [ML98, p. 161]. In particular, the functor category $\mathrm{Mod}_k^{\mathcal{C}}$ has tensor products and the tensor product of two functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathrm{Mod}_k$ is given object- and morphism-wise by $(\mathcal{F} \otimes \mathcal{G})(f) = \mathcal{F}(f) \otimes_k \mathcal{G}(f)$. That Mod_k is monoidal is discussed in [ML98, p. 159-160].

Let $\mathcal{F}$ and $\mathcal{G}$ be functors from a category $\mathcal{C}$ to the category Mod_k. We define their **tensor product** using the tensor product in Mod_k. For an object c of $\mathcal{C}$, we have

$$(\mathcal{F} \otimes \mathcal{G})\,(c) = \mathcal{F}(c) \otimes \mathcal{G}(c)$$

and on morphisms, one uses the tensor product of the morphisms,

$$(\mathcal{F} \otimes \mathcal{G})\,(f) = \mathcal{F}(f) \otimes \mathcal{G}(f)$$

Remark 2.13. There is a distinct notion of the tensor product of functors $\mathcal{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_k$ and $\mathcal{G} \colon \mathcal{C} \to \mathrm{Mod}_k$ as the coend of the bifunctor on $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ that sends an object (c, d) to $\mathcal{F}(c) \otimes \mathcal{G}(d)$. This is given by an object of Mod_k along with a universal property, rather than a functor.

A helpful way of thinking about this notion of tensor product is that for a left k-module M and a right k-module N, which can be described as functors $k^{\mathrm{op}} \to \mathrm{Ab}$ and $k \to \mathrm{Ab}$ on categories with one object, and the tensor product of these functors is the tensor product $M \otimes_k N$.

For a given $n \in \mathbb{N}$ and a functor $\mathcal{F} \colon \mathcal{C} \to \mathrm{Mod}_k$, one can take the **$n$th tensor power** of $\mathcal{F}$, as an iterated tensor product,

$$\mathrm{T}^n(\mathcal{F}) = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$$

Special case: $\mathcal{F} = \mathrm{id}_{\mathrm{Mod}_k}$, then $\mathrm{T}^n(\mathcal{F}) = \mathrm{T}^n$ which maps a k-module M to its nth tensor power, $\mathrm{T}^n(M) = M^{\otimes n}$.

Example 2.14. Picking out a simple tensor in $\mathrm{T}^n(M)$ For any simple tensor $x_1 \otimes \cdots \otimes x_n$ in $\mathrm{T}^n(M)$, we have a k-module homomorphism $h \colon k^n \to M$ that sends $\mathbf{e}_i$ to x_i for $i = 1, \ldots, n$.

Then,

$$\mathrm{T}^n(h)\,(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n) = h(\mathbf{e}_1) \otimes \cdots \otimes h(\mathbf{e}_n) = x_1 \otimes \cdots \otimes x_n$$

Showing that every member of a generating system of $\mathrm{T}^n(M)$ can be obtained as the image of the specific element $\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n$ of $\mathrm{T}^n(k^n)$ via a k-module homomorphism $k^n \to M$.

Chapter 3

GROUP ACTIONS AND COINVARIANTS

3.1 Group action on an object of a category

We set up our notation and conventions for group actions on objects of a category.

3.1.1 Definitions

There are familiar conceptions of group actions, such as actions of groups on sets or linear representations of groups. A common feature is that the group G maps into the group of automorphisms, structured by the category in which the object lives.

Definition 3.1. An **action** of a group G on an object c of a category $\mathcal{C}$ is a group homomorphism $\alpha\colon G \to \mathrm{Aut}_{\mathcal{C}}(c)$.

Remark 3.2. For any group G and object c of a category $\mathcal{C}$, the **trivial group action** is obtained using the group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(c)$ that sends every element of G to the identity automorphism of c.

Remark 3.3. A group G corresponds to a category $\mathcal{G}$ with one object, in which the elements of the group correspond to morphisms in that category. Then, a group action on an object c of $\mathcal{C}$ can be viewed as a functor $\delta\colon \mathcal{G} \to \mathcal{C}$ that sends the object of $\mathcal{G}$ to c.

Definition 3.4. For an object c of a category $\mathcal{C}$, there is a category GrpAct_c in which the objects are group actions on c, and a morphism from a group action α of a group G on c to a group action β of a group H on c consists of a group homomorphism $G \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} G & & \\ \downarrow & \overset{\alpha}{\searrow} & \\ H & \xrightarrow{\beta} & \mathrm{Aut}_{\mathcal{C}}(c) \end{array}$$

Alternatively, if you hold the group fixed and allow the object to vary, the structure preserving maps are equivariant.

Definition 3.5. Let G be a group with actions α and β on objects c and d in a category $\mathcal{C}$. A morphism $\varphi\colon c \to d$ is G-**equivariant** if for any $g \in G$ the following diagram commutes:

$$\begin{CD} c @>{\varphi}>> d \\ @V{\alpha(g)}VV @VV{\beta(g)}V \\ c @>{\varphi}>> d \end{CD}$$

We shall use the notation $\mathrm{Mor}_G(c, d)$ for the set of all G-equivariant morphisms from c to d.

Proposition 3.6. *Let G be a group along with actions α_1 and α_2 on objects c_1 and c_2 of a category $\mathcal{C}$. For a group homomorphism $\omega\colon H \to G$ there are induced actions $\beta_i = \alpha_i \circ \omega$ of H on c_i. With respect to these actions there is an inclusion of sets equivariant morphisms,*

$$\mathrm{Mor}_G(c_1, c_2) \subseteq \mathrm{Mor}_H(c_1, c_2)$$

Moreover, for all G-equivariant morphisms $\varphi\colon c_2 \to d$ the following diagram is well-defined and commutative,

$$\begin{CD} \mathrm{Mor}_G(c_1, c_2) @>{\varphi_*}>> \mathrm{Mor}_G(c_1, d) \\ @VV{\subseteq}V @VV{\subseteq}V \\ \mathrm{Mor}_H(c_1, c_2) @>{\varphi_*}>> \mathrm{Mor}_H(c_1, d) \end{CD} \tag{3.1}$$

Proof. Let $\psi\colon c_1 \to c_2$ be a G-equivariant morphism. Then, for any element h of H, the following diagram commutes,

$$\begin{CD} c_1 @>{\psi}>> c_2 \\ @VV{\beta_1(h)}V @VV{\beta_2(h)}V \\ c_1 @>{\psi}>> c_2 \end{CD}$$

so that f is an H-equivariant morphism. This shows that $\mathrm{Mor}_G(c_1, c_2)$ is a subset of $\mathrm{Mor}_H(c_1, c_2)$.

For G-equivariant φ from c_2 to an object d with action α, and all $g \in G$, the following diagram commutes,

$$\begin{array}{ccccc} c_1 & \xrightarrow{\psi} & c_2 & \xrightarrow{\varphi} & d \\ \downarrow{\scriptstyle \alpha_1(g)} & & \downarrow{\scriptstyle \alpha_2(g)} & & \downarrow{\scriptstyle \alpha(g)} \\ c_1 & \xrightarrow{\psi} & c_2 & \xrightarrow{\varphi} & d \end{array}$$

So, $\varphi \circ \psi \in \mathrm{Mor}_G(c_1, d)$. As we have shown, φ is also H-equivariant and it follows that $\varphi \circ \psi \in \mathrm{Mor}_H(c_1, d)$. This shows the morphisms of the diagram (3.1) are well-defined and that the diagram is commutative. □

3.2 Group Actions on functors $\mathcal{C} \to$ Mod$_k$

Proposition 3.7. *Given a group action α of G on a functor $\mathcal{F}\colon \mathcal{C} \to Mod_k$ the group homomorphism $\alpha\colon G \to \mathrm{Aut}(\mathcal{F})$ extends to a unique k-algebra homomorphism $\rho\colon kG \to \mathrm{End}(\mathcal{F})$, which is to say that the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & \mathrm{Aut}(\mathcal{F}) \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \subseteq} \\ kG & \overset{\exists!\rho}{\dashrightarrow} & \mathrm{End}(\mathcal{F}) \end{array}$$

Proof. By Corollary 2.8, $\mathrm{End}(\mathcal{F})$ has a structure of a k-algebra, in which the multiplication is composition of natural endomorphisms. The map ρ is uniquely determined, if it exists, because G is a basis for kG as vector space over k, and the formula is,

$$\rho\left(\sum_{i=1}^{r} a_i g_i\right) = \sum_{i=1}^{r} a_i \alpha(g_i)$$

It is immediate that ρ is a k-linear map. The fact that ρ preserves multiplication and multiplicative identity comes from the fact that α is a group homomorphism, as the following calculation reveals.

Let $u, v \in kG$. They can simultaneously be expressed using finitely many elements of G,

$$u = \sum_{i=1}^{r} a_i g_i \quad \text{and} \quad v = \sum_{i=1}^{r} b_i g_i$$

and the product is given by,

$$uv = \sum_{g \in G} c_g g \quad \text{where} \quad c_g = \sum_{\substack{1 \le i,j \le r \\ g_i g_j = g}} a_i b_j$$

Hence,

$$\rho(uv) = \sum_{g \in G} c_g \alpha(g)$$

Because it is the case that for all g and all i, j such that $g_i g_j = g$ we have $\alpha(g_i)\alpha(g_j) = \alpha(g)$, we get,

$$\rho(uv) = \sum_{i=1}^{r} a_i \alpha(g_i) \sum_{j=1}^{r} b_j \alpha(g_j) = \rho(u)\rho(v)$$

This establishes the existence and uniqueness of such an extension. □

3.3 Group Actions on Tensor power functors

Let k be a commutative unital ring. For $n \geq 0$, let $\mathrm{T}^n \colon \mathrm{Mod}_k \to \mathrm{Mod}_k$ be the nth tensor power functor. We consider the ring of natural endomorphisms of T^n, the units of which are especially interesting to us. Any permutation $\sigma \in \mathrm{S}_n$ induces an automorphism of T^n, by permutating components of simple tensors and extending linearly.

By Proposition 3.7, the action of S_n on T^n induces a k-algebra homomorphism

$$\rho \colon k\mathrm{S}_n \to \mathrm{End}(\mathrm{T}^n). \tag{3.2}$$

We prove Theorem 3.8, which states that ρ is an isomorphism. In Section 3.8 we describe how the functors Sym and $\wedge^n$ are coinvariants of T^n with respect to specific subgroups of kS_n.

Recall the ring homomorphism ρ of (3.2) which is induced by the action of S_n on the functor T. Given an element τ of the group algebra $k\mathrm{S}_n$, written out in terms of permutations σ_i as $\tau = \sum_{i=1}^{r} a_i \sigma_i$, a k-module M, and a simple tensor $x_1 \otimes \cdots \otimes x_n$ in the n-fold tensor product of M, the image of this simple tensor under $\rho(\tau)$ is given by,

$$\rho(\tau)_M(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{r} a_i x_{\sigma_i^{-1}(1)} \otimes \cdots \otimes x_{\sigma_i^{-1}(n)} \tag{3.3}$$

The following theorem states that every natural endomorphism of the functor T^n is of the form $\rho(\tau)$ for a unique τ.

Theorem 3.8. *The ring homomorphism ρ of (3.2) is an isomorphism.*

Proof. We first verify that ρ is injective. Let $\tau = \sum_{\sigma\in \mathrm{S}_n} a_\sigma\ \sigma$ be an element of $k\mathrm{S}_n$ and $\omega = \rho(\tau)$ the corresponding endomorphism of the functor T^n. For the standard basis $\mathbf{e}_1, \ldots,$ $\mathbf{e}_n$ of the free module k^n,

$$\omega_{k^n}\left(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n\right) = \sum_{\sigma\in \mathrm{S}_n} a_\sigma\ \mathbf{e}_{\sigma^{-1}(1)} \otimes \cdots \otimes \mathbf{e}_{\sigma^{-1}(n)}$$

as in (3.3). Note that the simple tensors $\mathbf{e}_{\sigma^{-1}(1)} \otimes \cdots \otimes \mathbf{e}_{\sigma^{-1}(n)}$ are distinct and thus are part of a free basis of $\mathrm{T}^n(M)$, by Example 2.12. So, if $\omega = 0$, then we would have $a_\sigma = 0$ for every σ, which implies $\tau = 0$. Having shown $\ker(\rho) = 0$, we conclude that ρ is injective.

We now show that ρ is surjective. Let $\eta \in \mathrm{End}(\mathrm{T}^n)$. We examine the element,

$$x_0 = \eta_{k^n}\left(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n\right) \in \mathrm{T}^n(k^n)$$

Expanding x_0 with respect to the basis of $\mathrm{T}^n(k^n)$ of Example 2.12, we have

$$x_0 = \sum_I a_I\ \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}$$

with the sum being taken over all tuples $I = (i_1, \ldots, i_n)$ of indices $i_j \in \{1, \ldots, n\}$ and coefficients $a_I \in k$. We shall show that for any tuple I with a repeat, $i_j = i_k$, the coefficients $a_I = 0$.

We do this using naturality of η with respect to the following k-module homomorphisms κ_i. For each i, let $\kappa_i \colon k^n \to k^n$ be the unique k-linear mapping such that

$$\kappa_i\left(\mathbf{e}_j\right) = \begin{cases} \mathbf{e}_j & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases}$$

Relevantly, $\mathrm{T}^n(\kappa_i)$ filters out basis elements of $\mathrm{T}^n(k^n)$ that include $\mathbf{e}_i$. Put more precisely, for an n-tuple $I = (i_1, \ldots, i_n)$, we have,

$$\mathrm{T}^n(\kappa_i)\left(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n\right) = \begin{cases} \mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n & \text{if } i_1 \neq i \text{ and } \ldots \text{and } i_n \neq i \\ 0 & \text{if } i_1 = i \text{ or } \ldots \text{or } i_n = i \end{cases} \tag{3.4}$$

By naturality of η, we have

$$\eta_{k^n} \circ \mathrm{T}^n(\kappa_i) = \mathrm{T}^n(\kappa_i) \circ \eta_{k^n}$$

and we compare the values of both sides on $\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n$. Using the description of $\mathrm{T}^n(\kappa_i)$ in (3.4), we have

$$\mathrm{T}^n(\kappa_i)\,(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n) = 0$$

and

$$\mathrm{T}^n(\kappa_i)\,(\eta_{k^n}(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n)) = \sum_I a_I\ \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}$$

where the sum is taken over those index sets I that do not include i.

We conclude that $a_I = 0$ for all I such that any given i does not occur. It follows that the only terms in the expression for $\eta_{k^n}\,(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n)$ that could be nonzero correspond to index tuples I that are permutations of $1, \ldots, n$.

Hence, we can write the image of $\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n$ in the following way,

$$\eta_{k^n}\,(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n) = \sum_{\sigma \in \mathrm{S}_n} a_\sigma\ \mathbf{e}_{\sigma^{-1}(1)} \otimes \cdots \otimes \mathbf{e}_{\sigma^{-1}(n)}$$

We finish by proving that

$$\eta = \rho\left(\sum_{\sigma \in \mathrm{S}_n} a_\sigma\ \sigma\right)$$

Let $x_1 \otimes \cdots \otimes x_n$ be a simple tensor in $\mathrm{T}^n(M)$ for a k-module M. We will use the map

$$\phi\colon k^n \to M : \mathbf{e}_i \mapsto x_i$$

to compute $\eta_M(x_1 \otimes \cdots \otimes x_n)$. By naturality of η, the following diagram commutes,

$$\begin{array}{ccc} \mathrm{T}^n(k^n) & \xrightarrow{\eta_{k^n}} & \mathrm{T}^n(k^n) \\ \big\downarrow{\scriptstyle \mathrm{T}^n(\phi)} & & \big\downarrow{\scriptstyle \mathrm{T}^n(\phi)} \\ \mathrm{T}^n(M) & \xrightarrow{\eta_M} & \mathrm{T}^n(M) \end{array}$$

By following the element $\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n$ we get

$$\begin{aligned}\eta_M(x_1 \otimes \cdots \otimes x_n) &= \mathrm{T}^n(\phi)\left(\sum_{\sigma \in \mathrm{S}_n} a_\sigma \; \mathbf{e}_{\sigma^{-1}(1)} \otimes \cdots \otimes \mathbf{e}_{\sigma^{-1}(n)}\right) \\ &= \sum_{\sigma \in \mathrm{S}_n} a_\sigma \; x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \\ &= \rho\left(\sum_{\sigma \in \mathrm{S}_n} a_\sigma \sigma\right)(x_1 \otimes \cdots \otimes x_n)\end{aligned}$$

as desired. Since $\mathrm{T}^n(M)$ is generated by simple tensors, we conclude $\eta_M = \rho\left(\sum_{\sigma \in \mathrm{S}_n} a_\sigma \sigma\right)_M$. Thus, η is in the range of ρ, showing that ρ is surjective. □

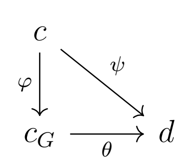
3.4 Coinvariants of an abstract group action on an object of a category

We give a definition of coninvariants for a group action on an object of a category. The definition is by a universal property, and the coinvariants need not exist. Under the hypothesis the category has all small colimits, then coinvariants do exist. For example, coinvariants in the category Set exist for all group actions, and are given by the set of orbits. In the category Mod_k coinvariants in our sense agree with a common definition as a quotient, stated in Corollary 3.19.

3.4.1 Definition of Coinvariants

Definition 3.9. Let G be a group with an action on an object c in a category $\mathcal{C}$. An object c_G of $\mathcal{C}$ along with a G-equivariant morphism $\varphi\colon c \to c_G$ where the codomain is equipped with trivial G-action are **coinvariants** of c with respect to G if the following universal property is satisfied:

For any G-equivariant morphism $\psi\colon c \to d$ in the category $\mathcal{C}$ for which d has the trivial G-action, there exists a unique morphism $\theta\colon c_G \to d$ in $\mathcal{C}$ such that the following diagram commutes:



3.4.2 Existence of coinvariants

We address the question of the existence of coinvariants for a group action by observing that c_G is a colimit of the diagram δ of Remark 3.3. This has the consequence that any category with all small colimits has coinvariants. In particular an abelian category that has arbitrary direct sums has coinvariants, and a description of coinvariants that is particular to that setting is provided in Section 3.6.

Proposition 3.10. *If one of the following constructions exists, then both of them exist and are canonically isomorphic:*

1. *The coinvariants* c_G

2. *For the diagram* δ *in* $\mathcal{C}$ *for the group action of* G *on* c*,* $\mathrm{colim}(\delta)$

Proof. For an object d of $\mathcal{C}$ with trivial G-action, the set $\mathrm{Mor}_G(c,d)$ is the set of all co-cones of δ under d. The relevant universal properties imply,

$$\mathrm{Mor}_{\mathcal{C}}(c_G, d) \simeq \mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}(\delta), d)$$

whence, c_G and $\mathrm{colim}(\delta)$ are naturally isomorphic. □

Corollary 3.11. *If* $\mathcal{C}$ *has all small colimits, then all coinvariants with respect to all groups and all objects of* $\mathcal{C}$ *exist.*

3.4.3 Functoriality

Let $\mathcal{C}$ be a category that has all small colimits, then the coinvariants with respect to any group action on any object of $\mathcal{C}$ exists in $\mathcal{C}$. The following proposition addresses functoriality of coinvariants, i.e. $G \mapsto c_G$ is a functor on the category of group actions on c.

Proposition 3.12. *Let* c *be an object of a category* $\mathcal{C}$ *that has all small colimits. Then, there is a covariant functor* $GrpAct_c \to \mathcal{C}$ *such that an action on* c *is mapped to its coinvariants. Moreover, a morphism of group actions is mapped to an epimorphism in* $\mathcal{C}$*.*

Proof. Let $\omega\colon H \to G$ be a morphism of group actions $\beta \to \alpha$ on an object c of $\mathcal{C}$. For any object d of $\mathcal{C}$, endowed with the trivial actions of G and H, Proposition 3.6 implies,

$$\mathrm{Mor}_G(c, d) \subseteq \mathrm{Mor}_H(c, d) \tag{3.5}$$

In fact, Proposition 3.6 shows the inclusion is natural with respect to d.

By the universal property of coinvariants, there is a natural monomorphism

$$\mathrm{Mor}(c_G, \cdot) \hookrightarrow \mathrm{Mor}(c_H, \cdot)$$

The Yoneda lemma implies there exists $c_H \to c_G$ that is an epimorphism by definition.

The fact that the mapping on morphisms preserves composition follows from the observation that the composition of inclusions of sets in (3.5) is an inclusion of sets. Preservation of identity morphisms of group actions is immediate. □

The following proposition shows that from the perspective of taking coinvariants, there is no difference between using a group G with action $\alpha\colon G \to \mathrm{Aut}(c)$ and using its image $\alpha(G)$.

Proposition 3.13. *Let c be an object of a category $\mathcal{C}$. Let $\alpha\colon G \to \mathrm{Aut}(c)$ and assume that the coinvariants c_G exists. Form the group $G' = \alpha(G)$, which has an action on c via the inclusion of G' in* $\mathrm{Aut}(c)$*. Then, there is a canonical isomorphism of coinvariants with respect to G and coinvariants with respect to G':*

$$c_G \simeq c_{G'}$$

Proof. Let $f\colon c \to d$ be a G'-equivariant morphism with trivial G' action on d. For any $g \in G$ we have $\alpha(g) \in G'$ and thus the following diagram commutes,

$$\begin{array}{ccc} c & \xrightarrow{f} & d \\ {\scriptstyle \alpha(g)}\downarrow & & \| \\ c & \xrightarrow{f} & d \end{array}$$

Let $\varphi\colon c \to c_G$ be the universal morphism for coinvariants with respect to G. By definition, there exists a unique morphism $\overline{f}\colon c_G \to d$ such that $\overline{f} \circ \varphi = f$. So, φ and c_G satisfy the universal property of coinvariants with respect to G'. By uniqueness up to unique isomorphism, we conclude that c_G and $c_{G'}$ are canonically isomorphic. $\square$

3.5 Properties of Coinvariants

Proposition 3.14. *If $G_1 \to G_2$ is a morphism of group actions on an object c of a category $\mathcal{C}$ and the coinvariants c_{G_i} exist for $i = 1, 2$, then the morphism on coinvariants $c_{G_1} \to c_{G_2}$ is an epimorphism.*

Proof. Let $h\colon G_1 \to G_2$ be a morphism of group actions on c. Let d be an object of $\mathcal{C}$, equipped with the trivial G_1 and G_2 actions. We have $\mathrm{Mor}_{G_2}(c, d) \subseteq \mathrm{Mor}_{G_1}(c, d)$ by Proposition 3.6. Since c_{G_1} and c_{G_2} represent the functors $\mathrm{Mor}_{G_1}(c, \cdot)$ and $\mathrm{Mor}_{G_2}(c, \cdot)$ from $\mathcal{C}$ to Set, we thus have a natural monomorphism of hom-functors,

$$\mathrm{Mor}_{\mathcal{C}}(c_{G_2}, \cdot) \hookrightarrow \mathrm{Mor}_{\mathcal{C}}(c_{G_1}, \cdot)$$

By the Yoneda lemma, Corollary 2.4, there is a uniquely determined morphism $c_{G_1} \to c_{G_2}$, which is an epimorphism by Definition 2.6. $\square$

Corollary 3.15. *If the G-coinvariants c_G exists for an object c equipped with a G-action, then the canonical morphism $c \to c_G$ is an epimorphism.*

Proof. Let G be a group with identity e. Then, the inclusion of the trivial subgroup $\{ e \}$ in G induces an epimorphism $c_{\{ e \}} \twoheadrightarrow c_G$, and $c = c_{\{ e \}}$. $\square$

Proposition 3.16. *Let H_1 and H_2 be conjugate subgroups of a group G that acts on an object c of $\mathcal{C}$. Assume $\mathcal{C}$ has all small colimits. Then, the coinvariants c_{H_1} and c_{H_2} are isomorphic.*

Proof. By assumption, there exists $\alpha \in G$ such that $\alpha^{-1} H_1 \alpha = H_2$.

Let $\delta_j\colon H_j \to \mathcal{C}$, for $j = 1, 2$, be the relevant diagrams. Let $\varphi\colon H_1 \to H_2$ be the functor that corresponds to the group homomorphism $h \mapsto \alpha^{-1} h \alpha$. Then, the automorphism $\alpha\colon c \to c$ induces a natural isomorphism $\delta_1 \to \delta_2 \circ \varphi$. Since φ is an isomorphism, $\varinjlim \delta_2 = \varinjlim (\delta_2 \circ \varphi)$. We have thus established $c_{H_1} \cong c_{H_2}$. $\square$

3.5.1 Coinvariants are inherited in functor categories

Let $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ be a functor and G a group that acts on $\mathcal{F}$ by natural transformations. Under the assumption that $\mathcal{D}$ has all colimits, the functor category $\mathcal{D}^{\mathcal{C}}$ inherits colimits. Thus, Corollary 3.11 implies that $\mathcal{F}_G$ exists.

Proposition 3.17. *Assume $\mathcal{D}$ has small colimits. Then, for every object c of $\mathcal{C}$ we have canonical isomorphisms*

$$\mathcal{F}_G(c) \simeq \mathcal{F}\,(c)_G$$

Proof. The assumption that $\mathcal{D}$ has small colimits implies the functor category $\mathcal{D}^{\mathcal{C}}$ has small colimits, using Proposition 2.9. The desired result is then a consequence of Proposition 3.10. □

3.6 Coinvariants as a cokernel

There is a familiar description of the kernel of the coinvariants map on modules, [Wei94, p. 160],

$$\ker(M \to M_G) = \langle\, x - \sigma(x) \,:\, x \in M, \sigma \in G\,\rangle \tag{3.6}$$

We show that this description holds, under the definition we have given. An abelian category $\mathcal{A}$ satisfies Grothendieck axiom (AB3) if it is closed under all infinite direct sums, introduced in [Gro57]. Proposition 3.18 is the analogue of (3.6) in the setting of abelian categories with arbitrary direct sums. We deduce the statements for Mod_k and categories of functors into Mod_k in Corollaries 3.19 and 3.21.

Let G be a group with an action on an object c of an additive category $\mathcal{C}$. We observe that for any morphism $f: c \to d$ of $\mathcal{C}$, it is G-equivariant for the trivial action on d if and only if for all $\sigma \in G$, we have $f \circ (\sigma - \mathrm{id}_c) = 0$. If $\mathcal{C}$ is abelian and satisfies (AB3), we can use a construction that imitates the sum of the images of morphisms $\sigma - \mathrm{id}_c$ as σ varies over the elements of G. Proposition 3.18 is borne out of this idea.

Proposition 3.18. *Let $\mathcal{C}$ be an abelian category that satisfies the axiom (AB3). Let G be an abstract group action on an object c of $\mathcal{C}$. Then, the coinvariants of c with respect to G arises as the following cokernel,*

$$c_G \simeq \operatorname{coker}\left(\bigoplus_{\sigma \in G} \operatorname{im}(\sigma - id_c) \to c\right)$$

Proof. Let $\varphi\colon \bigoplus_{\sigma \in G} \operatorname{im}(\sigma - \mathrm{id}_c) \to c$ be the canonical morphism. Let d be an object of $\mathcal{C}$ with trivial G-action. We first prove the equality,

$$\operatorname{Hom}_G(c, d) = \{\, f \in \operatorname{Hom}_{\mathcal{C}}(c, d) \,:\, f \circ \varphi = 0 \,\} \tag{3.7}$$

Suppose $f \circ \varphi = 0$. Given $\sigma_0 \in G$, observe that $\sigma_0 - \mathrm{id}_c$ is the composition

$$c \xrightarrow{\sigma_0 - \mathrm{id}_c} \operatorname{im}(\sigma_0 - \mathrm{id}_c) \xrightarrow{\iota} \bigoplus_{\sigma \in G} (\sigma - \mathrm{id}_c) \xrightarrow{\varphi} c$$

This implies $f \circ (\sigma_0 - \mathrm{id}_c) = 0$. We have thus shown that f is G-equivariant.

Conversely, suppose f is G-equivariant. This implies that $f \circ (\sigma - \mathrm{id}_c) = 0$ for every $\sigma \in G$. Since $\operatorname{im}(\sigma - \mathrm{id}_c) = \ker(\operatorname{coker}(\sigma - \mathrm{id}_c))$ and f factors through $\operatorname{coker}(\sigma - \mathrm{id}_c)$, the composition

$$\operatorname{im}(\sigma - \mathrm{id}_c) \xrightarrow{\iota} c \xrightarrow{f} d$$

is 0. This holds for all $\sigma \in G$, so $f \circ \varphi = 0$.

We have established (3.7). Exploiting the natural bijections,

$$\operatorname{Hom}_G(c, d) \simeq \operatorname{Hom}(c_G, d) \quad \text{and} \quad \{\, f\colon c \to d \,:\, f \circ \varphi = 0 \,\} \simeq \operatorname{Hom}(\operatorname{coker}(\varphi), d)$$

we have an isomorphism $\operatorname{coker}(\varphi) \simeq c_G$ by the Yoneda Lemma, Corollary 2.4. $\square$

Corollary 3.19. *For an action of a group G on a k-module M, the kernel of the coinvariants map takes the following form:*

$$\ker(M \to M_G) = \langle\, x - \sigma(x) \,:\, x \in M, \sigma \in G \,\rangle$$

Proof. Using Proposition 3.18, we have

$$\ker(M \to M_G) = \ker\left(\operatorname{coker}\left(\bigoplus_{\sigma\in G} \operatorname{im}(\sigma - \operatorname{id}_M) \to M\right)\right)$$

and that is the sum, in M, of the images of the maps $\sigma - \operatorname{id}_M$, as wanted. □

Corollary 3.20. *For a k-module M and a group G of (k-linear) automorphisms of M, let x_i, $i \in \mathscr{I}$, be a system of generators for the k-module M and σ_α, $\alpha \in \mathscr{A}$, generators for the group G. Then, for the coinvariants map $\varphi\colon M \to M_G$, the kernel is generated as a k-module by the set*

$$\{\, x_i - \sigma_\alpha(x_i) \,:\, i \in \mathscr{I}, \alpha \in \mathscr{A} \,\}$$

Proof. Let $K = \ker(\varphi)$. By Corollary 3.19,

$$K = \langle\, x - \sigma(x) \,:\, x \in M, \sigma \in G \,\rangle$$

Let L be the submodule of M,

$$L = \langle\, x_i - \sigma_\alpha(x_i) \,:\, i \in \mathscr{I}, \alpha \in \mathscr{A} \,\rangle$$

It is immediate that $L \subseteq K$. We show the inclusion $K \subseteq L$.

Consider the following subset H of G.

$$H = \{\, \sigma \in G \,:\, \forall x \in M, x - \sigma(x) \in L \,\}$$

Because L is closed under addition and scalar multiplication by elements of k, we have

$$\sigma_\alpha \in H \qquad (\alpha \in \mathscr{A}) \tag{3.8}$$

We proceed to show that H is closed under multiplication and inverses. Let $\sigma, \tau \in H$. For any $x \in M$, we have

$$x - \sigma(\tau(x)) = x - \tau(x) + \tau(x) - \sigma(\tau(x)) \in L$$

and

$$x - \sigma^{-1}(x) = -(\sigma^{-1}(x) - \sigma(\sigma^{-1}(x))) \in L$$

Therefore, H is a subgroup of G. Recalling that x_α is a system of generators of G and (3.8), we conclude

$$H = G$$

We conclude that $K \subseteq L$. □

Corollary 3.21. *Let $F\colon \mathcal{C} \to Mod_k$ be a functor into the category of k-modules, which admits an abstract group action by G. Let $\varphi\colon F \to F_G$ be the coinvariants map. Then, the functor* $\ker\varphi$ *exists and is given on objects c of $\mathcal{C}$ by*

$$(\ker\varphi)(c) = \ker(\varphi_c) = \langle x - \sigma x \mid x \in Fc, \sigma \in G\rangle \subseteq Fc$$

Proof. Coinvariants and kernels can be computed object-by-object using Proposition 3.17 and Proposition 2.9. Thus, we can appeal directly to Corollary 3.19. □

3.7 Coinvariants of functors $\mathcal{C} \to \mathbf{Mod}_k$ with respect to finite groups

Let α be a group action of a group G on a functor $\mathcal{F}\colon \mathcal{C} \to \mathrm{Mod}_k$. If $\alpha(G)$ is a finite group of automorphisms of $\mathcal{F}$ and its order is invertible in k, then the coinvariants of $\mathcal{F}$ with respect to G is a direct summand of $\mathcal{F}$. We use a construction that resembles the way in which Schur functors are defined using Young tableaux. By Proposition 3.13, we may replace G with its image in $\mathrm{Aut}(\mathcal{F})$.

Proposition 3.22. *Let $\mathcal{F}$ be a functor from a category $\mathcal{C}$ to Mod_k. Given a finite group G of automorphisms of $\mathcal{F}$ such that $|G|$ is invertible in k, there exists an idempotent endomorphism π_G of $\mathcal{F}$ such that*

$$\mathrm{im}(\pi_G) \simeq \mathcal{F}_G$$

In particular, $\mathcal{F}_G$ is a direct summand of $\mathcal{F}$.

Proof. We form π_G as a linear combination in $\mathrm{End}(\mathcal{F})$,

$$\pi_G = \frac{1}{|G|}\sum_{g\in G} g$$

Right multiplication by an element of the group G is a bijection, so

$$\pi_G g = \pi_G \qquad (\forall g \in G)$$

Then, it follows that $\pi_G \circ \pi_G = \pi_G$.

Now, the category of functors from $\mathcal{C}$ to Mod_k is Karoubian, meaning that it is preadditive and idempotent endomorphisms have kernels. By the proof of [Sta24, Lemma 09SH], the kernel of π_G is given by

$$\ker(\pi_G) = \mathrm{im}(\mathrm{id}_{\mathcal{F}} - \pi_G)$$

We can rewrite $\mathrm{id}_{\mathcal{F}} - \pi_G$ as $|G|^{-1} \sum_{g \in G} (\mathrm{id}_{\mathcal{F}} - g)$. Using Propositiion 3.18, for the universal morphism $\varphi \colon \mathcal{F} \to \mathcal{F}_G$, we have

$$\ker(\pi_G) = \ker(\varphi)$$

and we conclude that the image of π_G is isomorphic to the coinvariants $\mathcal{F}_G$. By [Sta24, Lemma 09SH] again, it follows that $\mathcal{F}_G$ is a direct summand of $\mathcal{F}$. □

3.8 Examples

3.8.1 Symmetric power functors

The nth symmetric power of a k-module M arises as the quotient of $\mathrm{T}^n(M)$ by the submodule,

$$N = \left\langle x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \;:\; x_1, \ldots, x_n \in M, \sigma \in \mathrm{S}_n \right\rangle$$

See [Lan02, p. 635]. By Corollary 3.19 N is the kernel of the coinvariants map $\mathrm{T}^n(M) \to \mathrm{T}^n_{\mathrm{S}_n}(M)$ in which we identify S_n with its image under ρ. We thus have a natural transformation,

$$\mathrm{T}^n_{\mathrm{S}_n} \simeq \mathrm{Sym}^n$$

3.8.2 Alternating power functors

If 2 is invertible in k, then we also obtain the nth alternating power of a k-module M as a module of coinvariants. The nth alternating power of a k-module M arises as the quotient of $\mathrm{T}^n(M)$ by the submodule,

$$A = \langle\, x_1 \otimes \cdots \otimes x_n \,:\, x_1, \ldots, x_n \in M, \exists i \neq j, x_i = x_j \,\rangle$$

See [Lan02, p. 731]. Consider the subgroup

$$H_n = \{\, \mathrm{sgn}(\sigma)\, \sigma \,:\, \sigma \in \mathrm{S}_n \,\} < U(kS_n)$$

We claim that $\wedge^n(M)$ arises as the coinvariants of $\mathrm{T}^n(M)$ with respect to H_n, and we approach it by analyzing the kernel of the coinvariants map,

$$\begin{aligned} A' &= \ker\bigl(\mathrm{T}^n(M) \to \mathrm{T}^n_{H_n}(M)\bigr) \\ &= \bigl\langle\, x_1 \otimes \cdots \otimes x_n - \mathrm{sgn}(\sigma)\, x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \,:\, x_1, \ldots, x_n \in M, \sigma \in \mathrm{S}_n \,\bigr\rangle \end{aligned}$$

Indeed, if σ is the transposition of i and j, with $i \neq j$, we have

$$\begin{aligned} &x_1 \otimes \cdots \otimes x_n + x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \\ &\quad = x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n + x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n \\ &\quad = \frac{1}{2}\,(x_1 \otimes \cdots \otimes (x_i + x_j) \otimes \cdots \otimes (x_i + x_j) \otimes \cdots \otimes x_n \\ &\qquad - x_1 \otimes \cdots \otimes (x_i - x_j) \otimes \cdots \otimes (x_i - x_j) \otimes \cdots \otimes x_n) \end{aligned}$$

with the sums and differences taking place in positions i and j. This shows that $A' \subseteq A$. On the other hand, for a tensor with x repeated at indices i and j, we have for the transposition τ of i and j,

$$x_1 \otimes \cdots \otimes x \otimes \cdots \otimes x \otimes \cdots \otimes x_n = \frac{1}{2}\,\bigl(x_1 \otimes \cdots \otimes x_n + x_{\tau^{-1}(1)} \otimes \cdots \otimes x_{\tau^{-1}(n)}\bigr)$$

which shows that $A \subseteq A'$.

Thus, we have $A = A'$ and this shows that in fact there is a natural isomorphism of functors,

$$\mathrm{T}^n_{H_n} \simeq \wedge^n$$

Chapter 4

AN EXAMPLE OF AN UNLIKELY ISOMORPHISM OF COINVARIANTS

4.1 Background on Units of Integral Group Rings

We provide a pair of subgroups $\mathrm{End}(\mathrm{T}^5)$ over $\mathbb{Z}$ where one subgroup is infinite while the other is finite such that they give rise to naturally isomorphic coinvariants of T^5. Since $\mathrm{End}(\mathrm{T}^5) \cong \mathbb{Z}\mathrm{S}_5$, we will benefit from the literature on group algebras, and – as it turns out – the theory of cyclotomic fields.

We develop an understanding of the units in integral group rings, $\mathbb{Z}G$, for finite abelian groups G. This exposition, which serves as a brief introduction while setting up our example, is based on the seminal work [Hig40] and the more recent book [PMS02].

Let G be a finite abelian group of order n. There are n inequivalent irreducible representations of G, with characters $\chi^0, \ldots, \chi^{n-1}\colon G \to \mathbb{C}^*$. As G has finite order, all of the characters map into the set of nth roots of unity. Let $G = \{g_0, \ldots, g_{n-1}\}$ with g_0 taken to be the identity. For notational convenience, we shall write χ_i^j for $\chi^j(g_i)$. Since $|\chi_i^j| = 1$, we have $\bar{\chi}_i^j = (\chi_i^j)^{-1}$.

We use h to denote the exponent of the abelian group G, which is the least common multiple of the orders of the elements of G. Let ζ be a primitive hth root of unity, and let $K = \mathbb{Q}(\zeta)$ the relevant cyclotomic field. Since $(\chi_i^j)^h = 1$ for all i and j, we have $\chi_i^j \in K$ for every i and j.

As a vector space over K, the group algebra KG has for a basis $g_0, \ldots, g_{n-1}$. However, there is a basis for KG over K with respect to which it straightforward to detect units in the algebraic closure R of $\mathbb{Z}$ in KG.

Lemma 4.1 ([Hig40])**.** *For a finite abelian group $G = \{g_0, \ldots, g_n\}$ and $K = \mathbb{Q}(\zeta)$ as above,* *consider* $\eta_0, \ldots \eta_{n-1} \in KG$ *where*

$$\eta_i = \frac{1}{n}\sum_{r=0}^{n-1} \chi_r^i g_r$$

Then, $\{\eta_i\}$ *is a basis for* KG *over* K *such that for all* i, j*, we have* $\eta_i \eta_j = \delta_{ij}\eta_i$*, written in terms of the Kronecker* δ*.*

Proof (Outline). The fact that $\eta_i\eta_j = \delta_{ij}\eta_i$ uses the orthogonality relations ([FH91, (2.10) on p. 16 and Exercise 2.21(i)] on the characters χ_i^j [Hig40, p. 234]). Moreover, the change of basis matrix from $g_0, \ldots, g_{n-1}$ to $\eta_0, \eta_1, \ldots, \eta_{n-1}$ is given by,

$$C = \begin{pmatrix} \bar{\chi}_0^0 & \bar{\chi}_0^1 & \cdots & \bar{\chi}_0^{n-1} \\ \bar{\chi}_1^0 & \bar{\chi}_1^1 & \cdots & \bar{\chi}_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\chi}_{n-1}^0 & \bar{\chi}_{n-1}^1 & \cdots & \bar{\chi}_{n-1}^{n-1} \end{pmatrix}$$

Specifically, given $x = a_0 g_0 + \cdots + a_{n-1}g_{n-1}$, then for $\mathbf{b} = C\mathbf{a}$, we have

$$x = b_0\eta_0 + b_1\eta_1 + \cdots + b_{n-1}\eta_{n-1}$$

So, $\eta_0, \ldots, \eta_{n-1}$ spans KG and it follows that it is a basis over K. □

We remark that it follows that $\sum \eta_i = 1$.

Before we fully take advantage of the basis from Lemma 4.1, we introduce some notation and terminology. For $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ there corresponds a permutation of $\{0, 1, \ldots, n-1\}$ which we will also denote by σ, as there is little room for confusion to arise. Given i, the composition $\sigma \circ \chi^i$ is a character. So, there exists an index $\sigma(i)$ such that

$$\sigma \circ \chi^i = \chi^{\sigma(i)}$$

We see that σ is a bijection on indices by observing that the mapping σ^{-1} on indices must be the inverse of σ. Call representations Γ^i and Γ^j **conjugate** if there exists σ that maps i to j.

After relabeling indices, there is a positive integer p such that $\Gamma^0, \Gamma^1, \ldots, \Gamma^{p-1}$ is a complete set of irreducible, mutually inequivalent and non-conjugate representations of G. Note that by doing this, we reorder the basis $\eta_0, \ldots, \eta_{n-1}$. For each $\alpha \in \{0, 1, \ldots, p-1\}$, let h_α be the least common multiple of the multiplicative orders of the roots of unity χ_i^α, as i ranges from 0 to $n-1$. We shall denote by ξ^α a choice of primitive root of unity of order h_α. Let $K_\alpha = \mathbb{Q}(\xi^\alpha)$.

Theorem 4.2 ([Hig40])**.** *In the above notation, there exists an isomorphism*

$$\mathbb{Q} \cdot G \cong \bigoplus_{\alpha=0}^{p-1} \mathbb{Q}(\xi^\alpha)$$

such that $x = a_0 g_0 + \cdots + a_{n-1} g_{n-1}$ *is sent to the tuple* $(b_0, \ldots, b_{p-1})$ *coming from the change of basis* $x = b_0 \eta_0 + \cdots + b_{p-1} \eta_{p-1} + \cdots + a_{n-1} b_{n-1}$.

Proof (Outline). Any $b_0, \ldots, b_{p-1}$ with $b_\alpha \in \mathbb{Q}(\xi^\alpha)$ extends to a unique sequence $b_0, \ldots, b_{n-1}$ satisfying $\sigma(b_i) = b_{\sigma(i)}$ for all i and all σ in the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$. That condition is, in turn, necessary and sufficient for $\sum_{r=0}^{n-1} b_r \eta_r \in \mathbb{Q}G$. One can prove that directly from formulas relating the coefficients a_i and b_i as in [Hig40, pp. 235-236]. □

Let R denote the integral closure of $\mathbb{Z}$ in $\mathbb{Q}G$. Then, $\mathbb{Z}G$ is a subring of R and thus we have a containment of groups $U(\mathbb{Z}G) \subseteq U(R)$. The latter can be understood using the isomorphism of Theorem 4.2.

Corollary 4.3 ([Hig40])**.**

$$U(R) \cong \prod_{\alpha=0}^{p-1} U\left(\mathcal{O}_{\mathbb{Q}(\xi^\alpha)}\right)$$

Finally, the preceeding work leads to a clean description of exactly which units in the integral group algebra $\mathbb{Z}G$ are of finite order.

Theorem 4.4 ([Hig40])**.** *A unit in* $\mathbb{Z}G$ *is of finite order if and only if it is of the form* $\pm g_i$.

Remark 4.5. Units of a group algebra RG of the form rg, with $r \in U(R)$ and $g \in G$, are referred to as **trivial** units. [Hig40, p.231]

Proof. The reverse implication follows from the fact that G is a finite group, so that all of its elements have finite order. We thus prove the forward implication. Let $x = \sum_{r=0}^{n-1} a_r g_r$ be a unit in $\mathbb{Z}G$ that has a finite order m. Expressing $x = \sum_{r=0}^{n-1} b_r \eta_r$, we observe that each coefficient b_r is an mth root of unity. For i such that $a_i \neq 0$, we thus have

$$|a_i| = \left| \frac{1}{n} \sum_{r=0}^{n-1} b_r \chi_i^r \right| \leq 1$$

In light of the fact that a_i is a nonzero integer, we must have $|a_i| = 1$. So, equality holds in the triangle inequality step. This means $a_i = b_j \chi_i^j$. which holds for all i such that $a_i \neq 0$ and every j. Hence,

$$b_j = \sum_{r=0}^{n-1} a_r \bar{\chi}_r^j = \#\{\, i \mid a_i \neq 0 \,\} \cdot b_j$$

Because x is nonzero, there exists j such that $b_j \neq 0$, which can be canceled in the above equation. This leaves us with just 1 index i such that $a_i \neq 0$. As we have observed, $|a_i| = 1$ so $x = \pm g_i$ as claimed. □

4.2 The group of units in $\mathbb{Z}C_5$ and an application to coinvariants

If $|G| < 5$, then Theorem 6 in [Hig40] implies that every unit of $\mathbb{Z}G$ is trivial, meaning that it is of the form $\pm g$ with $g \in G$. From this perspective, the first interesting case is when $G = \mathrm{C}_5$. Based on more recent work the group of units of $\mathbb{Z}\mathrm{C}_5$ has been understood.

Proposition 4.6 ((8.3.11) in [PMS02])**.** *Let C_5 be the group of order 5 with generator x. For $u = (1+x)^2 - 1 - x - x^2 - x^3 - x^4$, the group of units of $\mathbb{Z}C_5$ is given by*

$$U(\mathbb{Z}C_5) = \langle\, \pm x, u \,\rangle$$

Proof. Let R be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}C_5$. Let ζ be a primitive 5th root of unity. By Proposition 1.2 in [Was82], the ring of integers of $\mathbb{Q}(\zeta)$ is $\mathbb{Z}[\zeta]$. By Corollary 4.3, we thus have $U(R) \cong U(\mathbb{Z}) \times U(\mathbb{Z}[\zeta])$. We recall that the mapping is induced from the isomorphism $\varphi\colon \mathbb{Q}\mathrm{C}_5 \cong \mathbb{Q} \oplus \mathbb{Q}(\zeta)$ of Theorem 4.2. We find it convenient to index the characters of C_5 such that χ^1 sends x to ζ^4. By changing basis to η_i and extracting the coefficients of η_0 and η_1 we find that φ sends x to $(1, \zeta)$. Using Theorem 5.1 in [Lan78] and the fact that the real class number of $\mathbb{Q}(\zeta)$ is 1 (in other words the ring $\mathbb{Z}(\zeta + \zeta^{-1})$ is a PID), see p. 352 in [Was82], we deduce that $U(\mathbb{Z}[\zeta])$ is generated by $\pm\zeta$ and the **cyclotomic units** which are the units $(1-\zeta^a)/(1-\zeta)$ for integers a not divisible by 5. One cyclotomic unit is $1+\zeta$, for the choice $a = 2$. In fact $U(\mathbb{Z}[\zeta]) = \langle \pm\zeta, 1+\zeta \rangle$:

1. $\frac{1-\zeta^3}{1-\zeta} = 1 + \zeta + \zeta^2 = -\zeta^3 - \zeta^4 = -\zeta^3(1+\zeta)$, and

2. $\frac{1-\zeta^4}{1-\zeta} = 1 + \zeta + \zeta^2 + \zeta^3 = -\zeta^4$

Now, we transfer this fact to the situation of the group ring $\mathbb{Z}\mathrm{C}_5$. Let x be a generator of C_5. The mapping φ sends the element $u = (1+x)^2 - (1+x+x^2+x^3+x^4)$ to $2^2 - 5 = -1$ in the first component and $(1+\zeta)^2$ in the second. Let $G = \langle \pm x, u \rangle \subseteq U(\mathbb{Z}\mathrm{C}_5)$; it is immediate that the index of the image of G in $U(\mathbb{Z}[\zeta])$ is at most 2. Hence, the image of $U(\mathbb{Z}\mathrm{C}_5)$ in $U(\mathbb{Z}[\zeta])$ has index at most 2. Also, by (8.3.10) in [PMS02], the latter quantity is precisely 2. Therefore, $U(\mathbb{Z}\mathrm{C}_5) = G$. □

Equipped with an understanding of the group of units of $\mathbb{Z}\mathrm{C}_5$, we produce the example that is relevant for the setting of coinvariants.

Theorem 4.7. *The pair of subgroups H_1 and H_2 of the group algebra $\mathbb{Z}S_5$ given by*

$$H_1 = \langle (1) - (13524) - (14253) \rangle \quad \textit{and} \quad H_2 = \langle -(12345) \rangle$$

satisfy the following: $|H_1| = \infty$, $|H_2| = 10$, and for the tensor power functor $\mathrm{T}^5 \colon Mod_{\mathbb{Z}} \to Mod_{\mathbb{Z}}$, there is a natural isomorphism of coinvariants,

$$\mathrm{T}^5_{H_1} \cong \mathrm{T}^5_{H_2}$$

Proof. Given a generator x of C_5, the above description of $U(\mathbb{Z}\mathrm{C}_5)$ gives a generator $u = x(1 - x^2 - x^3)$ which implies $1 - x^2 - x^3$ is a unit in $\mathbb{Z}\mathrm{C}_5$. By Theorem 4.4, $1 - x^2 - x^3$ generates an infinite subgroup of $U(\mathbb{Z}\mathrm{C}_5)$. We view C_5 as a subgroup of S_5 by identifying x

with the permutation $\sigma = (12345)$. It is thus clear that H_1 is an infinite cyclic subgroup of $U(\mathbb{Z}\mathrm{C}_5)$.

For an abelian group M, let $K = \ker\big(\mathrm{T}^5(M) \to \mathrm{T}^5_{H_1}(M)\big)$. By Corollary 3.21, K is generated by the elements of the form

$$x - (x - \sigma^2 x - \sigma^3 x) \qquad x = x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5, x_i \in M$$

and that simplifies as $\sigma^2 x + \sigma^3 x$. Hence, we obtain relations,

$$x_4 \otimes x_5 \otimes x_1 \otimes x_2 \otimes x_3 + x_3 \otimes x_4 \otimes x_5 \otimes x_1 \otimes x_2$$

Given that the choice of $x_1, \ldots, x_5$ in M is arbitrary, it follows that K can equally well be described as follows,

$$K = \big\langle\, x + \sigma(x) \,:\, x \in \mathrm{T}^5(M) \,\big\rangle$$

Recalling our choice of H_2, it is immediate that $K = \ker\big(\mathrm{T}^5(M) \to \mathrm{T}^5_{H_2}(M)\big)$. The kernel functors associated with $\mathrm{T}^5 \to \mathrm{T}^5_{H_1}$ and $\mathrm{T}^5 \to \mathrm{T}^5_{H_2}$ are thus identical. By the universal property of coinvariants, there is a natural isomorphism,

$$\mathrm{T}^5_{H_1} \cong \mathrm{T}^5_{H_2}$$

as desired. $\square$

Chapter 5

COINVARIANTS FOR GRADED ALGEBRAS

5.1 What constitutes a compatible sequence of group actions

We notice that in the case of tensor powers, there are sequences of subgroups of $\mathrm{Aut}(T^n)$ that yield coinvariants which combine to form graded algebra quotients, viz. the symmetric algebra and exterior algebras associated to a k-module.

The examples of Sym and $\wedge$ suggest considering sequences of group actions on the graded components of T and forming coinvariants on each component. The resulting object is a functor with codomain $\mathrm{Mod}_k^{\mathbb{N}}$. We state a condition on the sequence of group actions, **lr-compatibility**, such that in any instance for which the condition is satisfied, the functor factors through the forgetful functor $\mathrm{GrAlg}_k \to \mathrm{Mod}_k^{\mathbb{N}}$.

Thus, the lr-compatibility condition provides a means to construct functors of graded algebras as coinvariants of the tensor algebra functor T. We work through the examples of the sequence of alternating groups A_n in S_n as well as the sequences of symmetric subgroups $S_{k_n} \subset S_n$.

5.1.1 The algebra of endomorphisms of the tensor algebra functor

In contrast to Theorem 3.8, the k-algebra of endomorphisms of the tensor algebra functor $\mathrm{T}\colon \mathrm{Mod}_k \to \mathrm{GrAlg}_k$ consists only of scalars in k. Morally, this is due to the universal property of the tensor algebra associated with a module, which implies that for k-modules M and N,

$$\mathrm{Hom}_{\mathrm{GrAlg}_k}(\mathrm{T}(M), \mathrm{T}(N)) \cong \mathrm{Hom}_{\mathrm{Mod}_k}\big(\mathrm{T}^1(M), \mathrm{T}^1(N)\big) = \mathrm{Hom}_{\mathrm{Mod}_k}(M, N)$$

Proposition 5.1. *The k-algebra* $\mathrm{End}(\mathrm{T})$ *is naturally isomorphic to k.*

Proof. Let $\lambda \in k$. Let $\widehat{\lambda}_M = \mathrm{T}(\lambda)$ be the endomorphism of $\mathrm{T}(M)$ which scalar multiplies a homogeneous degree n element of $\mathrm{T}(M)$ by λ^n. For a k-module homomorphism $f\colon M \to N$, we have,

$$\mathrm{T}(f) \circ \widehat{\lambda}_M = \mathrm{T}(f \circ \lambda) = \mathrm{T}(\lambda \circ f) = \widehat{\lambda}_N \circ \mathrm{T}(f)$$

and this shows that $\widehat{\lambda}$ is a natural endomorphism of T. We claim that all natural endomorphisms of T are of this form.

To this end, let α be a natural endomorphism of T. Note that $\mathrm{T}^1(k) \cong k$ and let $\mathbf{e}$ be its generator. There exists $\lambda \in k$ such that

$$\alpha_k(\mathbf{e}) = \lambda \mathbf{e}$$

Now, let M be a k-module and $x \in M$. Let $f\colon k \to M$ be the k-module homomorphism that sends 1 to x. We observe that $\mathrm{T}(f)\,(\mathbf{e}) = x$ in degree 1.

Considering the following commutative diagram,

$$\begin{array}{ccc} \mathrm{T}(k) & \xrightarrow{\mathrm{T}(f)} & \mathrm{T}(M) \\ {\scriptstyle \alpha_k}\downarrow & & \downarrow{\scriptstyle \alpha_M} \\ \mathrm{T}(k) & \xrightarrow{\mathrm{T}(f)} & \mathrm{T}(M) \end{array}$$

we find that $\alpha_M(x) = \lambda x$. By the universal property of tensor algebras, the endomorphism α_M is determined by the k-module homomorphism in degree 1. We have shown that in degree 1, α_M is scalar multiplication by λ. Therefore, $\alpha_M = \widehat{\lambda}_M$. □

This shows that interesting coinvariants of the functor T are not obtained directly via group actions on that functor. Instead, we use sequences of group actions that satisfy a compatibility condition to form new functors out of T.

5.2 *Compatibility of Group Actions on Components of a Functor to* $\mathbf{GrAlg}_k$

Let k be a commutative unital ring. Consider the category of $\mathbb{N}$-graded k-algebras, which we denote GrAlg_k. For each $i \in \mathbb{N}$, there is a functor

$$\pi_i\colon \mathrm{GrAlg}_k \to \mathrm{Mod}_k$$

which sends a graded k-algebra to its ith graded component. Taken together, the functors π_i constitute a functor

$$U\colon \mathrm{GrAlg}_k \to \mathrm{Mod}_k^{\mathbb{N}}$$

The functor U forgets the binary multiplication on a k-algebra, while retaining the scalar multiplication by k.

For a functor $\mathcal{F}$ from a category $\mathcal{C}$ into GrAlg_k the components of $\mathcal{F}$ are given by $\mathcal{F}_i = \pi_i \circ \mathcal{F}$. For an object c of the domain category $\mathcal{C}$, we have $\mathcal{F}_i(c)$ is the ith component of the graded algebra $\mathcal{F}(c)$, as expected.

For a given object c of $\mathcal{C}$, the multiplication map on the graded k-algebra $\mathcal{F}(c)$ gives componentwise k-module homomorphisms

$$\mu_c = (\mu_{i,j})_c\colon \mathcal{F}_i(c) \otimes_k \mathcal{F}_j(c) \to \mathcal{F}_{i+j}(c)$$

Because $\mathcal{F}$ maps morphisms of $\mathcal{C}$ to graded k-algebra homomorphisms, the morphisms $(\mu_{i,j})_c$ comprise natural transformations,

$$\mu_{i,j}\colon \mathcal{F}_i \otimes \mathcal{F}_j \to \mathcal{F}_{i+j}.$$

Let G_i be a group for each $i \in \mathbb{N}$ such that for each i, there is a group action of G_i on the ith component of a functor $\mathcal{F}\colon \mathcal{C} \to \mathrm{GrAlg}_k$. Formation of coinvariants on each component provides a functor $\mathcal{F}_{G.}\colon \mathcal{C} \to \mathrm{Mod}_k^{\mathbb{N}}$, so that

$$\pi_i \circ \mathcal{F}_{G.} = (\mathcal{F}_i)_{G_i}$$

We are particularly interested in the circumstance in which $\mathcal{F}_{G.}$ maps into the category of graded algebras.

Definition 5.2. A sequence of groups G_i, $i \in \mathbb{N}$ along with group actions on a functor $\mathcal{F}\colon \mathcal{C} \to \mathrm{GrAlg}_k$ is **compatible** (for $\mathcal{F}$) if there exists a functor $\mathcal{G}\colon \mathcal{C} \to \mathrm{GrAlg}_k$ such that

$$\mathcal{F}_{G.} = U \circ \mathcal{G}$$

i.e., the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C} & & \\ \downarrow{\scriptstyle \exists \mathcal{G}} & \searrow^{\mathcal{F}_{G.}} & \\ \mathrm{GrAlg}_k & \xrightarrow{U} & \mathrm{Mod}_k^{\mathbb{N}} \end{array}$$

5.3 lr-compatibility: A sufficient condition for compatibility

In order to work with compatibility, we introduce a sufficient condition that is relatively straightforword to check. We then show that this condition implies compatibility.

Definition 5.3. A sequence of groups G_i, $i \in \mathbb{N}$ is **lr-compatible** with respect to a functor $\mathcal{F} \colon \mathcal{C} \to \mathrm{GrAlg}_k$ if for all $i, j \in \mathbb{N}$ and all $g \in G_i$ there exist two elements g' and g'' of G_{i+j} such that the following diagroms commute,

$$\begin{array}{ccc} \mathcal{F}_i \otimes \mathcal{F}_j & \xrightarrow{\mu} & \mathcal{F}_{i+j} \\ \downarrow{\scriptstyle g \otimes \mathrm{id}} & & \downarrow{\scriptstyle g'} \\ \mathcal{F}_i \otimes \mathcal{F}_j & \xrightarrow{\mu} & \mathcal{F}_{i+j} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_j \otimes \mathcal{F}_i & \xrightarrow{\mu} & \mathcal{F}_{i+j} \\ \downarrow{\scriptstyle \mathrm{id} \otimes g} & & \downarrow{\scriptstyle g''} \\ \mathcal{F}_j \otimes \mathcal{F}_i & \xrightarrow{\mu} & \mathcal{F}_{i+j} \end{array}$$

Remark 5.4. We frequently work with sequences of subgroups of $\mathrm{Aut}(\mathcal{F}_i)$. From the perspective of lr-compatibility this makes no difference: if G_i is an arbitrary sequence of groups with actions α_i on $\mathcal{F}_i$, then G_i is lr-compatible for $\mathcal{F}$ if and only if the sequence of images G_i in $\mathrm{Aut}(\mathcal{F}_i)$ is lr-compatible. Indeed, the relevant diagrams pertain to $\alpha_i(g)$ and $\alpha_{i+j}(g')$, $\alpha_{i+j}(g'')$ which are the images of elements of G_i and G_{i+j} in the automorphism groups of $\mathcal{F}_i$ and $\mathcal{F}_{i+j}$.

What the diagrams in the definition of lr-compatibility are saying is that for objects c of $\mathcal{C}$ and elements $a \in \mathcal{F}_i(c)$ and $b \in \mathcal{F}_j(c)$, we have

$$g(a)b = g'(ab) \quad \text{and} \quad bg(a) = g''(ba)$$

We show that lr-compatibility is a sufficient condition for compatibility. The idea is to ensure that the kernel of the componentwise coinvariants maps is a functor of ideals of the given functor of graded algebras.

Proposition 5.5. *Any lr-compatible sequence of groups for a functor $\mathcal{F}$ is compatible for $\mathcal{F}$.*

Proof. Let G_i, $i \in \mathbb{N}$ be an lr-compatible sequence of groups for $\mathcal{F}$. We form a functor $\mathcal{I}\colon \mathcal{C} \to \mathrm{Mod}_k^{\mathbb{N}}$ by taking the kernel of the componentwise coinvariants map,

$$\mathcal{I} = \ker(U \circ \mathcal{F} \to \mathcal{F}_{G.})$$

We first show that for each object c of $\mathcal{C}$ the sequence of modules $\mathcal{I}(c)$ is an ideal of $\mathcal{F}(c)$.

We recall that $\mathcal{I}_i(c)$ is generated by elements of the form $a - g(a)$ for $a \in \mathcal{F}_i(c)$ and $g \in G_i$. Given any $j \in \mathbb{N}$, let g' and g'' be as in the definition of lr-compatibility. Then, for any $b \in \mathcal{F}_j(c)$,

$$(a - g(a))b = ab - g(a)b = ab - g'(ab) \in \mathcal{I}_{i+j}(c)$$

and

$$b(a - g(a)) = ba - bg(a) = ba - g''(ba) \in \mathcal{I}_{i+j}(c)$$

Since the condition for an ideal of a graded algebra can be checked on k-module generators and on components, by Corollary 3.20, we conclude that $\mathcal{I}(c)$ is an ideal of $\mathcal{F}(c)$.

We have thus shown that $\mathcal{F}_{G.}(c) \cong \mathcal{F}(c)/\mathcal{I}(c)$. In particular, $\mathcal{F}_{G.}(c)$ is a graded k-algebra. It remains to prove $\mathcal{F}_{G.}$ sends morphisms in $\mathcal{C}$ to graded k-algebra morphisms.

Let $f\colon c \to d$ be a morphism. Then $\mathcal{F}(f)$ is a graded k-algebra homomorphism from $\mathcal{F}(c)$ to $\mathcal{F}(d)$ and $\mathcal{F}(f)$ maps the ideal $\mathcal{I}(c)$ into $\mathcal{I}(d)$. Applying Lemma 5.6, we conclude that $\mathcal{F}_{G.}(f)$ is a graded k-algebra homomorphism. □

Lemma 5.6. *Let $h\colon A \to B$ be a graded k-algebra homomorphism and let I and J be graded ideals of A and B, respectively, such that $h(I) \subseteq J$. Then, h induces a graded k-algebra homomorphism on the quotients $\overline{h}\colon A/I \to B/J$.*

Proof. There exist k-module homomorphisms $\overline{h}_i$ from A_i/I_i to B_i/I_i by functoriality of cokernels in the category of k-modules. The condition that $h(I) \subseteq J$ implies $\overline{h}$ is a ring homomorphism. Hence, it is a graded ring homomorphism. □

5.3.1 lr-compatibility for tensor algebras

The lr-compatibility condition works for the tensor algebra functor T largely because the multiplication maps $\mu_{i,j}$ have a straightforward description. Indeed, the functors $\mathrm{T}^i \otimes \mathrm{T}^j$ and T^{i+j} are canonically isomorphic, via $\mu_{i,j}$.

For a given k-module M, the k-linear map $(\mu_{i,j})_M$ acts on simple tensors by concatenation,

$$(\mu_{i,j})_M\left((x_1 \otimes \cdots \otimes x_i) \otimes (y_1 \otimes \cdots \otimes yj)\right) = x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j$$

The universal property of the $i+j$th tensor power of M ensures that there is a k-linear map $\alpha\colon \mathrm{T}^{i+j}(M) \to \mathrm{T}^i(M) \otimes \mathrm{T}^j(M)$ with the property that

$$\alpha(x_1 \otimes \cdots \otimes x_{i+j}) = (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_{i+j})$$

and it is immediate that α is the inverse of μ. Because the natural transformation $\mu_{i,j}$ has object-wise inverses, it is invertible.

Note that while $\mu_{i,j}$ is invertible, we are not claiming that the multiplication on tensor algebras is invertible. That is because, more than one $\mu_{i,j}$ have the same target T^{i+j}.

A consequence of the invertibility of the componentwise multiplication maps is the uniqueness of the automorphisms g' and g'' in the lr-compatibility condition.

Let G_i, $i \in \mathbb{N}$ be lr-compatible with respect to the tensor algebra functor $\mathrm{T}\colon \mathrm{Mod}_k \to \mathrm{GrAlg}_k$. Using Theorem 3.8, we identify $\mathrm{End}(\mathrm{T}^n)$ with $k\mathrm{S}_n$. In light of Remark 5.4, we may view G_i as a subgroup of $U(k\mathrm{S}_i)$.

We select two group homomorphisms, $\lambda_i, \rho_i\colon \mathrm{S}_i \to \mathrm{S}_{i+1}$ such that for a permutation σ of $[i]$, the images $\lambda_i(\sigma)$ and $\rho_i(\sigma)$ are the permutations of $[i+1]$ that satisfy,

$$\lambda_i(\sigma)(m) = \begin{cases} \sigma(m) & \text{if } m \leq i \\ m & \text{if } m = i+1 \end{cases} \qquad \rho_i(\sigma)(m) = \begin{cases} m & \text{if } m = 1 \\ \rho(m-1)+1 & \text{if } m \geq 2 \end{cases}$$

The group homomorphisms λ_i and ρ_i have k-linear extensions as k-algebra homomorphisms

$$\widetilde{\lambda}_i, \widetilde{\rho}_i\colon k\mathrm{S}_i \to k\mathrm{S}_{i+1}$$

These k-algebra homomorphisms are used in describing lr-compatibility for sequences of group actions with respect to the tensor algebra functor.

Lemma 5.7. *For any $\tau \in kS_i$ the following diagrams commute,*

$$\begin{array}{ccc} \mathrm{T}^i \otimes \mathrm{T}^1 & \xrightarrow{\mu} & \mathrm{T}^{i+1} \\ \downarrow{\scriptstyle \tau\otimes id} & & \downarrow{\scriptstyle \widetilde{\lambda}_i(\tau)} \\ \mathrm{T}^i \otimes \mathrm{T}^1 & \xrightarrow{\mu} & \mathrm{T}^{i+1} \end{array} \qquad \begin{array}{ccc} \mathrm{T}^1 \otimes \mathrm{T}^i & \xrightarrow{\mu} & \mathrm{T}^{i+1} \\ \downarrow{\scriptstyle id\otimes \tau} & & \downarrow{\scriptstyle \widetilde{\rho}_i(\tau)} \\ \mathrm{T}^1 \otimes \mathrm{T}^i & \xrightarrow{\mu} & \mathrm{T}^{i+1} \end{array}$$

Proof. Expand $\tau = \sum_{\sigma \in S_i} a_\sigma \sigma$ with coefficients in k. Because $\otimes$ preserves k-linear combinations, and refencing Proposition 2.7, we have

$$\mu \circ (\tau \otimes \mathrm{id}) \circ \mu^{-1} = \sum_{\sigma \in S_i} \left(a_\sigma \mu \circ \sigma \circ \mu^{-1} \right)$$

Also, by the construction of $\widetilde{\lambda}_i$, we have

$$\widetilde{\lambda}_i(\tau) = \sum_{\sigma \in S_i} a_\sigma \lambda_i(\sigma)$$

So, we may assume $\tau \in S_i$. Then, the result for λ_i follows from

$$\begin{aligned} \lambda_i(\tau)(x_1 \otimes \cdots \otimes x_i \otimes x_{i+1}) &= x_{\tau^{-1}(1)} \otimes \cdots \otimes x_{\tau^{-1}(i)} \otimes x_{i+1} \\ &= (\tau \otimes \mathrm{id})(x_1 \otimes \cdots \otimes x_i \otimes x_{i+1}) \end{aligned}$$

The proof for $\widetilde{\rho}_i$ is essentially the same. □

Lemma 5.8. *A sequence G_i of group actions on the components of the tensor algebra functor* T *is lr-compatible if and only if for each $i \in \mathbb{N}$,*

$$\widetilde{\lambda}_i(G_i), \widetilde{\rho}_i(G_i) \subseteq G_{i+1}$$

Proof. Suppose G_i is an lr-compatible sequence of subgroups of kS_i. For each $g \in G_i$, there exist $g', g'' \in G_{i+1}$ such that the following diagrams commute,

$$\begin{array}{ccc} \mathrm{T}^i \otimes \mathrm{T}^1 & \xrightarrow{\mu_{i,1}} & \mathrm{T}^{i+1} \\ \downarrow{\scriptstyle g\otimes \mathrm{id}} & & \downarrow{\scriptstyle g'} \\ \mathrm{T}^i \otimes \mathrm{T}^1 & \xrightarrow{\mu_{i,1}} & \mathrm{T}^{i+1} \end{array} \qquad \begin{array}{ccc} \mathrm{T}^1 \otimes \mathrm{T}^i & \xrightarrow{\mu_{1,i}} & \mathrm{T}^{i+1} \\ \downarrow{\scriptstyle \mathrm{id}\otimes g} & & \downarrow{\scriptstyle g''} \\ \mathrm{T}^1 \otimes \mathrm{T}^i & \xrightarrow{\mu_{1,i}} & \mathrm{T}^{i+1} \end{array}$$

Because $\mu_{i,1}$ and $\mu_{1,i}$ are invertible, the automorphisms g' and g'' are uniquely determined by g. Hence, by Lemma 5.7, it follows that

$$g' = \widetilde{\lambda}_i(g) \quad \text{and} \quad g'' = \widetilde{\rho}_i(g)$$

it follows that $\widetilde{\lambda}_i(G_i)$ and $\widetilde{\rho}_i(G_i)$ are contained in G_{i+1}.

Conversely, assume that for each $i \in \mathbb{N}$, the images $\widetilde{\lambda}_i(G_i)$ and $\widetilde{\rho}_i(G_i)$ are contained in G_{i+1}. We will show that the sequence G_i is lr-compatible for the tensor algebra functor.

Let $i, j \in \mathbb{N}$ be arbitrary and let $g \in G_i$. Then,

$$g' = \left(\widetilde{\lambda_{i+j-1}} \circ \cdots \circ \widetilde{\lambda}_i\right)(g) \quad \text{and} \quad g'' = \left(\widetilde{\rho_{i+j-1}} \circ \cdots \circ \widetilde{\rho}_i\right)(g)$$

satisfy the lr-compatibility condition for g, using Lemma 5.7 and induction on j. □

Proposition 5.9. *Let G_i and H_i be lr-compatible sequences for the tensor power functor* T*. Assume that $G_i, H_i < U(kS_i)$ for each i. Then, $G_i \cap H_i$ is a lr-compatible sequence for* T.

Proof. We have

$$\widetilde{\lambda}_i(G_i \cap H_i) \subseteq \widetilde{\lambda}_i(G_i) \cap \widetilde{\lambda}_i(H_i) \subseteq G_{i+1} \cap H_{i+1}$$

and

$$\widetilde{\rho}_i(G_i \cap H_i) \subseteq \widetilde{\rho}_i(G_i) \cap \widetilde{\rho}_i(H_i) \subseteq G_{i+1} \cap H_{i+1}$$

The result follows from Lemma 5.8. □

5.4 Examples

- The sequence of symmetric groups $\mathrm{S}_i < U(kS_i)$ is lr-compatible for the tensor algebra functor. This is because λ_i and ρ_i map permutations to permutations. The functor that results from taking componentwise coinvariants is Sym.

- The groups $H_i = \{\, \mathrm{sgn}(\sigma)\, \sigma \,:\, \sigma \in \mathrm{S}_i \,\} < U(kS_i)$ are lr-compatible, because the functions λ_i and ρ_i preserve the sign of a permutation. Taking componentwise coinvariants, we get $\wedge$.

- We get a new example by intersecting the lr-compatible sequences S_i and H_i in $U(kS_i)$. The intersections are the alternating groups A_i, and there are natural transformations,

$$\mathrm{Sym} \leftarrow \mathrm{T}_{\mathrm{A.}} \rightarrow \wedge$$

5.4.1 Coinvariants of action of A_n in the case of finitely generated modules

We can say more about the example of $\mathrm{T}_{\mathrm{A.}}$. For a finitely generated k-module M, let $\mu(M)$ be the minimal number of generators. Then, $\wedge^n(M) = 0$ for all $n > \mu(M)$. In fact, for all such n, the maps $\mathrm{T}^n_{\mathrm{A.}}(M) \rightarrow \mathrm{Sym}^n(M)$ are isomorphisms.

Proposition 5.10. *Let M be a finitely generated k-module. For all $n > \mu(M)$, the number of generators of M, the natural map*

$$\mathrm{T}_{A_n}(M) \rightarrow \mathrm{T}_{S_n}(M) = \mathrm{Sym}^n(M)$$

is an isomorphism.

Proof. Let $m = \mu(M)$ and $x_1, \ldots, x_m$ be a system of generators for M. The k-module $\mathrm{T}^n(M)$ is generated by elements of the form $t = x_{i_1} \otimes \cdots \otimes x_{i_n}$ involving the specified generators of M. Let $\mathcal{T}$ be the set of all such tensors. Given that $n > m$, at least two of the indices must be repeated. Let j_1 and j_2 be distinct indices such that $i_{j_1} = i_{j_2}$.

By Corollary 3.19, we have, $\mathrm{T}^n_{\mathrm{A.}}(M) \cong \mathrm{T}^n(M)\,/I$ and $\mathrm{T}^n_{\mathrm{S.}}(M) \cong \mathrm{T}^n(M)\,/J$ for submodules

$$I = \langle\, t - \sigma(t) \,:\, t \in \mathrm{T}^n(M)\,, \sigma \in \mathrm{A}_n \,\rangle \quad \text{and} \quad J = \langle\, t - \tau(t) \,:\, t \in \mathrm{T}^n(M)\,, \tau \in \mathrm{S}_n \,\rangle$$

We have $I \subseteq J$ in general, and we aim to show the reverse inclusion. Let $t - \tau(t)$ be one of the generators of J. If τ is an even permutation, then $t - \tau(t) \in I$ by definition. We thus assume τ is an odd permutation. Write $t = x_{i_1} \otimes \cdots \otimes x_{i_n}$. Let $\sigma = \tau \cdot (jk)$ in cycle notation. Then, $\sigma(t) = \tau(t)$ whilst σ is an even permutation. Hence, $t - \tau(t) \in I$. □

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