Analysis of Exponential Filter Time Series Operators of Geometric Brownian Motion in Trading Strategies

Douglas Service

A thesis
submitted in partial fulfillment of the
requirements of the degree of

Master of Science in Computational Finance and Risk Management

University of Washington
2020

Committee:
Matthew Lorig
Daniel Hanson

Program Authorized to Offer Degree:
Applied Mathematics

University of Washington

Abstract

Analysis of Exponential Filter Time Series Operators of Geometric Brownian Motion in Trading Strategies

Douglas Service

Chair of the Supervisory Committee:
Matthew Lorig
Applied Mathematics

Trading strategies based on moving average indicators have been analyzed in the academic literature numerous times using historical data to make statistical inferences about various properties such as expected returns. In this work a deductive model is assumed where asset price dynamics are driven by a stochastic differential equation (SDE) in a continuous time setting under the assumption of a frictionless market. A number of properties about the structure of the stochastic processes which result from the application of an exponential time series operator to the solution of the asset price SDE and various algebraic combinations of such processes are proven. A trading strategy is proposed and analyzed in which the size and direction of the market position in the risky asset incorporates the difference of two exponential time series operators applied to the asset price process. The resulting portfolio SDE is proven to shown a positive expected return under specific assumptions.

Contents

List of Figures v

List of Tables vi

1 Introduction 1

1.1 Literature Review . . . 1

1.1.1 Technical Trading Strategies . . . 1

1.1.2 Mathematical Finance . . . 2

1.2 Asset Price Model . . . 5

1.3 Asset Price SDE Solution . . . 5

2 Filter Analysis 7

2.1 Exponential Filter Time Series Operator . . . 7

2.2 Exponentially Filtered Log Asset Price Υ_t . . . 7

2.2.1 Υ_t as Solution to Ornstein-Ulenbeck SDE . . . 12

2.2.2 Mean, Variance, and Distribution of Υ_t . . . 13

2.3 Difference of Filtered Log Price Ψ_t . . . 18

2.3.1 Mean and Variance of Ψ_t . . . 20

2.3.2 Distribution and Density Functions of Ψ_t . . . 23

3 Trading Strategy 27

3.1 Indicator . . . 27

3.2 Portfolio SDE . . . 28

3.3 Portfolio SDE Solution 31
3.4 Expected Log Returns r_t 34
3.5 Analysis of Portfolio SDE Solution 38

4 Conclusion and Future Work **43**

Bibliography **47**

Appendices **49**

List of Figures

2.1 Variance of Ψ_t: $\mathbb{V}[\Psi_t]$, $\frac{d\,\mathbb{V}[\Psi_t]}{d\,t}$, and $\frac{d^2\,\mathbb{V}[\Psi_t]}{d\,t^2}$ $\sigma = 0.1$, $\tau_1 = 5$ and $\tau_2 = 10$ 22

2.2 Variance of Ψ_t as a Function of τ_1 and τ_2: $\mathbb{V}[\Psi_t] = \frac{\sigma^2}{2}\left(\frac{(\tau_2-\tau_1)^2}{\tau_1+\tau_2}\right)$, $\mu = 0.001$, $\sigma = 0.1$ 25

3.1 Ratio of New to Old Position: $\frac{\nu_i}{\nu_{i-1}}$ vs ΔS 30

3.2 Portfolio Log Return: r_t vs ϱ with $\sigma = 0.1$ and $S_0 = S_t = 100$ 33

3.3 Plot of $\mathbb{E}\left[r_t(\tau_1, \tau_2)\right]_{t>t_{ss}}$ with $c = 1$, $\mu = 0.1$, $\sigma = 0.1$, Lower: $t = 50$, Upper: $t = 200$ . . 39

3.4 Plot of $\mathbb{E}\left[r_t(\tau_1, \tau_2)\right]_{t>t_{ss}}$ with $c = \kappa(5, .1, .1, \tau_1, \tau_2)$, $\mu = 0.1$, $\sigma = 0.1$, Lower: $t = 50$, Upper: $t = 200$ 42

List of Tables

3.1 SDE Specified Transactions Given ϱ, ΔS, and $R = \frac{\nu_i}{\nu_{i-1}}$ 30

Ackowledgements

First and foremost, I would like to thank Matthew Lorig for inspiring me through his classes and our conversations to take a deductive mathematical finance approach to the analysis of asset prices and trading strategies in the financial markets. Every one these encounters was a revelation to me and formed the impetus that culminated in this thesis. I would also like to thank Matt for taking time from his own busy research schedule to be my committee chairman.

Next, I would like to thank Daniel Hanson for taking the time to be a member of my committee, providing thoughtful feedback and always being available to discuss issues as they arise.

About half of this thesis was completed prior to 2016 but unfortunately balancing work, being the single father of a seven year old autistic child, family and health issues caused multiple delays. I would like to thank my family and friends who supported me through this time.

Chapter 1

Introduction

1.1 Literature Review

1.1.1 Technical Trading Strategies

One of the earliest references to technical trading strategies appears in Gartley 1935 who proposes that technical strategies with moving averages were developed by market participants as a means of understanding price movements when unpredictable lead or lag occurred between changes in prices and changes in underlying fundamental economic factors. He notes that moving averages allow the investor confirm the existence and direction of a trend. These technical trading techniques have been analyzed and debated in the finance literature from a statistical perspective numerous time without any clear resolution. In the 1960s Alexander 1961 tested percentage filter based strategies and reported significant positive returns. He explains his results as being consistent with the random walk hypothesis since the filter rules consider the magnitude of the movement over any time frame not just a uniform time period. Alexander states "...a move once initiated, tends to persist". Mandelbrot 1963 examined Alexander's return data and noted that price discontinuities and slippage are not considered. Alexander 1964 revised his calculations based on Mandelbrots comments and still reported significant returns. Fama and Blume 1966 did an extensive review of Alexander's work and noted a lack of consideration for the impact of dividends and commissions on returns. In the end, Fama and Blume's concluded the lack of consideration of these effects would cause Alexander's strategy to have no significant advantage over a buy and hold strategy

1.1.2 Mathematical Finance

On March 29^{th}, 1900, Louis Bachelier, a student of Henri Poincaré, defended his doctoral thesis *Theory of Speculation* (Bachelier 1900) which is considered to be the founding paper in the field Mathematical Finance Courtault et al. 2000. In this work Bachelier develops a mathematical model of the forward contracts and options traded on the Paris exchange in which he notes the "expecion of a speculator is nil", price changes follow a Gaussian law, and the "median spread...is proportional to the square root of the elapsed time". Bachelier thus provides the first description the price process as a stochastic diffusion which is now called Brownian motion. Bachelier's work is not only important because he was first to use this model in finance, but also because he established a framework for rigorous investigations in the financial markets.

Though Bachelier's thesis was published in the influential scientific scientific journal *Annales Scientifiques de lÉcole Normale Supérieure* and influenced a number of works on probability, Bachelier's most important impact came much later on the Russian mathematician Andrey Nikolaevich Kolmogorov Courtault et al. 2000. In in the introduction to his 1931 paper about analytic methods in probabilty, *Über die Grenzwertsatze der Wahrscheinlichkeitsrechnung* Kolmogorov 1931, [1] Kolmogorov cites three of Bachlier's works [2] and refers to to Bachelier as the first mathematician known to Kolmogorov who systematically studied continuous stochastic processes [3]. However, Kolmogorov notes that Bachelier's constructions do not meet a sufficient level of mathematical rigor. Kolomogorov's paper investigated the relationship between the conditional probabilities and transition probabilities of various points on certain stochastic processes which lead to the independent development of what is now called the Chapman-Kolmogorov equations Shiryaev 2011.

In 1933 Kolmogorov wrote his book on the theory of probability, *Grundbegriffe der Wahrscheinlichkeitsrechnung* Kolmogorov 1933 [4], in which he used Borel's measure theory and Henri Lebesgues'

[1] Originally published in German and later translated to Russian
[2] *Theorie de la speculation*, *Les probabilités à plusieurs variables*, and *Calcul des probabilités*
[3] Translated from Russian version,
[4] Originally published in German and later translated to Russian English Kolmogorgov 1956

theory of integration to axiomatize the theory of probability and place it on a solid analytical foundation (Shafer and Vovk 2006). Kolomogorov's formulation of probability has become the foundation of modern mathematical finance. It also shifted probability into the center of mathematics and established probability as a major field of research.

In 1944 Kiyosi Itô published his paper on an integral (Itô 1944) in which he defines an integral with respect to Brownian motion which has became known as the Itô integral. Itô proves a number of properties about this integral such as the Itô isometry which is extensively used in mathematical finance.

The modern era in mathematical finance began with the publication of Fischer Black and Myron Scholes's paper on *The Pricing of Options and Corporate Liabilities* (Black and Scholes 1973). In this work they derive a valuation formula for a European option under a specific set of market conditions using a portfolio consisting of a long position in the stock and a short position in the option. They construct a partial differential equation which expresses the evolution of the portfolio value which does not involve probability (Shreve 2004) and solve the equation using standard methods. Robert Merton in his paper *Theory of Rational Option Pricing* (Merton 1973) references the work of Bachelier noting that "Louis Bachelier deduced an option pricing formula based on the assumption that stock prices follow a Brownian motion with zero drift". Merton then provides an alternative derivation of the Black-Scholes model in the spirit of Bachelier's model using a stochastic differential equation (SDE) with constant drift and diffusion coefficients to model the instantaneous return on a stock, and an SDE with time varying drift and diffusion coefficients to model the instantaneous return on the bond.

The development of electronic exchanges like the NASDAQ in 1971 (NASDAQ 2020) linked with the rapid development and falling prices of computational platforms has given researches ever greater access to high-frequency data and computational power. Dacorogna et al. 2001 in their book *An Introduction to High-Frequency Finance* note that not only have these developments given researchers access to higher-frequency data but also changed the problems researchers can approach and complexity of the models they build. Researchers have moved from macroeconomic models on the monthly scale, to intraday models, and ultimately tick by tick microstructure models. Dacorogna et al. 2001 also note that analyzing high-frequency data require new tools. Zumbach and Müller 2001, in their paper *Operators on Inhomogeneous*

Time Series propose a framework for defining and analyzing linear time series convolution operators defined in continuous-time as an integral convolution with a filter kernel. Their focus is on developing and analyzing discrete-time versions of the these time-series operators for inhomogeneous and homogeneous time-series. They define the discrete-time exponential moving average filter kernel as their basic building block and use it to build more complex operators to analyze differentials, derivatives and volatilities.

Grebenkov and Serror 2013 study exponential moving averages in a discrete time setting considering how positively auto-correlated price variations are detectable by moving average strategies. In a Gaussian framework they derive the probability distributions of the profits and losses considering mean variance, skewness and kurtosis. Daily price variations are modeled as a discrete Ornstein-Uhlenbeck process. They show how the asymmetry of the distributions is exploited by an exponential moving average to produce many small short term losses and less frequent but larger profits.

Lorig, Zhou, and Zou 2019 investigated the logarithmic utility and long-term growth maximization problems of trading strategies based on exponential moving averages. They demonstrated closed form solutions when the drift of the asset price SDE is modeled as either an Ornstein-Ulenbeck process or a two-state continuous time Markov chain [5]

In this work we adopt a similar framework to Lorig, Zhou, and Zou 2019 and demonstrate a number of properties about various combinations of exponential moving averages applied to the asset price model that is controlled by a stochastic differential equation (SDE) where the drift is assumed to be constant. We then develop and analyze a trading strategy that uses the difference of two exponential time-series operators of the asset price to determine the instantaneous market position. Our goal is to develop some initial results and explore the potential of this configuration for continued future research [6].

[5] Lorig, Zhou, and Zou 2019 initiated their investigation after a review of a 2015 and 2016 version of this work as noted in their paper and documented in emails to Matthew Lorig dated February 25, 2016 and November 7th, 2016.

[6] The exposition proceeds as a set of propositions and proofs as a convenience to the the reader who may chose to skip details. We do not make any claims that the propositions rise to the level of a theorem or lemma

1.2 Asset Price Model

To reduce model complexity we assume a frictionless market where the difference between the bid and ask price is infinitesimal, transaction costs are zero, transaction execution is instantaneous, and buy (sell) transactions fully execute at the current ask (bid) price. The market contains a single risky asset whose price is denoted by S_t which does not pay dividends, and whose returns are normally distributed [7]. We also assume the market contains a single risk free bond whose price is denoted by B_t which pays a fixed rate of return r. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space where the instantaneous return $\frac{dS_t}{S_t}$ under the real world probability measure $\mathbb{P}$ is controlled by the sum of a deterministic linear drift component μdt plus a stochastic volatility component σdW_t.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1.1}$$

In equation 1.1 $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{\geq 0}$ are constant, and W_t is a one dimensional Brownian motion whose history up to time t is contained in the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For each $\omega \in \Omega$ there exits a function $W_t(\omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ where $W_0(\omega) = 0$, disjoint increments of $W_t(\omega)$ are independent, increments $W_{\Delta t}(\omega)$ are normally distributed with mean zero and variance Δt, and $W_t(\omega)$ is continuous on $[0, t]$ [8] Shreve 2004, p.94.

Multiplying equation 1.1 through by S_t gives the standard Geometric Brownin Motion (GBM) stochastic differential equation (SDE).

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1.2}$$

1.3 Asset Price SDE Solution

The well known solution to equation 1.2 is found by applying Itô's formula (Oksendal 2000, p.44) to calculate the differential of the natural logarithm of the asset price $d \ln S_t$.

[7] This is equivalent to trading an asset that pays a dividend at a frequency which is significantly greater than the dividend frequency, never holding a position overnight on the ex-dividend date, and adjusting historical calibration data for dividends.

[8] Given $0 = t_0 < t_1 < \cdots < t_m$ the increments $W_1(\omega) - W_0(\omega), \ldots, W_m(\omega) - W_{m-1}(\omega)$ are independent $\mathbb{E}\left[W_{i+1}(\omega) - W_i(\omega)\right] = 0$ and $\mathbb{V}\left[W_{i+1}(\omega) - W_i(\omega)\right] = t_{i+1} - t_i$

Let

$$f(t, s) = \ln s \tag{1.3}$$

The partial derivatives of $f(t, S_t)$ are

$$\frac{\partial f(t, S_t)}{\partial t} = 0, \quad \frac{\partial f(t, S_t)}{\partial s} = \frac{1}{S_t}, \quad \frac{\partial^2 f(t, S_t)}{\partial s^2} = -\frac{1}{S_t^2} \tag{1.4}$$

and $(dt)^2 = 0$, $(dt)(dW_t) = 0$, and $(dW_t)^2 = dt$ [9] Shreve 2004, p.138. Insert the results from equation 1.4 into Itô's formula and expand the differential dS_t.

$$\begin{aligned}
df(t, S_t) &= \frac{\partial f(t, S_t)}{\partial t} + \frac{\partial f(t, S_t)}{\partial s} dS_t + \frac{1}{2}\frac{\partial^2 f(t, S_t)}{\partial s^2} (dS_t)^2 \\
d\ln S_t &= \frac{1}{S_t} dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2 \\
d\ln S_t &= \frac{1}{S_t}\left(\mu S_t dt + \sigma S_t dW_t\right) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left(\sigma^2 S_t^2 dt\right) \\
d\ln S_t &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t
\end{aligned} \tag{1.5}$$

Integrating equation 1.5 from 0 to t and applying the exponential function to both sides of the resulting equation gives an analytical solution for the asset price S_t in equation 1.9 which is known as geometric Brownian motion.

$$\int_0^t d\ln S_u = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) du + \int_0^t \sigma dW_u \tag{1.6}$$

$$\ln S_t - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \tag{1.7}$$

$$\ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \tag{1.8}$$

$$S_t = S_0\, e^{(\mu - \frac{\sigma^2}{2})\, t + \sigma\, W_t} \tag{1.9}$$

We also assume the magnitude of the drift μ is relatively small; otherwise, the best strategy would most likely be a long buy and hold or sell short and hold position corresponding to the sign of μ.

[9] The formulas for dt and dW_t are implications of the non-zero quadratic variation of Brownian motion.

Chapter 2

Filter Analysis

2.1 Exponential Filter Time Series Operator

Definition 2.1.1 (Exponential Filter Time Series Operator)**.** An exponential filter time series operator $\Upsilon_t[X_t;\tau]$, where $\tau > 0$, is the convolution of the exponential kernel function $\frac{1}{\tau}e^{-\frac{1}{\tau}t}$ with the process X_t.

$$\Upsilon_t[X_t;\tau] = \int_0^t \frac{1}{\tau}e^{-\frac{1}{\tau}(t-u)}X_u du \tag{2.1}$$

Equation 2.1 implies the value of $\Upsilon_t[X_t;\tau]$ is path dependent since for a given t, $\Upsilon_t[X_t;\tau]$ depends on every value X_u on the interval $u \in [0,t]$. $\Upsilon_t[X_t;\tau]$ is also $\{\mathcal{F}_t\}_{t\geq 0}$ measurable since for a given t its value only depends at most on the history of X_u, $u \leq t$ up to time t. This is true for all $t > 0$; thus, $\Upsilon_t[X_t;\tau]$ is $\{\mathcal{F}_t\}_{t\geq 0}$ adapted. The derivation of the discrete-time version is given in the appendix on page 53.

2.2 Exponentially Filtered Log Asset Price Υ_t

First we construct the exponentially filtered logarithm of the asset price process $\Upsilon_t[\ln S_t, \tau]$ by substituting the logarithm of the SDE solution S_t from equation 1.8 for X_t in equation 2.1.

$$\Upsilon_t[\ln S_t, \tau] = \int_0^t \frac{1}{\tau} e^{-\frac{1}{\tau}(t-u)} \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) u + \sigma W_u \right] du \tag{2.2}$$

Proposition 2.2.1. Given an asset price process which is a solution of the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$, the value of the exponentially filtered logarithm of the price process $\Upsilon_t[\ln S_t, \tau]$ is equivalent to the difference a scaled version of the original log price process $\theta_1(t;\tau)$ minus an Itô process driven by the same Brownian motion.

$$\Upsilon_t[\ln S_t, \tau] = \theta_1(t;\tau) - \left[\theta_2(t;\tau) + \sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dW_u \right] \tag{2.3}$$

where

$$\theta_1(t;\tau) = \ln S_t - e^{-\frac{1}{\tau}t} \ln S_0$$
$$\theta_2(t;\tau) = \left(1 - e^{-\frac{1}{\tau}t}\right) \left(\mu - \frac{\sigma^2}{2} \right) \tau$$

Proof: (Proposition 2.2.1). First expand equation 2.2

$$\Upsilon_t[\ln S_t, \tau] = \int_0^t \frac{1}{\tau} e^{-\frac{1}{\tau}(t-u)} \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) u + \sigma W_u \right] du \tag{2.4}$$
$$= \frac{e^{-\frac{1}{\tau}t}}{\tau} \left[\ln S_0 \int_0^t e^{\frac{1}{\tau}u} du + \left(\mu - \frac{\sigma^2}{2} \right) \int_0^t u\, e^{\frac{1}{\tau}u} du + \sigma \int_0^t e^{\frac{1}{\tau}u} W_u du \right] \tag{2.5}$$

The first two integrals in equation2.5 are deterministic and have explicit solutions. The first integral is solved using the fundamental theorem of calculus.

$$\int_0^t e^{\frac{1}{\tau}u} du = \tau \left[e^{\frac{1}{\tau}u} \right]_0^t$$
$$= \tau \left(e^{\frac{1}{\tau}t} - 1 \right) \tag{2.6}$$

The second integral requires in addition to the fundamental theorem of calculus integration by parts where $\int x dy = xy - \int y dx$. Let $x = u$ which implies $dx = du$, and let

$$\begin{aligned} dy &= e^{\frac{1}{\tau}u} du \\ y &= \int e^{\frac{1}{\tau}u} du \\ &= \tau e^{\frac{1}{\tau}u} \end{aligned} \tag{2.7}$$

Using the results from equation 2.7 to complete the integration by parts solves the second integral in equation 2.5.

$$\begin{aligned} \int_0^t u\, e^{\frac{1}{\tau}u} du &= \tau \left[u e^{\frac{1}{\tau}u} \right]_0^t - \int_0^t \tau e^{\frac{1}{\tau}u} du \\ &= \tau t e^{\frac{1}{\tau}t} - \tau^2 \left[e^{\frac{1}{\tau}u} \right]_0^t \\ &= \tau \left[t e^{\frac{1}{\tau}t} - \tau \left(e^{\frac{1}{\tau}t} - 1 \right) \right] \end{aligned} \tag{2.8}$$

Inserting the results of equations 2.6 and 2.8 into equation 2.5 gives an equation with a deterministic component and a Riemann integral of scaled Brownian motion with respect to time.

$$\begin{aligned} \Upsilon_t[\ln S_t, \tau] &= \frac{e^{-\frac{1}{\tau}t}}{\tau} \left[\tau \ln S_0 \left(e^{\frac{1}{\tau}t} - 1 \right) + \tau \left(\mu - \frac{\sigma^2}{2} \right) \left[t e^{\frac{1}{\tau}t} - \tau \left(e^{\frac{1}{\tau}t} - 1 \right) \right] + \sigma \int_0^t e^{\frac{1}{\tau}u} W_u du \right] \\ &= \ln S_0 \left(1 - e^{-\frac{1}{\tau}t} \right) + \left(\mu - \frac{\sigma^2}{2} \right) \left[t - \tau \left(1 - e^{-\frac{1}{\tau}t} \right) \right] + \int_0^t \frac{\sigma}{\tau} e^{-\frac{1}{\tau}(t-u)} W_u du \end{aligned} \tag{2.9}$$

In order to apply Itô calculus to equation 2.9, it must be converted from a Riemann integral of Brownian motion with respect to time to an Ito integral with respect to Brownian motion. Since the coefficient of Brownian motion inside the integral in equation 2.9 does not depend on $W_t(\omega)$ Theorem 4.1.5 on integration by parts from Oksendal 2000 can be applied. We motivate the transformation using the Itô product rule and the fact that the cross variation of a deterministic process and Brownian motion are zero. Let

$$\begin{aligned} d(f_u W_u) &= (df_u) W_u + f_u dW_u + \cancelto{0}{(df_u)(dW_u)} \\ &= f_u' du W_u + f_u dW_u \end{aligned}$$

which implies

$$\begin{aligned} f'_u W_u du &= d(f_u W_u) - f_u dW_u \\ \int_0^t f'_u W_u du &= \int_0^t d(f_u W_u) - \int_0^t f_u dW_u \\ \int_0^t f'_u W_u du &= [f_u W_u]_0^t - \int_0^t f_u dW_u \end{aligned} \tag{2.10}$$

Let

$$\frac{d\,f_u}{d\,u} = \frac{\sigma}{\tau} e^{-\frac{1}{\tau}(t-u)} \tag{2.11}$$

$$\int df_u = \int \frac{\sigma}{\tau} e^{-\frac{1}{\tau}(t-u)} du$$

$$f_u = \sigma e^{-\frac{1}{\tau}(t-u)} \tag{2.12}$$

Substituting the values for f_u and $\frac{d\,f_u}{d\,t}$ from equations 2.11 and 2.12 into equation 2.10 produces the required transformation.

$$\begin{aligned} \int_0^t f'_u W_u du &= [f_u W_u]_0^t - \int_0^t f_u dW_u \\ \int_0^t \frac{\sigma}{\tau} e^{-\frac{1}{\tau}(t-u)} W_u du &= \left[\sigma e^{-\frac{1}{\tau}(t-u)} W_u\right]_0^t - \int_0^t \sigma e^{-\frac{1}{\tau}(t-u)} dW_u \\ &= \sigma W_t - \sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dW_u \end{aligned} \tag{2.13}$$

Now the exponentially filtered logarithm of the price in equation 2.9 can be written as the difference of scaled Brownian motion and an Itô process.

$$\Upsilon_t[\ln S_t, \tau] = \ln S_0 \left(1 - e^{-\frac{1}{\tau}t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)\left[t - \tau\left(1 - e^{-\frac{1}{\tau}t}\right)\right] + \sigma W_t - \sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dW_u \tag{2.14}$$

$$= \left[\left(1 - e^{-\frac{1}{\tau}t}\right)\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] - \left[\left(1 - e^{-\frac{1}{\tau}t}\right)\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dW_u\right] \tag{2.15}$$

$$= \left[\ln S_t - e^{-\frac{1}{\tau}t}\ln S_0\right] - \left[\left(1 - e^{-\frac{1}{\tau}t}\right)\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dW_u\right]$$

$$= \theta_1(t;\tau) - \left[\theta_2(t;\tau) + \sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dW_u\right] \tag{2.16}$$

where

$$\theta_1(t;\tau) = \ln S_t - e^{-\frac{1}{\tau}t}\ln S_0 \tag{2.17}$$

$$\theta_2(t;\tau) = \left(1 - e^{-\frac{1}{\tau}t}\right)\left(\mu - \frac{\sigma^2}{2}\right)\tau \tag{2.18}$$

□

From an analytical perspective for large enough t, we assume the exponentials in equations 2.17 and 2.18 are insignificant and write

$$\Upsilon_t[\ln S_t, \tau] \approx \Upsilon_t^*[\ln S_t, \tau] = \ln S_t - \left[\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dW_u\right] \tag{2.19}$$

However during implementation of such a filter the transients determine an initialization period that must be completed before the value of the operator can be used in trading (See Zumbach and Müller 2001).

We see from equation 2.19 that the exponential filter smooths the the log price process by subtracting an exponentially scaled sum of the of the increments of Brownian motion (Itô process) at each value of t from the log price process. Subtracting Υ_t^* from the log price gives an exponentially scaled sum of the increments of Brownian motion plus a constant for the entire history of the process. Define

$$\begin{aligned}\vartheta_t\left[\ln S_t,\tau\right] &= \ln S_t - \Upsilon^*_t\left[\ln S_t,\tau\right]\\ &= \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\int_0^t e^{-\frac{1}{\tau}(t-u)}dW_u\end{aligned} \tag{2.20}$$

In the next section we demonstrate that ϑ_t is normally distributed and is a solution to the the Ornstein-Ulenbeck SDE.

$$\vartheta_t\left[\ln S_t,\tau\right] \sim \mathcal{N}\left[0,\frac{\sigma^2\tau}{2}\left(1-e^{-\frac{2}{\tau}t}\right)\right] \tag{2.21}$$

2.2.1 Υ_t as Solution to Ornstein-Ulenbeck SDE

Proposition 2.2.2. Given an asset price process which is a solution of the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$, the logarithm of the price process $\chi_t = \ln S_t$ minus the exponentially filtered logarithm of the price process $\Upsilon_t[\ln S_t,\tau]$ is the solution of the Ornstein-Uhlenbeck SDE. In precise notation

$$\vartheta_t = \chi_t - \Upsilon_t \tag{2.22}$$

is a solution of the SDE

$$d\vartheta_t = \frac{1}{\tau}\left[\left(\mu - \frac{\sigma^2}{2}\right) - \frac{1}{\tau}\vartheta_t\right]dt + \sigma dW_t \tag{2.23}$$

Proof: (Proposition 2.2.2). By definition

$$\begin{aligned}\Upsilon_t &= \int_0^t \frac{1}{\tau}e^{-\frac{1}{\tau}(t-u)}\chi_u du\\ &= e^{-\frac{1}{\tau}t}\int_0^t \frac{1}{\tau}e^{\frac{1}{\tau}u}\chi_u du\end{aligned} \tag{2.24}$$

Taking the differential of equation 2.24 gives

$$
\begin{aligned}
d\,\Upsilon_t &= d\left(e^{-\frac{1}{\tau}t}\int_0^t \frac{1}{\tau}e^{\frac{1}{\tau}u}\chi_u du\right)\\
&= \frac{1}{\tau}\left[e^{-\frac{1}{\tau}t}\,d\left(\int_0^t e^{\frac{1}{\tau}u}\chi_u du\right) + d\left(e^{-\frac{1}{\tau}t}\right)\int_0^t e^{\frac{1}{\tau}u}\chi_u du + \cancelto{0}{d\left(e^{-\frac{1}{\tau}t}\right) d\left(\int_0^t e^{\frac{1}{\tau}u}\chi_u du\right)}\right]\\
&= \frac{1}{\tau}\left[\left(e^{-\frac{1}{\tau}t}e^{\frac{1}{\tau}t}\chi_t\right) - \left(\frac{1}{\tau}e^{-\frac{1}{\tau}t}\int_0^t e^{\frac{1}{\tau}u}\chi_u du\right)\right]dt\\
&= \frac{1}{\tau}\left[\chi_t - \Upsilon_t\right]dt\\
&= \frac{1}{\tau}\left[\chi_t - \Upsilon_t\right]dt
\end{aligned}
\tag{2.25}
$$

Define

$$\vartheta = \chi_t - \Upsilon_t$$

thus

$$
\begin{aligned}
d\vartheta &= d\chi_t - d\Upsilon_t\\
&= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t - \frac{1}{\tau}\vartheta_t dt\\
&= \frac{1}{\tau}\left[\left(\mu - \frac{\sigma^2}{2}\right)\tau - \vartheta_t\right]dt + \sigma dW_t
\end{aligned}
\tag{2.26}
$$

□

2.2.2 Mean, Variance, and Distribution of Υ_t

Proposition 2.2.3. Given an asset price process which is a solution of the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$, the distribution of the exponentially filtered logarithm of the price process $\Upsilon_t[\ln S_t, \tau]$ is normal with expected value and variance:

$$\begin{aligned}&\Upsilon_t\left[\ln S_t,\tau\right]\sim\\&\quad\mathcal{N}\left[\ln S_0\left(1-e^{-\frac{1}{\tau}t}\right)+\left(\mu-\frac{\sigma^2}{2}\right)\left[t-\tau\left(1-e^{-\frac{1}{\tau}t}\right)\right],\sigma^2\left[t-\tau\left(\frac{3}{2}+\frac{1}{2}e^{-\frac{2}{\tau}t}-2e^{-\frac{1}{\tau}t}\right)\right]\right]\end{aligned} \tag{2.27}$$

Proof: (Proposition 2.2.3). First rewrite equation 2.14 as

$$\begin{aligned}\Upsilon_t[\ln S_t,\tau]&=\sigma W_t-\left[-\ln S_0\left(1-e^{-\frac{1}{\tau}t}\right)-\left(\mu-\frac{\sigma^2}{2}\right)\left[t-\tau\left(1-e^{-\frac{1}{\tau}t}\right)\right]+\sigma\int_0^t e^{-\frac{1}{\tau}(t-u)}dW_u\right]\\&=\sigma W_t-\left[g_{\tau,t}+\sigma\int_0^t e^{-\frac{1}{\tau}(t-u)}dW_u\right]\end{aligned} \tag{2.28}$$

where

$$g_{\tau,t}=-\ln S_0\left(1-e^{-\frac{1}{\tau}t}\right)-\left(\mu-\frac{\sigma^2}{2}\right)\left[t-\tau\left(1-e^{-\frac{1}{\tau}t}\right)\right]$$

From the definition $\Upsilon_t[\ln S_t,\tau]$ in equation 2.28, let

$$X_t=\sigma W_t \tag{2.29}$$

$$Y_t=\sigma\int_{t_1}^{t_2} e^{-\frac{1}{\tau}(t-u)}dW_u \tag{2.30}$$

such that

$$\Upsilon_t[\ln S_t,\tau]=X_t-(g_{\tau,t}+Y_t) \tag{2.31}$$

It immediately follows from the definition of Brownian motion that X_t is normally distributed.

$$X_t\sim\mathcal{N}\left[0,\sigma^2 t\right] \tag{2.32}$$

Since the integrand in the Itô integral on the right side of equation 2.30 is deterministic, the requirements of Theorem 4.4.9 Shreve 2004 are met and the theorem implies Y_t is normally distributed with $\mathbb{E}[Y_t] = 0$. Now consider the square integral of the integrand.

$$\begin{aligned}
\int_0^t \left[\sigma e^{-\frac{1}{\tau}(t-u)}\right]^2 du &= \sigma^2 \int_0^t e^{-\frac{2}{\tau}(t-u)} du \\
&= \sigma^2 \left[\frac{\tau}{2} e^{-\frac{2}{\tau}(t-u)}\right]_0^t \\
&= \frac{\sigma^2 \tau}{2}\left(1 - e^{-\frac{2}{\tau}t}\right) < \infty \quad \forall t \geq 0
\end{aligned} \tag{2.33}$$

Equation 2.33 shows the integrand is square integrable, and it is adapted to the filtration of the Brownian motion; thus, the requirements of Theorem 4.3.1 Shreve 2004 are met and the Itô isometry property of the theorem gives the expected value of Y_t^2 from which we can derive the the variance of Y_t.

$$\begin{aligned}
\mathbb{V}[Y_t] = \mathbb{E}\left[(Y_t - \mathbb{E}[Y_t])^2\right] &= \mathbb{E}\left[Y_t^2\right] \\
&= \mathbb{E}\left[\int_0^t \left[\sigma e^{-\frac{1}{\tau}(t-u)}\right]^2 du\right] \\
&= \frac{\sigma^2 \tau}{2}\left(1 - e^{-\frac{2}{\tau}t}\right)
\end{aligned} \tag{2.34}$$

Thus

$$Y_t \sim \mathcal{N}\left[0, \frac{\sigma^2 \tau}{2}\left(1 - e^{-\frac{2}{\tau}t}\right)\right] \tag{2.35}$$

The distribution of $\Upsilon_t[\ln S_t, \tau]$ is normal since it is the difference of two normal processes X_t and Y_t minus a deterministic function $g_{\tau,t}$; thus, the expected value of $\Upsilon_t[\ln S_t, \tau]$ is

$$\begin{aligned}
\mathbb{E}[\Upsilon_t[\ln S_t, \tau]] &= \mathbb{E}[X_t - (g_{\tau,t} + Y_t)] \\
&= \mathbb{E}[X_t] - \mathbb{E}[Y_t] - g_{\tau,t} \\
&= -g_{\tau,t}
\end{aligned} \tag{2.36}$$

In order to obtain the variance of $\Upsilon_t[\ln S_t, \tau]$ we need the covariance of X_t and Y_t. Let

$$\begin{aligned}\mathrm{Cov}[X_t, Y_t] &= \mathbb{E}[(X_t - \mathbb{E}[X_t])(Y_t - \mathbb{E}[Y_t])] \\ &= \mathbb{E}[X_t Y_t] \\ &= \mathbb{E}\left[(\sigma W_t)\left(\sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dW_u\right)\right]\end{aligned} \tag{2.37}$$

In equation 2.37, X_t and Y_t are dependent since the last value of the Brownian path of integration in Y_t must be equal to W_t in X_t, otherwise the rest of the path is independent of X_t. We will construct a process Y_t^* which is equal in distribution to Y_t where the dependency between the left and right terms in the product in equation 2.37 is explicit. First assume W_t is known, then construct a Brownian bridge with a second independent Brownian motion B_t which meets the requirement that $W_t = \int_0^t dB_u$. Define the Brownian bridge I_u as

$$I_u = B_u - \frac{u}{t} B_t + \frac{u}{t} W_t \tag{2.38}$$

where t is treated as a constant. The first two terms in equation 2.38 form a Brownian bridge from zero to zero on the interval $[0, t]$ and the last term converts it to a Brownian bridge from zero to W_t on the same interval. The differential of I_u is

$$dI_u = dB_u - \frac{1}{t} B_t du + \frac{1}{t} W_t du \tag{2.39}$$

Now form Y_t^* by replacing W_u in the right hand term of equation 2.37 with I_u.

$$\begin{aligned}\mathrm{Cov}[X_t, Y_t] &= \mathrm{Cov}[X_t, Y_t^*] \\ &= \mathbb{E}\left[(\sigma W_t)\left(\sigma \int_0^t e^{-\frac{1}{\tau}(t-u)} dI_u\right)\right] \\ &= \mathbb{E}\left[\sigma^2 W_t \int_0^t e^{-\frac{1}{\tau}(t-u)} \left(dB_u - \frac{1}{t} B_t du + \frac{1}{t} W_t du\right)\right] \\ &= \mathbb{E}\left[\sigma^2 W_t \int_0^t e^{-\frac{1}{\tau}(t-u)} dB_u - \frac{\sigma^2 W_t B_t}{t} \int_0^t e^{-\frac{1}{\tau}(t-u)} du + \frac{\sigma^2 W_t^2}{t} \int_0^t e^{-\frac{1}{\tau}(t-u)} du\right]\end{aligned}$$

W_t and B_t are independent by definition.

$$= \sigma^2 \mathbb{E}\left[W_t\right] \mathbb{E}\left[\int_0^t e^{-\frac{1}{\tau}(t-u)} dB_u\right] - \frac{\sigma^2}{t}\int_0^t e^{-\frac{1}{\tau}(t-u)} du \; \mathbb{E}\left[W_t\right] \mathbb{E}\left[B_t\right] + \frac{\sigma^2}{t}\int_0^t e^{-\frac{1}{\tau}(t-u)} du \; \mathbb{E}\left[W_t^2\right]$$

$$\mathbb{E}\left[W_t\right] = \mathbb{E}\left[B_t\right] = \mathbb{E}\left[\int_0^t e^{-\frac{1}{\tau}(t-u)} dB_u\right] = 0$$

$$\begin{aligned}
&= \frac{\sigma^2\tau}{t}\left(1 - e^{-\frac{1}{\tau}t}\right) \mathbb{E}\left[\left(W_t - 0\right)^2\right] \\
&= \frac{\sigma^2\tau}{t}\left(1 - e^{-\frac{1}{\tau}t}\right) \mathbb{E}\left[\left(W_t - \mathbb{E}\left[W_t\right]\right)^2\right] \\
&= \frac{\sigma^2\tau}{t}\left(1 - e^{-\frac{1}{\tau}t}\right) \mathbb{V}\left[W_t\right] \\
&= \sigma^2\tau\left(1 - e^{-\frac{1}{\tau}t}\right) \qquad (2.40)
\end{aligned}$$

Using basic probability theory we know the variance of a sum of random variables

$$\begin{aligned}
\mathbb{V}\left[\Upsilon_t\right] &= \mathbb{V}\left[X_t\right] + \mathbb{V}\left[-Y_t\right] + 2\mathrm{Cov}\left[X_t, -Y_t,\right] \\
&= \mathbb{V}\left[X_t\right] + \mathbb{V}\left[Y_t\right] - 2\mathrm{Cov}\left[X_t, Y_t,\right] \\
&= \sigma^2 t + \frac{\sigma^2\tau}{2}\left(1 - e^{-\frac{2}{\tau}t}\right) - 2\sigma^2\tau\left(1 - e^{-\frac{1}{\tau}t}\right) \\
&= \sigma^2\left[t - \tau\left(\frac{3}{2} + \frac{1}{2}e^{-\frac{2}{\tau}t} - 2e^{-\frac{1}{\tau}t}\right)\right] \qquad (2.41)
\end{aligned}$$

Thus from equations 2.36 and 2.40, $\Upsilon_t\left[\ln S_t, \tau\right]$ is normally distributed with the following parameters.

$$\begin{aligned}
&\Upsilon_t\left[\ln S_t, \tau\right] \sim \\
&\quad \mathcal{N}\left[\ln S_0\left(1 - e^{-\frac{1}{\tau}t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)\left[t - \tau\left(1 - e^{-\frac{1}{\tau}t}\right)\right], \sigma^2\left[t - \tau\left(\frac{3}{2} + \frac{1}{2}e^{-\frac{2}{\tau}t} - 2e^{-\frac{1}{\tau}t}\right)\right]\right] \qquad (2.42)
\end{aligned}$$

□

The distribution of $\Upsilon_t\left[\ln S_t, \tau\right]$ in equation 2.42 contains a number of exponential terms with negative exponents which quickly decay and are insignificant to the long term analysis.

$$\Upsilon_t\left[\ln S_t, \tau\right] \approx \Upsilon_t^*\left[\ln S_t, \tau\right] \sim \mathcal{N}\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)(t-\tau), \sigma^2\left(t - \frac{3\tau}{2}\right)\right] \tag{2.43}$$

Comparing the variance of the log price process and the variance of the exponentially filtered log price process from equation 2.43,

$$\frac{\mathbb{V}\left[\Upsilon_t^*\left[\ln S_t, \tau\right]\right]}{\mathbb{V}\left[\ln S_t\right]} \approx \frac{\sigma^2\left(t - \frac{3}{2}\tau\right)}{\sigma^2 t} = \left(1 - \frac{3}{2}\frac{\tau}{t}\right) \tag{2.44}$$

we see the exponential filter converts the log price process into a process with a smaller variance by a constant value.

2.3 Difference of Filtered Log Price Ψ_t

Now we construct the difference indicator $\Psi_t[\ln S_t, \tau_1, \tau_2]$ from two versions of $\Upsilon_t\left[\ln S_t, \tau\right]$ with different values of τ applied to the log price process $\ln S_t$.

Proposition 2.3.1. Given an asset price process which is a solution of the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$, and $0 < \tau_1 < \tau_2$, the difference of two exponentially filtered logarithm of the asset price processes $\Upsilon_t[\ln S_t, \tau_1]$ and $\Upsilon_t[\ln S_t, \tau_2]$ produces an Itô process $\Psi_t[\ln S_t, \tau_1, \tau_2]$ driven by the same Brownian motion.

$$\Psi_t[\ln S_t, \tau_1, \tau_2] = \Upsilon_t[\ln S_t, \tau_1] - \Upsilon_t[\ln S_t, \tau_2] \tag{2.45}$$

$$= \Theta(t; \tau_1, \tau_2) + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right) dW_u \tag{2.46}$$

where

$$\Theta(t; \tau_1, \tau_2) = \left[\ln S_0 - \left(\mu - \frac{\sigma^2}{2}\right)\right]\left(e^{-\frac{1}{\tau_2}t} - e^{-\frac{1}{\tau_1}t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) \tag{2.47}$$

Proof: (Proposition 2.2.1). Substituting the results from equation 2.3 of proposition 2.2.1 into equation 2.45 and simplifying proves the proposition.

$$\begin{aligned}
&\Psi_t[\ln S_t, \tau_1, \tau_2] \\
&\quad = \Upsilon_t[\ln S_t, \tau_1] - \Upsilon_t[\ln S_t, \tau_2] \\
&\quad = \theta_1(t;\tau_1) - \left(\theta_2(t;\tau_1) + \sigma \int_0^t e^{-\frac{1}{\tau_1}(t-u)} dW_u\right) - \left[\theta_1(t;\tau_2) - \left(\theta_2(t;\tau_2) + \sigma \int_0^t e^{-\frac{1}{\tau_2}(t-u)} dW_u\right)\right] \\
&\quad = \theta_1(t;\tau_1) - \theta_1(t;\tau_2) - (\theta_2(t;\tau_1) - \theta_2(t;\tau_2)) + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right) dW_u \\
&\quad = \Theta_{\tau_1,\tau_2,t} + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right) dW_u \qquad (2.48)
\end{aligned}$$

where

$$\begin{aligned}
\Theta_{\tau_1,\tau_2,t} &= [\theta_1(t;\tau_1) - \theta_1(t;\tau_2)] - [\theta_2(t;\tau_1) - \theta_2(t;\tau_2)] \\
&= \left[\left(\cancel{\ln S_t} - e^{-\frac{1}{\tau_1}t} \ln S_0\right) - \left(\cancel{\ln S_t} - e^{-\frac{1}{\tau_2}t} \ln S_0\right)\right] \\
&\qquad - \left[\left(1 - e^{-\frac{1}{\tau_1}t}\right)\left(\mu - \frac{\sigma^2}{2}\right)\tau_1 - \left(1 - e^{-\frac{1}{\tau_2}t}\right)\left(\mu - \frac{\sigma^2}{2}\right)\tau_2\right] \qquad (2.49) \\
&= \ln S_0 \left(e^{-\frac{1}{\tau_2}t} - e^{-\frac{1}{\tau_1}t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)\left[\tau_2\left(1 - e^{-\frac{1}{\tau_2}t}\right) - \tau_1\left(1 - e^{-\frac{1}{\tau_1}t}\right)\right] \\
&= \left[\ln S_0 - \left(\mu - \frac{\sigma^2}{2}\right)\right]\left(e^{-\frac{1}{\tau_2}t} - e^{-\frac{1}{\tau_1}t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) \qquad (2.50)
\end{aligned}$$

□

The important transformation that occurs in the proof of proposition 2.3.1 in equation 2.49 is the log price processes cancel and what is left is an exponentially scaled sum of the Brownian motion increments with a deterministic term that converges to a constant with increasing t.

From this point forward let $\Psi_t = \Psi_t[\ln S_t, \tau_1, \tau_2]$. Assuming that steady state behavior has been reached $t > t_{ss}$, the difference indicator Ψ_t from equation 2.48 is now written with the deterministic term in its steady state form.

$$\Psi_t = \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right) dW_u \qquad (2.51)$$

Looking at equation 2.51 it is clear that Ψ_t consists of a constant term plus the difference between two exponentially smoothed filters of diffusion component. The constant term is positive or negative depending on whether or not $\mu > \frac{\sigma^2}{2}$ or $\mu < \frac{\sigma^2}{2}$ and its magnitude is affected by the magnitude of the difference between τ_1 and τ_2. The integral is the difference of two exponential filters of the brownian motion with different filter constants. This should result in a greater smoothing effect than the the single filter of the Brownian motion in Υ_t.

2.3.1 Mean and Variance of Ψ_t

Let M_t represent the Itô integral contained in the definition of Ψ_t in equation 2.51.

$$\begin{aligned} M_t &= \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right) \, dW_u \\ &= \int_0^t \Gamma_{(u;t)} \, dW_u \quad \text{where} \quad \Gamma_{(u;t)} = \sigma \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right) \end{aligned} \tag{2.52}$$

Consider the expected value of the integral of $\Gamma^2_{(u;t)}$ from zero to t.

$$\begin{aligned} \mathbb{E}\left[\int_0^t \Gamma^2_{(u;t)} \, du\right] &= \mathbb{E}\left[\sigma^2 \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right)^2 \, du\right] \\ &= \sigma^2 \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)} \right)^2 \, du \\ &= \sigma^2 \int_0^t \left(e^{-\frac{2}{\tau_2}(t-u)} - 2e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)(t-u)} + e^{-\frac{2}{\tau_1}(t-u)} \right) \, du \\ &= \sigma^2 \left(\left[\frac{\tau_2}{2} e^{-\frac{2}{\tau_2}(t-u)}\right]_0^t - \left[\frac{2}{\frac{1}{\tau_1}+\frac{1}{\tau_2}} e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)(t-u)}\right]_0^t + \left[\frac{\tau_1}{2} e^{-\frac{2}{\tau_1}(t-u)}\right]_0^t \right) \\ &= \sigma^2 \left(\frac{\tau_2}{2}\left(1 - e^{-\frac{2}{\tau_2}t}\right) - \frac{2}{\frac{1}{\tau_1}+\frac{1}{\tau_2}} \left(1 - e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)t}\right) + \frac{\tau_1}{2}\left(1 - e^{-\frac{2}{\tau_1}t}\right) \right) \end{aligned} \tag{2.53}$$

Equation 2.53 shows the expected value of the integral of $\Gamma^2_{(u;t)}$ is square integrable, $\mathbb{E}\left[\int_0^t \Gamma^2_{(u;t)} \, du\right] < \infty$. In addition, for all $u \leq t$, $\Gamma_{(u;t)}$ is $\mathcal{F}_t$ adapted; thus, the conditions of Theorem 4.3.1 Shreve 2004 are met and M_t is a martingale. This implies that given $0 < s < t$

$$\mathbb{E}\left[M_t | \mathcal{F}_s\right] = M_s \tag{2.54}$$

$\Gamma_{(0;0)=0}$ and $W_0 = 0$ implies that $M_0 = 0$ which combined with the martingale property expressed in equation 2.54 implies for all $t > 0$ $\mathbb{E}\left[M_t\right] = 0$. Thus, taking the expected value of

Ψ_t

$$\begin{aligned}\mathbb{E}\left[\Psi_t\right] &= \mathbb{E}\left[\left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right) dW_u\right] \\ &= \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) \end{aligned} \tag{2.55}$$

results in the constant function of time in equation 2.55. Given $0 < \tau_1 < \tau_2$ by assumption, the sign of the expected value of Ψ_t is controlled by the relationship between μ and σ^2.

$$\mathbb{E}\left[\Psi_t\right] \begin{cases} > 0 & \mu > \sigma^2/2 \\ = 0 & \mu = \sigma^2/2 \\ < 0 & \mu < \sigma^2/2 \end{cases} \tag{2.56}$$

The results of equation 2.53, the fact that $\mathbb{E}\left[M_t\right] = 0$ and Theorem 4.3.1 Shreve 2004, whose conditions have been met, implies the variance of Ψ_t is equal to the variance of M_t which is given by Itô's Isometry.

$$\begin{aligned}\mathbb{V}\left[M_t\right] &= \mathbb{E}\left[\left(M_t - \mathbb{E}\left[M_t\right]\right)^2\right] = \mathbb{E}\left[M_t^2\right] \\ &= \mathbb{E}\left[\int_0^t \Gamma^2_{(u;t)}\, du\right] \\ &= \sigma^2 \left(\frac{\tau_2}{2}\left(1 - e^{-\frac{2}{\tau_2}t}\right) - \frac{2}{\frac{1}{\tau_1} + \frac{1}{\tau_2}}\left(1 - e^{-\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)t}\right) + \frac{\tau_1}{2}\left(1 - e^{-\frac{2}{\tau_1}t}\right)\right) && (2.57) \\ &= \sigma^2 \left[\left(\frac{\tau_2}{2} - \frac{2}{\frac{1}{\tau_1} + \frac{1}{\tau_2}} + \frac{\tau_1}{2}\right) + \left(\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right) e^{-\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)t} - \frac{\tau_2}{2} e^{-\frac{2}{\tau_2}t} - \frac{\tau_1}{2} e^{-\frac{2}{\tau_1}t}\right)\right] && (2.58)\end{aligned}$$

The graph of the variance of Ψ_t for parameter values $\sigma = 0.1$, $\tau_1 = 5$ and $\tau_2 = 10$ is displayed in figure 2.1.

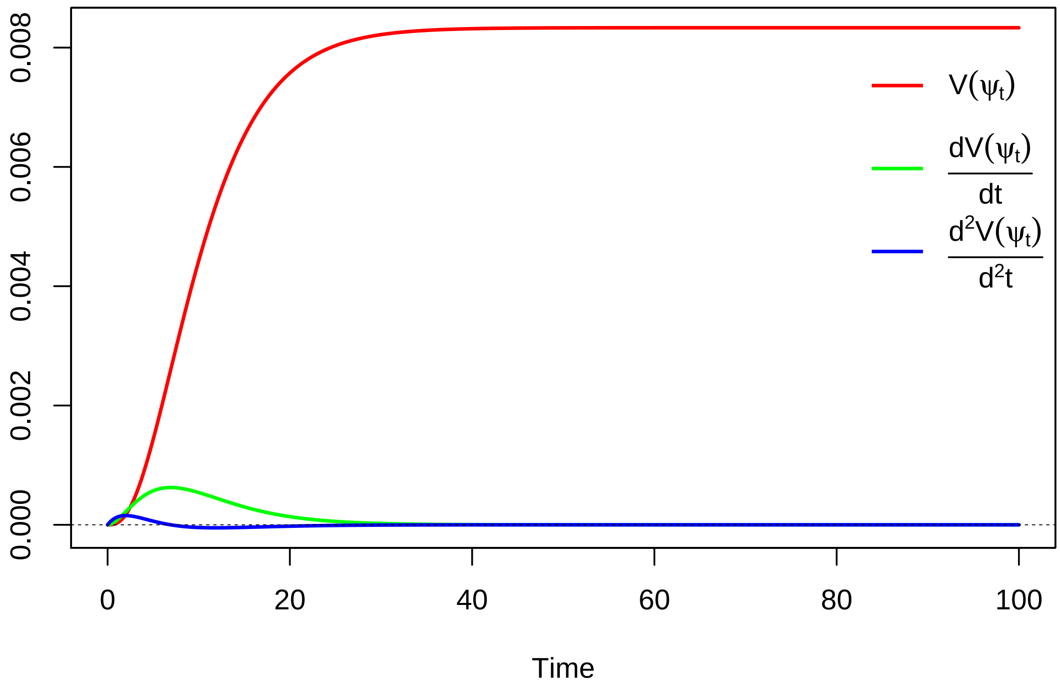

Figure 2.1: Variance of Ψ_t: $\mathbb{V}\left[\Psi_t\right]$, $\frac{d\,\mathbb{V}[\Psi_t]}{d\,t}$, and $\frac{d^2\,\mathbb{V}[\Psi_t]}{d\,t^2}$ $\sigma = 0.1$, $\tau_1 = 5$ and $\tau_2 = 10$

When $t = 0$ in equation 2.57, the variance of M_t is equal to zero which is expected since $W_0 = 0$. Consider the limit of equation 2.57 as t goes to infinity.

$$
\begin{aligned}
\lim_{t\to\infty} \mathbb{V}\left[M_t\right] &= \lim_{t\to\infty}\left[\sigma^2\left(\frac{\tau_2}{2}\left(1-e^{-\frac{2}{\tau_2}t}\right)-\frac{2}{\frac{1}{\tau_1}+\frac{1}{\tau_2}}\left(1-e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)t}\right)+\frac{\tau_1}{2}\left(1-e^{-\frac{2}{\tau_1}t}\right)\right)\right] \\
&= \sigma^2\left(\frac{\tau_2}{2}-\frac{2}{\frac{1}{\tau_1}+\frac{1}{\tau_2}}+\frac{\tau_1}{2}\right) \\
&= \sigma^2\left(\frac{\tau_1+\tau_2}{2}-\frac{2\tau_1\tau_2}{\tau_1+\tau_2}\right) \\
&= \sigma^2\left(\frac{\left(\tau_1+\tau_2\right)^2-4\tau_1\tau_2}{2(\tau_1+\tau_2)}\right) \\
&= \sigma^2\left(\frac{\left(\tau_1^2+2\tau_1\tau_2+\tau_2^2\right)-4\tau_1\tau_2}{2(\tau_1+\tau_2)}\right) \\
&= \sigma^2\left(\frac{\tau_1^2-2\tau_1\tau_2+\tau_2^2}{2(\tau_1+\tau_2)}\right) \\
&= \frac{\sigma^2}{2}\left(\frac{\left(\tau_2-\tau_1\right)^2}{\tau_1+\tau_2}\right)
\end{aligned}
\tag{2.59}
$$

Since $0 < \tau_1 < \tau_2$ the steady state variance of Ψ_t is always greater than zero which is expected as it should only be zero if the two exponentially filtered stochastic process have the same value of τ. Based on the proofs in the appendix on page **??**, we assume that for every $S_0 > 1$ and $0 < \tau_1 < \tau_2$ there exists a t_{ss} and δ such that for all $t > t_{ss}$

$$
\left|\lim_{t\to\infty} \mathbb{V}\left[M_t\right] - \mathbb{V}\left[M_t\right]\big|_{t=t_{ss}}\right| < \delta
\tag{2.60}
$$

and the transient behavior of the exponential terms in equation 2.57 is insignificant to the analysis.

2.3.2 Distribution and Density Functions of Ψ_t

Since M_t is an Itô integral with respect to Brownian motion with a deterministic integrand $\Gamma_{(u;t)}$, by Theorem 4.4.9 Shreve 2004 for each $t > 0$, M_t is normally distributed with expected value zero and variance $\int_0^t \Gamma^2_{(u;t)}\, du$. Thus Ψ_t from equation 2.51 is normally distributed with the expected value u and variance s given by equations 2.55 and 2.59.

$$\begin{aligned} \Psi_t &\sim \mathcal{N}\left[u, s^2\right] \\ &\sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1), \frac{\sigma^2}{2}\left(\frac{(\tau_2 - \tau_1)^2}{\tau_1 + \tau_2}\right)\right] \end{aligned} \tag{2.61}$$

Equation 2.61 shows the variance can change dramatically based on the difference between τ_2 and τ_1. We also note that once steady state is reached both the mean and variance of Ψ_t are independent of time.

Figure 2.2 provides an interesting view of the variance of Ψ_t as a function of τ_1 and τ_2. In this 3D plot $\mu = 0.001$ $\sigma = 0.1$ and $0 < \tau_1 < \tau_2$. Holding τ_2 constant and increasing τ_1 decreases the variance, whereas holding τ_1 constant and increasing τ_2 increases variance.

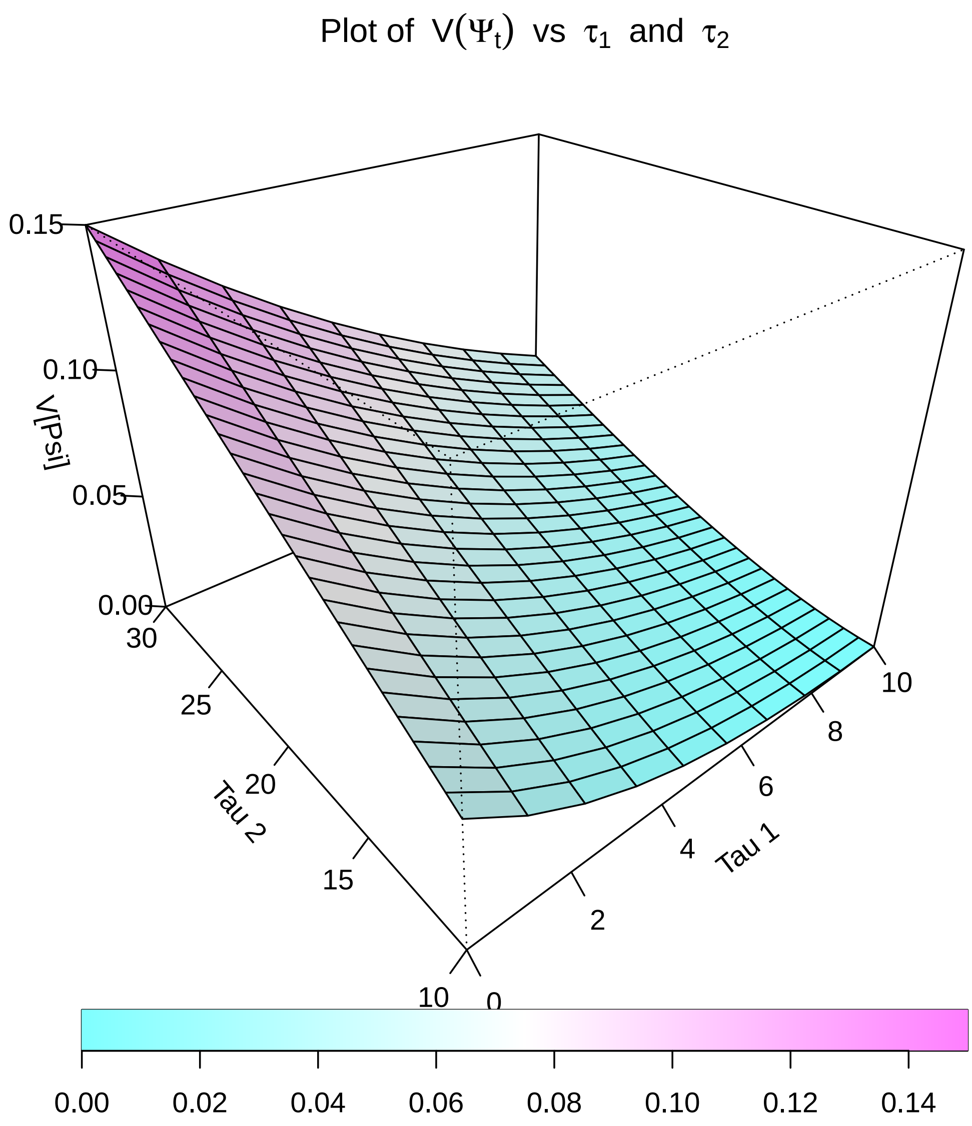

Figure 2.2: Variance of Ψ_t as a Function of τ_1 and τ_2: $\mathbb{V}\left[\Psi_t\right] = \frac{\sigma^2}{2}\left(\frac{(\tau_2-\tau_1)^2}{\tau_1+\tau_2}\right)$, $\mu = 0.001$, $\sigma = 0.1$

Given that Ψ_t is normal with mean u and variance s^2 the cumulative distribution function (CDF) F_Ψ and probability density function (PDF) f_Ψ for Ψ_t are

$$F_\Psi = \Phi\left(\frac{\psi - u}{s}\right) \tag{2.62}$$

$$f_\Psi = \frac{d\,F_\Psi}{d\,\psi} = \frac{1}{s}\phi\left(\frac{\psi - u}{s}\right) \tag{2.63}$$

where Φ is the standard normal CDF and ϕ is the standard normal PDF. Substitution of u and s from equation 2.61 into equation 2.63 gives the full density function for Ψ_t. Once the steady state regime is reached the density function has no dependence on t.

$$\begin{aligned} f_\Psi(\psi) &= \frac{1}{\sqrt{2\pi\left[\frac{\sigma^2}{2}\frac{(\tau_2-\tau_1)^2}{\tau_1+\tau_2}\right]}}\exp\left[-\frac{\left[\psi - \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1)\right]^2}{2\left[\frac{\sigma^2}{2}\frac{(\tau_2-\tau_1)^2}{\tau_1+\tau_2}\right]}\right] \\ &= \frac{1}{\sigma\left(\tau_2 - \tau_1\right)\sqrt{\frac{\pi}{(\tau_1+\tau_2)}}}\exp\left[-\frac{\left[\psi - \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1)\right]^2}{\frac{\sigma^2(\tau_2-\tau_1)^2}{(\tau_1+\tau_2)}}\right] \end{aligned} \tag{2.64}$$

Chapter 3

Trading Strategy

In this chapter we define and analyze the structural elements of a trading strategy which uses the difference of two exponentially filtered log asset price in determining the market postion. Terminology is drawn from the framework developed in Peterson 2015.

3.1 Indicator

Indicators (as defined in Peterson 2015) are quantitative transformations of real-time market and economic data streams that are designed to reveal specific market phenomena which can be exploited by an automated trading system. In this work, indicators will be various algebraic combinations of exponential filter time series operators 2.1.1 applied to market log price data streams.

Definition 3.1.1 (Difference Indicator)**.** The difference indicator $\Psi_t : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the difference of two exponential filter time series operators as defined in equation 2.51 applied to a stochastic process X_t where $0 < \tau_1 < \tau_2$.

$$\Psi_t[X_t; \tau_1, \tau_2] = \Upsilon_t[X_t; \tau_2] - \Upsilon_t[X_t; \tau_1] \tag{3.1}$$

We examine the case where the stochastic process is the natural logarithm of the asset price $X_t = \ln S_t$.

The indicator which specifies the percentage and direction (long/short) of the portfolio to be invested in the market asset at time t will be designated ϱ_t.

Definition 3.1.2 (Position Indicator)**.** The position indicator $\varrho_t = c\,\Psi_t$

In the mathematical finance literature ϱ_t is oftern referred to as the investors strategy.

3.2 Portfolio SDE

The instantaneous change in the value of the portfolio is represented by the SDE in equation 3.2 [1]. The position indicator is a function ϱ_t that represents the percentage and direction (long or short) of the portfolio to be invested in the market asset at time t, Π_t represents the portfolio value at time t, and $\frac{\Pi_t}{S_t}$ represents the number of shares which could be bought or sold short at time t if the entire value of the portfolio was invested long or used as margin against a short postion in the single market asset. $\frac{\Pi_t}{B_t}$ represents the number of risk free bonds that can be bought if the entire portfolio was invested in the risk free bond.

$$d\,\Pi_t = \varrho_t \frac{\Pi_t}{S_t} dS_t + r\,(1-\varrho_t)\,\frac{\Pi_t}{B_t} dt \tag{3.2}$$

Our goal is the investigate the how the long term expected log returns

$$\lim_{t->\infty} \mathbb{E}\left[\ln\left(\frac{\Pi_t}{\Pi_0}\right)\right] \tag{3.3}$$

is affected by the choice of parameters c, τ_1, and τ_2 to the indicator (strategy) function ϱ_t.

When the position in the risky asset is short $\varrho < 0$, cash comes into the portfolio account due to the short position; thus, the amount of cash in the portfolio account is greater than the value of the portfolio Π_t which is the initial value of the portfolio Π_0 plus all of the incremental changes due to changes in the risky asset price and the time value of the risk free bond. When the full value of the portfolio is used as margin aginst a short position, that is $\varrho_t = -1$ which implies that $(1-\varrho_t) = 2$, the amount of cash in the portfolio account is twice the value of the portfolio and all of the cash is available to invest in the risk free bond.

[1] The portfolio SDE used in this section appears in both Bäuerle and Rieder 2004 and Lorig, Zhou, and Zou 2019.

Using all of the available cash in the portfolio account to invest in the risk free bound would only increase the portfolio returns; however, to gain a better understaning of the trading strategy returns independent of investing in the risk free bond and gain analytical tractability, we assume that $r = 0$ and simplify equation 3.2 by removing the right hand term on the right side.

$$d\,\Pi_t = \varrho_t \frac{\Pi_t}{S_t} dS_t \tag{3.4}$$

We note that if $\|\varrho\| > 1$ leverage would be required the achive the constraints of equation 3.4.

Assume $\varrho_t = \varrho > 0$ is constant. When the change in the asset price dS_t is positive, the change in the value of the portfolio is positive, and when the change in the asset price is negative, the change in the value of the portfolio is negative. Likewise assume $\varrho_t = \varrho < 0$ is constant. When the change in the asset price dS_t is negative, the change in the value of the portfolio is positive, and when the change in the asset price is positive, the change in the value of the portfolio is negative. Writing equation 3.4 in discrete form makes the dynamics more intuitive.

$$\begin{aligned} \Pi_i - \Pi_{i-1} &= \varrho_{i-1} \frac{\Pi_{i-1}}{S_{i-1}} \left(S_i - S_{i-1}\right) \\ \Pi_i &= \Pi_{i-1} + \varrho_{i-1} \frac{\Pi_{i-1}}{S_{i-1}} \left(S_i - S_{i-1}\right) \end{aligned} \tag{3.5}$$

Assume $\varrho_i = \varrho_{i-1} = \varrho$ is constant. Multiplying both sides of equation 3.5 by $\frac{\varrho}{S_i}$ results in the value of the portfolio to be invested in the risky asset at the next time step.

$$\varrho \, \frac{\Pi_i}{S_i} = \varrho \, \frac{\Pi_{i-1}}{S_i} \left(1 + \varrho \frac{S_i - S_{i-1}}{S_{i-1}}\right) \tag{3.6}$$

Let $\nu_i = \varrho \frac{\Pi_i}{S_i}$. The ratio of new number of shares in the risky asset to the old one is calculated by dividing equation 3.6 by $\nu_{i-1} = \varrho \frac{\Pi_{i-1}}{S_{i-1}}$.

$$\begin{aligned} \frac{\nu_i}{\nu_{i-1}} &= \frac{S_{i-1} + \varrho \left(S_i - S_{i-1}\right)}{S_i} \\ &= \frac{S_{i-1}}{S_i} + \varrho \left(1 - \frac{S_{i-1}}{S_i}\right) \end{aligned} \tag{3.7}$$

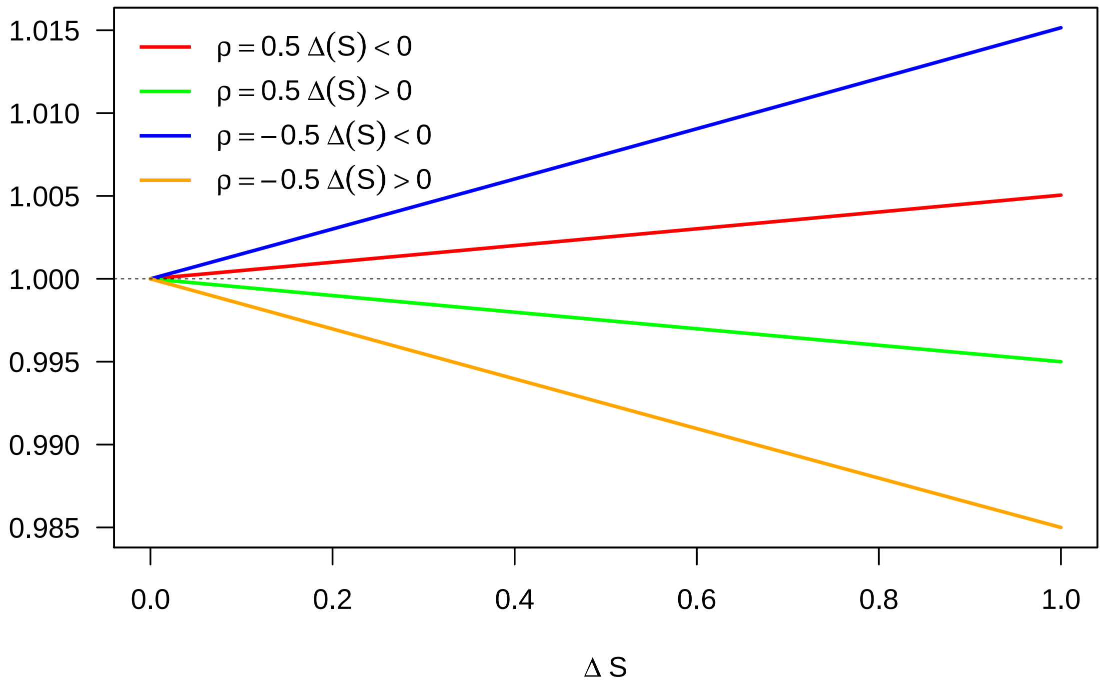

Figure 3.1: Ratio of New to Old Position: $\frac{\nu_i}{\nu_{i-1}}$ vs ΔS

Equation 3.7 is liner as a function of $\Delta S = S_i - S_{i-1}$ with intercept zero and slope ϱ. Figure 3.1 is a plot of $\frac{\nu_i}{\nu_{i-1}}$ vs $\Delta S_i = S_i - S_{i-1}$ from equation 3.7 for a few values of ϱ and ΔS.

Position (ϱ)	ΔS	$R = \frac{\nu_i}{\nu_{i-1}}$	Transaction
Long: $0 < \varrho < 1$	Decreasing price: $\Delta S < 0$	$R > 1$	Buy Long
Long: $0 < \varrho < 1$	Increasing price: $\Delta S > 0$	$R < 1$	Sell Long
Short: $-1 < \varrho < 0$	Decreasing price: $\Delta S < 0$	$R > 1$	Sell Short
Short: $-1 < \varrho < 0$	Increasing price: $\Delta S > 0$	$R < 1$	Buy - Cover Short

Table 3.1: SDE Specified Transactions Given ϱ, ΔS, and $R = \frac{\nu_i}{\nu_{i-1}}$

The ratio $\frac{\nu_i}{\nu_{i-1}}$ and the sign of ϱ specify the required market transaction to be completed moving from time step $i-1$ to i to satisfy the new value of the $\varrho\frac{\Pi_i}{S_i}$. Table 3.1 lists these market transactions for specific values of ϱ and ΔS. When the position indicator ϱ is not constant and changes sign between step $i-1$ and step i, the current position must liqudated and new position taken in the opposite direction. That is if ϱ transitions from positive to negative, the long position in step $i-1$ must be sold and new short position taken in step i. Likewise if ϱ transitions from negative to positive, the short position in step $i-1$ must be covered and a new long position taken during step i.

Long and short aspects of the strategy are not symetrical under constant ϱ. When $\varrho > 0$ is constant, the strategy is contrarian and sells into a rising market and buys in a falling market. However, when $\varrho < 0$ is constant, the strategy sells short into a falling market and covers the short (buys) in a rising market. In order to be symetrical when $\varrho < 0$ is constant and be contrarian, the strategy would sell short into a rising market and cover the short (buy) into a falling market.

3.3 Portfolio SDE Solution

To solve for the value of the portfolio as a function time, we first apply Itô's lemma to the SDE in equation 3.4 with $f(\pi) = \ln \pi$.

$$
\begin{aligned}
d \ln \Pi_t &= \frac{\partial f(\Pi_t)}{\partial t} + \frac{\partial f(\Pi_t)}{\partial \Pi_t} d\Pi_t + \frac{1}{2} \frac{\partial^2 f(\Pi_t)}{\partial \Pi_t^2} d\left\langle \Pi_t \right\rangle_t \\
&= \frac{1}{\Pi_t} d\,\Pi_t + \frac{1}{2}\left(-\frac{1}{\Pi_t^2}\right) d\left\langle \Pi \right\rangle_t \\
&= \frac{1}{\Pi_t}\left[\varrho_t \frac{\Pi_t}{S_t} dS_t\right] - \frac{1}{2\Pi_t^2}\left[\varrho_t \frac{\Pi_t}{S_t} dS_t\right]^2 \\
&= \frac{\varrho_t}{S_t} dS_t - \frac{\varrho_t^2}{2S_t^2}\left\langle dS_t \right\rangle_t \\
&= \frac{\varrho_t}{S_t}\left[\mu S_t dt + \sigma S_t dW_t\right] - \frac{\varrho_t^2}{2S_t^2}\,\sigma^2 S_t^2\, dt \\
&= \varrho_t \left[\mu dt + \sigma dW_t\right] - \frac{\varrho_t^2 \sigma^2}{2} dt \\
&= \varrho_t \left[d \ln S_t + \frac{\sigma^2}{2} dt\right] - \frac{\varrho_t^2 \sigma^2}{2} dt \\
&= \frac{\sigma^2}{2}\,\varrho_t\left(1 - \varrho_t\right) dt + \varrho_t\, d\ln S_t
\end{aligned}
\tag{3.8}
$$

Equation 3.8 gives the instantaneous change in the log value of the portfolio as a function the position indicator ϱ_t, the instantaneous change in the log price $\ln S_t$, and the diffusion coefficient. Integrating equation 3.8 from 0 to t results in the portfolio value process Π_t that solves the SDE in equation 3.4.

$$\begin{aligned}
\int_0^t d\ln\Pi_u &= \frac{\sigma^2}{2}\int_0^t \varrho_u\,(1-\varrho_u)\,du + \int_0^t \varrho_u\,d\ln S_u \\
\ln\Pi_t - \ln\Pi_0 &= \frac{\sigma^2}{2}\int_0^t \varrho_u\,(1-\varrho_u)\,du + \int_0^t \varrho_u\,d\ln S_u \qquad (3.9)\\
\Pi_t &= \Pi_0\exp\left[\frac{\sigma^2}{2}\int_0^t \varrho_u\,(1-\varrho_u)\,du + \int_0^t \varrho_u\,d\ln S_u\right] \qquad (3.10)
\end{aligned}$$

Equation 3.9 gives the log return of the portfolio on the interval $[0,t]$. The portfolio log return process is given by equation 3.9 where $r_t = \ln\Pi_t - \ln\Pi_0$. Assuming the position indicator is constant $\varrho_t = \varrho_0$ gives

$$\begin{aligned}
r_t &= \frac{\sigma^2}{2}\varrho_0\,(1-\varrho_0)\int_0^t du + \varrho_0\int_0^t d\ln S_u \\
&= \frac{\sigma^2}{2}\varrho_0\,(1-\varrho_0)\,t + \varrho_0\,(\ln S_t - \ln S_0) \qquad (3.11)
\end{aligned}$$

Let $g(\varrho_0) = \varrho_0\,(1-\varrho_0)$ then $g(\varrho_0) = \varrho_0\,(1-\varrho_0) = 0$ implies $g(\varrho)$ has zeros at $\varrho_0 = \{0,1\}$. $\frac{d\,g}{d\,\varrho_0} = 1 - 2\varrho_0 = 0$ implies $g(\varrho)$ has a potential local maximum of 0.25 at $\varrho_0 = \frac{1}{2}$ which is confirmed by $\frac{d^2\,g}{d\,\varrho_0^2} = -2$ which implies $g(\varrho_0)$ is concave downwards on $[-1,1]$.

Now substitute $\varrho_0 = \{-1, 0.5, 1\}$, if $\varrho_0 = -1$ then

$$r_t\big|_{\varrho_0=-1} = \ln S_0 - \ln S_t - \sigma^2 t \qquad (3.12)$$

which is the portfolio log return on $[0,t]$ for a short sell and hold position. If $\varrho_0 = 0.5$ then

$$r_t\big|_{\varrho_0=0.5} = \frac{1}{2}\,(\ln S_t - \ln S_0) + \frac{\sigma^2}{8}t \qquad (3.13)$$

If $\varrho_0 = 1$ then

$$r_t\big|_{\varrho_0=1} = \ln S_t - \ln S_0 \qquad (3.14)$$

which is the expected log return for a investing the entire portfolio in a buy and hold position on the interval $[0, 1]$. Equation 3.13 shows that trading according to the SDE in equation 3.4 with a constant $\varrho = 0.5$ reaps a volatility benefit to the log returs of $\frac{\sigma^2}{8}t$; whereas, equation 3.12 shows trading according to the SDE with a full short position incurs a volatility penalty of $\sigma^2 t$ in the log returns. Equation 3.14 shows that trading according to the SDE with full long postion receives neither a penalty and benefit from volatility.

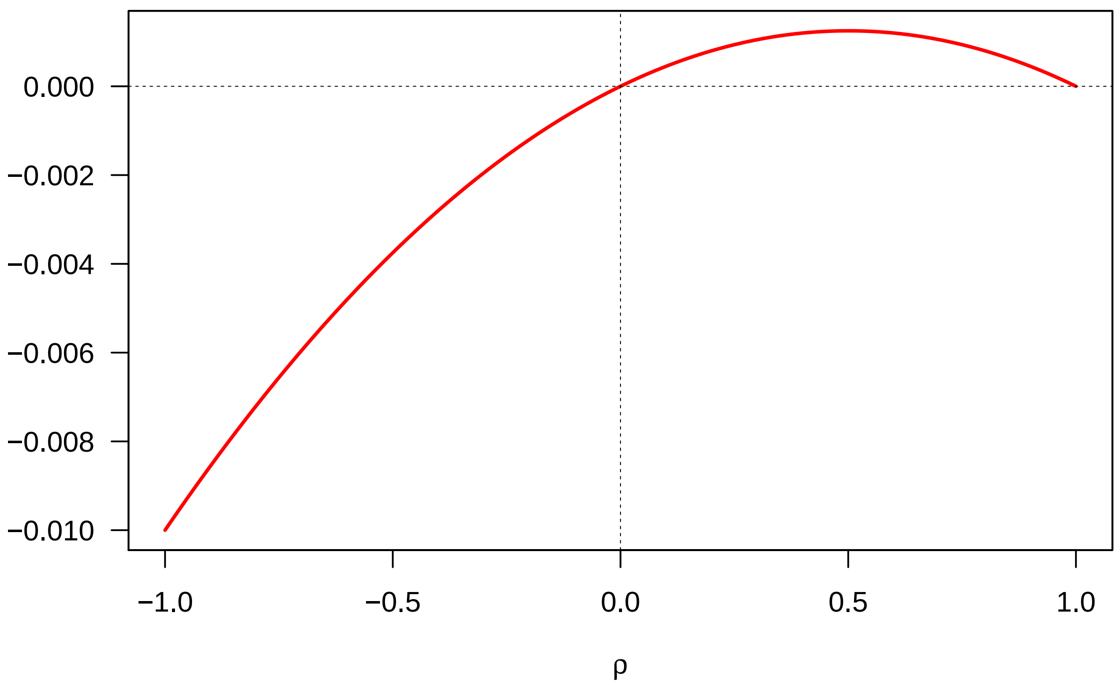

Figure 3.2: Portfolio Log Return: r_t vs ϱ with $\sigma = 0.1$ and $S_0 = S_t = 100$

Figure 3.2 summarize the results of this analysis for constant $\varrho \in [-1, 1]$ when the asset price has changed over the interval due to volitility but is equal at the end points $S_t = S_0$. Positive constant values of ϱ in the interval $(0, 1)$ reap a volatility benefit; whereas, constant negative values of ϱ in the interval $[-1, 0)$ inccur a volatility penalty.

Expanding equation 3.9 gives log return r_t as a function of the drift coefficient μ, diffusion coefficient σ, the position indicator ϱ_t, and the underlying Brownian motion W_t.

$$
\begin{aligned}
r_t &= \int_0^t \varrho_u \left[\left(\mu - \frac{\sigma^2}{2}\right) du + \sigma dW_u\right] + \frac{\sigma^2}{2}\int_0^t \varrho_u \left(1 - \varrho_u\right) du \\
&= \int_0^t \left[\left(\mu - \frac{\sigma^2}{2}\right) \varrho_u + \frac{\sigma^2}{2}\varrho_u \left(1 - \varrho_u\right)\right] du + \sigma \int_0^t \varrho_u \, dW_u \\
&= \int_0^t \left(\mu - \frac{\sigma^2}{2}\varrho_u\right) \varrho_u \, du + \sigma \int_0^t \varrho_u \, dW_u \qquad (3.15)
\end{aligned}
$$

3.4 Expected Log Returns r_t

In this section we develop a closed form solution for the expected log returns of the strategy. The full position indicator from definition 3.1.2 and equation 2.51 is

$$
\begin{aligned}
\varrho_t &= c\left[\left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) + \sigma \int_0^t \left(e^{-\frac{1}{\tau_2}(t-u)} - e^{-\frac{1}{\tau_1}(t-u)}\right) \, dW_u\right] \qquad (3.16) \\
&= c\left[\Theta + \sigma \int_0^t \Gamma_{t,u} dW_u\right] \qquad (3.17)
\end{aligned}
$$

The expected log returns of the strategy using the indicator function ϱ_t is calculated by taking the expected value of equation 3.15.

$$
\begin{aligned}
\mathbb{E}\left[r_t\right] &= \mathbb{E}\left[\int_0^t \left(\mu - \frac{\sigma^2}{2}\varrho_u\right) \varrho_u \, du + \sigma \int_0^t \varrho_u \, dW_u\right] \\
&= \mathbb{E}\left[\mu \int_0^t \varrho_u \, du\right] - \mathbb{E}\left[\frac{\sigma^2}{2} \int_0^t \varrho_u^2 \, du\right] + \cancelto{0}{\mathbb{E}\left[\sigma \int_0^t \varrho_u \, dW_u\right]} \\
&= \mathbb{E}\left[\mu \int_0^t \varrho_u \, du\right] - \mathbb{E}\left[\frac{\sigma^2}{2} \int_0^t \varrho_u^2 \, du\right] \qquad (3.18)
\end{aligned}
$$

The expected value of the first term in equation 3.18.

$$
\begin{aligned}
\mathbb{E}\left[\mu \int_0^t \varrho_u \, du\right] &= \mu \, \mathbb{E}\left[\int_0^t c\left(\Theta + \sigma \int_0^u \Gamma_{u,v} dW_v\right) \, du\right] \\
&= c\mu \, \Theta \, \mathbb{E}\left[\int_0^t du\right] + c\mu\sigma \int_0^t \cancelto{0}{\mathbb{E}\left[\int_0^u \Gamma_{u,v} dW_v\right]} du \\
&= c\mu \left(\mu - \frac{\sigma^2}{2}\right)(\tau_2 - \tau_1) \, t \qquad (3.19)
\end{aligned}
$$

The expected value of the second term in equation 3.18.

$$
\begin{aligned}
\mathbb{E}\left[\frac{\sigma^2}{2}\int_0^t \varrho_u^2\,du\right] &= \frac{\sigma^2}{2}\mathbb{E}\left[\int_0^t c^2\left(\Theta+\sigma\int_0^u \Gamma_{u,v}dW_v\right)^2 du\right] \\
&= \frac{c^2\sigma^2}{2}\mathbb{E}\left[\int_0^t \left(\Theta^2+2\Theta\sigma\int_0^u \Gamma_{u,v}dW_v+\left(\sigma\int_0^u \Gamma_{u,v}dW_v\right)^2\right) du\right] \\
&= \frac{c^2\sigma^2}{2}\mathbb{E}\left[\int_0^t \Theta^2 du\right]+c^2\Theta\sigma^3\mathbb{E}\left[\int_0^t\int_0^u \Gamma_{u,v}dW_v du\right] \\
&\quad+\frac{c^2\sigma^4}{2}\mathbb{E}\left[\int_0^t\left(\int_0^u \Gamma_{u,v}dW_v\right)^2 du\right] \\
&= \frac{c^2\sigma^2}{2}\left(\mu-\frac{\sigma^2}{2}\right)^2(\tau_2-\tau_1)^2\,\mathbb{E}\left[\int_0^t du\right] \\
&\quad+c^2\Theta\sigma^3\int_0^t \cancelto{0}{\mathbb{E}\left[\int_0^u \Gamma_{u,v}dW_v\right]}du \\
&\qquad+\frac{c^2\sigma^4}{2}\int_0^t \mathbb{E}\left[\left(\int_0^u \Gamma_{u,v}dW_v\right)^2\right] du \\
&= \frac{c^2\sigma^2}{2}\left(\mu-\frac{\sigma^2}{2}\right)^2(\tau_2-\tau_1)^2\,t+\frac{c^2\sigma^4}{2}\int_0^t\int_0^u \Gamma_{u,v}^2 dv\,du
\end{aligned}
\tag{3.20}
$$

The value of the last term in equation 3.20.

$$
\begin{aligned}
\frac{c^2\sigma^4}{2}\int_0^t\int_0^u \Gamma_{u,v}^2 dv\,du &= \frac{c^2\sigma^4}{2}\int_0^t\int_0^u\left(e^{-\frac{1}{\tau_2}(u-v)}-e^{-\frac{1}{\tau_1}(u-v)}\right)^2 dv\,du \\
&= \frac{c^2\sigma^4}{2}\int_0^t\int_0^u\left(e^{-\frac{2}{\tau_2}(u-v)}-2e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)(u-v)}+e^{-\frac{2}{\tau_1}(u-v)}\right) dv\,du
\end{aligned}
\tag{3.21}
$$

Breaking out each integral in equation 3.21 and solving.

$$\begin{aligned}\frac{c^2\sigma^4}{2}\int_0^t\int_0^u e^{-\frac{2}{\tau_2}(u-v)}dv\,du &= \frac{c^2\sigma^4}{2}\int_0^t e^{-\frac{2}{\tau_2}u}\int_0^u e^{\frac{2}{\tau_2}v}dv\,du\\ &= \frac{c^2\sigma^4}{2}\int_0^t e^{-\frac{2}{\tau_2}u}\left[\frac{\tau_2}{2}e^{\frac{2}{\tau_2}v}\right]_0^u du\\ &= \frac{\tau_2 c^2\sigma^4}{4}\int_0^t e^{-\frac{2}{\tau_2}u}\left(e^{\frac{2}{\tau_2}u}-1\right)du\\ &= \frac{\tau_2 c^2\sigma^4}{4}\int_0^t \left(1-e^{-\frac{2}{\tau_2}u}\right)du\\ &= \frac{\tau_2 c^2\sigma^4}{4}\left(t-\left[-\frac{\tau_2}{2}e^{-\frac{2}{\tau_2}u}\right]_0^t\right)\\ &= \frac{\tau_2 c^2\sigma^4}{4}\left(t+\frac{\tau_2}{2}\left(e^{-\frac{2}{\tau_2}t}-1\right)\right) \end{aligned} \tag{3.22}$$

$$\begin{aligned}c^2\sigma^4\int_0^t\int_0^u \left(e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)(u-v)}\right)dv\,du &= c^2\sigma^4\int_0^t e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)u}\int_0^u e^{\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)v}dv\,du\\ &= c^2\sigma^4\int_0^t e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)u}\left[\frac{1}{\frac{1}{\tau_1}+\frac{1}{\tau_2}}e^{\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)v}\right]_0^u dv\,du\\ &= c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}\int_0^t e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)u}\left(e^{\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)u}-1\right)du\\ &= c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}\int_0^t\left(1-e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)u}\right)du\\ &= c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}\left(t-\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)u}\right]_0^t\right)\\ &= c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}\left(t+\frac{\tau_1\tau_2}{\tau_1+\tau_2}\left[e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)t}-1\right]\right)\end{aligned} \tag{3.23}$$

$$\frac{c^2\sigma^4}{2}\int_0^t\int_0^u e^{-\frac{2}{\tau_1}(u-v)}dv\,du = \frac{\tau_1 c^2\sigma^4}{4}\left(t+\frac{\tau_1}{2}\left(e^{-\frac{2}{\tau_1}t}-1\right)\right) \tag{3.24}$$

Substituting the expected value of the log returns from equations 3.19, 3.20, 3.21, 3.22, 3.23, and 3.24 back into equation 3.18.

$$\begin{aligned}\mathbb{E}\left[r_t\right] = c\mu\left(\mu-\frac{\sigma^2}{2}\right)\left(\tau_2-\tau_1\right)t - \frac{c^2\sigma^2}{2}\left(\mu-\frac{\sigma^2}{2}\right)^2\left(\tau_2-\tau_1\right)^2 t \\ -\frac{\tau_2 c^2\sigma^4}{4}\left(t+\frac{\tau_2}{2}\left(e^{-\frac{2}{\tau_2}t}-1\right)\right)+c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}\left(t+\frac{\tau_1\tau_2}{\tau_1+\tau_2}\left[e^{-\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)t}-1\right]\right) \\ -\frac{\tau_1 c^2\sigma^4}{4}\left(t+\frac{\tau_1}{2}\left(e^{-\frac{2}{\tau_1}t}-1\right)\right)\end{aligned} \tag{3.25}$$

As a check we note that the expected log returns is zero when t is zero $\mathbb{E}\left[r_t\right]_{t=0} = 0$. If we assume a steady state where $t = t_{ss}$ the exponential terms go to zero in equation 3.25.

$$\begin{aligned}\mathbb{E}\left[r_t\right]_{t>t_{ss}} &= c\mu\left(\mu-\frac{\sigma^2}{2}\right)\left(\tau_2-\tau_1\right)t - \frac{c^2\sigma^2}{2}\left(\mu-\frac{\sigma^2}{2}\right)^2\left(\tau_2-\tau_1\right)^2 t \\ &\quad -\frac{\tau_2 c^2\sigma^4}{4}\left(t-\frac{\tau_2}{2}\right)+c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}\left(t-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\right)-\frac{\tau_1 c^2\sigma^4}{4}\left(t-\frac{\tau_1}{2}\right) \qquad (3.26) \\ &= \left[c\mu\left(\mu-\frac{\sigma^2}{2}\right)\left(\tau_2-\tau_1\right) - \frac{c^2\sigma^2}{2}\left(\mu-\frac{\sigma^2}{2}\right)^2\left(\tau_2-\tau_1\right)^2\right. \\ &\qquad \left. -\frac{\tau_2 c^2\sigma^4}{4}+c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}-\frac{\tau_1 c^2\sigma^4}{4}\right]t \\ &\qquad +\frac{\tau_2^2 c^2\sigma^4}{8}-c^2\sigma^4\left(\frac{\tau_1\tau_2}{\tau_1+\tau_2}\right)^2+\frac{\tau_1^2 c^2\sigma^4}{8} \\ &= \left[c\mu\left(\mu-\frac{\sigma^2}{2}\right)\left(\tau_2-\tau_1\right) - \frac{c^2\sigma^2}{2}\left(\mu-\frac{\sigma^2}{2}\right)^2\left(\tau_2-\tau_1\right)^2\right. \\ &\qquad \left. -\frac{c^2\sigma^4}{4}\left(\tau_1+\tau_2\right)+c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}\right]t \\ &\qquad +\frac{c^2\sigma^4}{8}\left(\tau_2^2+\tau_1^2\right)-c^2\sigma^4\left(\frac{\tau_1\tau_2}{\tau_1+\tau_2}\right)^2 \qquad (3.27)\end{aligned}$$

Equation 3.27 is linear in t. Rewriting equation 3.27 in terms of τ_1 and τ_2 and holding t constant gives the expected returns as a function of τ_1 and τ_2.

$$\begin{aligned}
\mathbb{E}\left[r_t(\tau_1,\tau_2)\right]_{t>t_{ss}} &= \left[\frac{c^2\sigma^4}{8} - \frac{c^2\sigma^2}{2}\left(\mu - \frac{\sigma^2}{2}\right)^2 t\right]\tau_2^2 + \left[c\mu\left(\mu - \frac{\sigma^2}{2}\right) - \frac{c^2\sigma^4}{4}\right] t\,\tau_2 \\
&\quad + \left[\frac{c^2\sigma^4}{8} - \frac{c^2\sigma^2}{2}\left(\mu - \frac{\sigma^2}{2}\right)^2 t\right]\tau_1^2 - \left[c\mu\left(\mu - \frac{\sigma^2}{2}\right) + \frac{c^2\sigma^4}{4}\right] t\,\tau_1 \\
&\quad + c^2\sigma^2\left(\mu - \frac{\sigma^2}{2}\right)^2 t\,\tau_1\tau_2 + c^2\sigma^4 t\frac{\tau_1\tau_2}{\tau_1+\tau_2} - c^2\sigma^4\frac{\tau_1^2\tau_2^2}{\left(\tau_1+\tau_2\right)^2} \\
&= \frac{c^2\sigma^2}{2}\left[\frac{\sigma^2}{4} - \left(\mu - \frac{\sigma^2}{2}\right)^2 t\right]\tau_2^2 + \left[c\mu\left(\mu - \frac{\sigma^2}{2}\right) - \frac{c^2\sigma^4}{4}\right] t\,\tau_2 \\
&\quad + \frac{c^2\sigma^2}{2}\left[\frac{\sigma^2}{4} - \left(\mu - \frac{\sigma^2}{2}\right)^2 t\right]\tau_1^2 - \left[c\mu\left(\mu - \frac{\sigma^2}{2}\right) + \frac{c^2\sigma^4}{4}\right] t\,\tau_1 \\
&\quad + c^2\sigma^2\left(\mu - \frac{\sigma^2}{2}\right)^2 t\,\tau_1\tau_2 + c^2\sigma^4\frac{\tau_1\tau_2}{\tau_1+\tau_2}\left(t - \frac{\tau_1\tau_2}{\tau_1+\tau_2}\right) \qquad (3.28)
\end{aligned}$$

Equation 3.28 reveals the expected log returns are quadratic in τ_1 and τ_2.

3.5 Analysis of Portfolio SDE Solution

In this section we examine two configurations of the solution where the indicator function ϱ_t is either unscaled or scaled by the inverse of the standard deviation of ϱ_t.

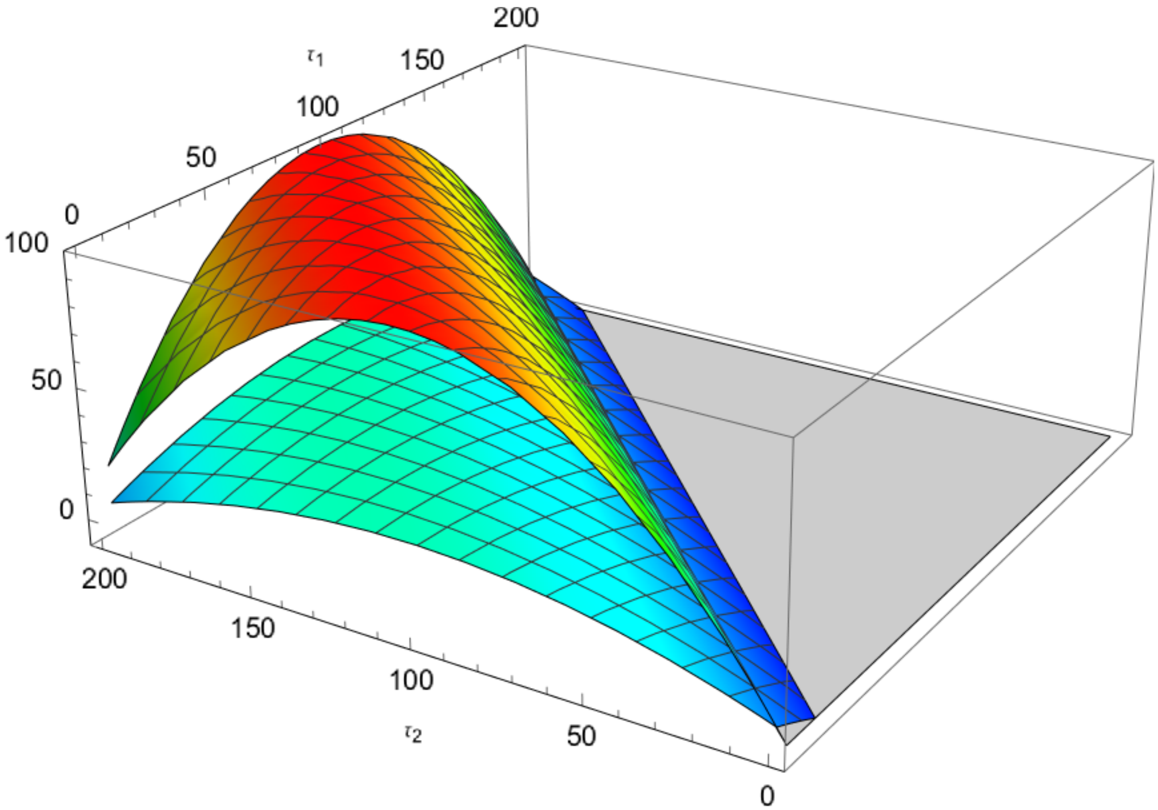

Figure 3.3: Plot of $\mathbb{E}\left[r_t(\tau_1, \tau_2)\right]_{t>t_{ss}}$ with $c = 1$, $\mu = 0.1$, $\sigma = 0.1$, Lower: $t = 50$, Upper: $t = 200$

Figure 3.3 shows the expected log returns as a function of τ_1 and τ_2 where $c = 1$, $\mu = 0.1$, and $\sigma = 0.1$ [2]. The lower surface is $t = 50$ and the upper surface is $t = 200$. The region of maximum return is parallel to the $x = y$ plane.

The partial derivatives of expected log returns with respect to τ_1 and τ_2.

$$\frac{\partial\, \mathbb{E}\left[r_t(\tau_1, \tau_2)\right]_{t>t_{ss}}}{\partial\, \tau_2} = c^2\sigma^2 \left[\frac{\sigma^2}{4} - \left(\mu - \frac{\sigma^2}{2}\right)^2 t\right] \tau_2 + \left[c\mu \left(\mu - \frac{\sigma^2}{2}\right) - \frac{c^2\sigma^4}{4}\right] t$$
$$+ c^2\sigma^2 \left(\mu - \frac{\sigma^2}{2}\right)^2 t\,\tau_1 + c^2\sigma^4 t \left(\frac{\tau_2}{\tau_1 + \tau_2}\right)^2 - 2c^2\sigma^4 \frac{\tau_1^3 \tau_2}{\left(\tau_1 + \tau_2\right)^3} \qquad (3.29)$$

[2] The Mathematica code used in plots and numerical calculations appears in the appendix. See appendix page 55

$$\frac{\partial\,\mathbb{E}\left[r_t(\tau_1,\tau_2)\right]_{t>t_{ss}}}{\partial\,\tau_1} = c^2\sigma^2\left[\frac{\sigma^2}{4}-\left(\mu-\frac{\sigma^2}{2}\right)^2 t\right]\tau_1 - \left[c\mu\left(\mu-\frac{\sigma^2}{2}\right)+\frac{c^2\sigma^4}{4}\right]t$$
$$+\,c^2\sigma^2\left(\mu-\frac{\sigma^2}{2}\right)^2 t\,\tau_2 + c^2\sigma^4 t\left(\frac{\tau_1}{\tau_1+\tau_2}\right)^2 - 2c^2\sigma^4\frac{\tau_1\tau_2^3}{(\tau_1+\tau_2)^3} \quad (3.30)$$

The roots of the partial derivitive equations 3.29 and 3.30 can then be used to determine the values of τ_1 and τ_2 which result in the maximum return for a given μ and σ. For example, when $\tau_1 = 10$, $c = 1$, $\mu = 0.1$, and $\sigma = 0.1$ the positive solution of $\frac{\partial\,\mathbb{E}[r_t(\tau_1,\tau_2)]_{t>t_{ss}}}{\partial\,\tau_2} = 0$ is about 115 which matches figure 3.3 well. The expected log returns $\frac{\mathbb{E}[r_t(\tau_1,\tau_2)]_{t>t_{ss}}}{t} \approx .49.9$ which is about five times the expected value of a buy and hold position over the same time frame, however the three standard deviation evelope around $\mathbb{E}\left[\varrho_t\right]$ implies that approximately an eight to twelve times leverage is required to achieve this result.

$$\varrho_t \in \left[\mathbb{E}\left[\varrho_t\right]-3\sqrt{\mathbb{V}\left[\varrho_t\right]},\mathbb{E}\left[\varrho_t\right]+3\sqrt{\mathbb{V}\left[\varrho_t\right]}\right] \approx [7.98, 11.96] \quad (3.31)$$

The drift coefficient μ only appears in the first two terms of the derivative of the expected log returns with respect to τ_2 in equation 3.29. The rest of the terms are written as K_1. The coefficients of μ are extracted from the first two terms and the remaining terms are written as K_2

$$\frac{\partial\,\mathbb{E}\left[r_t(\tau_1,\tau_2)\right]_{t>t_{ss}}}{\partial\,\tau_2} = c^2\sigma^2\left[\frac{\sigma^2}{4}-\left(\mu-\frac{\sigma^2}{2}\right)^2 t\right]\tau_2 + \left[c\mu\left(\mu-\frac{\sigma^2}{2}\right)-\frac{c^2\sigma^4}{4}\right]t + K_1$$
$$= \left(ct - c^2 t\sigma^2\tau_2\right)\mu^2 + \left(ct\sigma^4\tau_2 - ct\frac{\sigma^2}{2}\right)\mu + K_1 + K_2$$
$$= c\,t\left(1-\sigma^2\tau_2\right)\mu^2 + c\,t\sigma^2\left(\sigma^2\tau_2 - \frac{1}{2}\right)\mu + K_1 + K_2 \quad (3.32)$$

Equation 3.32 shows the partial derivative of the expected log returns with respect to τ_2 is quadratic with respect to μ which means a change of sign in μ should not change the solution significantly while holding all of the other parameters constant. Taking the partial derivative of equation 3.32 gives the second order mixed partial derivative the expected log returns first with respect to τ_2 then with respect to μ.

$$\frac{\partial\,\mathbb{E}\left[r_t(\tau_1,\tau_2)\right]_{t>t_{ss}}}{\partial\,\tau_2\partial\,\mu} = c\,t\left[2\left(1-\sigma^2\tau_2\right)\mu + \sigma^2\left(\sigma^2\tau_2 - \frac{1}{2}\right)\right] \tag{3.33}$$

This mixed partial tells us how sensitive an optimal value (zero) of τ_2 is to changes in μ. Given that both the slope of μ and the constant term grow linearly with respect to c, t, and τ_2 and they are approximate equal in magnitude but posite in sign changes in τ_2 due changes in μ should be small. Using the earlier example with $\tau_1 = 10$, $\tau_2 = 115$, $c = 1$, $\mu = 0.1$, and $\sigma = 0.1$ gives $\frac{\partial\,\mathbb{E}[r_t(\tau_1,\tau_2)]_{t>t_{ss}}}{\partial\,\tau_2\partial\,\mu} \approx -2.35$.

Next we try scaling ϱ_t by the inverse of the scaled standard deviation of ϱ to understand how that affects the solution. Let

$$\kappa(k,\sigma,\tau_1,\tau_2) = \frac{1}{k\sqrt{\mathbb{V}\left[\varrho_t\right]}} \tag{3.34}$$

Figure 3.4 is a plot of the solution with ϱ_t scaled by $c = \kappa(5,\mu,\sigma,\tau_1,\tau_2)$ where $\mu = 0.1$, $\sigma = 0.1$. Clearly the scaling factor κ improves the stability of the solution with respect to the choice of τ_1 and τ_2 by giving a large flat region in the area above the line $\tau_1 > 1.2\tau_2$ in the xy-plane at the cost of some return where $\frac{\mathbb{E}[r_t(\tau_1,\tau_2)]_{t>t_{ss}}}{t} \approx .393$. Given that

$$\begin{aligned}\mathbb{P}\left[\varrho_t \in \left[\mathbb{E}\left[\varrho_t\right] - 2\sqrt{\mathbb{V}\left[\varrho_t\right]}, \mathbb{E}\left[\varrho_t\right] + 2\sqrt{\mathbb{V}\left[\varrho_t\right]}\right]\right] &\approx \mathbb{P}\left[\varrho \in [0.258, 1.058]\right] \\ &\approx .954\end{aligned} \tag{3.35}$$

we consider further investigations of optimal solutions to this strategy to warranted considering various functions of the form investigated in Lorig, Zhou, and Zou 2019.

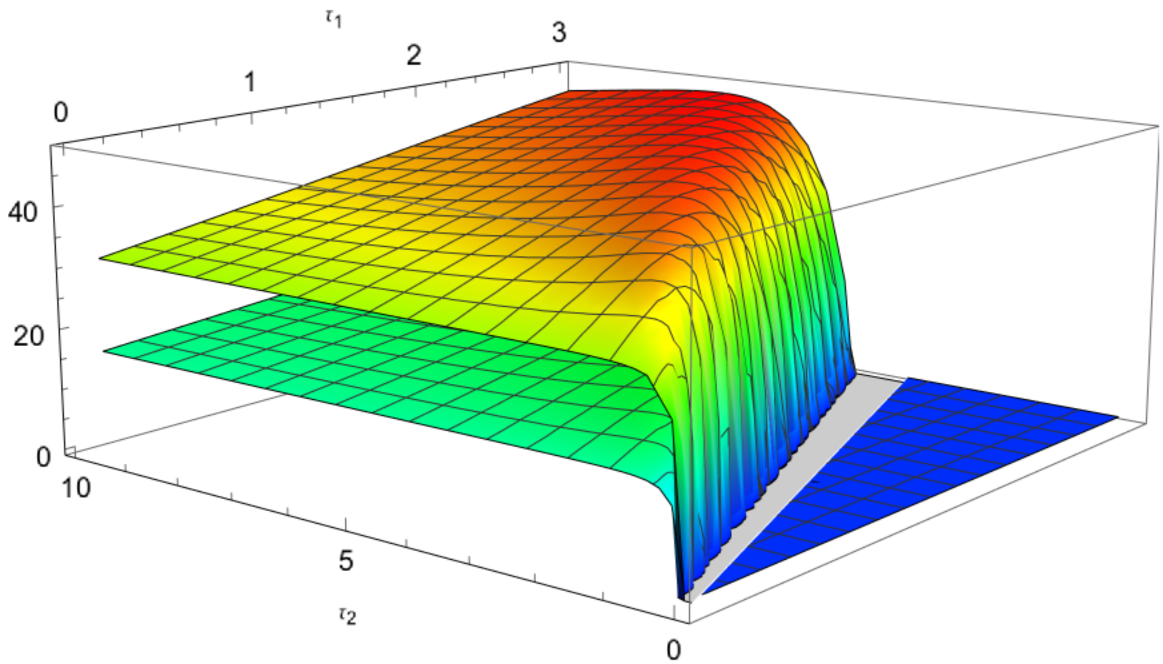

Figure 3.4: Plot of $\mathbb{E}\left[r_t(\tau_1, \tau_2)\right]_{t>t_{ss}}$ with $c = \kappa(5, .1, .1, \tau_1, \tau_2)$, $\mu = 0.1$, $\sigma = 0.1$, Lower: $t = 50$, Upper: $t = 200$

Chapter 4

Conclusion and Future Work

In our analysis of the application of the exponential filter time series operator to geometric Brownian motion we derived a simple closed form solution for both the single filter case and the the difference of two filters. We then calculated the mean and variance of both. The time decaying transients appear in both results. When considering long term behavior, they can be treated as insignificant. However, these transients need to be further analyzed to understand the implications for implementors of automated strategies employing such filters as they can have significant startup effects.

In our analysis of the portfolio SDE, we showed that given a constant position indicator $\varrho_t = \varrho$ a long position in the range between $(0, 1)$ gains a volatility benefit while a short position incurs a volatility penalty. Given this asymmetry between long and short positions, it may be worth considering a different portfolio equation that teats both long and short positions symmetrically. Given that short positions increase the amount of cash available in the account that can be invested in risk free assets, and that brokers typically treat investments in money market accounts or government bonds as counting towards the margin requirements of a short position, the version of the portfolio SDE which includes the risk free asset should also be explored.

The SDE assumptions about μ and σ being constant are unrealistic. Lorig, Zhou, and Zou 2019 found optimal solutions for linear algebraic functions when μ_t is either an Ornstein-Uhlenbeck process or a two state continuous time Markov chain and the diffusion coefficient is constant. Implications of both a time varying drift and diffusion coefficient and should be considered.

The portfolio SDE does not model the jumps that frequently occur between market open and close. Such a jump could cause the investor strategy (indicator function) to require a very large leveraged position.

The continual trading assumption of fractional shares is highly unrealistic. The minimum size for stocks trades is 100 shares if the order is to go directly to the market and not be held at the broker until a 100 share order can be assembled from multiple market participants. A very large initial portfolio Π_0 would be required to enable treating 100 shares as the unit quantity. Futures contracts can be traded in single units but typically represent a 20x to 100x leveraged position with a corresponding increase in risk. The strategy also does not consider trading costs which could be significant when the strategy is trading frequently in order to replicate continuous-time assumptions as closely as possible.

Bibliography

[20] *Indicator Reference: Exponential Moving Average*. FM Labs. 2020. URL: `https://www.fmlabs.com/reference/default.htm?url=ExpMA.htm` (cit. on p. 55).

[Ale61] Sidney S. Alexander. "Price Movements in Speculative Markets: Trends or Random Walks". In: *Industrial Management Review* 2 (May 1961), pp. 7–26 (cit. on p. 1).

[Ale64] Sidney S. Alexander. "Price Movements in Speculative Markets–Trends or Random Walks, Number 2". In: *Industrial Management Review* 5.2 (1964), p. 25 (cit. on p. 1).

[Bac00] Louis Bachelier. "The Theory of Speculation". Translated by D. May from Annales scientifiques de l'Ecole Normale Superieure, Ser.3, 17 (1900), p 21-86. PhD thesis. Academy of Paris, 1900 (cit. on p. 2).

[BR04] Nicole Bäuerle and Ulrich Rieder. "Portfolio Optimization With Markov-Modulated Stock Prices and Interest Rates". In: *IEEE Transactions on Automatic Control* 49.3 (2004) (cit. on p. 28).

[BS73] Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities". In: *Journal of Political Ecomon* (1973) (cit. on p. 3).

[Cou+00] Jean-Michel Courtault et al. "Louis Bachelier on the Centenary of the Theorie de la Speculation". In: *Mathematical Finance* 10.3 (July 2000), p 341–353 (cit. on p. 2).

[Dac+01] Michel M. Dacorogna et al. *An Introduction to High-Frequency Finance*. 2001 (cit. on pp. 3, 53).

[FB66] Eugene F Fama and Marshall E Blume. "Filter Rules and Stock-Market Trading". In: *The Journal of Business* 39.1 (Jan. 1966) (cit. on p. 1).

[Gar35] H. M. Gartley. *Profits in the Stock Market*. Lambert-Gann Publishing Co, Pomeroy WA, 1935 (cit. on p. 1).

[GS13] Denis S. Grebenkov and Jeremy Serror. "Following a trend with an exponential moving average:Analytical results for a Gaussian model". In: *Physica A* (2013) (cit. on p. 4).

[Itô44] K. Itô. "Stochastic Integral". In: *Proceedings of the Japan Academy* 20.8 (1944), pp. 519–524. URL: `https://projecteuclid.org/euclid.pja/1195572786` (cit. on p. 3).

[Kol31] A. N. Kolmogorov. „Uber die analytisehen Methoden in der Wahrseheinliehkeitsreehnung". Deutsch. In: *Mathematische Annalen* 104 (1931). Translated to Russian in **Kolmogorov1938**, S. 415–458 (siehe S. 2).

[Kol33] A. N. Kolmogorov. *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Deutsch. Translated to Russian in **Kolmogorov1936** and English in Kolmogorgov 1956. Verlag Von Julikus Springer, 1933 (siehe S. 2, 46).

[Kol56] A. N. Kolmogorgov. *Theory of Probability*. Trans. by Nathan Morrision. Originally published in German in Kolmogorov 1933. Chelsea Publishing Copmpany, 1956 (cit. on pp. 2, 46).

[LZZ19] Matthew Lorig, Zhou Zhou, and Bin Zou. "A Mathematical Analysis of Technical Analysis". In: *Applied Mathematical Finance* (2019) (cit. on pp. 4, 28, 41, 43).

[Man63] Benoit Mandelbrot. "The Variation of Certain Speculative Prices". In: *Journal of Business* 36.4 (Oct. 1963), pp. 394–419 (cit. on p. 1).

[Mer73] Robert C. Merton. "Theory of Rational Option Pricing". In: *The Bell Journal of Economics and Mangement Science* 4.1 (Spring 1973), pp. 141–183 (cit. on p. 3).

[NAS20] NASDAQ. *NASDAQ*. 2020. URL: `https://en.wikipedia.org/wiki/NASDAQ` (cit. on p. 3).

[Oks00] Bernt Oksendal. *Stochastic Differential Equations: An Introduction with Applications*. 5th. Springer-Verlag, 2000 (cit. on pp. 5, 9).

[Pet15] Brian G. Peterson. "Developing and Backtesting Systematic Trading Strategies". June 2015. URL: https://r-forge.r-project.org/scm/viewvc.php/*checkout*/pkg/quantstrat/sandbox/backtest_musings/strat_dev_process.pdf?root=blotter (cit. on p. 27).

[Shi11] A. N. Shiryaev. *Kolmogorov-Chapman equation*. 2011. URL: https://encyclopediaofmath.org/wiki/Kolmogorov-Chapman_equation (cit. on p. 2).

[Shr04] Steven E. Shreve. *Stochastic Calculus for Finance: Continous-Time Models*. Vol. II. Springer Finance Textbook. Springer, 2004 (cit. on pp. 3, 5, 6, 14, 15, 20, 21, 23).

[SV06] Glenn Shafer and Bladimir Vovk. "The Sources of Kolmogorov's Grundbegriffe". In: *Statistical Science* 21.1 (2006), pp. 70–98 (cit. on p. 3).

[ZM01] Gilles Zumbach and Ulrich Müller. "Operators on Inhomogeneous Time Series". In: *International Journal of Theoretical and Applied Finance* (2001) (cit. on pp. 3, 11).

Appendices

Asset Price Transition Density, Expectation and Variance

The conditional transition density of S_t is needed to extract the model parameters σ and μ using maximum likelihood methods from market data. Given the log form solution to the GBM SDE in equation 1.8

$$\ln S_t - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \tag{1}$$

and by assumption that $\mathbb{E}\left[W_t\right] = 0$, the expected value of the log change in the price from time 0 to t, is equal to

$$\begin{aligned} \mathbb{E}\left[\ln S_t - \ln S_0\right] &= \mathbb{E}\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right] \\ &= \mathbb{E}\left[\left(\mu - \frac{\sigma^2}{2}\right) t\right] + \sigma \cancelto{0}{\mathbb{E}\left[W_t\right]} \\ &= \left(\mu - \frac{\sigma^2}{2}\right) t \end{aligned} \tag{2}$$

Similarly the variance of log change in price from time 0 to t given by definition $\mathbb{V}\left[W_t\right] = t - 0 = t$ is equal to

$$\begin{aligned} \mathbb{V}\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right] &= \cancelto{0}{\mathbb{V}\left[\left(\mu - \frac{\sigma^2}{2}\right) t\right]} + \sigma^2 \mathbb{V}\left[W_t\right] \\ &= \sigma^2 t \end{aligned} \tag{3}$$

Let

$$\begin{aligned} X_t &= \ln S_t \\ x_0 &= \ln S_0 \\ \eta &= \left(\mu - \frac{\sigma^2}{2}\right) t \\ \varsigma^2 &= \sigma^2 t \end{aligned} \tag{4}$$

By assumption the transition probability of W_t is normal; thus, at time t the log price X_t is normal with mean $x_0 + m$ due to the determistic drift component of the price and the variance is ς^2 due to the stochastic volatility component

$$X_t \sim \mathcal{N}\left[x_0 + \eta, \varsigma^2\right] \tag{5}$$

and the conditional density function for X_t given x_0 can be written.

$$f_{X_t|x_0}(x|x_0) = \frac{1}{\varsigma\sqrt{2\pi}} e^{-\frac{[x-(x_0+\eta)]^2}{2\varsigma^2}} \tag{6}$$

From basic probability theory the conditional expectation of a function of a random variable, in this case $g(X_t) = e^{X_t}$, can be written as the integral of the function of the random variable and the density over the domain of the function.

$$\begin{aligned}\mathbb{E}\left[S_t|S_0\right] &= \mathbb{E}\left[e^{X_t}\middle|x_0\right] \\ &= \int_{-\infty}^{\infty} e^x f_{X_t|x_0}(x|x_0)dx \\ &= \frac{1}{\varsigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^x e^{-\frac{[x-(x_0+\eta)]^2}{2\varsigma^2}} dx \end{aligned} \tag{7}$$

Combining the exponents of the integrand in equation 7, completing the square and simplifying gives.

$$x - \frac{[x-(x_0+\eta)]^2}{2\varsigma^2} = (x_0 + m) + \frac{\varsigma^2}{2} - \frac{\left[x-(x_0+\eta+\varsigma^2)\right]^2}{2\varsigma^2} \tag{8}$$

Replacing the exponents in the integrand of equation 7 with the results in equation 8 gives the expected value of e^{X_t}.

$$\begin{aligned}
\mathbb{E}\left[S_t|S_0\right] &= \mathbb{E}\left[e^{X_t}\middle|x_0\right] \\
&= e^{(x_0+\eta)+\frac{\varsigma^2}{2}} \cancelto{1}{\frac{1}{\varsigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\left[x-(x_0+\eta+\varsigma^2)\right]^2}{2\varsigma^2}} dx} \\
&= e^{\ln S_0+\left(\mu-\frac{\sigma^2}{2}\right)t+\frac{\sigma^2}{2}t} \\
&= S_0 e^{\mu t} \qquad (9)
\end{aligned}$$

Similarly we will need the expected value of $\left(e^{X_t}\right)^2$

$$\begin{aligned}
\mathbb{E}\left[\left(e^{X_t}\right)^2\middle|x_0\right] &= \int_{-\infty}^{\infty} e^{2x} f_{X_t|x_0}(x|x_0)dx \\
&= \frac{1}{\varsigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2x} e^{-\frac{[x-(x_0+\eta)]^2}{2\varsigma^2}} dx \qquad (10)
\end{aligned}$$

Combining the exponents of the integrand in equation 10, completing the square and simplifying gives.

$$2x - \frac{[x-(x_0+\eta)]^2}{2\varsigma^2} = 2\left[(x_0+\eta)+\varsigma^2\right] - \frac{\left[x-(x_0+\eta+2\varsigma^2)\right]^2}{2\varsigma^2} \qquad (11)$$

Replacing the exponents in the integrand of equation 10 with the results in equation 11 gives the expected value of $\left(e^{X_t}\right)^2$.

$$\begin{aligned}
\mathbb{E}\left[\left(e^{X_t}\right)^2\middle|x_0\right] &= e^{2\left[(x_0+\eta)+\varsigma^2\right]} \cancelto{1}{\frac{1}{\varsigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\left[x-(x_0+\eta)+2\varsigma^2\right]^2}{2\varsigma^2}} dx} \\
&= e^{2\left[\ln S_0+\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma^2 t\right]} \\
&= S_0^2 e^{2\left(\mu+\frac{\sigma^2}{2}\right)t} \qquad (12)
\end{aligned}$$

The variance is calculated using the well known expansion of the variance as the difference of the expected value of the square and the square of the expected value and inserting the values from equations 9 and 12.

$$
\begin{aligned}
\mathbb{V}\left[S_t|S_0\right] &= \mathbb{V}\left[e^{X_t}|x_0\right] \\
&= \mathbb{E}\left[\left(e^{X_t}\right)^2\Big|x_0\right] - \left(\mathbb{E}\left[e^{X_t}|x_0\right]\right)^2 \\
&= e^{2\left[(x_0+\eta)+\varsigma^2\right]} - e^{2\left[(x_0+\eta)+\frac{\varsigma^2}{2}\right]} \\
&= e^{2\left[\ln S_0+\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma^2 t\right]} - e^{2\left[\ln S_0+\left(\mu-\frac{\sigma^2}{2}\right)t+\frac{\sigma^2}{2}t\right]} \\
&= S_0^2 e^{2\left(\mu+\frac{\sigma^2}{2}\right)t} - S_0^2 e^{2\mu t} \\
&= S_0^2 e^{2\mu t}\left(e^{\sigma^2 t}-1\right) \\
&= \left(S_0 e^{\mu t}\right)^2\left(e^{\sigma^2 t}-1\right)
\end{aligned} \tag{13}
$$

Combining equations 9 and 13 gives log normal distribution parameters for geometric Brownian motion transition density.

$$
S_t \sim \mathcal{LN}\left[S_0 e^{\mu t}, S_0^2 e^{2\mu t}\left(e^{\sigma^2 t}-1\right)\right] \tag{14}
$$

Using the definition of the cumulative conditional distribution function and the condition density function for the log asset price in equation 6, the condition density function is derived.

$$
\begin{aligned}
F(x) &= \mathbb{P}\left[S_t < u|S_0\right] \\
&= \mathbb{P}\left[e^{X_t} < u|e^{x_0}\right] \\
&= \mathbb{P}\left[X_t < \ln u|x_0\right] \\
&= \int_{-\infty=\ln 0}^{\ln u} f_{X_t|x_0}(x|x_0)dx \\
&= \frac{1}{\varsigma\sqrt{2\pi}}\int_{\ln 0}^{\ln u} e^{-\frac{[x-(x_0+\eta)]^2}{2\varsigma^2}}dx \\
&= \frac{1}{\sigma\sqrt{2\pi t}}\int_0^u e^{-\frac{\left[\ln v-\ln S_0-\left(\mu-\frac{\sigma^2}{2}\right)t\right]^2}{2\sigma^2 t}}\frac{1}{v}dv
\end{aligned}
$$

which implies

$$f(s)_{S_t|s_0} = \frac{1}{\sigma s\sqrt{2\pi t}} e^{-\frac{\left[\ln s - \ln s_0 - \left(\mu - \frac{\sigma^2}{2}\right)t\right]^2}{2\sigma^2 t}} \tag{15}$$

Digital Exponential Time Series Operator

In this section the homogeneous discrete time version of the continuous time exponential filter time series operator is derived from definition 2.1.1 based on Dacorogna et al. 2001. Let $\Delta t = t_i - t_{i-1}$ and assume the X_t is constant during the i^{th} time step of length Δt denoted as x_i. At the first time step

$$\begin{aligned} y_0 &= \int_{t_0}^{t_1} \frac{1}{\tau} e^{-\frac{1}{\tau}(t_1 - u)} X_u du \\ &= e^{-\frac{t_1}{\tau}} x_0 \int_{t_0}^{t_1} \frac{1}{\tau} e^{\frac{u}{\tau}} du \\ &= e^{-\frac{t_1}{\tau}} x_0 \frac{1}{\tau} \left[\tau e^{\frac{u}{\tau}}\right]_{t_0}^{t_1} \\ &= e^{-\frac{t_1}{\tau}} x_0 \left(e^{\frac{t_1}{\tau}} - e^{\frac{t_0}{\tau}}\right) \\ &= \left(1 - e^{-\frac{t_1 - t_0}{\tau}}\right) x_0 \\ &= \left(1 - e^{-\frac{\Delta t}{\tau}}\right) x_0 \\ &= \alpha x_0 \end{aligned} \tag{16}$$

where

$$\alpha = 1 - e^{-\frac{\Delta t}{\tau}} \tag{17}$$

which implies

$$1 - \alpha = e^{-\frac{\Delta t}{\tau}} \tag{18}$$

At the end of the second time step, the output from the first time-step y_0 scaled by $(1-\alpha)$ is added to scaled version of the current input x_1.

$$\begin{aligned} y_1 &= \alpha x_1 + (1-\alpha)y_0 \\ &= \alpha x_1 + (1-\alpha)(\alpha x_0) \\ &= \alpha x_1 + \alpha(1-\alpha)x_0 \end{aligned}$$

The third time step the process iterates as in the second time-step.

$$\begin{aligned} y_2 &= \alpha x_2 + (1-\alpha)y_1 \\ &= \alpha x_2 + (1-\alpha)(\alpha x_1 + \alpha(1-\alpha)x_0) \\ &= \alpha x_2 + \alpha(1-\alpha)x_1 + \alpha(1-\alpha)^2 x_0 \\ &= \alpha\left[(1-\alpha)^0 x_2 + (1-\alpha)^1 x_1 + (1-\alpha)^2 x_0\right] \end{aligned} \tag{19}$$

The n^{th} time step.

$$\begin{aligned} y_n &= \alpha\left[(1-\alpha)^0 x_n + (1-\alpha)^1 x_{n-1} + \cdots + (1-\alpha)^{n-1} x_1 + (1-\alpha)^n x_0\right] \\ &= \alpha \sum_{i=0}^{n} (1-\alpha)^{n-i} x_i \\ \operatorname{ema}(x_n) &= \left(1 - e^{-\frac{\Delta t}{\tau}}\right) \sum_{i=0}^{n} e^{-\frac{\Delta t}{\tau}(n-i)} x_i \end{aligned} \tag{20}$$

The relationship between α and τ (*Indicator Reference: Exponential Moving Average* 2020).

$$\begin{aligned} \alpha &= 1 - e^{-\frac{\Delta t}{\tau}} \\ 1 - \alpha &= e^{-\frac{\Delta t}{\tau}} \\ \ln(1-\alpha) &= -\frac{\Delta t}{\tau} \\ \tau &= -\frac{\Delta t}{\ln(1-\alpha)} \end{aligned} \tag{21}$$

For a k period discrete exponential filter time series operator $\alpha = \frac{2}{k+1}$.

$$\tau = -\frac{\Delta t}{\ln\left(1 - \frac{2}{k+1}\right)} \tag{22}$$

Note that τ has the same units as Δt.

Mathematica Code

In this section the follow variables are used in Mathematica code.

Symbol	Variable	Meaning
t	t	time
c	c	scaling constant
μ	m	drift coefficient
σ	s	diffusion coefficient
τ_1	t1	filter time coefficient one
τ_2	t2	filter time coefficient two

Portfolio Expected Log Retuns

$\mathbb{E}[r_t]$

```
Ert[t_, c_, m_, s_, t1_, t2_] :=
```

```
c*m*(m - s^2/2)*(t2 - t1)*t - (c^2*s^2/2)*(m - s^2/2)^2*(t2 - t1)^2*t
- (t2*c^2*s^4/4)*(t + (t2/2)*(Exp[-2*t/t2] - 1))
+ c^2*s^4*((t1*t2)/(t1 + t2))*(t + ((t1*t2)/(t1 + t2))*(Exp[-(1/t1 + 1/t2)] - 1))
- (t1* c^2*s^4/4)*(t + (t1/2)*(Exp[-2*t/t1] - 1))
```

Portfolio Expected Log Retuns (Steady State)

$\mathbb{E}\left[r_t(\tau_1, \tau_2)\right]_{t>t_{ss}}$

```
Ertss[t_, c_, m_, s_, t1_, t2_] :=
  (c*m*(m - s^2/2)*(t2 - t1) - (c^2*s^2/2)*((m - s^2/2)^2)*(t2 - t1)^2
  - (t2*c^2*s^4/4) + c^2*s^4*((t1*t2)/(t1 + t2)) - (t1*c^2*s^4/4))*t
  + (t2^2*c^2*s^4/8) - c^2*s^4*((t1*t2)/(t1 + t2))^2 + (t1^2*c^2*s^4/8)
```

Derivative of Expected Log Retuns with Respect to τ_1 (Steady State)

$\frac{\partial\, \mathbb{E}[r_t(\tau_1,\tau_2)]_{t>t_{ss}}}{\partial\, \tau_1}$

```
DErt1[t_, c_, m_, s_, t1_, t2_] :=
  c^2*s^2*(s^2/4 - (m - s^2/2)^2*t)*t1
  - (c*m*(m - s^2/2) + c^2*s^4/4)*t
  + c^2*s^2*(m - s^2/2)^2*t*t2
  + c^2*s^4*t*(t2/(t1 + t2))^2
  - 2*c^2*s^4*((t1*t2^3)/(t1 + t2)^3)
```

Derivative of Expected Log Retuns with Respect to τ_2 (Steady State)

$\frac{\partial\, \mathbb{E}[r_t(\tau_1,\tau_2)]_{t>t_{ss}}}{\partial\, \tau_2}$

```
DErt2[t_, c_, m_, s_, t1_, t2_] :=
  c^2*s^2*(s^2/4 - (m - s^2/2)^2*t)*t2
  + (c*m*(m - s^2/2) - c^2*s^4/4)*t
  + c^2*s^2*(m - s^2/2)^2*t*t1
  + c^2*s^4*t*(t1/(t1 + t2))^2
```

```
- 2*c^2*s^4*((t1^3*t2)/(t1 + t2)^3)
```